Bayesian Inference On General Order Statistic Models

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Abstract

In the present article, we consider the Bayesian inference of the unknown population size N, along with the other model parameters based on a general order statistics model. The inference is carried out for (i) exponential, (ii) Weibull, and (iii) generalized exponential lifetime distributions. It is observed that under the standard squared error loss function, the Bayes estimators cannot be obtained explicitly. The Bayes estimator of N and its credible interval are obtained using the Markov Chain Monte Carlo technique. The Bayesian methods can be implemented very easily and it avoids the difficulties of the classical inference. In this case, there is a positive probability that the maximum likelihood estimator of N is not finite. An extensive Monte Carlo simulation experiments have been performed to observe the behavior of the proposed Bayesian method. The Bayes factors and the predictive likelihood values have been used for choosing the correct model. The analysis of one real data set has been performed to illustrate the proposed method.

Key Words and Phrases: Generalized order statistics; exponential distribution; Weibull model; Generalized exponential model; Gibbs sampling; Bayes Factor; Bayes prediction.

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1 Introduction

Suppose there is a closed population of size N. In this paper, we consider the estimation of N based on Type-I censored data. The problem is the following. Let $T_1, \ldots, T_N$ be a random sample of a positive random variable having the positive probability density function (PDF) at x equals to $f(x;\delta)$. In this case, N and δ both are unknown, moreover δ may be vector valued also. Let T^* be a pre-fixed time, denote the period of observations. Suppose we observe $0 < t_{(1)} < \ldots < t_{(r)} < T^*$ within the observation period T^*. There is no failure between $t_{(r)}$ and T^*. We would like to draw the inference on N and δ, based on the above Type-I censored sample.

This is known as the general order statistics (GOS) model, and it has several applications in different areas. Consider the example of a population cited by Hoel (1968), where some members of a given population are exposed to disease or radiation at a given time. Let N be the number of individuals exposed to radiation. It is assumed that the times from exposure to detection of these individuals are random, and they are independent and identically distributed (i.i.d.) random variables, say $T_1, \ldots, T_N$, from a PDF $f(x;\delta)$. Based on the first r ordered sample until time point T^*, the problem is to estimate N and δ.

Similar problem can occur in software reliability, see for example Jelinski and Moranda (1972). Here one is interested in estimating the number of faults or bugs in a software from the initial failure times, $t_{(1)} < \ldots < t_{(r)}$, observed during a observation period T^*. Anscombe (1961) provided an interesting example of this model in estimating the sells of a company's product in a particular market. The main aim is to predict the average sales of the product in the future based on the information obtained during a short period after its penetration into the market. Osborne and Severini (2000) also considered the same problem to estimate the size N of a closed population based on the available observations upto a fixed length

$T^* > 0$. They have considered the estimation of N under an exponential general order statistic model.

Johnson (1962) and Hoel (1968) gave the methods for discriminating between two values of N based on the likelihood ratio and the sequential probability ratio test, respectively, when underlying lifetime distributions are completely known. Blumenthal and Marcus (1975) provided the maximum likelihood estimator (MLE) of N assuming the underlying probability distribution as exponential. Jelinski and Moranda (1972), Forman and Singpurwalla (1977), Meinhold and Singpurwalla (1983), Jewell (1985), Joe and Reid (1985) and Joe (1989) also considered this problem in the context of software reliability where the problem is to estimate the number of faults N in a software. The models proposed by them are the extensions of the model originally proposed by Jelinski and Moranda (1972), where the underlying lifetime distribution is exponential. Raftery (1987) considered estimation of the unknown population size N under general order statistic model and adopted an empirical Bayes approach. In particular, Raftery (1987) considered single parameter Weibull and Pareto order statistic models, which mainly belong to the exponential family. Although the performance of point estimators was not satisfactory, the interval estimators can be obtained and they might be useful for practical purposes. The problem has a close resemblance to the problem of estimating n for a binomial random variable. Extensive work has been done in estimating n of a binomial population, see for example DasGupta and Herman (2005) in this respect.

Kuo and Yang (1995, 1996) also considered similar models. They have provided an interesting connection between GOS models and non-homogeneous Poisson processes. They have also adopted Bayesian inference, but their problems of interests are slightly different than ours. They have mainly considered in details the Bayesian prediction and model selection, not the estimation of N.

It may be mentioned that the estimation of N is a non-trivial problem. The point estimator of N (say $\widehat{N}_{MLE}$) obtained by maximizing the likelihood function has several unusual features. It is observed that $P(\widehat{N} = \infty) > 0$. It is well known that both the mean and medians are biased estimators but in opposite direction. With a very high probability $\widehat{N}$ can take large values and it falls below the actual parameter value quite frequently. Further, it is quite unstable, and a small change in the data can lead to a large change in $\widehat{N}_{MLE}$.

Our proposed method is purely Bayesian in nature mainly for two purposes. First of all, it avoids the problem of finding an estimator which is not finite. Secondly, although the exponential GOS model has been studied quite extensively by several authors, not much attention has been paid for other distributions. It is observed that if the lifetime distribution is not exponential, analytically it becomes a challenging problem in the frequentist set up. It seems that for many lifetime distributions, the implementation of the Bayesian analysis is quite straight forward. Here, we have considered several lifetime distributions, namely (i) exponential, (ii) Weibull and (iii) generalized exponential models. Suitable theories and proper implementation procedures have been developed for point and highest posterior density (HPD) credible interval estimation of N and other unknown parameters.

The choice of prior plays an important role in any Bayesian inference problem. An independent Poisson prior has been assigned on N and for three different lifetime distributions, quite flexible priors on the unknown parameters of the distribution of T have been assumed. Based on the prior distributions and data, posterior distributions are obtained. All the estimates are obtained under the squared error loss (SEL) function. The Bayes estimators under the SEL function cannot be obtained explicitly. Hence, Markov Chain Monte Carlo (MCMC) technique has been used to compute the Bayes estimates and the associated credible intervals. Extensive simulation experiments have been performed to assess the effectiveness of the proposed methods. The performances are quite satisfactory. The analysis of one real data set has been presented to illustrate the proposed methods.

We have organized the remaining chapter as follows. The models and the priors have been presented in Section 2. In Section 3, the posterior analysis under different lifetime distributions has been provided. Monte Carlo simulation results have been presented in Section 4. In Section 5, we provides the analysis of a real data set. Finally in Section 6, the conclusions have appeared.

2 Model Assumptions and Prior Selection

2.1 Model Assumptions

Suppose $T_1, \ldots, T_N$ is a random sample of a positive random variable with PDF $f(x;\delta)$, and cumulative distribution function (CDF) $F(x;\delta)$. Let the first r order statistics, $t_{(1)} < \ldots < t_{(r)} < T^*$ be observed within the observation period T^*. The likelihood function is then given by

$$L(N,\delta|data) = \frac{N!}{(N-r)!}\left(\prod_{i=1}^{r} f(t_{(i)};\delta)\right)(1-F(T^*;\delta))^{N-r}, \; N = r, r+1, \ldots \tag{1}$$

The problem is to estimate N and δ and we have assumed the following different parametric forms of $f(x:\delta)$.

Exponential Model: It is the most commonly used lifetime distribution. Analytically, it is the most tractable lifetime distribution. In this chapter we have assumed the following PDF of an exponential distribution for $\lambda > 0$.

$$f_{EX}(t;\lambda) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases} \tag{2}$$

Weibull Model: Although the exponential distribution is used quite extensively as a lifetime distribution, it has a decreasing PDF and a constant hazard function. These are serious limitations for an exponential distribution. The Weibull distribution has two parameters;

one shape parameter and one scale parameter. The presence of the shape parameter makes it a very flexible distribution. The Weibull distribution has a decreasing or an unimodal density function. If the shape parameter is less than or equal to one, it has a decreasing PDF. Otherwise, the PDF is an unimodal function. Further, the hazard function also can take various shapes namely increasing, decreasing or constant. It can be used quite successfully to analyze lifetime data. The Weibull distribution has the following PDF for $\alpha > 0$ and $\lambda > 0$;

$$f_{WE}(t;\alpha,\lambda) = \begin{cases} \alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases} \tag{3}$$

Generalized Exponential Model: Gupta and Kundu (1999) introduced the generalized exponential distribution which behaves very similarly as the Weibull or gamma distribution. For details, see the survey article by Nadarajah (2011). The generalized exponential distribution has the following PDF for $\alpha > 0$ and $\lambda > 0$;

$$f_{GE}(t;\alpha,\lambda) = \begin{cases} \alpha\lambda e^{-\lambda t}(1-e^{-\lambda t})^{\alpha-1} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0. \end{cases} \tag{4}$$

2.2 Prior Selection

A Poisson random variable X with mean μ (POI(μ)), has the following probability mass function (PMF);

$$P(X = i) = \frac{e^{-\mu}\mu^i}{i!}; \quad i = 0, 1, \ldots. \tag{5}$$

The PDF of a gamma random variable with the shape parameter $a > 0$ and the scale parameter $b > 0$ (GA(a, b)) is:

$$f_{GA}(x;a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases} \tag{6}$$

The following prior assumptions have been made. In all the three cases considered here; N follows ($\sim$) POI(θ). In case of exponential distribution $\delta = \lambda$, and it is assumed that $\lambda \sim$ GA(c, d). Moreover, N and λ are independently distributed. For two-parameter Weibull and two-parameter generalized exponential distributions, $\delta = (\alpha, \lambda)$, and in both the cases $\alpha \sim$ GA(a, b). The prior on λ is the same as before, and all priors are assumed to be independent.

3 Posterior Analysis of Different GOS Models

3.1 Exponential GOS Model

The likelihood function (1) for $N = r, r+1, \ldots$ and $\lambda > 0$, becomes;

$$L_{EX}(N, \lambda|data) = \frac{N!}{(N-r)!}\lambda^r \left(e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)} + (N-r)T^*\right)}\right). \tag{7}$$

Hence, based on the prior distributions of N and λ, the posterior distribution of N and λ, is

$$\pi_{EX}(N, \lambda|data) \propto \frac{\theta^N}{(N-r)!}\lambda^{c+r-1} \left(e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)} + (N-r)T^* + d\right)}\right). \tag{8}$$

Let $M = N - r$, then the joint posterior distribution of M and λ is

$$\pi_{EX}(M, \lambda|data) \propto \frac{\theta^{M+r}}{M!}\lambda^{c+r-1} \left(e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)} + MT^* + d.\right)}\right). \tag{9}$$

If $g(M, \lambda)$ is a function of M and λ, then under SEL function

$$\widehat{g}_B(M, \lambda) = E(g(M, \lambda)) = \sum_{m=0}^{\infty} \int_0^{\infty} g(m, \lambda)\pi_{EX}(m, \lambda|data)d\lambda, \tag{10}$$

is the Bayes estimate of $g(M, \lambda)$.

It is clear that (10) cannot be expressed in explicit form, hence we use the Monte Carlo simulation to approximate (10). First observe that the joint posterior density function (9) is

$$\pi_{EX}(M, \lambda|data) = \pi_{EX}(\lambda|M, data) \times \pi_{EX}(M|data). \tag{11}$$

Here

$$\pi_{EX}(\lambda|M, data) \sim \text{GA}(c + r, \sum_{i=1}^{r} t_{(i)} + d + MT^*) \tag{12}$$

and

$$\pi_{EX}(M = m|data) = K \frac{\theta^{m+r}}{m!(\sum_{i=1}^{r} t_{(i)} + d + mT^*)^{a+r}}; \quad m = 0, 1, \ldots, \tag{13}$$

where

$$K^{-1} = \sum_{m=0}^{\infty} \frac{\theta^{m+r}}{m!(\sum_{i=1}^{r} t_{(i)} + d + mT^*)^{a+r}}.$$

For $\theta > 0$, $K < \infty$, all the moments of $\pi_{EX}(M|data)$ are finite. Since $\{M|data\}$ is a discrete distribution, a random sample from the probability mass function (13) can be easily generated. Therefore, the generation of samples from (9) can be performed as follows. First generate M from the discrete distribution with the probability mass function (13), and for a given $M = m$, λ can be generated from a $\text{GA}\Big(a + r, \sum_{i=1}^{r} t_{(i)} + b + mT^*\Big)$. Based on the generated samples, Bayes estimates and HPD credible intervals can be easily constructed.

Alternatively, since here the full conditionals have well-known distributions, the Gibbs sampling is more convenient to use to compute the Bayes estimates and to construct the credible intervals. It can be easily observed that

$$\pi_{EX}(M|\lambda, data) \sim \text{POI}(\theta e^{-\lambda T^*}), \tag{14}$$

whereas, $\pi(\lambda|M, data)$ has already been provided in (12). The following algorithm can be used for the above purpose.

ALGORITHM 1:

- Step 1: Choose λ_0 and m_0, initial values of λ and M, respectively.
- Step 2: For $i = 1, \ldots, B$, generate λ_i and m_i from $\pi(\lambda|M_{i-1}, data)$ and $\pi(M|\lambda_{i-1}, data)$, respectively.

- Step 3: Choose a suitable burn-in-period B^*, and discard the initial B^* values of λ_i and m_i.

- Step 4: If we denote $g_i = g(m_i, \lambda_i)$, for $i = B^*+1, \ldots, B$, the Bayes estimate of $g(M, \lambda)$ can be approximated as

$$\widehat{g}(M, \lambda) \approx \frac{1}{B - B^*} \sum_{i=B^*+1}^{B} g_i$$

- Step 5: To construct the $100(1-\beta)\%$ HPD credible intervals of $g(M, \lambda)$, first order g_i for $i = B^* + 1, \ldots, B$, as $g_{(B^*+1)} < \ldots < g_{(B)}$, then construct all the $100(1-\beta)\%$ credible intervals of $g(M, \lambda)$ as

$$(g_{(B^*+1)}, g_{(B^*+1+(1-\beta)B)}), \ldots, (g_{(\beta B)}, g_{(B)}).$$

Choose that interval which has the smallest length, see for example Chen and Shao (1999).

3.2 Weibull GOS Model

The likelihood function (1) is

$$L_{WE}(N, \alpha, \lambda | data) = \frac{N!}{(N-r)!} \alpha^r \lambda^r \prod_{i=1}^{r} t_{(i)}^{\alpha-1} e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)}^{\alpha} + (N-r)(T^*)^{\alpha}\right)}. \qquad (15)$$

Therefore, the joint posterior density function of N, α and λ is

$$\pi_{WE}(N, \alpha, \lambda | data) \propto \frac{\theta^N}{(N-r)!} \alpha^{a+r-1} e^{-\alpha(b - \sum_{i=1}^{r} \ln t_{(i)})} \lambda^{c+r-1} e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)}^{\alpha} + (N-r)(T^*)^{\alpha} + d\right)}. \qquad (16)$$

Similarly as before, the joint posterior density of $M = N - r$, α and λ becomes

$$\pi_{WE}(M, \alpha, \lambda | data) \propto \frac{\theta^M}{M!} \alpha^{a+r-1} e^{-\alpha(b - \sum_{i=1}^{r} \ln t_{(i)})} \lambda^{c+r-1} e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)}^{\alpha} + M(T^*)^{\alpha} + d\right)}. \qquad (17)$$

Moreover, under SEL function

$$\widehat{g}_B(M, \alpha, \lambda) = E(g(M, \alpha, \lambda)) = \frac{\sum_{m=0}^{\infty} \int_0^{\infty} \int_0^{\infty} g(m, \alpha, \lambda) \pi_{WE}(m, \alpha, \lambda | data) d\alpha d\lambda}{\sum_{m=0}^{\infty} \int_0^{\infty} \int_0^{\infty} \pi_{WE}(m, \alpha, \lambda | data) d\alpha d\lambda}, \qquad (18)$$

is the the Bayes estimate of $g(M, \alpha, \lambda)$. In general, it does not have a compact form and hence, we use MCMC technique to evaluate (18).

The full conditional distribution of α is

$$\pi_{WE}(\alpha|M, \lambda, data) \propto \alpha^{a+r-1} e^{-\alpha(b-\sum_{i=1}^{r} \ln t_{(i)})} e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)}^{\alpha} + M(T^*)^{\alpha}\right)}, \tag{19}$$

and we have the following result regarding the shape of (19).

Lemma 1: The full conditional PDF of α as given in (19) is log-concave.

Proof: See in the Appendix. ∎

Random samples can be easily generated from (19), see for example Devroye (1984) or Kundu (2008), Further, the full conditional distributions of λ and M are

$$\pi_{WE}(\lambda|\alpha, M, data) \sim \text{GA}\left(c + r, \sum_{i=1}^{r} t_{(i)}^{\alpha} + M(T^*)^{\alpha} + d\right) \tag{20}$$

and

$$\pi_{WE}(M|\alpha, \lambda, data) \sim \text{POI}\left(\theta e^{-\lambda(T^*)^{\alpha}}\right). \tag{21}$$

Therefore, we will be able to generate random samples from the full conditionals of α, λ and M. Hence using the Gibbs sampling procedure as in Algorithm 1, the Bayes estimates and the credible intervals can be constructed.

3.3 Generalized Exponential GOS Model

In this case for $N = r, r+1, \ldots$, $\lambda > 0$ and $\alpha > 0$, the likelihood function (1) becomes:

$$L_{GE}(N, \alpha, \lambda|data) = \frac{N!}{(N-r)!} \alpha^r \lambda^r e^{-\lambda \sum_{i=1}^{r} t_{(i)}} \prod_{i=1}^{r} (1 - e^{-\lambda t_{(i)}})^{\alpha-1} \left(1 - (1 - e^{-\lambda T^*})^{\alpha}\right)^{N-r}.$$

The posterior distribution of α, λ and N is

$$\begin{aligned}\pi_{GE}(N, \alpha, \lambda|data) \ \propto\ & \frac{\theta^N}{(N-r)!} \alpha^{a+r-1} e^{-b\alpha} e^{(\alpha-1)\sum_{i=1}^{r} \ln(1-e^{-\lambda t_{(i)}})} \lambda^{c+r-1} e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)} + d\right)} \\ & \times \left\{1 - (1 - e^{-\lambda T^*})^{\alpha}\right\}^{N-r}.\end{aligned}$$

The joint posterior distribution of $M = N - r$, α and λ is

$$\begin{aligned}\pi_{GE}(M, \alpha, \lambda|data) \;\; &\propto \;\; \frac{\theta^M}{M!}\alpha^{a+r-1}e^{-\alpha\left(b-\sum_{i=1}^{r}\ln(1-e^{-\lambda t_{(i)}})\right)}\lambda^{c+r-1}e^{-\lambda(d+\sum_{i=1}^{r}t_{(i)})} \\ &\quad \times e^{-\sum_{i=1}^{r}\ln(1-e^{-\lambda t_{(i)}})}\left\{1-(1-e^{-\lambda T^*})^{\alpha}\right\}^M . \end{aligned} \quad (22)$$

The Bayes estimate of any function $g(M, \alpha, \lambda)$ under the SEL function cannot be obtained in explicit form. Importance sampling can be used for constructing the Bayes estimate and the credible interval of $g(M, \alpha, \lambda)$. After some calculations, it can be seen that

$$\pi_{GE}(M, \alpha, \lambda|data) \propto \pi_{GE}(M|\alpha, \lambda, data) \times \pi_{GE}(\alpha|\lambda, data) \times \pi_{GE}(\lambda|data) \times h(m, \alpha, \lambda, data), \quad (23)$$

where

$$\begin{aligned} M|\alpha, \lambda, data \;\; &\sim \;\; \text{POI}(\theta(1-(1-e^{-\lambda T^*})^{\alpha})) \\ \alpha|\lambda, data \;\; &\sim \;\; \text{GA}\left(a+r, b-\sum_{i=1}^{r}\ln\left(1-e^{-\lambda t_{(i)}}\right)\right) \\ \lambda|data \;\; &\sim \;\; \text{GA}\left(c+r, \sum_{i=1}^{r}t_{(i)}+d\right) \end{aligned}$$

and

$$h(m, \alpha, \lambda, data) = \left(e^{\theta(1-(1-e^{-\lambda T^*})^{\alpha})-\sum_{i=1}^{r}\ln(1-e^{-\lambda t_{(i)}})}\right) \times \frac{1}{\left(b-\sum_{i=1}^{r}\ln\left(1-e^{-\lambda t_{(i)}}\right)\right)^{a+r}}.$$

It follows that if $b > 0$ and $c > 0$, then the right hand side of (23) is integrable. Moreover, it may be noted that $\left(b-\sum_{i=1}^{r}\ln\left(1-e^{-\lambda t_{(i)}}\right)\right)^{a+r}$ is always positive. We can use the following importance sampling procedure for the purpose.

Algorithm 2:

- Step 1: Generate λ_1, α_1 and m_1, where

$$\lambda_1 \;\sim \pi_{GE}(\lambda|data), \quad \alpha_1 \;\sim \pi_{GE}(\alpha|\lambda_1, data) \text{ and } \; m_1 \;\sim \pi_{GE}(M|\alpha_1, \lambda_1, data).$$

- Step 2: Repeat Step 1 B times and obtain $(m_1, \alpha_1, \lambda_1), \ldots, (m_B, \alpha_B, \lambda_B)$.

- Step 3: Obtain the Bayes estimate of $g(M, \alpha, \lambda)$ as

$$E(g(M, \alpha, \lambda)) = \frac{\sum_{i=1}^{B} g_i h(m_i, \alpha_i, \lambda_i, data)}{\sum_{i=1}^{B} h(m_i, \alpha_i, \lambda_i, data)}, \quad \text{where} \quad g_i = g(m_i, \alpha_i, \lambda_i).$$

- Step 4: To compute the HPD credible interval of $g(M, \alpha, \lambda)$, first compute

$$w_i = \frac{h(m_i, \alpha_i, \lambda_i, data)}{\sum_{i=1}^{B} h(m_i, \alpha_i, \lambda_i, data)}; \quad i = 1, 2, \ldots, B.$$

  Arrange $\{(g_1, w_1), \ldots, (g_B, w_B)\}$ as $\{(g_{(1)}, w_{[1]}), \ldots, (g_{(B)}, w_{[B]})\}$, where $g_{(1)} < \ldots < g_{(B)}$, and $w_{[i]}$'s are associated with $g_{(i)}$'s.

- Step 5: Then a $100(1-\beta)\%$ credible interval of $g(M, \alpha, \lambda)$ is $(g_{(L_\gamma)}, g_{(U_\gamma)})$, for $0 < \gamma < \beta$, where

$$\sum_{i=1}^{L_\gamma} w_{[i]} \le \gamma < \sum_{i=1}^{L_\gamma+1} w_{[i]} \quad \text{and} \quad \sum_{i=1}^{U_\gamma} w_{[i]} \le 1 + \gamma - \beta < \sum_{i=1}^{U_\gamma+1} w_{[i]}.$$

- Step 6: The $100(1-\beta)\%$ HPD credible interval of $g(M, \alpha, \lambda)$ is $(g_{(L_{\gamma^*})}, g_{(U_{\gamma^*})})$, such that the length of the credible interval is minimum, see for example Chen and Shao (1999).

4 Simulation Experiments

Extensive simulation experiments have been performed for different parameter values and for different termination points (T^*) under all the proposed models. To generate samples from a given distribution function, for a fixed N and for fixed parameter values, first we generate a sample of size N from the given distribution using the simple inverse transformation method. We order the generated sample, and consider those points which are less than or equal to T^*, and that is the required sample. We have adopted the Gibbs sampling algorithm as suggested in Section 3 for exponential and Weibull GOS models, and in the case of the generalized exponential GOS model, importance sampling technique has been incorporated. The results have been obtained based on 5000 replications. For Gibbs sampling procedure, for each replication, 2500 iterations are performed, and the initial 500 iterations are taken as burn-in. So, for a particular sample, the Bayes estimate and the credible interval are obtained based on 2000 sample values from the posterior density function. For importance sampling, 2000 sample values have been used for each replication.

The results are presented in Tables 2 - 7. For each parameter, we present the average estimates, the average lower and upper limits of the 95% HPD credible intervals, the average lengths and the coverage percentages. For each model, the results are presented in two tables. In the first table, T^* is varied while other parameters are kept fixed, and in the second table, T^* is fixed, where the other parameters are changing. We have considered both informative and non-informative priors. In the case of non-informative priors, we take $a = b = c = d = 0$, and it is improper.

It is observed that in all the cases considered here, the average Bayes estimates are very close to the true parameter values. From Tables 2, 4 and 6, it is observed that as T^* increases the length of the HPD credible intervals decrease, as expected. From Tables 3, 5 and 7, it is observed that for fixed N, the performance of the Bayes estimates does not depend much on the hyper-parameters. In the case of non-informative priors, the average Bayes estimates of N are slightly smaller than the true N, for all the models, although the length of the average HPD credible intervals is also smaller than the rest. In all the cases considered, it is observed that the coverage percentages are very close to the corresponding nominal value. Overall, the performance of the proposed methods is quite satisfactory.

5 Data Analysis

We use the following data set presented in Table 1. The data points represent the failure

Table 1: Data Set.

9	21	32	36	43	45	50	58	63	70	71	77
78	87	91	92	95	98	104	105	116	149	156	247
249	250	337	384	396	405	540	798	814	849		

times for errors detected during the development of software and time units are in days. See for example Osborne and Severini (2000) for the detailed description of the data. This data set has been analyzed by several authors, for example by Goel and Okumoto (1979), Jelinski and Moranda (1972), Raftery (1987) and Osborne and Severini (2000) using exponential GOS model. In Table 8, we have presented the results compiled by Osborne and Severini (2000) on different point estimates of N, and also two different confidence intervals. Various estimators of N are obtained by taking different stopping time T^*, and using exponential distribution.

We have analyzed the data set using three different GOS distributions namely: (i) Exponential GOS model (M_1), (ii) Weibull GOS model (M_2) and (iii) Generalized exponential GOS model (M_3). Since there is no prior information available, we have assumed improper priors in all these cases. Point estimates of the different parameters are presented in Table 9 and 95% HPD credible intervals for different parameters are presented in Tables 10 and 11.

Next we consider selection of appropriate model out of three models under study. Bayes factor will be used for the purpose of selection of correct model from the set of models. If we have two models, say, M_i and M_j, then the Bayes factor is given by $2\ln B_{ij}$, where

$$B_{ij} = \frac{Prob(data|M_i)}{Prob(data|M_j)}.$$

Here $Prob(data|M_i)$ and $Prob(data|M_j)$ are the marginal probabilities of data under M_i and M_j, respectively. To compute $Prob(data|M_i)$, we adopt the following procedure. Observe that

$$\begin{aligned} Prob(data|M_1) &= \sum_{N=0}^{\infty} \int_0^{\infty} P(data, N, \lambda|M_1) d\lambda \\ &= \sum_{N=r}^{\infty} \int_0^{\infty} P(data|N, \lambda, M_1) P(N, \lambda|M_1) d\lambda \\ &= \frac{d^c}{\Gamma(c)} \sum_{N=r}^{\infty} \int_0^{\infty} \frac{e^{-\theta}\theta^N}{(N-r)!} \lambda^{r+c-1} e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)} + (N-r)T^* + d\right)} d\lambda \\ &= \frac{d^c \theta^r}{\Gamma(c)} \sum_{M=0}^{\infty} \frac{e^{-\theta}\theta^M}{M!} \frac{\Gamma(r+c)}{\left(d + \sum_{i=1}^{r} t_{(i)} + MT^*\right)^{r+c}}. \end{aligned}$$

Therefore, for non-informative prior

$$Prob(data|M_1) = e^{-r} r^r \Gamma(r) \sum_{M=0}^{\infty} \frac{r^M}{M! \left(\sum_{i=1}^{r} t_{(i)} + MT^*\right)^{r+c}}.$$

For non-informative prior, it can be shown after some calculations that

$$\begin{aligned} Prob(data|M_2) &= e^{-r} r^r \Gamma(r) \sum_{M=0}^{\infty} \frac{r^M}{M!} \times \int_0^{\infty} \frac{\alpha^{r-1} \prod_{i=1}^{r} t_{(i)}^{\alpha-1}}{\left(\sum_{i=1}^{r} t_{(i)}^{\alpha} + M(T^*)^{\alpha}\right)^r} d\alpha \\ &= e^{-r} r^r \Gamma(r) \sum_{M=0}^{\infty} \frac{r^M}{M!} \times A(M), \end{aligned}$$

where

$$A(M) = \int_0^{\infty} \frac{\alpha^{r-1} \prod_{i=1}^{r} t_{(i)}^{\alpha-1}}{\left(\sum_{i=1}^{r} t_{(i)}^{\alpha} + M(T^*)^{\alpha}\right)^r} d\alpha.$$

Note that $A(M)$ can be easily computed using importance sampling technique. Similarly, $Prob(data|M_3)$ can also be obtained using importance sampling technique.

For the given data set, we have provided the Bayes factors namely $2\ln(B_{12})$, $2\ln(B_{13})$ and $2\ln(B_{23})$ in Table 12. From Table 12, it is immediate that for all values of T^*, the exponential distribution is a clear choice over the Weibull or generalized exponential distributions. We have also computed the log-predictive likelihood values for the three different models as it has been suggested by Kuo and Yang (1996) for the model choice. We have used the first 26 failures as the training sample and the last 5 failures as the predictive sample. Based on the same non-informative prior, we obtained the log-predictive likelihood values for exponential, Weibull and generalized exponential as -37.15, -37.94 and -38.13, respectively. Therefore, it also indicates the exponential distribution.

It is clear that Bayes estimator under non-informative priors behave very similarly to the different estimators obtained based on the frequentist approach. On the other hand, the proposed method has an advantage that it is quite simple to implement, and even for small T^*, it produces estimators which are finite and with finite credible intervals.

6 Conclusion

In this paper we have considered Bayesian estimation of N and other unknown parameters in GOS models. We have considered three different lifetime distributions namely (i) exponential, (ii) Weibull and (iii) generalized exponential distributions. Based on fairly general priors, the Bayesian inferences are obtained for the unknown parameters. It may be mentioned that Raftery (1987) considered the same problem and provided a Bayesian solution. But the author has mainly restricted the attention on the exponential family. Although he has indicated the generalization to the non-exponential family, no proper method has been suggested for construction of credible intervals of the unknown parameters. In our case although we have considered only three models, but the method can be extended for other distributions also. Since the implementation of the proposed method is quite straight forward, it can be used quite conveniently in practice in a very general set up.

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Appendix

Proof of Lemma 1: The full conditional of α is

$$\pi_{WE}(\alpha|M,\lambda,data) = k\alpha^{a+r-1}e^{-\alpha(b-\sum \ln t_{(i)})}e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)}^{\alpha}+M(T^*)^{\alpha}\right)},$$

where

$$k^{-1} = \int_0^{\infty} \alpha^{a+r-1}e^{-\alpha(b-\sum \ln t_{(i)})}e^{-\lambda\left(\sum_{i=1}^{r} t_{(i)}^{\alpha}+M(T^*)^{\alpha}\right)}d\alpha.$$

Now,

$$\ln \pi(\alpha|M,\lambda,data) = \ln k + (\alpha+r-1)\ln\alpha - \alpha(b-\sum \ln t_{(i)}) - \lambda g(\alpha),$$

where

$$g(\alpha) = \sum_{i=1}^{r} t_{(i)}^{\alpha} + M(T^*)^{\alpha}.$$

Hence,

$$\frac{d^2 \ln \pi(\alpha|M,\lambda,data)}{d\alpha^2} = -\frac{a+r-1}{\alpha^2} - \lambda\frac{d^2 g(\alpha)}{d\alpha^2}.$$

Now $\frac{d^2 g(\alpha)}{d\alpha^2} = \sum_{i=1}^{r}(\ln t_{(i)})^2 t_{(i)}^{\alpha} + M(\ln T^*)^2(T^*)^{\alpha} > 0$ and $a+r-1 > 0$, so $\dfrac{d^2 \ln \pi(\alpha|M,\lambda,data)}{d\alpha^2} < 0$. Hence $\pi_{WE}(\alpha|M,\lambda,data)$ is log-concave.

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Table 2: Exponential GOS Model Results (Fixed Parameters).

Parameters		N=30, $\theta = 30$, $\lambda = 2$, $a = 5$, $b = 2.5$				
		$T^* = 0.25$	$T^* = 0.5$	$T^* = 0.75$	$T^* = 1.0$	$T^* = 5.0$
r	Observed (Average)	11.43	18.87	23.42	26.11	30.01
N	Estimate	30.07	30.19	30.27	30.31	30.05
	HPD	(21.37, 40.23)	(22.63, 39.23)	(24.67, 37.56)	(26.25, 36.24)	(28.23, 31.89)
	(Av. Length)	(18.86)	(16.60)	(12.89)	(9.99)	(3.66)
	(Cov. Perc.)	0.92	0.93	0.93	0.94	0.94
λ	Estimate	2.02	2.10	2.09	2.03	2.03
	HPD	(0.98, 3.18)	(1.06, 3.24)	(1.11, 3.16)	(1.13, 3.09)	(1.41, 2.75)
	(Av. Length)	(2.20)	(2.18)	(2.05)	(1.96)	(1.34)
	(Cov. Perc.)	0.93	0.93	0.94	0.95	0.94

Table 3: Exponential GOS Model Results (Fixed T^*).

Parameters		T^*=0.75				
		N=30, θ=r $\lambda = 2$, $a = 0$ $b = 0$	$N = 30 = \theta$ $\lambda = 3$, $a = 15$ $b = 5$	$N = 30 = \theta$ $\lambda = 2$, $a = 5$ $b = 2.5$	$N = 30 = \theta$ $\lambda = 2$, $a = 10$ $b = 5$	$N = 100 = \theta$ $\theta = 100$, $\lambda = 2$ $a = 5$, $b = 2.5$
r	Observed (Average)	23.41	26.79	23.16	23.25	77.51
N	Estimate HPD (Av. Length) (Cov. Per.)	27.18 (23.47, 32.33) (8.86) 0.93	30.39 (26.54, 34.75) (8.21) 0.92	30.25 (24.62, 37.55) (12.93) 0.92	30.22 (24.54, 37.12) (12.58) 0.93	100.17 (87.54, 115.12) (27.58) 0.94
λ	Estimate HPD (Av. Length) (Cov. Per.)	2.67 (1.27, 4.18) (2.91) 0.94	3.06 (1.93, 4.18) (2.25) 0.95	2.03 (1.09, 3.18) (2.09) 0.94	2.03 (1.21, 2.99) (1.78) 0.94	2.06 (1.40, 2.75) (1.36) 0.95

Table 4: Weibull GOS Model Results (Fixed Parameters).

Parameters		N=30, $\theta = 30$, $\alpha = 2$, $\lambda = 1$, $a = 5$, $b = 2.5$, $c = 2$, $d = 2$				
		$T^* = 0.5$	$T^* = 1.0$	$T^* = 1.25$	$T^* = 1.5$	$T^* = 2.0$
r	Observed (Average)	6.77	18.98	23.19	26.89	29.51
N	Estimate HPD (Av. Length) (Cov. Per.)	30.11 (20.78, 40.96) (20.19) 0.91	29.88 (21.80, 39.48) (17.67) 0.93	30.19 (24.26, 37.89) (13.63) 0.95	30.56 (26.56, 36.82) (10.26) 0.94	30.88 (29.33, 33.97) (4.64) 0.95
α	Estimate HPD (Av. Length) (Cov. Per.)	2.03 (1.03, 3.05) (2.02) 0.92	2.14 (1.39, 2.91) (1.52) 0.92	2.09 (1.36, 2.91) (1.51) 0.94	2.15 (1.42, 2.81) (1.40) 0.94	2.01 (1.39, 2.68) (1.29) 0.95
λ	Estimate HPD (Av. Length) (Cov. Per.)	1.11 (0.25, 2.04) (1.79) 0.92	1.07 (0.47, 1.88) (1.41) 0.94	1.07 (0.48, 1.75) (1.27) 0.94	1.03 (0.53, 1.58) (1.05) 0.95	0.99 (0.55, 1.45) (0.90) 0.95

Table 5: Weibull GOS Model Results (Fixed T^*).

Parameters		T^*=1.25				
		$N = 30,\ \theta =$ r $\alpha = 2, \lambda = 1$ $a = 0, b = 0$ $c = 0, d = 0$	$N = 30 = \theta$ $\alpha = 1, \lambda = 1$ $a = 5, b = 5$ $c = 2, d = 2$	$N = 30 = \theta$ $\alpha = 2, \lambda = 1$ $a = 5, b = 2.5$ $c = 2, d = 2$	$N = 30 = \theta$ $\alpha = 2, \lambda = 1$ $a = 10, b = 5$ $c = 5, d = 5$	$N = 100 = \theta$ $\alpha = 2, \lambda = 1$ $a = 5, b = 2.5$ $c = 2, d = 2$
r	Observed (Average)	23.67	21.38	23.54	23.76	79.46
N	Estimate HPD (Av. Length) (Cov. Per.)	26.99 (23.65, 32.23) (8.58) 0.93	29.89 (22.69, 38.76) (16.07) 0.93	30.11 (24.19, 37.97) (13.78) 0.94	30.18 (24.51, 37.69) (13.18) 0.95	99.91 (86.13, 116.24) (30.11) 0.95
α	Estimate HPD (Av. Length) (Cov. Per.)	2.24 (1.48, 3.21) (1.73) 0.91	1.09 (0.69, 1.45) (0.76) 0.95	2.08 (1.43, 2.83) (1.40) 0.94	2.11 (1.37, 2.71) (1.34) 0.92	2.05 (1.63, 2.48) (0.85) 0.95
λ	Estimate HPD (Av. Length) (Cov. Per.)	1.45 (0.67, 2.27) (1.60) 0.93	1.09 (0.45, 1.86) (1.41) 0.93	1.07 (0.50, 1.77) (1.28) 0.94	1.06 (0.51, 1.62) (1.11) 0.94	1.03 (0.66, 1.45) (0.79) 0.94

Table 6: Generalized Exponential GOS Model Results (Fixed Parameters).

Parameters		N=30, $\theta = 30$, $\alpha = 2$, $\lambda = 1$, $a = 5$, $b = 2.5$, $c = 2$, $d = 2$				
		$T^* = 0.5$	$T^* = 1.0$	$T^* = 1.5$	$T^* = 2.0$	$T^* = 5.0$
r	Observed (Average)	4.39	12.15	18.27	22.11	29.76
N	Estimate	29.24	29.18	29.21	30.12	30.10
	HPD	(19.49,38.51)	(20.15,36.28)	(21.55,37.75)	(23.59, 38.68)	(29.58, 33.10)
	(Av. Length)	(18.02)	(16.13)	(16.20)	(15.09)	(3.52)
	(Cov. Per.)	0.92	0.92	0.94	0.93	0.93
α	Estimate	2.08	2.10	2.08	2.08	2.02
	HPD	(1.05, 3.29)	(1.17, 3.20)	(1.20, 3.22)	(1.18, 3.22)	(1.22, 2.75)
	(Av. Length)	(2.24)	(2.03)	(2.20)	(2.04)	(1.53)
	(Cov. Per.)	0.93	0.94	0.95	0.95	0.94
λ	Estimate	1.01	1.07	1.07	1.06	0.97
	HPD	(0.41, 1.92)	(0.59, 1.84)	(0.58, 1.77)	(0.57, 1.67)	(0.60, 1.19)
	(Av. Length)	(1.51)	(1.25)	(1.19)	(1.10)	(0.59)
	(Cov. Per.)	0.93	0.94	0.94	0.95	0.95

Table 7: GENERALIZED EXPONENTIAL GOS MODEL RESULTS (FIXED T^*).

Parameters		T^*=1.25				
		$N=30$, $\theta=$ r $\alpha=2$, $\lambda=1$ $a=0$, $b=0$ $c=0$, $d=0$	$N=30=\theta$ $\alpha=1$, $\lambda=1$ $a=5$, $b=5$ $c=2$, $d=2$	$N=30=\theta$ $\alpha=2$, $\lambda=1$ $a=5$, $b=2.5$ $c=2$, $d=2$	$N=30=\theta$ $\alpha=2$, $\lambda=1$ $a=10$, $b=5$ $c=5$, $d=5$	$N=100=\theta$ $\alpha=2$, $\lambda=1$ $a=5$, $b=2.5$ $c=2$, $d=2$
r	Observed (Average)	18.18	23.11	18.23	18.17	60.51
N	Estimate	22.57	29.31	29.78	29.88	94.12
	HPD	(18.17, 27.43)	(24.19, 34.37)	(21.54, 37.74)	(21.56, 38.59)	(81.23, 104.45)
	(Av. Length)	(9.26)	(10.18)	(16.20)	(16.03)	(23.22)
	(Cov. Per.)	0.93	0.91	0.94	0.95	0.94
α	Estimate	2.91	1.03	2.10	2.06	2.05
	HPD	(1.33, 4.44)	(0.67, 1.61)	(1.18, 3.28)	(1.27, 2.94)	(1.53, 2.60)
	(Av. Length)	(3.11)	(0.94)	(2.10)	(1.67)	(1.07)
	(Cov. Per.)	0.93	0.94	0.94	0.92	0.95
λ	Estimate	1.75	1.12	1.11	1.01	1.04
	HPD	(0.84, 2.56)	(0.77, 2.03)	(0.68, 1.88)	(0.62, 1.62)	(0.88, 1.49)
	(Av. Length)	(1.72)	(1.26)	(1.20)	(1.00)	(0.61)
	(Cov. Per.)	0.94	0.93	0.94	0.94	0.95

Table 8: ESTIMATES OF N COMPILED BY OSBORNE AND SEVERINI (2000).

Stopping Time (T^*)	r	Point Estimates				95% Conf. Int.	
		$\widehat{N}_{ML}$	$\widehat{N}_{JR}$	$\widehat{N}_{1/2}$	$\widehat{N}_U$	LR	ILR
50	7	∞	25	17	17	$(8, \infty)$	(7, 248)
100	18	∞	116	82	65	$(30, \infty)$	(24, 954)
250	26	31	34	30	33	(26, 94)	(27, 81)
550	31	31	32	31	32	(31,36)	(31, 38)
800	32	32	32	32	32	(32,33)	(32,34)
850	34	34	35	34	34	(34, 37)	(34, 39)

$\widehat{N}_{ML}$ is the maximum likelihood estimate of N.
$\widehat{N}_{JR}$ is the estimator proposed by Joe and Reid (1985).
$\widehat{N}_{1/2}$ is the integrated likelihood estimator (midpoint prior) by Osborne and Severini (2000).
$\widehat{N}_U$ is the integrated likelihood estimator (uniform prior) by Osborne and Severini (2000).
LR is the likelihood ratio confidence interval.
ILR is the integrated likelihood ratio confidence interval.

Table 9: POINT ESTIMATES OF PARAMETERS FOR DIFFERENT MODELS.

Stopping Time (T^*)	r	M1		M2			M3		
		$\widehat{N}$	$\widehat{\lambda}$	$\widehat{N}$	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{N}$	$\widehat{\alpha}$	$\widehat{\lambda}$
50	7	9.84	0.020	10.25	1.233	0.025	9.17	3.79	0.046
100	18	24.6	0.010	25.87	1.412	0.022	23.50	3.24	0.024
250	26	30.59	0.007	37.25	1.319	0.017	27.38	2.80	0.016
550	31	32.55	0.006	44.55	1.278	0.011	32.02	1.37	0.007
800	32	32.54	0.0056	45.18	1.118	0.009	32.40	1.22	0.006
850	34	35.24	0.0043	48.89	1.178	0.007	35.05	1.02	0.004

Table 10: 95% HPD Credible intervals of N under different models.

Stopping Time	r	M1	M2	M3
50	7	(7,14)	(7,14)	(7,14)
100	18	(20,32)	(21,33)	(18,30)
250	26	(26,36)	(31,46)	(26,33)
550	31	(31,36)	(37,53)	(31,37)
800	32	(32,34)	(38,55)	(32,34)
850	34	(34,48)	(40,56)	(34,38)

Table 11: 95% HPD Credible intervals of other parameters under different models.

Stopping Time (T^*)	r	M1	M2		M3	
		$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\lambda}$
50	7	(0.004,0.038)	(0.56,4.231)	(0.011,0.068)	(0.79,8.23)	(0.012,0.084)
100	18	(0.005,0.016)	(0.79,3.891)	(0.009,0.059)	(1.38,4.64)	(0.009,0.033)
250	26	(0.004,0.012)	(0.82,3.167)	(0.008,0.045)	(1.11,3.68)	(0.006,0.020)
550	31	(0.003,0.009)	(0.89,2.671)	(0.003,0.021)	(0.76,2.18)	(0.004,0.011)
800	32	(0.0035,0.0079)	(0.91,2.567)	(0.001,0.018)	(0.69,1.96)	(0.003,0.009)
850	34	(0.0025,0.0061)	(0.94,1.789)	(0.001,0.011)	(0.65,1.52)	(0.003,0.007)

Table 12: Bayes factor for model selection.

Stopping Time (T^*)	r	2 $\log(B_{12})$	2 $\log(B_{13})$	2 $\log(B_{23})$
50	7	24.12	12.4	11.7
100	18	57.17	23.8	32.78
250	26	74.67	14.8	59.59
550	31	77.21	31.4	46.61
800	32	80.89	29.7	50.38
850	34	77.19	31.9	45.23