C. Radhakrishna Rao: A Century in Statistical Science

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Abstract

In this article, the authors provide a glimpse into the legendary career of centenarian Professor C. Radhakrishna Rao, a career that is inextricably linked to the history of modern statistics from Pearson and Fisher to the age of big data and Artificial Intelligence. A scientist of extraordinary vision, his ground-breaking contributions have influenced not only the field of statistics, but the mathematical, social, biological, and economic sciences. An outstanding researcher, a gifted and inspirational teacher who could make complex concepts come to life in a classroom, and a mentor who demanded excellence from his students, Professor Rao continues to inspire generations of statisticians and scientists around the world.

1 Initial Years: 1941-1950

Calyampudi Radhakrishna Rao was born on September 10, 1920 in Huvanna Hadagali in what was then the Madras Presidency, and is now a town in the Indian state of Karnataka. As the eighth of ten children, he was named Radhakrishna after the Hindu deity Krishna, also the eight-born according to Hindu mythology. His father, C. D. Naidu, an Inspector in the Criminal Investigation Department, recognized Rao's gift for mathematics and encouraged him to pursue a career in research. His mother instilled the discipline and work ethic that have stayed with him throughout his extraordinary career. Rao's dedication in his book Statistics and Truth: Putting Chance to Work [43], acknowledges his mother's influence on his life: *"For instilling in me the quest for*

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knowledge, I owe to my mother, A. Laxmikanthamma, who, in my younger days, woke me up every day at four in the morning and lit the oil lamp for me to study in the quiet hours of the morning when the mind is fresh."

Rao obtained the equivalent of a Master's degree in Mathematics from Andhra University in 1940, graduating with top honours. With jobs in Mathematics hard to find because of the war, Rao decided to apply for a position with the military in the Department of Survey. Rao travelled to Calcutta (now Kolkata) for the interview, a journey that would completely change the course of his career.

In what can only be described as a moment of serendipity, Rao struck up a conversation with a young man who had travelled to Calcutta to enroll in a training program in statistics at the Indian Statistical Institute (ISI). After meeting the research staff and learning about the many ongoing projects, Rao was convinced that the training program would not only enhance his job prospects, but also provide an opportunity to pursue research. In an interview published by the Indian Academy of Sciences, *Wise Decisions Under Uncertainty* [6], Rao writes "I went back to Visakhapatnam and told my mother that the only alternative for me was to get admitted to ISI for training in statistics, and it would cost me Rs.30 a month to stay in Kolkata. She said that she would raise the money somehow and that I should go to Kolkata to join ISI. I travelled to Kolkata with Rs.30 in my pocket and joined ISI on 1st January 1941."

Figure 1: Rao at ages 15 and 19

2 Mahalanobis and ISI – Early Influences

No biography of Rao can be complete without reference to one of the most important and influential figures in his life – the legendary scientist, mathematician, and statistician Prasanta Chandra Mahalanobis, known in ISI as "Professor". While at Cambridge studying physics, Mahalanobis came across a copy of Biometrika, the journal founded by Karl Pearson with Francis Galton and Raphael Weldon. Statistics was emerging as a new discipline, and Mahalanobis was quick to recognize its potential for significant impact in many application domains. Mahalanobis returned to India and started the Statistical Laboratory at Presidency College, eventually leading to the establishment of the Indian Statistical Institute (ISI) in 1931. The Institute offered a training program in statistics, the first of its kind in India, attracting students from different disciplines as well as government officials interested in using statistical techniques for data analysis and modelling.

While Rao found the program lacking rigor and the teaching to be "somewhat disorganized", he was fortunate to interact with the outstanding research faculty at ISI which at the time included R. C. Bose, S. N. Roy, K. R. Nair, and A. Bhattacharya. Rao's first paper with K. R. Nair [24] on confounded designs was published in 1941 within a few months of his joining ISI! As Rao was completing his training program, Calcutta University launched India's first Master's program in statistics with Mahalanobis selected to head the new department. Mahalanobis recognized Rao's incredible talent and encouraged him to register for the program. Rao received the M.A. degree in statistics standing first in his class with a score of 87%, a record at Calcutta University that remains unbroken until this day.

Rao's Master's thesis was by all accounts, an extraordinary piece of work for a 23 year old. In an interview with Anil Bera [5], Rao writes "Looking through my thesis, forwarded by Professor P.C. Mahalanobis on June 18, 1943, to the controller of examinations of Calcutta University, I find that I must have been working hard during the period 1941- 1943. The thesis was in three parts, the first with 119 pages on design of experiments, the second with 28 pages on multivariate tests, and the third with 42 pages on bivariate distributions."

One of the reviewers of his thesis would later remark that the work was "almost equivalent to a Ph.D. degree." Rao's thesis contained original contributions, including a solution to a characterization problem formulated by Ragnar Frisch, the Norwegian economist. The solution would be published in 1947 in *Econometrica* [33].

After completing his Master's degree, Rao joined ISI as a Technical Apprentice, conducting research and teaching part-time at Calcutta University. Rao credits his early research success to two of his mentors, the eminent statisticians R. C. Bose and S. N. Roy. Referring to the period from 1944 – 1946 as the "most eventful of his research career", Rao would publish articles in a number of different areas including combinatorics in experimental design (with R. C. Bose), linear estimation, and multivariate analysis.

2.1 1945- The Breakthrough Article that Created the Legend

Rao's 1945 paper "Information and accuracy attainable in the estimation of statistical parameters", Bulletin of the Calcutta Mathematical Society, Vol.37, No.3, pp.81–91, 1945 [28] would cement his legacy as one of the legends of modern statistics. Simple in its exposition with elegant proofs, the ten-page article presented two foundational results in statistical inference and planted the seeds that would grow into the field of information geometry.

Cramér- Rao Inequality-the Information Bound

In the early 1920's, R.A. Fisher [15], [16] introduced the concept of Fisher Information, $I(\theta)$, a measure of the amount of information that a random sample $X_1, \ldots, X_n$ contains about an underlying population parameter θ. $I(\theta)$ measures the expected curvature of the log-likelihood function around θ; the more curved the log-likelihood, the more information the data provide about the parameter. Fisher proved the information inequality that established a lower bound for the asymptotic variance of a consistent estimator in the context of maximum likelihood estimation.

In his 1945 paper, Rao proved that any unbiased estimator of a parameter has a variance that is bounded below by the reciprocal of the Fisher information. The result obtained using a simple application of the Cauchy-Schwarz inequality would become one of the most recognized theorems in Statistics. Unbeknownst to Rao, the Swedish mathematician, Harold Cramér, independently established the information inequality which would appear in his 1946 book, Mathematical Methods of Statistics [8]. While similar results were also established by Darmois and Frechet, the inequality would be known as the Cramér- Rao bound.

The Cramér- Rao lower bound (CRLB) is ubiquitous in the signal processing literature and has found applications in almost every field of science and engineering. There are numerous extensions of the result including the quantum CRLB [18] and the Bayesian CRLB [18]. Dembo, Cover and Thomas [10] provide a review of various inequalities in information theory and their connections to inequalities found in other areas of mathematics and physics. The authors illustrate how the Weyl-Heisenberg uncertainty principle can be derived from the Cramér- Rao inequality.

Figure 2: Rao with Cramér (left) and Blackwell (right)

Rao-Blackwell Theorem

The second fundamental result established in the 1945 paper involved the use of sufficient statistics to improve the efficiency of an estimator. Rao writes "if a sufficient statistic and an unbiased estimate exist for θ, then the best unbiased estimate of θ is an explicit function of the sufficient statistic." Formally, the result states that if $g(X)$ is an estimator of an unknown parameter θ, then the estimator constructed by taking the conditional expectation of $g(X)$ given the sufficient statistic $T(X)$ has a smaller mean squared error (MSE). David Blackwell [7] proved the same result independently in a 1947 paper. The result is known as the Rao-Blackwell theorem, and the process of replacing an estimator by its conditional expectation is referred to as Rao-Blackwellization. The classic result has found a new home in modern applications. Doucet et al. [11] consider the use of Rao-Blackwellisation to improve the efficiency of particle filtering for different dynamic Bayesian networks. Robert and Roberts [47] discuss the application of Rao-Blackwellization in the context of Gibbs sampling and in a more general MCMC framework.

Differential Geometry and Statistics: the Genesis of Information Geometry

Rao's 1945 paper introduced the notion of distance or divergence between probability distributions parameterized by a q-dimensional vector θ. The paper was one of the earliest to apply differential geometric approaches to probability models, creating the framework for the new field of Information Geometry. Rao viewed the parametric family as a Riemannian manifold with the Fisher Information Matrix (FIM) as the associated Riemannian metric tensor. He proposed the geodesic distance induced by this Riemannian metric as a measure of dissimilarity between two probability distributions. The Riemannian geodesic metric distance has been named the Fisher–Rao distance [26]. The quadratic differential metric has some desirable properties including invariance to transformation of the variables as well as the parameters.

It would be the 1970's before differential geometric methods made their way into the mainstream statistical literature with the publication of Efron's article on statistical curvature [12], followed by the monographs of Amari [1] and Amari et al. [3]. In recent years, there has been tremendous interest in Information Geometry and its applications to Optimization, Machine Learning, and Deep Learning [11]. Amari [2] provides an overview of the field and Rao's contributions in this issue of the *International Statistical Review*.

Even as Rao continued to make fundamental contributions to mathematical statistics, Mahalanobis tapped him to lead a project analysing multivariate anthropometric data collected during the 1941 census in Uttar Pradesh. The experience instilled in Rao a deep appreciation of the interplay between applications and theory, a central tenet of his statistical journey. While Rao was working on this project, J. C Trevor, an anthropologist at the Duckworth Laboratory at Cambridge contacted Mahalanobis and asked if he could depute someone from ISI to help with the analysis of ancient skeletal remains from gravesites unearthed by a British expedition in Jebel Moya in the Sudan. Another twist of fate would open a new chapter in Rao's life.

3 Cambridge – The Fisher Years

Rao arrived in England in August 1946 and enrolled in King's College, Cambridge, registering for a Ph.D. under Ronald Aylmer Fisher, recognized as the founder of modern-day Statistics. Fisher's 1925 paper, Theory of Statistical Estimation, had laid out the foundations of inference introducing concepts like consistency, sufficiency, efficiency, Fisher information, maximum likelihood estimation, and optimality. Efron in his article, The Statistical Century [14], states "I mark 1925 as the year statistics went from a collection of ingenious techniques to a coherent discipline."

By the 1940's, Fisher had moved to Cambridge and was named the Balfour Chair of Genetics. Fisher agreed to be Rao's thesis supervisor under the condition that Rao work in the genetics lab where Fisher was breeding mice to map genes on the mouse chromosome. Rao's days were packed: mornings spent at the Museum of Archaeology and Ethnology analysing skeletal data, evenings at the genetics lab, and the few remaining hours of the day devoted to his Ph.D.

3.1 1948: The Score Test – The Second Breakthrough Article

At the Genetics Lab, Fisher was mapping mice chromosomes to investigate genetic linkages. Rao considered the problem of estimating the recombination fractions for different chromosome segments using linkage data from different experiments (possibly from different crossings or litters), with each dataset containing information about one or more of the segments. While combining the datasets would yield more efficient estimates, Rao recognized the need for a formal test of homogeneity and introduced a statistic based on the score function and the Information matrix.

Rao recalls showing Fisher a draft note on the score test for linkage analysis. Fisher refused to read the article, insisting that Rao include numerical results illustrating the test's performance on real data. Following his recommendation, Rao applied the score test to linkage data for Primula sinensis and submitted the paper to *Heredity* [35], the journal founded by Fisher. A second paper developing the theoretical foundations and properties of the score test was published in 1948 [30]in the Proceedings of the Cambridge Philosophical Society.

While it would be several years before the test found its way into the mainstream statistical literature, Rao's score test is now part of every statisticians and practitioners toolkit. The Neyman and Pearson Likelihood Ratio, Wald, and Rao's Score tests are often referred to as the Holy Trinity.

Rao was determined to make the most of his two-year stay at Cambridge, learning genetics from Fisher, taking a course in Stochastic Processes from Bartlett, and even participating in an economics reading group studying Von Neumann and Morgenstern's book *Theory of Games and Economic Behaviour*. Rao also found time to attend the Cambridge Debating Society events, listen to talks by some of the leading scientists, philosophers, and political figures of the time, and audit a series of lectures by Bertrand Russell!

While Fisher's direct involvement with Rao's work was limited, his influence on Rao's career was second only to Mahalanobis. Fisher, like Mahalanobis, recognized the importance of data and applications in the development of statistical methodology. Efron [14] writes "Nobody was ever better at both the inside (mathematical foundations) and outside (methodology) of statistics than Fisher, nor better at linking them together. His theoretical structures connected seamlessly to important applications (and as a matter of fact, caused those applications to increase dramatically in dozens of fields)." In a tribute to Rao on the occasion of his 100th birthday, Efron writes "Rao *really* was Fisher's student, in the sense of carrying on the Fisherian statistical tradition"

Mahalanobis and Fisher were towering figures, passionate about their work but often uncompromising. They believed that statistical research should be motivated by practical applications, understood how applications could often lead to the development of new theory, and recognized the importance of data quality. Larger than life personalities, Rao refers to them as the "guiding forces in his life".

In the Bulletin of the Institute of Mathematical Statistics (2010), Terry Speed writes "The 1940s were ungrudgingly Rao's. His 1945 paper, which contains the Cramér–Rao bound, Rao–Blackwell theorem, and the beginning of differential geometry of parameter space will guarantee that even had he done nothing else – but there was much else."

Rao returned to the Indian Statistical Institute in August 1948 and was offered a professorship in July 1949, just a few months short of his 29th birthday!

(a) Rao and Mahalanobis

(b) Rao and Fisher

Figure 3: Rao with his Mentors

(a) Rao in Fisher's Genetics Lab

(b) Rao: Cambridge Graduation

Figure 4: Rao at Cambridge

4 1950 – 1978: Rao and the Indian Statistical Institute

With India gaining independence from the British in 1947, the economic and social challenges facing the fledgling nation were monumental. Mahalanobis believed that data and statistical methods were key to addressing these challenges, enabling policy makers and government officials to make informed decisions. Jawaharlal Nehru, the first Prime Minister of India and a long-time friend, agreed with his assessment and appointed him honorary statistical advisor to the Indian Government. Nehru's support was instrumental in ISI being declared an Institute of national importance by an Act of Parliament in 1959.

Under Mahalanobis's leadership, ISI had established itself as a world-class institution, promoting research and training at the interface of statistics and other disciplines, including the physical, biological and social sciences. He was an excellent judge of talent and recruited the brightest minds to come work at ISI. Mahalanobis also understood the value of global collaboration, inviting eminent researchers from around the world to come visit the Institute. It was an exciting and intellectually stimulating environment, and Rao appreciated "the freedom to pursue one's own ideas, to participate in the projects of the Institute, to accept administrative responsibilities or to devote time to research." With R. C. Bose and S. N. Roy leaving for the United States, Rao would become the de facto second in command at ISI in 1950 and would play a key role in the Institute's growth.

Rao's first project after his return was to complete work on his book *Advanced Statistical Methods in Biometric Research* [37], a primer on multivariate methods and its applications based on his work at Cambridge. Rao also started advising Ph.D. students, recruiting D. Basu as his first student in 1950. Basu completed his Ph.D. in 1953 and would go on to make fundamental contributions to statistical inference, including the famous theorem that bears his name.

At any given time at ISI, Rao would have 5-6 Ph.D. students working with him, often in completely different areas of Statistics. These included Survey Sampling (A.C. Das, Des Raj), Multivariate Analysis (J. Roy), Quality Control (A. Mathai), Characterization (R. G. Laha), Experimental Design (I. M. Chakraborti), Directional Data (J.S. Rao), Combinatorics and Graph Theory (U.S. R. Murthy, S. B. Rao). Leveraging the knowledge and valuable insights he gained during his time at Cambridge, he also supervised students in Statistical Genetics (D. C. Rao, K. Dronamraju, Ranajit Chakraborty), Probability Theory and Stochastic Processes (S. R. S. Varadhan, V. Varadarajan, K. R. Parthasarathy), and Game Theory (T. Parthasarathy, T.E.S. Raghavan). In areas that were new to him, he would ask students to review the relevant literature in detail and provide an overview. He had an incredible mind that could intuitively draw connections between different fields and a unique approach to problem solving.

The period from 1930 – 1960 is often referred to as the "Golden Age of Statistics" in India, with ISI as the epicentre. The Institute played host to a series of eminent visitors including Norbert Weiner, Ragnar Frisch, John Kenneth Galbraith, J. B. S. Haldane, Walter Shewhart, A. N. Kolmogorov, and J. L. Doob. Rao encouraged the Ph.D. students and faculty to take advantage of these opportunities, gaining exposure to new and emerging areas in Statistics and allied fields. In the interview with Anil Bera [5], Rao notes "there was no parallel to the intellectual atmosphere that existed at the ISI during the 1950s and 1960s."

(a) Rao at the black board (b) Rao: Indian Statistical Institute (ISI)

Figure 5: Rao: The ISI Years

Rao revamped the training program at ISI to address the needs of government and industry, adding new course on statistical quality control and industrial statistics. ISI started a new division of Statistical Quality Control in 1953 to provide consulting service to industry, and set up branches in different parts of India. One of the visitors to ISI during this period was Genichi Taguchi, a Japanese engineer and statistician who along with Edward Deming helped Japanese companies adopt the principles of quality control post World War II. Rao's work on orthogonal arrays [34] would play a key role in the development of robust designs and principles for industrial experiments referred to as Taguchi methods.

Rao also directed the training programs at the International Statistical Educational Center (ISEC), a partnership between the International Statistical Institute and the Indian Statistical Institute with the support of UNESCO and the Indian Government. ISEC has continued to play a critical role in training and capacity building in the developing world, offering courses in theoretical and applied statistics to participants from the Middle East, South-East Asia, the Far East, and Africa. As Honorary Statistical Adviser to the Government, Mahalanobis was instrumental in setting up the Central Statistical Organisation (CSO) to coordinate statistical activities and ensure the development of standards. Rao played a key role in setting up statistical bureaus in different states and developing a network of district-level statistical agencies for data collection. He was involved with designing surveys and developing the protocols for tabulation and data collection as part the National Sample Survey (NSS), a multi-purpose socio-economic survey.

Rao was one of the founders of the Indian Econometric Society in 1960 to promote research and training in econometrics and quantitative economics. Ragnar Frisch, the Norwegian economist and the co-recipient of the first Nobel Memorial Prize in Economic Sciences in 1969 with Jan Tinbergen was one of the founders of the discipline of Econometrics in the 1930's. One of Rao's early papers published in *Econometrica* [33] provided a solution to a problem posed by Frisch on characterization problems in the context of structural equations. Many of Rao's fundamental results in mathematical statistics, linear models, and characterization have found their way into the economics literature. See Kumar, Vinod and Deman [22] for a review of his contributions to the economic sciences.

In 1953, Rao spent a year as a visiting research professor at the University of Illinois at Urbana-Champaign (UIUC) soon after UIUC had built and commissioned one of the first supercomputers known as ILLIAC-I. Rao took courses in programming using machine language and was given access to ILLIAC to develop programs for statistical computations. ISI had moved from using "human computers" in the 1930's to manual calculating machines by the fifties, and in 1956 installed the Hollerith Electronic Computer (model 2M), the first electronic digital computer in India. Rao returned to ISI eager to leverage his programming skills and accelerate the use of computers in India. Unfortunately, concerns about loss of jobs due to automation resulted in significant delays. In an interview with DeGroot [9], Rao says "That was a big disappointment for me. We lost the opportunity to do certain kinds of research or develop methodology based on complex computations."

By 1960, ISI had started offering Bachelor's, Master's, and Ph.D. degrees in Statistics. Rao was tasked with creating the framework for the new programs, coordinating the development of an innovative, interdisciplinary curriculum that focused on both mathematical foundations and applications, and recruiting teaching staff. These new programs attracted the brightest students from all over India, many of whom would go on to Ph.D. programs at top institutions around the world.

Despite shouldering a heavy administrative load, Rao continued to find time for research. He also started work on a new book that would focus on the foundations of Statistics. The book, *Linear Statistical Inference and Its Applications* (LSI), was published in 1965 and is required reading for any serious student of the subject. Translated into six major languages, LSI is one of the most cited books in the literature and remains as relevant today even 50 years after its publication. Rao was also asked to help with the editorial work for *Sankhya*, and would become the Editor after Mahalanobis.

After Mahalanobis's death in 1972, Rao assumed the mantle of leadership as Secretary and Director of the Indian Statistical Institute. In a tribute to Mahalanobis, Rao writes "Developing statistics was like exploring a new territory. It needed a pioneer like Mahalanobis, with his indomitable courage and tenacity to fight all opposition, clear all obstacles and throw open wide pastures of new knowledge for the advancement of science and society."

Rao recognized that for ISI to evolve and grow as a research institution, the administrative procedures and top-down approach needed to change. A new constitution was adopted and a new five-year term for the Director approved. Rao stepped down as Director and was named the Jawaharlal Nehru Professor. Another chance encounter at Pittsburgh would once again change the course of his career.

(a) Rao in Phillipines with students

(b) Rao with Mrs. Indira Gandhi, Prime Minister of India, at ISI Delhi

Figure 6: Rao: In good company

5 The U.S. Years

Rao spent 1978 as Mellon Professor at the University of Pittsburgh and later as a Visiting Professor at the Ohio State University. Both institutions offered him permanent positions, leaving Rao with a difficult choice. He was close to 60, a mandatory age of retirement in most Indian institutions;

accepting either offer would allow him to focus on his research and write the next chapter of his career.

As chance would have it, Rao's son Veerendra decided to transfer to the University of Pittsburgh to complete his Bachelor's degree in engineering. Rao accepted the University of Pittsburgh's offer of a University Professorship in the Department of Mathematics and Statistics in the Fall of 1979.

While the Department had a very strong Mathematics program with several top researchers on the faculty, there were only a handful of faculty members with expertise in Statistics in the early 1970's. In a move to expand the Statistics program and develop a strong research core, the Department hired P. R. Krishnaiah in 1976. Krishnaiah, one of the leading experts in Multivariate Analysis, had spent several years at the Wright-Patterson Air Force Base as a mathematical statistician. He was the founding editor of the Journal of Multivariate Analysis (JMVA), editor of the Handbook in Statistics volumes, and had organized several international conferences. Krishnaiah, who counted S. N. Roy as one of his mentors, would play a key role in recruiting Rao to Pittsburgh.

Rao and Krishnaiah would revamp the Master's and Ph.D. programs and make the University of Pittsburgh a top choice for international students. With support from the U.S. Airforce and Navy, the Center for Multivariate Analysis (CMA) was established in 1982. Krishnaiah and Rao were able to bring in several large Air Force research grants to support foundational and applied research in the areas of multivariate analysis, pattern recognition, and signal processing. The grants supported Ph. D. students, postdoctoral fellows, and visitors from the US and around the world. Free from administrative responsibilities, Rao could focus on his research and supervise Ph.D. students. His presence at Pittsburgh attracted a steady stream of visitors every summer.

Rao and Krishnaiah had forged a close personal bond during their time in Pittsburgh and Krishnaiah's untimely death due to cancer in 1987 affected Rao deeply. He had to step in and take over the responsibilities of managing the Center, the large Air Force and Navy grants, as well as the editorial activities of JMVA. Gutti Jogesh Babu, a Professor at Penn State, and a frequent visitor to Pittsburgh sensed that Rao would be open to an offer to leave Pittsburgh. The Department of Statistics at Penn State was able to convince Rao to join the Department as the first Eberly Chair in Statistics. Rao moved to Penn State in 1988 along with 6 Ph.D. students and several visitors. He would also move the Center of Multivariate Analysis to Penn State and become Director of the Center.

While the Department of Statistics at Penn State was highly ranked with outstanding faculty, Rao's arrival certainly increased the Department's visibility, attracting top researchers from around the world. James Rosenberger, Professor Emeritus and Former Chair of the Department writes "When Professor C.R. Rao joined Penn State in 1988, he brought international recognition to the department, inviting the most prominent leaders in the profession as speakers and visitors, including every newly elected member of the National Academy of Sciences. His influence helped raise the rankings of our department from the 30's to the top 20's in 1995."

Rao held the Eberly Chair in Statistics from 1988 until his retirement in 2001 and became an Emeritus Holder of the Eberly Family Chair in Statistics in 2009. Following his retirement, Rao moved to Buffalo where he serves as research professor in the Department of Biostatistics.

6 Research Contributions: Some Highlights

Rao's pioneering contributions span almost all areas of theoretical and applied statistics, and a number of different disciplines including economics, electrical engineering, anthropology, and genetics. This section highlights his contributions in four different areas that the authors hope will provide a glimpse into his legendary career. Other than these areas he has made some significant contributions in Design of Experiments, Sample Survey and Linear Models also.

6.1 Characterization of Probability Distributions

Rao's interest in characterization problems was piqued when he came across a paper by H. V. Allen in Statistical Research Memoirs, a volume edited by Neyman. The paper proposed a solution to the following problem posed by Ragnar Frisch, the Norwegian economist: if X_1 and X_2 are two variables such that

$$X_1 = a\zeta + \alpha, X_2 = b\zeta + \beta,$$

what are the conditions under which the regression of X_1 on X_2 is linear, ζ, α, and β being independent variables and a and b being constants but unknown. Rao's more general solution established the existence of probability distributions "with infinite variances for ζ and β and with a little more freedom for α, satisfying the necessary and sufficient conditions for the linearity of regression". The solution, part of his Master's thesis submitted to Calcutta University in 1943, was later published in *Econometrica* in 1947 [33]. He continued to work on characterization problems after he returned from Cambridge, and supervised Ph.D. students including R. G. Laha. The research yielded important characterization of Poisson, gamma, univariate, multivariate normal and infinite variance stable distributions.

Rao had an opportunity to discuss his work with Yuri Linnik, the Russian mathematician/ probabilist during his visit to ISI in 1955. Rao visited Russia several times to collaborate with Linnik and his student Abram Kagan. The collaboration would yield several publications and one of the first books on Characterization Problems [20]. Diaconis, Olkin and Ghurye in a review published in the Annals of Statistics write "the sheer amount of material covered by the authors makes this an essential and indispensable book to the specialist."

After moving to the U.S., Rao and D. N. Shanbhag would work on applying Choquet-Deny type functional equations and their variants to establish characterization results. Several characterizations of well-known distributions such as the exponential, Weibull, and geometric distributions based on order statistics, hazard function, and the mean residual life can be established using these results. The research would result in several publications and a book *Choquet-Deny Type Functional Equations with Applications to Stochastic Model*s.

6.2 Generalized Inverses

During his Master's program, Rao took a course in linear models and design of experiments from R.C. Bose where he was introduced to the concept of estimability of parametric functions. Rao would build on this idea and consider a general theory of least squares where the matrix of normal equations is singular. In his 1955 paper [38], Rao proposed computational techniques to obtain a solution to the normal equations using what he called a "pseudoinverse" of a singular matrix. The same year, Roger Penrose, the eminent physicist and 2020 Nobel laureate, published his paper "A generalized inverse for matrices" in the Proceedings of the Cambridge Philosophical Society. The inverse would be named the Moore-Penrose inverse, acknowledging the contributions of E. H. Moore in 1920.

While Rao's pseudoinverse did not satisfy the Moore-Penrose restrictions, it provided a solution to the problem of least squares estimation under a more general framework. Rao would extend the results obtained by Penrose and others to problems encountered in statistics in his 1962 paper [40]. He considered a generalized inverse (or g-inverse) using a weaker definition than the one proposed by Moore and Penrose and presented a "unified theory of least squares theory".

Rao and Sujit Kumar Mitra would continue to work in this area, developing new classes of g-inverses and their applications to multivariate analysis. The collaboration would result in the publication of a monograph *Generalized Inverse of Matrices and its Applications* in 1971.

Rao's interest in matrix theory and linear algebra are evident to anyone who has read his book Linear Statistical Inference and its Applications. In addition to his work on g-inverses, Rao and C. G. Khatri would propose extensions and generalizations of the Kantorovich inequality in the context of least squares. Rao and M. B. Rao would also write a book *Matrix Algebra and Its Applications to Statistics and Econometrics* in 1998.

6.2.1 A Historical Note about the Pseudoinverse

The US National Academy of Sciences - National Research Council was asked to undertake a long-range study of the medical and biological effects on survivors of the atomic bombings of Hiroshima and Nagasaki. The study began in 1946 with the Genetics Program that was tasked with collecting and analysing survey and anthropometric data on the characteristics of children born to survivors, and assessing possible genetic effects. The presence of extraneous (concomitant) variation and differing numbers of observations in the various exposure cells made the analysis challenging. The authors of the report [25] would use Rao's analysis of dispersion approach and the computational technique he developed based on the pseudoinverse to compute the standard errors of differences in the estimates. In the introduction, the authors write that the "statistical analysis profited greatly from discussions and correspondence" with Rao and a small group of eminent statisticians and geneticists.

Puntanen and Styan [27] state that Rao's 1955 paper was motivated in part by the atomic bombing data. They write "In 1954 C. R. Rao received some data collected in Japan in order to study the long-term effects of radiation on atom-bomb casualties in Hiroshima and Nagasaki. The statistical analysis involved finding a matrix to replace the inverse of $X'X$, where X is the model or design matrix in the usual linear model; here the matrix $X'X$ was singular and so the inverse was not defined."

6.3 Multivariate Analysis: Theory and Applications

6.3.1 Anthropometry and Mahalanobis D^2

In 1941, Mahalanobis and Dhirendra Nath Majumdar, an eminent anthropologist, designed and conducted a large-scale anthropometric survey in the United Provinces (now Uttar Pradesh) to investigate affinity and diversity between different groups. The data included 12 morphological measurements made on a large number of individuals belonging to over 20 different groups (tribe, caste, religion). Mahalanobis asked Rao to lead the project and oversee the statistical analysis using the Mahalanobis D^2 statistic, a measure of divergence/ dissimilarity between two populations:

$$D_p^2 = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2),$$

where $\boldsymbol{\Sigma}$ denotes the common dispersion matrix of the p measurements, and $\boldsymbol{\mu}_i$ is the mean vector for the ith population.

The computation of the D^2 statistic required inverting a large order covariance matrix, a task that was not feasible given the tools available at the time. To simplify the computation, Rao developed a method for selecting a subset of the r "best variables", where $r < p$, using a test based on a weighted average of all possible D_r^2 values computed from different pairs of populations [23].. The so-called *perimeter statistic* was developed by Rao as part of his Master's thesis.

While reducing the number of variables addressed the computational burden, it was also important to understand the resulting loss of information. Using the notion of projections, Rao [29] developed an exact test to examine whether the extra information gained by including q additional variables to an existing set of p variables is statistically significant. The test has the following form:

$$U = \frac{n - (p+q) - 1}{q} \frac{n_1 n_2 (D_{p+q}^2 - D_p^2)}{n(n-2) + n_1 n_2 D_p^2},$$

where n_1 and n_2 denote the number of samples from the two populations, D_{p+q}^2 and D_p^2 denote the Mahalanobis distances based on $(p+q)$ and p variables respectively, and $n = n_1 + n_2$. Under the null hypothesis, the test statistic has a F distribution. The test, referred to as the U-test by Kshirsagar [21], has been widely used in the econometric and sociology literature.

During the two years working on the project, Rao writes [42], "I was able to establish credentials on my expertise in multivariate analysis." Recognizing this expertise, Mahalanobis sent Rao to Cambridge to assist J. C. Trevor, an anthropologist, with the analysis of human skeletal data excavated from Jebel Maya in North Africa. Rao's work at the Museum of Archaeology and Ethnology analysing skeletal data led him to explore different techniques from Multivariate Analysis. He developed a general classification tool for more than two populations [31], extending Fisher's work. In an interview with DeGroot [9], Rao writes about the approach: "I used essentially Bayesian techniques. That's the appropriate thing to do in problems of that kind. Later on I described what you should do when you could not know the prior probabilities. I also created a kind of doubtful region where you can take the position that you are unable to say whether an observation belongs to this group or that group, and described how to operate with that region." In an attempt to interpret the D^2 values and identify clusters, Rao introduced the idea of "canonical variates" to "roughly indicate the relative positions of the various groups on a two-dimensional chart using the D^2 values." The canonical variates were ordered linear combinations $l_1, l_2, \ldots$ of the variables, with the first k providing the "best representation of the groups in a k dimensional subspace. This was one of the early graphical tools for multivariate data. These results were presented as a discussion paper at the Royal Statistical Society meeting and published in the Journal of the Royal Statistical Society [31].

Rao's Ph.D. thesis, *Statistical Problems of Biological Classification*, included theoretical results obtained while working on the anthropology project as well as extensions of Fisher's discriminant function to higher dimensions.

6.3.2 Multivariate Analysis of Variance (MANOVA)

In his 1948 *Biometrika* [32] paper, Rao developed a general framework he called analysis of dispersion, for hypothesis testing, extending the univariate analysis of variance to the multivariate case. Rao defines the analysis of dispersion as "the technique of analysing the variances and covariances of multiple correlated variables", what we now refer to as MANOVA. One of the important problem in MANOVA is to test the equality of the mean vectors of q different p-variate multivariate normal populations. Let $\boldsymbol{B}$ and $\boldsymbol{W}$ denote the sums of squares and products due to treatment (between groups) and error (within groups), respectively. The test statistic

$$\Lambda = \frac{|\boldsymbol{W}|}{|\boldsymbol{B}+\boldsymbol{W}|},$$

is known as Wilk's lambda, and Rao developed one of the most widely used approximations for the distribution of the statistic. Rao's 1948 paper also developed a general testing framework using the multivariate linear model approach.

6.3.3 Rao's paradox and the Curse of Dimensionality

While analyzing some bivariate data to compare the population means, Rao observed that while tests based on the Mahalanobis D^2 or Hotelling's T^2 showed no significance, the two-sample t-tests based on each of the variables were highly significant. Healy [19] refers to this as Rao's paradox. Rao refers to this as "the first example of what is called the *curse of dimensionality* in multivariate analysis", a phenomenon encountered routinely in the analysis of high-dimensional data.

6.3.4 Analysis of Diversity

Rao's interest in anthropometry led him to investigate different measures of diversity within populations using the notion of dissimilarity and entropy functions [41]. He was interested in developing methods to ascertain (a) how much of the diversity between individuals of a population is due to different factors (the decomposition of diversity problem), and (b) how much of the diversity in a composite population can be attributed to the diversity "within populations" and how much due to "between populations" (the apportionment of diversity problem).

Rao's analysis of diversity (ANODIV) is a generalization of ANOVA that is applicable to both qualitative and quantitative data and uses measures other than variance. Rao's paper provided a general framework for ANODIV of m-way classified data using different dissimilarity measures and entropy functions. Rao also introduced quadratic entropy as a measure of diversity of probability distributions. While most of the well known entropy functions like Shannon, Gini-Simpson, and Renyi cannot be used beyond two-way ANODIV, Rao and his collaborators showed that quadratic entropy satisfies can be used to analyze m-way classified data, see [44]. These diversity measures have been used in a variety of applications in ecology, genetics, anthropology, sociology, and economics.

6.3.5 Factor Analysis

During Rao's sabbatical at the University of Illinois at Urbana Champaign, he met Charles Wrigley, a psychologist working in the area Factor Analysis. In psychometrics, the factor analytic model assumes the observed vector U is partitioned into two parts, an unobserved systematic component and an unobserved error component. The model is represented as

$$\boldsymbol{U} = \boldsymbol{L}\boldsymbol{F} + \boldsymbol{G},$$

here $\boldsymbol{F}^\top = (F_1, \ldots, F_m)$ is the factor matrix and $\boldsymbol{G}^\top = (G_1, \ldots, G_p)$ are uncorrelated random variables. Here $\boldsymbol{L}$ is a $p \times m$ matrix of unknown constants called the loading matrix.

The covariance matrix of $\boldsymbol{U}$, $\boldsymbol{\Sigma}$ may be written as

$$\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{\Delta}\boldsymbol{L}^{'} + \boldsymbol{\Gamma},$$

where $\boldsymbol{\Delta}$ represents the covariance matrix of $\boldsymbol{F}$ and $\boldsymbol{\Gamma}$ is the covariance matrix of the error term.

One of the important problems in factor analysis is to determine the minimum value m so that the above structural relation holds under suitable assumptions on $\boldsymbol{F}$ and $\boldsymbol{G}$. Rao' paper in *Psychometrika* [39] addresses the problem of estimation and hypothesis testing in factor analysis. He considered the canonical correlations between the observed variable $\boldsymbol{U}$ and the latent factors, and developed a method he referred to as canonical factor analysis to distinguish it from the approach using principal components. Using the form of the covariance matrix $\boldsymbol{\Sigma}$, Rao proposed a method for estimating m that involved testing the equality of the smallest characteristic roots of the dispersion matrix. This idea has been used several decades later in signal processing to estimate the number of signals in the Direction of Arrival model.

In addition to the areas described above, Rao has made fundamental contributions to the analysis of growth curves, principal component analysis, and estimation of variance components including the widely-used minimum norm quadratic unbiased estimation (MINQUE) technique, see Fujikoshi [17]. Rao was awarded the Samuel S. Wilks Memorial Medal in 1989, in part for his major contributions to the theory of multivariate statistics and applications to problems in biometry.

6.4 Statistical Signal Processing

During Rao's tenure at the University of Pittsburgh, he and Krishnaiah received funding from the U.S. Air Force to support theoretical and methodological research in signal processing. The Air Force was interested in detection and estimation problems in signal processing, including the development of efficient statistical methods to analyze noisy signals received at multiple sensors with applications to radar and sonar processing.

The research focused on the development of a robust statistical framework that included parameter estimation, hypothesis testing,model selection as well as efficient computational approaches for data analysis. Rao, his Ph.D. students, and collaborators considered two of the most widely used models in the signal processing literature: (i)superimposed exponential model (also known as the sum of sinuoids model) and (ii) Direction of Arrival (DOA) model. These models have been used in telecommunications, radio location of objects, seismic signal processing, image processing and computer assisted medical diagnostics. Both authors were fortunate to work in the area of signal processing for their Ph.D.'s under Rao's supervision.

6.4.1 Superimposed Exponential Model

The superimposed exponential model may be written as

$$y(t) = \sum_{k=1}^{p} A_k e^{i\omega kt} + e(t); \quad t = 1, \ldots, N.$$

Here A_k's are complex amplitudes, ω_k's are frequencies, p represents the number of components, and $e(t)$'s are additive noise. Given a sample $\{y(1), \ldots, y(N)\}$ of size N, the problem is to estimate the unknown parameters (amplitudes, frequencies, and model order).

Given that any periodic signal can be represented in the above form, the estimation problem has received significant amount of attention in the signal processing literature. The problem is a non-linear regression problem and standard techniques available in the statistical literature cannot provide an adequate solution. Rao and his collaborators developed efficient estimation procedures based on nonlinear least squares, and established the large sample properties of the estimators.

6.4.2 Direction of Arrival (DOA) Model

Consider a linear array of p sensors which receives signals from q sources. The signals arrive at the array at angles $\theta_1, \ldots, \theta_q$ with respect to the line of array. The DOA model can be expressed as follows:

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t); \quad t = 1, \ldots, N.$$

Here, at time t, $\boldsymbol{x}(t)$ is the $p \times 1$ complex vector of observations received by the p uniformly spaced sensors, $\boldsymbol{s}(t)$ is the $q \times 1$ complex vector of unobservable signals emitted by the q sources, and $\boldsymbol{n}(t)$

is a $p \times 1$ complex valued noise vector. The matrix $\boldsymbol{A}(\boldsymbol{\theta})$ is a $p \times q$ Vandermonde matrix that depends on the parameters $\theta_i, i = 1, \ldots, q$. Here the number of sources is assumed to be smaller than the number of sensors. The problem is to estimate the number of signals q and the parameters $\theta_1, \ldots, \theta_q$, the directions of arrival of the signals. The DOA model is widely used in radar and sonar processing, and can be modified to include a tracking component.

The two models are similar: the superimposed exponential model resembles the fixed effects model encountered in linear models, and the DOA model resembles the random effects model. Estimation in the DOA model makes use of spectral analytic techniques, properties of patterned matrices, and also leverages factor analysis techniques developed by Rao in the 1950's.

Working closely with electrical engineering researchers at Pittsburgh and Penn State, Rao and his collaborators would develop a rigorous statistical framework for inference in signal processing, leading to new advances in both Electrical Engineering and Statistics. For an overview of Rao's fundamental contributions, the reader may refer to the following articles [4], [46] and the references cited therein.

7 Rao as a Teacher and Mentor: Some personal memories

"It should be one of the functions of a teacher to open vistas before his pupils, showing them the possibility of activities that will be as delightful as they are useful." Bertrand Russell

Rao was an exemplary teacher: he could make abstract mathematical and statistical ideas come to life in a classroom. Starting with the big picture, Rao would use examples from different disciplines to motivate the theoretical and methodological developments. For many of us used to a standard (passive) lecture format, Rao's use of didactic and Socratic approaches was eye-opening. He would ask students to come to the board to solve a problem or prove a theorem. While the student was busy providing a lengthy (often convoluted) proof, Rao would quietly move to the other side of the room and sketch a simple and elegant solution.

At the University of Pittsburgh, Rao's graduate courses were a major draw with first-year and senior Ph.D. students, Ph.D. students from different departments (often accompanied by their Ph.D. advisors), and senior faculty members from across campus in attendance. Despite the diversity of expertise and statistical background in the room, Rao had the unique ability to engage with each student, drawing connections between theory and applications, and deconstructing complex ideas.

Rao's mentoring style was greatly influenced by his experience working with Fisher. Like his mentor, Rao encouraged his Ph.D. students to find their own research problems. Working with Rao was a unique experience - you were constantly exposed to new areas of Statistics and had the opportunity to interact with leading experts from around the world. He taught us how to think critically about problems, to appreciate the synergy between statistical theory and methodology, and most importantly how to be not just a statistician but a scientist.

Asked what achievement he is most proud of, Rao writes "it is the outstanding contributions my students are making to statistical theory and practice." Rao has supervised 51 students, who in turn have produced 690 Ph.D.'s as of 2021. (See https://www.mathgenealogy.org/id.php?id=40000 for a complete list of Rao's doctoral students and their descendants).

8 Awards and Honours

During his legendary career spanning eight decades, Rao has received numerous awards and honours from countries around the world. He is the recipient of the Guy Medal in Gold (2011) of the Royal Statistical Society, India Science Award (2010), the highest award in a scientific field presented by the Government of India, International Mahalanobis Prize (2003) of the International Statistical Institute, Srinivas Ramanujan Medal (2003) of the Indian National Science Academy, National Medal of Science (2002) awarded by the Government of the United States of America, Padma Vibhushan (2001), the second highest civilian honours bestowed by the Government of India, Mahalanobis Centenary Gold Medal of the Indian Science Congress (1993), Wilks Memorial Award (1989) of the American Statistical Association, Megnadh Saha Medal (1969) of the Indian National Science Academy, Guy Medal in Silver (1965) of the Royal Statistical Society, and the S. S. Bhatnagar Award (1963) of Council of Scientific and Industrial Research, Government of India. He is a Fellow of the Royal Society and a Member of the American Academy of Arts and Sciences. Rao has received 31 Honorary Doctoral Degrees from universities in 18 countries spanning six continents.

(a) National Medal of Science

(b) Guy Medal: Royal Statistical Society

Figure 7: Awards and Honours

(a) Padma Vibhushan - Government of India

(b) Mahalanobis Prize: International Statistical Institute

Figure 8: Awards and Honours

9 Rao the Visionary- Statistics in the Age of Big Data

Through a career spanning almost eight decades, Rao has not only witnessed the evolution of Statistics as an independent scientific discipline, he is one of a small group of individuals who helped establish its foundations. Rao understood that for Statistics to remain an influential discipline, it must adapt to the rapidly changing world, to remain a "powerful tool in acquiring knowledge in any field of enquiry." Quoting Savage, Rao writes: "The foundations of statistics are shifting, not only in the sense that they have always been, and will doubtless long continue to be, changing, but also in the idiomatic sense that no known system is quite solid."

In the first half of the 20th century, the foundations of statistics were developed using powerful mathematical and probabilistic tools. Referring to this period as "the golden age of statistical theory", Efron in his article, the *Statistical Century* published in the Royal Statistical Society

News [13], writes "Men of the intellectual calibre of Fisher, Neyman, Pearson, Hotelling, Wald, Cramer and Rao were needed to bring statistical theory to maturity."

Working with Mahalanobis and Fisher, Rao understood the importance of working with data and the need for research to be motivated by real applications. As early as the 1950's, Rao recognized that computers and computational methods would play a critical role in the development of statistical methodology. A scientist of extraordinary prescience, Rao predicted the Big Data revolution and the growth of new disciplines at the interface of mathematics, statistics, and computer science. In 2007, he established the C.R. Rao Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS), an institute that seeks to pursue research at the interface of these disciplines and address applications that enable social good. Rao has continued to push the boundaries of Statistics to address emerging challenges in this era of Big Data and Artificial Intelligence.

On the occasion of his 100th birthday, the Government of India issued a citation for Rao's "path-breaking contributions to the field of Statistics, for recognizing and facilitating the critical role of data and computing to handle scientific and societal challenges, for mentoring, inspiring and nurturing generations of students and researchers, and for developing a world-class statistical infrastructure in India." A "prophet of a new age", Rao continues to inspire new generations of statisticians around the world.

Figure 9: Rao the visionary

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