Inference for a Step-stress Model with Competing Risks for Failure from the Generalized Exponential Distribution under Type-I Censoring

David Han[1*] **and Debasis Kundu**[2]

[1] *Department of Management Science & Statistics, University of Texas at San Antonio, Texas, USA 78249*

[2] *Department of Mathematics & Statistics, Indian Institute of Technology at Kanpur, India 208016*

Abstract – In a reliability experiment, accelerated life-testing allows higher-than-normal stress levels on test units. In a special class of accelerated life tests known as step-stress tests, the stress levels are increased at some pre-planned time points, allowing the experimenter to obtain information on the lifetime parameters more quickly than under normal operating conditions. Also, when a test unit fails, there are often several risk factors associated with the cause of failure (*i.e.*, mechanical, electrical, etc.). In this article, the step-stress model under Type-I censoring is considered when the different risk factors have s-independent generalized exponential lifetime distributions. With the assumption of cumulative damage, the point estimates of the unknown scale and shape parameters of the different causes are derived using the maximum likelihood approach. Using the asymptotic distributions and the parametric bootstrap method, we also discuss the construction of confidence intervals for the parameters. The precision of the estimates and the performance of the confidence intervals are assessed through extensive Monte Carlo simulations, and lastly, the methods of inference discussed

*Corresponding author: *david.han@utsa.edu*

here is illustrated with examples.

Index Terms – accelerated life-testing, competing risks, confidence interval, cumulative damage model, generalized exponential distribution, maximum likelihood estimation, step-stress model, Type-I censoring

ACRONYM[1]

ALT	accelerated life test
BCa	bias-corrected and accelerated
CDF	cumulative distribution function
CI	confidence interval
GE	generalized exponential distribution
MLE	maximum likelihood estimate (or estimator)
MSE	mean squared error
PDF	probability density function
RAB	relative absolute bias
$s-$	implies: statistical(ly)

NOTATION

k	number of stress levels used in the test
x_i	i-th stress level for $i = 0, 1, \ldots, k$
τ_i	i-th pre-fixed stress change time point
Δ_i	step duration at the stress level x_i (*viz.*, $\Delta_i = \tau_i - \tau_{i-1}$)
r	number of competing risk factors

[1] The singular and plural forms of an acronym are always spelled the same.

α_j	shape parameter of the GE distribution for the risk factor j, $j = 1, 2, \dots, r$
λ_{ij}	scale parameter of the GE distribution for the risk factor j at the stress level x_i
T_j	failure time by the risk factor j
$G_j(t)$	CDF of T_j
$g_j(t)$	PDF of T_j
$h_j(t)$	hazard rate function of the risk factor j
$h_j^{\pi}(t)$	hazard proportion of the risk factor j at time $t > 0$
π_{ij}	relative risk imposed on a test unit at the stress level x_i due to the risk factor j
T	overall failure time of a test unit $\big(viz., T = \min\{T_1, T_2, \dots, T_r\}\big)$
$S(t)$	survival function of T
$F(t)$	CDF of T
$f(t)$	PDF of T
C	indicator variable for the cause of failure, $1 \le C \le r$
$f_{T,C}(t, j)$	joint distribution of (T, C)
n_{ij}	(observed) number of failures at the stress level x_i due to the risk factor j
$n_{i\oplus}$	(observed) total number of failures at the stress level x_i (*i.e.*, in time interval $[\tau_{i-1}, \tau_i)$)
$n_{\oplus j}$	(observed) total number of failures by the risk factor j
$n_{\oplus\oplus}$	(observed) accumulated number of failures until the censoring time τ_k
n	initial sample size
B	bootstrap sample size
$\delta(\cdot)$	indicator function that takes on the value of 1 if the argument is true and 0 otherwise

1 INTRODUCTION

With the continuous improvement in manufacturing design and technology, modern products are becoming highly reliable with substantially long life-spans. This in turn, however, makes life-testing under normal operating conditions an extremely difficult or even undesirable process for collecting sufficient information about the failure time distribution of the products, as they requires time-consuming and expensive tests. Hence, the standard life-testing methods are often inappropriate, especially when developing prototypes of new products. One effective solution to this problem is ALT wherein the test units are subjected to more severe stress levels than normal in order to cause rapid failures. ALT allows the experimenter not only to collect enough failure data in a shorter period of time but also to draw inference about the relationship of lifetime with the external stress variables.

1.1 *Step-stress ALT*

There are three stress loading schemes applied in ALT: constant-stress, step-stress, and progressive-stress. A constant-stress ALT is the most common type where each test unit is subjected to only one chosen stress level until its failure or the termination of the test, whichever occurs first. On the other hand, a progressive-stress ALT lets the stress level to increase linearly and continuously on any surviving test units. A step-stress loading scheme is somewhere between the constant-stress and the progressive-stress schemes since it allows the stress level to increase gradually at some pre-planned time points during the test for flexibility and adjustability. The step-stress ALT has attracted great attention in the reliability literature and there are now three fundamental models for the effect of increased stress levels on the lifetime distribution of a test unit. These are the tampered random variable model, the tampered hazard model, and the cumulative exposure model. The *cumulative exposure* or *cumulative damage* model has been further discussed and generalized by Bagdonavicius [1] and Nelson [2]. It relates the lifetime distribution at one stress level to the lifetime distribution at the next stress level by assuming that the residual life of a unit depends only on the cumulative exposure that unit had experienced with no memory of how this exposure was accumulated. Using this model, Miller & Nelson [3] studied the optimum plans for a simple step-stress ALT under complete sampling. Bai, Kim & Lee [4] then extended the result to the Type-I censoring case while Bai & Chun [5], and Liu & Qiu [6] extended it with s-independent competing risks. A Bayes model was developed by van Dorp *et al.* [7] for a step-stress ALT under exponential distribution while Yuan, Liu & Kuo [8] used the Bayesian approach for planning an optimal step-stress ALT. Recently, exact conditional inference for a step-stress model with exponential competing risks was developed by Balakrishnan & Han [9], and Han & Balakrishnan [10]. Gouno, Sen & Balakrishnan [11], and Balakrishnan & Han [12] discussed the problem of determining the optimal stress duration under progressive Type-I censoring; see also Han *et al.* [13] for some related comments. More recently, Han & Ng [14] quantified the advantage of using the step-stress ALT relative to the constant-stress ALT under several optimality criteria in the situations of complete sampling and Type-I censoring.

1.2 *Competing Risks Model*

In reliability analysis, often a test unit is exposed to several fatal risk factors and its failure is associated with one of such. In practice, it is usually not possible to study each risk factor under an isolating condition. Thus, it is rather necessary to assess each risk factor in the presence of other risk factors; a process known as the competing risks analysis. For analyzing a competing risks model, each complete observation must be in a bivariate format composed of the failure time and the corresponding cause of failure. The causes of failure can be assumed independent or dependent. In most situations, the analysis of competing risks data assumes s-independent causes of failure. Although a dependent risk structure might be more realistic, there is a concern about the identifiability of the underlying model. As discussed by Crowder [15], and Kalbfleisch & Prentice [16], without the information on covariates, it is not possible to test the assumption of s-independent risks. Prentice *et al.* [17] also summarized the two approaches of modeling the competing risks data: the cause-specific hazard functions and the latent failure times for each risk factor. Berkson & Elveback [18], Cox [19], Crowder [20], and Park [21] have all investigated the competing risks models with each risk factor having some specific parametric lifetime distributions. In this paper, we consider the case when the lifetime distribution of each risk factor is two-parameter GE.

1.3 *Generalized Exponential Distribution*

The GE distribution is a member of the three-parameter exponentiated Weibull distribution and was introduced by Mudholkar & Srivastava [22] as an alternative to the popular Weibull, gamma, and log-normal distributions. Gupta & Kundu [23] observed that this particular distribution can be used quite effectively to analyze many lifetime data, providing a better fit (in terms of the lower Kolmogorov-Smirnov distance or higher log-likelihood), particularly in place of the two-parameter Weibull or gamma distribution. Since then, it has received a considerable attention in the literature. From the literature survey, the preferred method of parameter estimation in many recent studies was found to be the MLE with the support of Gupta & Kundu [23]. From extensive simulation studies to compare the performances of different estimators, they observed that for large sample sizes, all the estimators behaved in a similar manner but for small sample sizes, the performances of the MLE were better than the rest; see Chen & Lio [24] for related comments. For a concise review of some recent developments for the GE distribution, readers may refer to Gupta & Kundu [25].

In this paper, we consider the problem of point and interval estimations for a general step-stress model under Type-I censoring when the lifetime distributions of the different risk factors are s-independent GE. The rest of the paper is organized as follows. Using the cumulative damage model for the effect of changing stress in step-stress ALT, Section 2 describes the model under study and derives the MLE of the scale and shape parameters of different risk factors. Based on the asymptotic distributions of the MLE, we construct the confidence intervals for the unknown parameters as well as the confidence intervals by a parametric bootstrap method in Section 3. In Section 4, the precision of the estimates and the performance of the confidence intervals are investigated in terms of bias, MSE, and probability coverage via extensive Monte Carlo simulations. Section 5 presents a real dataset as well as a numerical example to illustrate the methods of inference developed in this article, and Section 6 is devoted to some concluding remarks and future works in this direction.

2 MODEL DESCRIPTION AND MLE

Let us first define $(x_0 <)\ x_1 < x_2 < \ldots < x_k$ to be the k (≥ 2) ordered stress levels with x_0 being the normal use stress level, and let $0 \equiv \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \infty$ to be the pre-fixed stress change time points being used in the step-stress ALT. A random sample of n identical units is placed on the test under the initial stress level x_1 (or x_0 for a partially accelerated life test, PALT). Each failure time is then successively recorded along with the information about the risk factor that caused each failure. At the first pre-fixed time τ_1, the stress level is increased to x_2 and the life test continues until the next pre-fixed time τ_2 at which the stress level is increased to x_3. The life test continues in this fashion until the pre-specified censoring time τ_k. When all n units fail before τ_k or when τ_k is unbounded (*viz.*, $\tau_k \to \infty$), then a complete set of failure observations would result for this step-stress test (*viz.*, no censoring). Suppose each unit fails by one of r (≥ 2) fatal risk factors and the time-to-failure by each competing risk has an s-independent GE distribution which obeys the cumulative damage model. With a constant shape parameter $\alpha_j > 0$ for the risk factor j across the stress levels being used, let $\lambda_{ij} > 0$ be the scale parameter for the risk factor j at the stress level x_i for $1 \leq i \leq k$ and $1 \leq j \leq r$. Then, the CDF of the lifetime T_j due to the risk factor j is given by

$$G_j(t) = G_j(t; \boldsymbol{\lambda}_{\star j}, \alpha_j) = \left[1 - \exp\left(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\right)\right]^{\alpha_j}$$

$$\text{if } \begin{cases} \tau_{i-1} \leq t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \leq t < \infty & \text{for } i = k \end{cases} \tag{2.1}$$

for $1 \le j \le r$ where $\boldsymbol{\lambda}_{\star j} = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$, and $\Delta_i = \tau_i - \tau_{i-1}$ is the step duration at the stress level x_i. The corresponding PDF of T_j is given by

$$
\begin{aligned}
g_j(t) = g_j(t; \boldsymbol{\lambda}_{\star j}, \alpha_j) &= \alpha_j \lambda_{ij} \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big) \\
&\quad \times \left[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\right]^{\alpha_j - 1} \\
&\quad \text{if} \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases}
\end{aligned} \tag{2.2}
$$

for $1 \le j \le r$. Since only the smallest of $T_1, T_2, \ldots, T_r$ is observed, let $T = \min\{T_1, T_2, \ldots, T_r\}$ denote the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$
\begin{aligned}
F(t) = F(t; \boldsymbol{\lambda}, \boldsymbol{\alpha}) &= 1 - S(t) = 1 - \prod_{j=1}^{r} \big(1 - G_j(t)\big) \\
&= 1 - \prod_{j=1}^{r} \left\{ 1 - \left[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\right]^{\alpha_j} \right\} \\
&\quad \text{if} \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases}
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
f(t) = f(t; \boldsymbol{\lambda}, \boldsymbol{\alpha}) &= \left[\sum_{j=1}^{r} h_j(t)\right] \left[\prod_{j=1}^{r} \big(1 - G_j(t)\big)\right] = \left[\sum_{j=1}^{r} h_j(t)\right] S(t) \\
&= \left[\sum_{j=1}^{r} h_j(t)\right] \prod_{j=1}^{r} \left\{ 1 - \left[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t - \tau_{i-1})\Big)\right]^{\alpha_j} \right\} \\
&\quad \text{if} \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases}
\end{aligned} \tag{2.4}
$$

respectively, where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\star 1}, \boldsymbol{\lambda}_{\star 2}, \ldots, \boldsymbol{\lambda}_{\star r})$ with $\boldsymbol{\lambda}_{\star j} = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, and $h_j(t)$ is the hazard rate function of the risk factor j defined by

$$\begin{aligned} h_j(t) &= h_j(t; \boldsymbol{\lambda}_{\star j}, \alpha_j) = \frac{g_j(t)}{1 - G_j(t)} \\ &= \frac{\alpha_j \lambda_{ij} \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t-\tau_{i-1})\Big)\Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t-\tau_{i-1})\Big)\Big]^{\alpha_j - 1}}{1 - \Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t-\tau_{i-1})\Big)\Big]^{\alpha_j}} \\ & \qquad \text{if } \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases} \end{aligned} \tag{2.5}$$

for $1 \le j \le r$. Furthermore, let C denote the indicator variable for the cause of failure. Then, the joint distribution of (T, C) is given by

$$\begin{aligned} f_{T,C}(t,j) &= g_j(t) \prod_{\substack{j'=1 \\ j' \neq j}}^{r} \big(1 - G_{j'}(t)\big) = h_j(t)\ S(t) \\ &= \alpha_j \lambda_{ij} \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t-\tau_{i-1})\Big)\Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj}\Delta_l - \lambda_{ij}(t-\tau_{i-1})\Big)\Big]^{\alpha_j - 1} \\ & \qquad \times \prod_{\substack{j'=1 \\ j' \neq j}}^{r} \Big\{1 - \Big[1 - \exp\Big(-\sum_{l=1}^{i-1} \lambda_{lj'}\Delta_l - \lambda_{ij'}(t-\tau_{i-1})\Big)\Big]^{\alpha_{j'}}\Big\} \\ & \qquad \text{if } \begin{cases} \tau_{i-1} \le t < \tau_i & \text{for } i = 1, 2, \ldots, k-1 \\ \tau_{k-1} \le t < \infty & \text{for } i = k \end{cases} \end{aligned} \tag{2.6}$$

for $t > 0$ and $1 \le j \le r$. Based on these, the relative risk imposed on a test unit at the stress level x_i due to the risk factor j is given by

$$\begin{aligned} \pi_{ij} = Pr\big\{C = j \big| \tau_{i-1} < T < \tau_i\big\} &= [S(\tau_{i-1}) - S(\tau_i)]^{-1} \int_{\tau_{i-1}}^{\tau_i} h_j(t) S(t)\ dt \\ &= E\big[h_j^{\pi}(T) \big| \tau_{i-1} < T < \tau_i\big] \end{aligned}$$

for $1 \le i \le k$ and $1 \le j \le r$ where $h_j^{\pi}(t)$ is the hazard proportion of the risk factor j at time $t > 0$ defined by

$$h_j^{\pi}(t) = h_j(t) \Big/ \sum_{j'=1}^{r} h_{j'}(t), \qquad t > 0.$$

Hence, the relative risks are simply the s-expected proportions of each hazard rate for the corresponding risk factor in the given time frame of stress level.

With the step-stress ALT scheme described as above, the following ordered failure times will be observed:

$$\Big\{\tau_{i-1} < t_{i;1} < t_{i;2} < \cdots < t_{i;n_{i\oplus}} < \tau_i\Big\}$$

for $i = 1, 2, \ldots, k$ where $n_{i\oplus}$ denotes the (observed) total number of units failed at the stress level x_i (*i.e.*, in time interval $[\tau_{i-1}, \tau_i)$) and $t_{i;l}$ denotes the l-th ordered failure time of $n_{i\oplus}$ units at the stress level x_i, $l = 1, 2, \ldots, n_{i\oplus}$. Let n_{ij} denote the (observed) number of units failed at the stress level x_i due to the risk factor j and let $n_{\oplus j}$ denote the (observed) total number of units failed by the risk factor j. Also, let $n_{\oplus\oplus}$ ($\leq n$) denote the (observed) accumulated number of failures until the censoring time τ_k according to the testing scheme such that $n_{i\oplus} = \displaystyle\sum_{j=1}^{r} n_{ij}$, $n_{\oplus j} = \displaystyle\sum_{i=1}^{k} n_{ij}$, and $n_{\oplus\oplus} = \displaystyle\sum_{i=1}^{k} n_{i\oplus} = \sum_{j=1}^{r} n_{\oplus j} = \sum_{i=1}^{k}\sum_{j=1}^{r} n_{ij}$. Since each failure time is also accompanied by the corresponding cause of failure, for the observed failure times $\boldsymbol{t} = (t_{1;1}, t_{1;2}, \ldots, t_{k;n_{k\oplus}})$, let $\boldsymbol{c} = (c_{1;1}, c_{1;2}, \ldots, c_{k;n_{k\oplus}})$ be the corresponding observed sequence of the cause of failure. Whenever appropriate, no notational distinction will be made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$. Using (2.1)–(2.6), the likelihood function of $\boldsymbol{\theta} = (\boldsymbol{\lambda}, \boldsymbol{\alpha})$ based on this Type-I censored data is then formulated as

$$L(\boldsymbol{\theta}) = L\big(\boldsymbol{\theta}|(\boldsymbol{t}, \boldsymbol{c})\big) = \frac{n!}{(n - n_{\oplus\oplus})!}\left\{\prod_{i=1}^{k}\prod_{l=1}^{n_{i\oplus}} f_{T,C}\left(t_{i;l}, c_{i;l}\right)\right\}\Big\{1 - F(\tau_k)\Big\}^{n-n_{\oplus\oplus}} \tag{2.7}$$

and the corresponding log-likelihood function of $\boldsymbol{\theta}$ is obtained from (2.7) as

$$\begin{aligned} l(\boldsymbol{\theta}) &= l\big(\boldsymbol{\theta}|(\boldsymbol{t}, \boldsymbol{c})\big) = \log L(\boldsymbol{\theta}) \\ &= \left\{\sum_{i=1}^{k}\sum_{l=1}^{n_{i\oplus}} \log g_{c_{i;l}}(t_{i;l})\right\} + \left\{\sum_{i=1}^{k}\sum_{l=1}^{n_{i\oplus}} \sum_{\substack{j=1 \\ j\neq c_{i;l}}}^{r} \log\big(1 - G_j(t_{i;l})\big)\right\} \\ &\qquad + (n - n_{\oplus\oplus})\left\{\sum_{j=1}^{r} \log\big(1 - G_j(\tau_k)\big)\right\}. \end{aligned} \tag{2.8}$$

After differentiating $l(\boldsymbol{\theta})$ in (2.8) with respect to λ_{ij} and α_j, we obtain the likelihood equations as

$$\begin{aligned}
0 = \frac{\partial}{\partial \lambda_{ij}} l(\boldsymbol{\theta}) \quad &= \quad \frac{n_{ij}}{\lambda_{ij}} - U_{ij} + \sum_{i'=1}^{k} \sum_{l=1}^{n_{i'\oplus}} (\alpha_j - 1) \frac{1 - \left[G_j(t_{i';l})\right]^{1/\alpha_j}}{\left[G_j(t_{i';l})\right]^{1/\alpha_j}} \Big[\Delta_i \delta(i' > i) \\
&\quad + (t_{i;l} - \tau_{i-1}) \delta(i' = i)\Big] \delta(c_{i';l} = j) - \sum_{i'=1}^{k} \sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})}{\lambda_{i'j}[1 - G_j(t_{i';l})]} \Big[\Delta_i \delta(i' > i) \\
&\quad + (t_{i;l} - \tau_{i-1}) \delta(i' = i)\Big] \delta(c_{i';l} \neq j) - (n - n_{\oplus\oplus}) \frac{g_j(\tau_k)}{\lambda_{kj}[1 - G_j(\tau_k)]} \Delta_i, \qquad (2.9) \\
0 = \frac{\partial}{\partial \alpha_j} l(\boldsymbol{\theta}) \quad &= \quad \frac{n_{\oplus j}}{\alpha_j} + \sum_{i=1}^{k} \sum_{l=1}^{n_{i\oplus}} \frac{\log G_j(t_{i;l})}{\alpha_j} \delta(c_{i;l} = j) - \sum_{i=1}^{k} \sum_{l=1}^{n_{i\oplus}} \frac{G_j(t_{i;l})}{1 - G_j(t_{i;l})} \frac{\log G_j(t_{i;l})}{\alpha_j} \delta(c_{i;l} \neq j) \\
&\quad - (n - n_{\oplus\oplus}) \frac{G_j(\tau_k)}{1 - G_j(\tau_k)} \frac{\log G_j(\tau_k)}{\alpha_j} \qquad (2.10)
\end{aligned}$$

for $1 \leq i \leq k$ and $1 \leq j \leq r$ where

$$U_{ij} = \Delta_i \sum_{i'=i+1}^{k} n_{i'j} + \sum_{l=1}^{n_{i\oplus}} (t_{i;l} - \tau_{i-1}) \delta(c_{i;l} = j) \qquad (2.11)$$

and $\delta(\cdot)$ is an indicator function that takes on the value of 1 if the argument is true and 0 otherwise. Note that U_{ij} in (2.11) is precisely the *Total Time on Test* statistic at the stress level x_i for the risk factor j. The MLE $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\lambda}}, \widehat{\boldsymbol{\alpha}})$ are then obtained as simultaneous solutions to the above system of nonlinear equations. There is no closed form solution to the above equations and thus, some iterative search procedures such as the bisection method, Newton-Raphson method or Brent's method should be used to find the numerical solutions. As noted by Kalbfleisch [26], the Newton-Raphson algorithm is often a convenient method to obtain the MLE when there are two or more unknown parameters. The method works well if the likelihood is close to normal in shape. The asymptotic likelihood theory ensures the normal shape for large sample sizes, in which case this method is expected to be efficient; see [16]. Further analysis of the likelihood equations in (2.9) reveals that the MLE of λ_{ij} does not exist if $n_{ij} = 0$. That is, at least one failure caused by each risk factor must be observed at each stress level so that $\boldsymbol{\lambda}$ can be estimated simultaneously. Consequently, the acceptable sample size needs to be much larger than the product of the number of stress levels implemented and the number of fatal risk factors under consideration in the planning stage of the experiment.

Remark 2.1. *In the model considered above, we have not assumed any relationships among the scale parameters* $\boldsymbol{\lambda}_{\star j} = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{kj})$ *of each risk factor. A popular log-linear relationship between the stress level and the scale parameter was not assumed either since it can be too restrictive when the physical stress-response relationship is not clear. The objective here is to estimate the parameters of each risk factor at each stress level (i.e., a full model) in order to investigate and formulate a plausible stress-response relationship which can be tested in the subsequent stage of analysis and incorporated into a reduced model. In certain situations, however, we may know that some particular relationships hold among the scale parameters; for instance,* $\lambda_{ij} = \rho_j \lambda_{(i-1)j}$ *with known* ρ_j*. In that case, the MLE of* λ_{ij} *exists whenever at least one failure occurs by the risk factor* j *(viz.,* $n_{\oplus j} > 0$*). One might also use the likelihood ratio test statistic in order to test the multiple hypotheses* $H_0: \ \lambda_{ij} = \rho_j \lambda_{(i-1)j}$ *with specified* ρ_j*'s.*

Remark 2.2. *The model proposed in Section 2 accommodates multiple stress levels and multiple competing risks. In fact, the model under consideration is general since it includes its marginal models as special cases. For instance, when* $k = 2$*,* $j = 2$*, and* $\alpha_1 = \alpha_2 = 1$*, the failure of a test unit will be caused by one of two competing risks from the exponential distributions in a simple step-stress ALT under Type-I censoring, which was considered by Han & Balakrishnan [10]. Consequently, when* $k = 2$*,* $j = 2$*, and* $\alpha_1 = \alpha_2 = 1$*, the distributional results obtained above reduce to those in [10]. On the other hand, when* $\boldsymbol{\lambda}_{\star j} \to \mathbf{0}_k$ *for* $j = 2, 3, \ldots, r$*, the failure of a test unit will be caused by a single risk factor with probability* 1*. Hence, the limiting case of the proposed model is the step-stress model under Type-I censoring without the competing risk structure. If we let* $\tau_1 \to \infty$*, then the model developed here converges to the usual single stress model (i.e., one stress level only) with multiple competing risks.*

3 INTERVAL ESTIMATIONS

In this section, we discuss the methods of constructing CI for the unknown parameters $\boldsymbol{\theta}$. Since there is no closed form solution to the likelihood equations given in (2.9) and (2.10), it is not possible to derive the exact distributions of the MLE. Hence, we construct the approximate CI for the parameters based on the asymptotic distributions of the estimators, and also present the CI using the parametric bootstrap approach for the purpose of comparison in simulation studies in Section 4.

3.1 *Approximate Confidence Intervals*

As shown in Section 2, $\widehat{\boldsymbol{\theta}}$ is non-linear functions of random quantities, which make it virtually impossible to find their exact marginal/joint distributions for exact inference. Nevertheless, with a growing sample size, the MLE exhibit asymptotically optimal characteristics and hence, statistical inference for $\boldsymbol{\theta}$ can be based on the asymptotic distributional result of the MLE. That is, the vector $\widehat{\boldsymbol{\theta}}$ is approximately distributed as a multivariate normal with mean vector $\boldsymbol{\theta}$ and variance-covariance matrix $\boldsymbol{I}_n^{-1}(\boldsymbol{\theta})$, where $\boldsymbol{I}_n^{-1}(\boldsymbol{\theta})$ is the inverse of the Fisher information matrix $\boldsymbol{I}_n(\boldsymbol{\theta})$. As seen, the MLE are asymptotically unbiased and s-efficient since their biases tend to zero and their variances achieve the Cramer-Rao lower bounds with the sample size approaching infinity. In this section, an approximate method to construct the CI for $\boldsymbol{\theta}$ is presented using the asymptotic normality of the MLE with large sample sizes. As noted in [9]–[10], the approximate method brings the computational ease and also provides good probability coverage (close to the nominal level) as the initial sample size gets larger. This finding is further discussed through a numerical study in Section 4.

Let us now denote the (s-expected) Fisher information matrix of $\boldsymbol{\theta}$ by

$$\boldsymbol{I}_E(\boldsymbol{\theta}) = E\left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta_1 \partial\theta_2} \right]_{\theta_1,\theta_2 \in \Omega} \tag{3.1}$$

where $\Omega = \{\lambda_{ij}, \alpha_j\}_{1\le i\le k,\ 1\le j\le r}$ is the complete set of the model parameters. The second partials of the log-likelihood in (2.8) are presented in Appendix. By substituting $\widehat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$, the elements of $\boldsymbol{I}_E(\boldsymbol{\theta})$

in (3.1) can be approximated by those of the observed Fisher information matrix given by

$$\boldsymbol{I}_O(\boldsymbol{\theta}) = \left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta_1 \partial\theta_2} \right]_{\theta_1,\theta_2\in\Omega} \Bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}. \tag{3.2}$$

Upon inverting this matrix in (3.2), we obtain the observed variance-covariance matrix of $\widehat{\boldsymbol{\theta}}$. Let $\theta \in \Omega$ and $\widehat{\theta}$ be the corresponding MLE of θ. Also, let V be the diagonal element of $\boldsymbol{I}_O^{-1}(\boldsymbol{\theta})$ corresponding to $\widehat{\theta}$. Since $\widehat{\theta}$ is asymptotically unbiased for θ, we can then use $(\widehat{\theta} - \theta)/\sqrt{V}$ as a pivotal quantity for θ to construct the two-sided $100(1-\gamma)\%$ approximate CI for θ, which is given by

$$\left(\max\left\{0, \widehat{\theta} - z_{\gamma/2}\sqrt{V}\right\},\ \widehat{\theta} + z_{\gamma/2}\sqrt{V} \right) \tag{3.3}$$

where $z_{\gamma/2}$ is the upper $\gamma/2$-th quantile of a standard normal distribution.

3.2 *Bootstrap Confidence Intervals*

This section presents the method of constructing the CI for $\boldsymbol{\theta}$ using a parametric bootstrap method, namely the BCa percentile bootstrap method; see Efron & Tibshirani [27], and Hall [28] for details. As pointed out by [9]–[10], the BCa percentile bootstrap CI are known to perform better in comparison to the ordinary percentile bootstrap CI or the Studentized-t bootstrap CI. Kundu *et al.* [29] also observed that the nonparametric bootstrap method does not work well for competing risks data. Building the BCa percentile bootstrap intervals for $\boldsymbol{\theta}$, the very first step is to generate the bootstrap sample. We implement the following algorithm to obtain the bootstrap sample of size B based on the original Type-I censored sample of size $n_{\oplus\oplus}$:

Step 1 Given the stress change time points $\tau_1, \tau_2, \ldots, \tau_{k-1}$, the right censoring time point τ_k, the initial sample size n, and the original Type-I censored sample of size $n_{\oplus\oplus}$, calculate $\widehat{\boldsymbol{\theta}}$ by solving the system of the likelihood equations in (2.9) and (2.10).

Step 2 Generate a random sample of $\boldsymbol{U} = (U_1, U_2, \ldots, U_r)$ of size n, where U_j's are independently from the standard uniform distribution with the range $(0,1)$. Set the counter $i = 1$ and $\eta = n$.

Step 3 Transform each $\boldsymbol{U} = (U_1, U_2, \ldots, U_r)$ in the sample into a vector $(T_{i1}, T_{i2}, \ldots, T_{ir})$ via

$$T_{ij} = -\frac{1}{\widehat{\lambda}_{ij}} \left[\log\left(1 - U_j^{1/\widehat{\alpha}_j}\right) + \sum_{l=1}^{i-1} \widehat{\lambda}_{lj} \Delta_l \right] + \tau_{i-1}$$

for $j = 1, 2, \ldots, r$ so that T_{ij} is a shifted generalized exponential variate with the scale parameter $\widehat{\lambda}_{ij}$ and the shape parameter $\widehat{\alpha}_j$. For each vector of $(T_{i1}, T_{i2}, \ldots, T_{ir})$, take the minimum of the elements as well as the corresponding index of the minimum (*e.g.*, record 3 if T_{i3} is the smallest). Let $\boldsymbol{T}_i$ be the vector of the minima collected and $\boldsymbol{C}_i$ be the vector of the corresponding indices, both of the dimension η.

Step 4 Sort the elements of $\boldsymbol{T}_i$ in an ascending order and permute the elements of $\boldsymbol{C}_i$ in a corresponding manner. Let $v_{1:\eta} < v_{2:\eta} < \cdots < v_{\eta:\eta}$ denote the ordered elements of $\boldsymbol{T}_i$ and let $w_1, w_2, \ldots, w_\eta$ denote the corresponding elements of $\boldsymbol{C}_i$.

Step 5 Find $n^*_{i\oplus}$ such that $v_{n^*_{i\oplus}:n} < \tau_i \le v_{n^*_{i\oplus}+1:n}$. Then, for $1 \le l \le n^*_{i\oplus}$, set $t^*_{i;l}$ to be the value of $v_{l:\eta}$ and set $c^*_{i;l}$ to be the value of w_l. Also, set n^*_{ij} to be the number of j's in the first $n^*_{i\oplus}$ elements of the permuted $\boldsymbol{C}_i$ for $j = 1, 2, \ldots, r$ so that $\sum_{j=1}^{r} n^*_{ij} = n^*_{i\oplus}$.

Step 6 From the sample of $\boldsymbol{U}$ of size η, remove $\boldsymbol{U}$'s corresponding to each $t^*_{i;l}$ for $1 \le l \le n^*_{i\oplus}$ so that the reduced sample now has the new size of $\eta = n - \sum_{l=1}^{i} n^*_{l\oplus}$. Update the counter $i = i + 1$ and repeat Steps 3–6 until i hits $k + 1$.

Step 7 Define $n^*_{\oplus j} = \sum_{i=1}^{k} n^*_{ij}$ and $n^*_{\oplus\oplus} = \sum_{i=1}^{k} n^*_{i\oplus} = \sum_{j=1}^{r} n^*_{\oplus j}$. Based on $\tau_1, \tau_2, \ldots, \tau_k$, n, n^*_{ij}'s and the ordered observations $\boldsymbol{t}^* = (t^*_{1;1}, t^*_{1;2}, \ldots, t^*_{k;n^*_{k\oplus}})$ with the corresponding vector of the cause $\boldsymbol{c}^* = (c^*_{1;1}, c^*_{1;2}, \ldots, c^*_{k;n^*_{k\oplus}})$, calculate the new MLE of $\boldsymbol{\theta}$, denoted by $\widehat{\boldsymbol{\theta}}^*$ from (2.9) and (2.10).

Step 8 Repeat Steps 2–7 B times. Then, for each $\theta \in \Omega$, arrange all the values of $\widehat{\theta}^*$ in an ascending order to obtain the bootstrap sample

$$\left\{ \widehat{\theta}^{*[1]} < \widehat{\theta}^{*[2]} < \cdots < \widehat{\theta}^{*[B]} \right\}.$$

Using the bootstrap sample generated by the algorithm given above, the two-sided $100(1-\gamma)\%$ BCa percentile bootstrap CI for $\theta \in \Omega$ is obtained as

$$\left(\widehat{\theta}^{*[\beta_1 B]},\ \widehat{\theta}^{*[\beta_2 B]}\right)$$

where

$$\beta_1 = \Phi\left(\widehat{z}_0 + \frac{\widehat{z}_0 - z_{\gamma/2}}{1-\widehat{a}\big[\widehat{z}_0 - z_{\gamma/2}\big]}\right) \qquad \text{and} \qquad \beta_2 = \Phi\left(\widehat{z}_0 + \frac{\widehat{z}_0 + z_{\gamma/2}}{1-\widehat{a}\big[\widehat{z}_0 + z_{\gamma/2}\big]}\right).$$

Here, $\Phi(\cdot)$ denotes the CDF of the standard normal distribution and the value of the bias-correction $\widehat{z}_0$ is estimated by

$$\widehat{z}_0 = \Phi^{-1}\left(\frac{\sum_{b=1}^{B} \delta\left(\widehat{\theta}^{*[b]} < \widehat{\theta}\right)}{B}\right)$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal CDF and $\delta(\cdot)$ is an indicator function as defined before. According to [27], a suggested estimate of the acceleration factor $\widehat{a}$ is

$$\widehat{a} = \frac{\sum_{l=1}^{L}\left(\widehat{\theta}^{(l)} - \widehat{\theta}^{(\cdot)}\right)^3}{6\left\{\sum_{l=1}^{L}\left(\widehat{\theta}^{(l)} - \widehat{\theta}^{(\cdot)}\right)^2\right\}^{3/2}}$$

where

$$\widehat{\theta}^{(\cdot)} = \frac{1}{L}\sum_{l=1}^{L}\widehat{\theta}^{(l)}.$$

For $\theta = \lambda_{ij}$, $L = n_{ij}$ and $\widehat{\theta}^{(l)}$ is the MLE of λ_{ij} based on the original Type-I censored sample with the l-th observation deleted from the failures that occurred at the stress level x_i by the risk factor j (*i.e.*, the jackknife estimate) for $l = 1, 2, \ldots, n_{ij}$. Similarly, when $\theta = \alpha_j$, $L = n_{\oplus j}$ and $\widehat{\theta}^{(l)}$ is the MLE of α_j based on the original sample with the l-th observation deleted from the failures that occurred throughout the test by the risk factor j for $l = 1, 2, \ldots, n_{\oplus j}$.

4 NUMERICAL STUDY

An extensive Monte Carlo simulation study was conducted to assess the performance of the estimation methods discussed in the previous sections, and its results are presented herewith. For the purpose of illustration, the case of the simple step-stress model (*viz.*, two stress levels, $k = 2$) with two competing risks (*viz.*, $r = 2$) is presented here since other various parameter settings we considered exhibited a similar pattern of the results. The values of the parameters were chosen to be $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, and $\alpha_2 = 2.0$. This particular scenario also describes a system with two components connected in series where the first component consists of three identical and independently operating sub-components connected in parallel while the second component consists of two identical and independently operating sub-components connected in parallel. The lifetime distribution of each sub-component is exponential with mean λ_{ij}^{-1}. Under this particular parameter setting, when the stress level increases, there is 50% loss in the mean time to failure of any sub-component. Also, before or after the change in the stress levels, the failure likelihood of a sub-component in the component 1 is twice as high as that of a sub-component in the component 2.

In order to explore the effects of several experimental parameters on the performance of estimation, we chose the initial sample size n to be 25, 50, and 100 while a range of different values was selected for the stress change time point τ_1 with the censoring time point τ_2 fixed at 1.0. Given the parameter values, a random sample was then generated by using Steps 2–7 of the algorithm for generating a bootstrap sample. Based on 1000 Monte Carlo simulations with $B = 1000$ bootstrap replications, the actual coverage probabilities of the 90%, 95%, and 99% intervals for each model parameter were determined empirically as well as the bias, RAB[2], and MSE associated with the estimator. Tables 1-6 present the results of this simulation along with the estimated mean widths of CI.

Tables 1-6 are to be inserted about here.

As noted in [9]–[10], a major issue associated with the approximate CI is that the proper parameter space is not taken into account when the interval is estimated. This is the price for its simplicity owing to the *asymptotic* normality of the MLE. Since no built-in procedure is available to prevent this

[2] $RAB = E\left[\left|\frac{\widehat{\theta} - \theta}{\theta}\right|\right]$ for $\theta \in \Omega$

without a suitable parameter transformation, we observed that the lower limits of the approximate intervals frequently hit below zero as a result, especially for small sample sizes and/or for large confidence coefficients. Since the parameters $\boldsymbol{\theta}$ can take on only positive values, the negative lower limits produced from the simulation were all replaced by zero according to (3.3) so that all the intervals are mathematically sensible.

From Tables 1, 3, and 5, we see that with fixed censoring time point τ_2, as the stress change time point τ_1 increases, the biases, RAB, MSE of the estimators for λ_{11}, λ_{12}, α_1, α_2 all decrease while those for λ_{21} and λ_{22} increase in most cases. The primary reason for this is that when Δ_1 gets larger or equivalently, when Δ_2 gets smaller with increasing τ_1, we expect a relatively large number of failures to occur before τ_1 (*i.e.*, at the first stress level), resulting in lower variability in the estimation of λ_{11} and λ_{12}. On the other hand, a relatively small number of failures will occur after τ_1 (*i.e.*, at the second stress level), resulting in higher variability in the estimation of λ_{21} and λ_{22}. The same intuition applies to explain the observation from Tables 2, 4, and 6 that with increasing τ_1, the widths of CI for λ_{11}, λ_{12}, α_1, α_2 all decrease while those for λ_{21} and λ_{22} increase.

From Tables 1, 3, and 5, it is also observed that the biases, RAB, MSE of the estimators for α_j are mostly much higher than those for λ_{ij}, illustrating the difficulty of estimating the shape parameters with good accuracy and precision. This resulted in much wider CI for α_j compared to those for λ_{ij} in Tables 2, 4, and 6. We also see that the MLE overestimate the corresponding parameters on average since their biases are all positive with varying degrees. As the sample size n increases, however, the performance of the MLE gets better as the biases, RAB, MSE associated with the estimators all decrease along with the actual coverage probabilities of the CI getting closer to the nominal levels.

From Tables 2, 4, and 6, we observe that in comparison to the BCa bootstrap method, the approximate method consistently provides narrower intervals overall. This seems to explain somewhat better performance of the BCa bootstrap CI relative to those obtained from the approximate method, especially for small sample sizes, as observed in Table 1. Nevertheless, actual coverage probabilities of both the approximate intervals and the BCa bootstrap intervals are eventually improved with larger sample sizes as evident from Tables 3 and 5. From this numerical study, we experienced that as the initial sample size grows substantially, the bootstrap method requires a long computational time for generating a bootstrap sample, which can be a problematic issue in some time-sensitive situations. Hence, based on more exhaustive simulation study, it is our recommendation to use the BCa bootstrap approach for small sample sizes because its coverage probabilities are quite satisfactory being close to the nominal levels even for small sample sizes. When the initial sample size is considerably large (*e.g.*, $n >> 30$), the approximate method seems to be a suitable choice since its computation is much more convenient to obtain the CI and its performance is also reasonably good with respect to the probability coverage.

5 ILLUSTRATIVE EXAMPLES

In this section, we illustrate the methods of inference described in the preceding sections, using a real dataset and a simulated dataset. Both datasets are Type-I censored samples from simple step-stress ALT with two known competing risks.

5.1 *Electronic Device Dataset*

A simple step-stress test was conducted under time constraint in order to assess the reliability characteristics of a solar lighting device, which has two dominant failure modes (capacitor failure and controller failure). Here, temperature is the stress factor whose level was changed during the test in the range of 293K to 353K with the normal operating temperature at 293K. It is assumed that at any constant temperature, the lifetime distributions of the two risk factors for the device failure are s-independent GE. The stress change time point was $\tau_1 = 5$ (in hundred hours) and the censoring time point was $\tau_2 = 6$ (in hundred hours). The dataset consists of total $n_{\oplus\oplus} = 31$ failure times (in hundred hours) from the initial sample size of $n = 35$ prototypes (*i.e.*, 11.4% right censoring); see Table 7.

Table 7 is to be inserted about here.

From this dataset, we have $n_{11} = 3$, $n_{12} = 13$, $n_{21} = 10$, $n_{22} = 5$, and the observed MLE of the GE parameters are estimated from (2.9) and (2.10) to be

$$\widehat{\lambda}_{11} = 0.008,\ \widehat{\lambda}_{12} = 0.200,\ \widehat{\lambda}_{21} = 0.932,\ \widehat{\lambda}_{22} = 0.727,\ \widehat{\alpha}_1 = 0.720,\ \widehat{\alpha}_2 = 1.960.$$

The observed standard errors of the estimators are obtained from (3.2) to be

$$\widehat{\sqrt{V}}_{\lambda_{11}} = 0.014,\ \widehat{\sqrt{V}}_{\lambda_{12}} = 0.078,\ \widehat{\sqrt{V}}_{\lambda_{21}} = 0.525,\ \widehat{\sqrt{V}}_{\lambda_{22}} = 0.315,\ \widehat{\sqrt{V}}_{\alpha_1} = 0.319,\ \widehat{\sqrt{V}}_{\alpha_2} = 0.744.$$

The CI for each parameter is also presented in Table 8 using the methods described in Section 3.

Table 8 is to be inserted about here.

From Table 8, it is observed that the lengths of the approximate CI and the BCa bootstrap CI are quite similar although the bootstrap CI for the shape parameters get much wider for high levels of confidence.

5.2 *Simulated Dataset*

The dataset generated by Han & Balakrishnan [10] is a Type-I censored sample from a simple step-stress test with two known competing risks along with the stress change time point $\tau_1 = 3$ and the censoring time point $\tau_2 = 6$ for an equal step duration. It consists of total $n_{\oplus\oplus} = 23$ failure times from the initial sample size of $n = 25$ (*i.e.*, 8% right censoring). To be self-contained, the dataset is reproduced in Table 9 for easy reference.

Table 9 is to be inserted about here.

From this dataset, we have $n_{11} = 7$, $n_{12} = 5$, $n_{21} = 5$, $n_{22} = 6$, and the observed MLE of the GE parameters are estimated from (2.9) and (2.10) to be

$$\widehat{\lambda}_{11} = 0.085,\ \widehat{\lambda}_{12} = 0.167,\ \widehat{\lambda}_{21} = 0.229,\ \widehat{\lambda}_{22} = 0.373,\ \widehat{\alpha}_1 = 0.802,\ \widehat{\alpha}_2 = 1.548.$$

The observed standard errors of the estimators are obtained from (3.2) to be

$$\widehat{\sqrt{V}}_{\lambda_{11}} = 0.065,\ \widehat{\sqrt{V}}_{\lambda_{12}} = 0.125,\ \widehat{\sqrt{V}}_{\lambda_{21}} = 0.120,\ \widehat{\sqrt{V}}_{\lambda_{22}} = 0.154,\ \widehat{\sqrt{V}}_{\alpha_1} = 0.294,\ \widehat{\sqrt{V}}_{\alpha_2} = 0.799.$$

Since the true parameter values are given as

$$\lambda_{11} = 0.112,\ \lambda_{12} = 0.082,\ \lambda_{21} = 0.223,\ \lambda_{22} = 0.246,\ \alpha_1 = \alpha_2 = 1,$$

the observed biases of the estimates are

$$\text{Bias}_{\lambda_{11}} = -0.027,\ \text{Bias}_{\lambda_{12}} = 0.085,\ \text{Bias}_{\lambda_{21}} = 0.006,\ \text{Bias}_{\lambda_{22}} = 0.126,\ \text{Bias}_{\alpha_1} = -0.198,\ \text{Bias}_{\alpha_2} = 0.548.$$

The CI for each parameter is also presented in Table 10 using the methods described in Section 3.

Table 10 is to be inserted about here.

From Table 10, we observe that relative to the BCa bootstrap CI, the approximate method consistently provides narrower CI although every CI contains the respective true parameter value in this example. As expected, the CI get wider as the nominal level of confidence increases. As in Table 8, it is also observed that the lower bounds of the approximate CI are zero, especially for small λ_{ij}'s and/or for high levels of confidence since they were hitting below zero in such cases. We also make a general observation from Table 10 that the CI for the scale parameters are narrower than those for the shape parameters. Since the CI for α_j's contain one, there is no sufficient statistical evidence to reject $H_0 :\ \alpha_j = 1$ with at most 10% level of significance. That is, the lifetime distribution of each risk factor is exponential, which is the correct decision in this situation.

6 CONCLUDING REMARKS

In this article, the step-stress model under Type-I censoring was discussed in the competing risks framework with the lifetimes of different risk factors having s-independent GE distributions. Under the assumption of a cumulative damage model, point and interval estimations of the model parameters were then discussed using the maximum likelihood approach. The performance of all these procedures was evaluated via a simulation study and finally, real and simulated datasets were provided to illustrate the methods of inference developed in this work. In the case of moderate to large sizes, the estimators give relatively accurate estimation of the parameters. Examining the outcomes of an extensive simulation study, our recommendation is to use the BCa percentile bootstrap method when constructing CI for $\theta \in \Omega$, especially for small sample sizes. Nevertheless, for larger sample sizes, the approximate method seems more suitable since it enjoys not only the computational ease but also the improved probability coverage getting close to the nominal levels. Based on our best knowledge, this study is the first to introduce the Type-I censoring to the GE distribution under the competing risk framework on the step-stress ALT. For future research, we will develop the Bayesian estimation method and a procedure for discriminating between the Weibull and GE distributions. We will also explore the extension of the methods developed for other distributions such as two-parameter Birnbaum-Saunders distribution.

APPENDIX

The second partials of the log-likelihood in (2.8) are expressed as

$$
\begin{aligned}
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \lambda_{ij}^2} &= \frac{n_{ij}}{\lambda_{ij}^2} + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} (\alpha_j - 1)\frac{1-\left[G_j(t_{i';l})\right]^{1/\alpha_j}}{\left[G_j(t_{i';l})\right]^{2/\alpha_j}}\Big[\Delta_i\delta(i'>i) + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]^2\delta(c_{i';l}=j) \\
&\quad + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})\Big[\alpha_j\Big(1-\left[G_j(t_{i';l})\right]^{1/\alpha_j}\Big) - \Big(1-G_j(t_{i';l})\Big)\Big]}{\lambda_{i'j}\left[G_j(t_{i';l})\right]^{1/\alpha_j}\left[1-G_j(t_{i';l})\right]^2}\Big[\Delta_i\delta(i'>i) \\
&\quad + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]^2\delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{g_j(\tau_k)\Big[\alpha_j\Big(1-\left[G_j(\tau_k)\right]^{1/\alpha_j}\Big) - \Big(1-G_j(\tau_k)\Big)\Big]}{\lambda_{kj}\left[G_j(\tau_k)\right]^{1/\alpha_j}\left[1-G_j(\tau_k)\right]^2}\Delta_i^2, \\
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \lambda_{ij}\partial\lambda_{i''j}} &= \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} (\alpha_j - 1)\frac{1-\left[G_j(t_{i';l})\right]^{1/\alpha_j}}{\left[G_j(t_{i';l})\right]^{2/\alpha_j}}\Big[\Delta_i\delta(i'>i) + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\Big[\Delta_{i''}\delta(i'>i'') \\
&\quad + (t_{i'';l}-\tau_{i''-1})\delta(i'=i'')\Big]\delta(c_{i';l}=j) \\
&\quad + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})\Big[\alpha_j\Big(1-\left[G_j(t_{i';l})\right]^{1/\alpha_j}\Big) - \Big(1-G_j(t_{i';l})\Big)\Big]}{\lambda_{i'j}\left[G_j(t_{i';l})\right]^{1/\alpha_j}\left[1-G_j(t_{i';l})\right]^2}\Big[\Delta_i\delta(i'>i) \\
&\quad + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\Big[\Delta_{i''}\delta(i'>i'') + (t_{i'';l}-\tau_{i''-1})\delta(i'=i'')\Big]\delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{g_j(\tau_k)\Big[\alpha_j\Big(1-\left[G_j(\tau_k)\right]^{1/\alpha_j}\Big) - \Big(1-G_j(\tau_k)\Big)\Big]}{\lambda_{kj}\left[G_j(\tau_k)\right]^{1/\alpha_j}\left[1-G_j(\tau_k)\right]^2}\Delta_i\Delta_{i''}, \\
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \lambda_{ij}\partial\alpha_j} &= -\sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{1-\left[G_j(t_{i';l})\right]^{1/\alpha_j}}{\left[G_j(t_{i';l})\right]^{1/\alpha_j}}\Big[\Delta_i\delta(i'>i) + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\delta(c_{i';l}=j) \\
&\quad + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{g_j(t_{i';l})[1-G_j(t_{i';l})+\log G_j(t_{i';l})]}{\alpha_j\lambda_{i'j}\left[1-G_j(t_{i';l})\right]^2}\Big[\Delta_i\delta(i'>i) \\
&\quad + (t_{i;l}-\tau_{i-1})\delta(i'=i)\Big]\delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{g_j(\tau_k)[1-G_j(\tau_k)+\log G_j(\tau_k)]}{\alpha_j\lambda_{kj}\left[1-G_j(\tau_k)\right]^2}\Delta_i, \\
-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \alpha_j^2} &= \frac{n_{\oplus j}}{\alpha_j^2} + \sum_{i'=1}^{k}\sum_{l=1}^{n_{i'\oplus}} \frac{G_j(t_{i';l})}{\left[1-G_j(t_{i';l})\right]^2}\left[\frac{\log G_j(t_{i';l})}{\alpha_j}\right]^2\delta(c_{i';l}\neq j) \\
&\quad + (n-n_{\oplus\oplus})\frac{G_j(\tau_k)}{\left[1-G_j(\tau_k)\right]^2}\left[\frac{\log G_j(\tau_k)}{\alpha_j}\right]^2, \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \lambda_{ij}\partial\lambda_{ij''}} &= \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \lambda_{ij}\partial\lambda_{i''j''}} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \lambda_{ij}\partial\alpha_{j''}} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \alpha_j\partial\alpha_{j''}} = 0
\end{aligned}
$$

for $1 \le i, i'' \le k$ and $1 \le j, j'' \le r$ with $i \neq i''$ and $j \neq j''$.

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David Han, M.Sc., Ph.D.

Assistant Professor

University of Texas at San Antonio

One UTSA Circle

San Antonio, Texas, USA 78249–0632

phone: +1-210-458-7895

eMail: David.Han@utsa.edu

David Han is an Assistant Professor in the Department of Management Science & Statistics at the University of Texas at San Antonio, TX. He received two Honours B.Sc. degrees, one in Biochemistry and the other in Computer Science & Statistics, both from McMaster University in Canada. Continuing his studies at McMaster, he received the M.Sc. and Ph.D. in Statistics. His research interests include the statistical inference for accelerated life testing in reliability and survival analysis, analyses of censored data, optimal censoring plans, competing risks analyses, and nonparametric statistics.

Debasis Kundu is a Professor of Statistics in the Department of Mathematics & Statistics at the Indian Institute of Technology Kanpur. He received the B.Stat in 1982, and M.Stat in 1984 both from the Indian Statistical Institute; M.A. (Mathematics) in 1985 from the University of Pittsburgh; and the Ph.D. in 1989 from the Pennsylvania State University. He has worked as an Assistant Professor at the University of Texas at Dallas from 1989 to 1990, and then joined Indian Institute of Technology Kanpur as an Assistant Professor in 1990. He became Professor in 1999. His research interests include statistical signal processing, reliability analysis, statistical computing, and competing risks models.

Table 1: Estimated biases, RAB, MSE, and coverage probabilities (in %) based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 25$, $\tau_2 = 1.0$, and $B = 1000$

		Nominal CL			90%		95%		99%	
Parameter	τ_1	Bias	RAB	MSE	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	1.057	0.911	2.726	84.0	87.6	88.5	93.2	94.6	97.8
	0.5	0.642	0.619	1.264	83.7	89.8	89.2	95.1	96.5	98.6
	0.7	0.330	0.353	0.923	90.4	89.1	94.3	97.0	98.0	98.5
λ_{12}	0.3	1.258	1.346	2.909	87.8	90.6	89.6	94.8	92.6	99.0
	0.5	0.723	0.887	1.287	85.9	87.9	89.5	92.9	93.3	96.9
	0.7	0.270	0.652	0.862	86.2	91.0	90.4	93.1	94.7	97.8
λ_{21}	0.3	0.140	0.242	1.518	88.7	87.6	93.6	94.8	99.0	98.8
	0.5	0.131	0.250	1.725	90.5	88.7	93.1	93.9	97.9	95.9
	0.7	0.490	0.388	1.928	86.9	88.8	93.0	92.5	97.5	96.6
λ_{22}	0.3	0.033	0.333	0.732	92.0	89.0	93.7	95.0	96.4	95.9
	0.5	0.112	0.404	1.074	88.1	92.0	92.9	96.3	95.2	99.5
	0.7	0.337	0.493	1.818	93.3	91.4	95.5	96.0	99.5	94.8
α_1	0.3	2.658	1.862	9.258	88.2	90.5	89.3	93.5	93.2	97.8
	0.5	1.957	1.058	5.877	89.6	87.9	93.0	94.0	94.5	96.6
	0.7	1.353	0.634	4.386	94.0	92.4	96.4	97.1	99.0	99.5
α_2	0.3	2.819	1.973	9.549	90.5	90.7	94.6	94.6	95.0	98.7
	0.5	2.080	1.670	8.312	90.3	87.9	93.0	93.1	95.0	97.2
	0.7	1.950	1.252	7.594	90.4	93.0	93.4	96.0	97.3	98.5

Table 2: Average widths of confidence intervals based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 25$, $\tau_2 = 1.0$, and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ_1	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	8.330	9.160	9.353	11.021	11.205	13.419
	0.5	4.088	4.701	4.752	5.446	5.833	6.727
	0.7	2.801	3.663	3.321	4.240	4.249	4.814
λ_{12}	0.3	7.778	8.181	8.742	10.109	10.535	12.911
	0.5	3.718	5.389	4.193	6.305	5.009	7.000
	0.7	2.427	2.870	2.755	3.405	3.304	4.043
λ_{21}	0.3	3.668	4.922	4.364	5.709	5.698	6.607
	0.5	4.107	3.745	4.886	4.332	6.387	7.830
	0.7	5.788	6.874	6.846	7.710	8.703	8.734
λ_{22}	0.3	2.932	3.198	3.419	3.644	4.254	4.201
	0.5	3.268	3.603	3.804	4.231	4.686	4.973
	0.7	4.344	4.798	4.955	5.476	5.896	6.397
α_1	0.3	30.929	32.670	33.933	35.119	37.737	38.502
	0.5	17.777	19.742	22.310	22.887	27.016	29.174
	0.7	12.862	13.857	13.027	14.968	14.710	17.097
α_2	0.3	28.536	29.902	29.929	33.388	34.372	36.281
	0.5	24.689	25.115	27.713	30.698	30.444	32.629
	0.7	22.926	23.089	23.609	25.650	26.692	28.971

Table 3: Estimated biases, RAB, MSE, and coverage probabilities (in %) based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 50$, $\tau_2 = 1.0$, and $B = 1000$

		Nominal CL			90%		95%		99%	
Parameter	τ_1	Bias	RAB	MSE	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	1.032	0.872	2.406	84.6	94.1	88.6	97.0	95.6	98.5
	0.5	0.260	0.326	0.786	88.5	90.0	93.1	95.0	98.4	98.0
	0.7	0.170	0.251	0.419	83.4	89.6	91.0	93.8	98.0	97.8
λ_{12}	0.3	0.801	1.252	2.901	84.4	87.8	88.8	91.9	94.7	96.9
	0.5	0.322	0.632	0.797	87.0	90.0	92.2	95.2	96.5	99.5
	0.7	0.136	0.441	0.332	87.5	90.1	94.0	95.0	97.1	97.5
λ_{21}	0.3	0.086	0.157	0.665	88.0	90.5	94.2	95.5	99.0	98.0
	0.5	0.099	0.172	0.768	90.5	89.0	95.3	94.0	99.5	97.7
	0.7	0.222	0.228	1.373	93.1	88.8	95.0	92.8	98.1	96.8
λ_{22}	0.3	0.031	0.296	0.594	85.0	87.9	93.0	93.3	96.0	97.9
	0.5	0.111	0.297	0.551	89.3	87.6	93.6	93.5	97.4	96.9
	0.7	0.090	0.365	0.894	89.2	88.0	95.5	93.2	97.5	97.0
α_1	0.3	1.348	0.958	8.352	88.4	93.0	92.8	95.5	94.6	99.0
	0.5	0.949	0.502	5.077	92.0	88.6	95.0	93.6	97.3	97.7
	0.7	0.546	0.358	3.167	92.0	89.5	95.0	94.5	97.5	99.0
α_2	0.3	1.401	0.975	8.403	88.6	88.7	93.3	94.9	94.8	96.8
	0.5	0.936	0.666	7.856	92.0	91.0	94.0	95.0	97.5	99.0
	0.7	0.519	0.471	2.848	93.2	92.3	94.2	95.5	97.2	98.0

Table 4: Average widths of confidence intervals based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 50$, $\tau_2 = 1.0$, and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ_1	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	5.684	7.856	6.488	9.259	7.789	10.787
	0.5	2.740	3.619	3.251	4.209	4.155	4.784
	0.7	1.870	2.621	2.228	3.020	2.925	3.418
λ_{12}	0.3	4.278	5.841	4.810	7.039	5.771	9.025
	0.5	2.305	2.931	2.650	3.394	3.192	4.061
	0.7	1.667	1.973	1.951	2.251	2.416	2.570
λ_{21}	0.3	2.592	4.030	3.088	4.731	4.059	5.680
	0.5	2.793	4.555	3.328	5.452	4.373	6.188
	0.7	3.856	5.089	4.595	5.815	6.037	6.597
λ_{22}	0.3	2.279	2.713	2.692	3.143	3.435	3.558
	0.5	2.305	2.854	2.744	3.237	3.551	3.603
	0.7	2.904	3.402	3.418	3.562	4.275	4.010
α_1	0.3	8.654	14.434	9.156	16.370	10.681	22.338
	0.5	6.209	12.558	7.256	15.214	8.875	20.034
	0.7	4.172	5.991	4.964	7.347	6.460	9.127
α_2	0.3	8.944	10.028	9.406	13.677	10.559	17.198
	0.5	5.302	8.999	6.099	10.112	7.339	15.403
	0.7	3.744	7.591	4.381	9.181	5.441	11.231

Table 5: Estimated biases, RAB, MSE, and coverage probabilities (in %) based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 100$, $\tau_2 = 1.0$, and $B = 1000$

		Nominal CL			90%		95%		99%	
Parameter	τ_1	Bias	RAB	MSE	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	0.394	0.511	1.944	89.4	89.9	95.5	95.5	98.0	99.0
	0.5	0.135	0.240	0.374	89.5	93.0	93.9	96.2	98.4	97.9
	0.7	0.096	0.151	0.152	92.1	87.9	95.1	94.9	99.0	98.4
λ_{12}	0.3	0.517	0.913	2.281	88.9	89.5	94.5	95.3	97.8	98.5
	0.5	0.128	0.386	0.250	92.5	93.0	95.4	96.0	97.7	97.9
	0.7	0.092	0.306	0.152	89.0	90.2	93.9	94.6	96.9	99.1
λ_{21}	0.3	0.062	0.112	0.238	88.8	89.5	94.7	95.2	98.5	98.7
	0.5	0.080	0.115	0.326	93.1	88.9	96.1	95.0	100.0	98.0
	0.7	0.112	0.158	0.628	91.5	89.9	94.8	95.1	99.1	99.0
λ_{22}	0.3	0.030	0.206	0.283	88.7	89.0	95.3	95.1	97.9	98.6
	0.5	0.101	0.219	0.294	87.9	88.5	95.0	97.0	99.0	97.9
	0.7	0.115	0.244	0.393	89.2	90.1	96.0	96.0	98.8	99.5
α_1	0.3	1.054	0.577	6.840	90.5	89.7	95.5	93.9	97.9	99.1
	0.5	0.403	0.296	1.669	90.0	90.5	94.7	96.0	98.3	98.0
	0.7	0.245	0.213	0.711	93.2	89.8	98.0	94.8	99.0	96.9
α_2	0.3	1.328	0.876	5.248	89.0	89.6	94.7	94.7	98.7	98.9
	0.5	0.271	0.326	1.028	92.5	92.0	95.0	96.0	98.2	98.1
	0.7	0.222	0.257	0.503	90.0	89.9	94.8	93.8	98.5	99.5

Table 6: Average widths of confidence intervals based on 1000 simulations with $\lambda_{11} = 2.0$, $\lambda_{12} = 1.0$, $\lambda_{21} = 4.0$, $\lambda_{22} = 2.0$, $\alpha_1 = 3.0$, $\alpha_2 = 2.0$, $n = 100$, $\tau_2 = 1.0$, and $B = 1000$

Nominal CL		90%		95%		99%	
Parameter	τ_1	Approx	Boot	Approx	Boot	Approx	Boot
λ_{11}	0.3	3.677	4.598	4.285	5.213	5.265	5.890
	0.5	1.842	2.601	2.195	2.986	2.881	3.298
	0.7	1.299	2.116	1.548	2.545	2.034	2.887
λ_{12}	0.3	2.974	3.235	3.371	3.819	4.026	4.389
	0.5	1.568	1.909	1.847	2.182	2.309	2.472
	0.7	1.148	1.484	1.366	1.707	1.777	1.917
λ_{21}	0.3	1.834	3.217	2.186	4.035	2.873	4.976
	0.5	1.976	3.259	2.355	4.335	3.094	5.154
	0.7	2.657	4.338	3.166	4.975	4.161	5.691
λ_{22}	0.3	1.645	2.133	1.960	2.491	2.566	2.929
	0.5	1.612	2.117	1.921	2.539	2.523	2.984
	0.7	1.989	2.654	2.369	3.044	3.094	3.399
α_1	0.3	6.894	12.849	7.948	13.994	9.690	14.835
	0.5	3.435	4.791	4.093	5.680	5.374	6.596
	0.7	2.592	4.006	3.089	4.843	4.059	5.589
α_2	0.3	6.654	8.458	7.626	10.447	9.297	12.952
	0.5	2.672	4.251	3.160	5.402	4.073	6.523
	0.7	2.090	3.235	2.490	3.892	3.272	4.537

Table 7: Type-I censored dataset from $n = 35$ prototypes of a solar lighting device on a simple step-stress test with two failure modes, $\tau_1 = 5$ and $\tau_2 = 6$

Temperature Level 1 (before $\tau_1 = 5$)		**Temperature Level 2** (after $\tau_1 = 5$)	
Failure Time	Failure Cause	Failure Time	Failure Cause
0.140	1	5.002	1
0.783	2	5.022	2
1.324	2	5.082	2
1.582	1	5.112	1
1.716	2	5.147	1
1.794	2	5.238	1
1.883	2	5.244	1
2.293	2	5.247	1
2.660	2	5.305	1
2.674	2	5.337	2
2.725	2	5.407	1
3.085	2	5.408	2
3.924	2	5.445	1
4.396	2	5.483	1
4.612	1	5.717	2
4.892	2		
$n_{1\oplus} = 16$		$n_{2\oplus} = 15$	

Table 8: Interval estimation based on the Type-I censored step-stress data in Table 7 with $B = 1000$

Parameter	CL	Approximate CI	BCa Bootstrap CI
	90%	(0.000, 0.032)	(0.000, 0.047)
λ_{11}	95%	(0.000, 0.037)	(0.000, 0.078)
	99%	(0.000, 0.045)	(0.000, 0.194)
	90%	(0.072, 0.329)	(0.090, 0.335)
λ_{12}	95%	(0.047, 0.354)	(0.070, 0.367)
	99%	(0.000, 0.402)	(0.000, 0.415)
	90%	(0.068, 1.796)	(0.320, 1.834)
λ_{21}	95%	(0.000, 1.962)	(0.257, 2.093)
	99%	(0.000, 2.285)	(0.167, 2.990)
	90%	(0.209, 1.245)	(0.274, 1.305)
λ_{22}	95%	(0.110, 1.345)	(0.203, 1.425)
	99%	(0.000, 1.539)	(0.000, 1.685)
	90%	(0.195, 1.244)	(0.423, 1.559)
α_1	95%	(0.094, 1.345)	(0.381, 2.377)
	99%	(0.000, 1.541)	(0.292, 8.214)
	90%	(0.734, 3.183)	(1.202, 4.584)
α_2	95%	(0.500, 3.418)	(1.135, 6.026)
	99%	(0.041, 3.876)	(0.992, 11.501)

Table 9: Type-I censored sample from $n = 25$ units on a simple step-stress test with two competing risks, $\tau_1 = 3$ and $\tau_2 = 6$

Stress Level 1 (before $\tau_1 = 3$)		**Stress Level 2** (after $\tau_1 = 3$)	
Failure Time	Failure Cause	Failure Time	Failure Cause
0.011	1	3.246	2
0.273	2	3.362	2
0.395	1	3.498	1
1.173	1	3.774	2
1.477	1	3.879	1
1.608	2	4.024	1
1.890	1	4.169	2
2.066	2	4.438	2
2.133	2	4.882	2
2.577	1	5.343	1
2.706	1	5.670	1
2.787	2		
$n_{1\oplus} = 12$		$n_{2\oplus} = 11$	

Table 10: Interval estimation based on the Type-I censored step-stress data in Table 9 with $B = 1000$

Parameter	CL	Approximate CI	BCa Bootstrap CI
λ_{11}	90%	(0.000, 0.191)	(0.018, 0.277)
	95%	(0.000, 0.212)	(0.014, 0.349)
	99%	(0.000, 0.252)	(0.009, 0.408)
λ_{12}	90%	(0.000, 0.372)	(0.027, 0.503)
	95%	(0.000, 0.412)	(0.019, 0.675)
	99%	(0.000, 0.489)	(0.010, 1.081)
λ_{21}	90%	(0.032, 0.426)	(0.080, 0.459)
	95%	(0.000, 0.464)	(0.072, 0.546)
	99%	(0.000, 0.538)	(0.048, 0.800)
λ_{22}	90%	(0.119, 0.626)	(0.167, 0.739)
	95%	(0.070, 0.675)	(0.138, 0.901)
	99%	(0.000, 0.770)	(0.081, 0.940)
α_1	90%	(0.319, 1.285)	(0.477, 2.031)
	95%	(0.226, 1.377)	(0.457, 3.049)
	99%	(0.045, 1.558)	(0.410, 3.970)
α_2	90%	(0.235, 2.862)	(0.785, 7.697)
	95%	(0.000, 3.114)	(0.732, 9.923)
	99%	(0.000, 3.606)	(0.646, 12.090)