# Bayesian and Maximum Likelihood Estimations of the Inverse Weibull Parameters Under Progressive Type-II Censoring

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**Abstract**

In this paper, the statistical inference of the unknown parameters of a two-parameter inverse Weibull (IW) distribution based on the progressive Type-II censored sample has been considered. The maximum likelihood estimators cannot be obtained in explicit forms, hence the approximate maximum likelihood estimators are proposed, which are in explicit forms. The Bayes and generalized Bayes estimators for the IW parameters and the reliability function based on the squared error and Linex loss functions are provided. The Bayes and generalized Bayes estimators cannot be obtained explicitly, hence Lindley's approximation is used to obtain the Bayes and generalized Bayes estimators. Further the highest posterior density credible intervals of the unknown parameters based on Gibbs sampling technique are computed, and using an optimality

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criterion the optimal censoring scheme has been suggested. Simulation experiments are performed to see the effectiveness of the different estimators, and two data sets have been analyzed for illustrative purposes.

*Keywords:* Bayes estimation; Lindley approximation; maximum likelihood estimation; reliability function; squared error and Linex loss function, estimated risk and Monte Carlo simulation; Gibbs samples.

## 1 Introduction

The Weibull distribution has been used quite extensively to analyze lifetime data. The main reason of its popularity is due to wide variety of shapes it can assume by varying its shape parameter. Weibull distribution was introduced by Weibull in 1935, and since then extensive work has been done both from the frequentist and Bayesian point of view on this distribution, see for example the excellent review by Johnson *et al.* (1995). Although, the Weibull distribution can have increasing, decreasing or constant hazard function depending on the shape parameter, it cannot have non-monotone hazard function. In many practical situations it is known that the data are coming from a distribution which cannot have a non-monotone hazard function, and in that case it is not possible to use the Weibull distribution to analyze those data. Therefore, if the empirical study indicates that the hazard function of the underlying distribution is not monotone, and it is unimodal, then inverse Weibull (IW) distribution may be used to analyze such data set. A brief description of a IW distribution is presented in Section 2. Extensive work has been done on the IW distribution, see for example Keller and Kamath (1982), Calabria and Pulcini (1989, 1990, 1992, 1994), Etro (1989), Jiang, Ji and Xiao (2003) Mahmoud, Sultan and Amer (2003) Maswadah (2003) Kundu and Howlader (2010) and the references cited therein.

Type-I and Type-II censoring schemes are the two most popular censoring schemes which have been used in practice. Unfortunately, none of these censoring schemes allow the removal of any experimental units during the experiment. Type-I and Type-II progressive censoring schemes allow the removal of experimental units during the experiment. Due to this flexibility progressive censoring scheme has received considerable attention in the applied statistics literature for the last 10-12 years. A Type-II progressively censored experiment can be briefly described as follows. For $m < n$, choose $R_1, \ldots, R_m$, $m$ non-negative integers such that

$$R_1 + \ldots + R_m = n - m. \tag{1.1}$$

Consider an experiment, in which $n$ identical units are put on a test. It is assumed that the lifetime distribution of the $n$ units are independent and identically distributed random variables with the common distribution function $F$. At the time of the first failure, say $x_{1:m:n}$, $R_1$ surviving units are chosen at random to be removed. Similarly, at the time of the second failure, say $x_{2:m:n}$, $R_2$ surviving units are removed and so on. Finally, at the time of the $m$-th failure, say $x_{m:m:n}$, all the remaining units are removed. Therefore, for a given Type-II progressive censoring scheme, $(n, m, R_1, \ldots, R_m)$, one observes the following sample;

$$x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}. \tag{1.2}$$

In the last few years, progressive censoring scheme has received considerable attention, see for example the book by Balakrishnan and Aggarwala (2000) and also the relatively recent review article by Balakrishnan (2007) in this respect.

Although, extensive work has been done on the statistical inference of the unknown parameters of different parametric models based on progressively censored observation in the frequentist setup, not that much work has been done in the Bayesian inference. Kundu (2008) considered the Bayesian inference of the unknown parameters of a two-parameter Weibull model based on progressively censored data. It is observed that the Bayes estimates and the associated credible intervals cannot be obtained in closed form, and he proposed to use the Markov Chain Monte Carlo (MCMC) technique to compute the Bayes estimates of the unknown parameters and also to construct the associated credible intervals. Kim, Jung and Kim (2009) considered the Bayesian inference of the unknown parameters of a three-parameter exponentiated Weibull distribution, based on Type-II progressively censored sample. They computed the Bayes estimates under various loss functions such as squared error and Linex loss functions, and compared their performances with the maximum likelihood estimators (MLEs). It is observed that the performance of Bayes estimators are better than the MLEs in many situations. Kundu and Pradhan (2009) considered the Bayesian inference of the unknown parameters for a two-parameter generalized exponential distribution based on importance sampling technique.

The main aim of this paper is to consider the frequentist and Bayesian inference of the unknown parameters of a two-parameter IW distribution under Type-II progressively censoring. It is observed that the MLEs of the unknown parameters cannot be obtained in closed form, as expected, and they have to obtained by solving two non-linear equations simultaneously. To avoid that problem, we propose to use the approximate MLEs, by making the Taylor series approximation of the normal equations, and they can be obtained in explicit forms. We further consider the Bayesian inference of the unknown parameters based on fairly flexible priors. It is observed that the Bayes estimators cannot be obtained in closed form, and we provide the Lindley's approximation of the Bayes estimates. Although, Lindleys' approximation can be used to compute approximate Bayes estimates, it cannot be used to construct associated highest posterior density credible intervals. We propose to use importance sampling technique to compute Bayes estimates and also to construct the associated highest posterior density credible intervals. Monte Carlo simulations are conducted to compare the performances among different methods considered, and two data sets under different progressive censoring schemes are analyzed for illustrative purposes.

The rest of the article is organized as follows: In Section 2, we provide the model assumptions and derive MLEs and the approximate MLEs. In Section 3, we develop the Bayesian inference. Monte Carlo simulation results and the analysis of data sets are presented in Section 4. An optimal criterion was presented in Section 5, and finally in Section 6, we conclude the paper.

# 2 Model Assumptions and Estimation

## 2.1 Model Assumptions

A random variable $X$ is said to have a two-parameter IW distribution if it has the following probability density function (PDF);

$$f(x) = \begin{cases} \alpha\lambda x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}}, & x \geq 0, \alpha, \lambda > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

From now on it will be denoted by IW$(\alpha, \lambda)$. Here $\alpha$ and $\lambda$ are the shape and scale parameters respectively. If $X \sim$ IW$(\alpha, \lambda)$, then the cumulative distribution function (CDF), reliability function and hazard function are

$$F(x) = \begin{cases} e^{-\lambda x^{-\alpha}}, & x \geq 0, \alpha, \lambda > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

$$R(x) = 1 - e^{-\lambda x^{-\alpha}}, \quad x \geq 0, \alpha, \lambda > 0, \tag{2.3}$$

and

$$h(x) = \frac{\alpha\lambda x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}}}{1 - e^{-\lambda x^{-\alpha}}}, \quad x \geq 0, \alpha, \lambda > 0. \tag{2.4}$$

respectively.

It is assumed that $n$ identical units are placed on a test and each of them has IW$(\alpha, \lambda)$ lifetime distribution. Based on a Type-II progressive censoring scheme

$(n, m, R_1, \ldots, R_m)$, we have the following observations; $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$. Then the joint probability density function of $x_{1:m:n}, x_{2:m:n}, ..., x_{m:m:n}$ is

$$\begin{aligned} f_{X_{1:m:n},X_{2:m:n},...,X_{m:m:n}}(x_1, x_2, ..., x_m) &= C \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i}, \\ & \quad 0 < x_1 < ... < x_m < \infty, \end{aligned} \tag{2.5}$$

where $f(.)$ and $F(.)$ are, respectively, the PDF and CDF given in (2.1) and (2.2) and

$$C = n(n - R_1 - 1)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1). \tag{2.6}$$

For more details, see Balakrishnan and Aggarwala (2000). Throughout this article it is assumed that $n$, $m$ and $R_1, \ldots, R_m$ are fixed in advance, and

$$\mathcal{D} = (x_{1:m:n}, \ldots, x_{m:m:n}). \tag{2.7}$$

## 2.2 Maximum Likelihood Estimation

Based on the observations $\mathcal{D}$ as given in (2.7), the likelihood functions of $\alpha$ and $\lambda$ can be written as

$$L(\alpha, \lambda) \propto \alpha^m \lambda^m \prod_{i=1}^{m} x_{i:m:n}^{-(\alpha+1)} \exp\{-\lambda x_{i:m:n}^{-\alpha}\} \Big[1 - \exp\{-\lambda x_{i:m:n}^{-\alpha}\}\Big]^{r_i}, \tag{2.8}$$

hence, the log likelihood function $l(\alpha, \lambda) = \log L(\alpha, \lambda)$ becomes

$$\begin{aligned} l(\alpha, \lambda) \propto m \log \alpha \quad &+ \quad m \log \lambda - (\alpha + 1) \sum_{i=1}^{m} \log x_{i:m:n} \\ &- \quad \lambda \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{m} r_i \log \Big[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}\Big]. \end{aligned} \tag{2.9}$$

Differentiating the log-likelihood function $l(\alpha, \lambda)$ partially with respect to $\alpha$ and $\lambda$ and then equating to zero we have

$$\begin{aligned} \frac{\partial l}{\partial \alpha} \quad &= \quad \frac{m}{\alpha} - \sum_{i=1}^{m} \log x_{i:m:n} \\ &+ \quad \lambda \sum_{i=1}^{m} \left( x_{i:m:n}^{-\alpha} \log x_{i:m:n} - \frac{r_i x_{i:m:n}^{-\alpha} \log x_{i:m:n} e^{-\lambda x_{i:m:n}^{-\alpha}}}{[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]} \right) = 0, \end{aligned} \tag{2.10}$$

and

$$\frac{\partial l}{\partial \lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} + \sum_{i=1}^{m} \frac{r_i x_{i:m:n}^{-\alpha} e^{-\lambda x_{i:m:n}^{-\alpha}}}{[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]} = 0. \tag{2.11}$$

The maximum likelihood estimates (MLEs) of $\alpha$ and $\lambda$, say $\hat{\alpha}$ and $\hat{\lambda}$ ,respectively, are the solution of the equations (2.10) and (2.11). Unfortunately, analytic solutions for $\alpha$ and $\lambda$ are not available. We propose to use the Newton Raphson algorithm to solve the equations (2.10) and (2.11), simultaneously, to get the desired MLEs of $\alpha$ and $\lambda$. Now, let us consider

$$\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} &= \frac{-m}{\alpha^2} - \lambda \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} (\log x_{i:m:n})^2 \\
&- \lambda \sum_{i=1}^{m} \frac{r_i x_{i:m:n}^{-\alpha} (\log x_{i:m:n})^2 e^{-\lambda x_{i:m:n}^{-\alpha}} (\lambda x_{i:m:n}^{-\alpha} + e^{-\lambda x_{i:m:n}^{-\alpha}} - 1)}{[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^2}, \qquad (2.12) \\
\frac{\partial^2 l}{\partial \lambda^2} &= \frac{-m}{\lambda^2} - \sum_{i=1}^{m} \frac{r_i (x_{i:m:n}^{-\alpha})^2 e^{-\lambda x_{i:m:n}^{-\alpha}}}{[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^2}, \qquad (2.13) \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda} &= \sum_{i=1}^{m} x_{i:m:n}^{-\alpha} \log x_{i:m:n} \\
&+ \sum_{i=1}^{m} \frac{r_i x_{i:m:n}^{-\alpha} \log x_{i:m:n} e^{-\lambda x_{i:m:n}^{-\alpha}} (e^{-\lambda x_{i:m:n}^{-\alpha}} + \lambda x_{i:m:n}^{-\alpha} e^{-\lambda x_{i:m:n}^{-\alpha}} - 1)}{[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^2}. \qquad (2.14)
\end{aligned}$$

The elements of the Fisher information matrix can be obtained from (2.12), (2.13) and (2.14). Using invariance property of MLEs, the MLEs of $\hat{R}(t)$ and $\hat{h}(t)$ can be easily obtained by replacing the parameters with the corresponding MLEs.

## 2.3 Approximate MLEs

The likelihood equations based on progressively Type-II censored samples, as discussed in the previous section, do not provide explicit estimators for the parameters. Hence, it may be desirable to develop approximations to the likelihood equations which provide us with explicit estimators of the unknown parameters. These explicit estimators may also provide us with an excellent starting value for the iterative solution of the likelihood equations. The idea of approximating the likelihood equations is certainly not new, there have been several solutions discussed in the book by Tiku *et al* (1986).

Let $T \sim IW(\alpha, \beta)$ with pdf

$$f(t) = \beta \alpha^{\beta} t^{-(\beta+1)} \exp\left\{ -\left(\frac{\alpha}{t}\right)^{\beta} \right\}, t \geq 0, \quad \alpha,\ \beta > 0,$$

and

$$F(t) = \exp\left\{ -\left(\frac{\alpha}{t}\right)^{\beta} \right\}.$$

Let $Y = \log(T)$, then

$$F(y) = F(Y \leq y) = F(\log(T) \leq y) = \exp\left\{-\left(\frac{e^y}{\alpha}\right)^{-\beta}\right\}.$$

By setting $\alpha = \exp(\mu), \beta = 1/\sigma$, then

$$F(y) = \exp\left\{-\exp\left[-\left(\frac{y-\mu}{\sigma}\right)\right]\right\}.$$

It implies that $Y \sim$ Extreme value distribution with $(\mu, \sigma)$.

Balakrishnan *et al.* (2004) have calculated the approximate maximum estimators of the extreme value parameters under progressive Type-II censoring as follows: Let

$$\alpha_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \frac{j + r_{m-j+1} + ... + r_m}{j + 1 + r_{m-j+1} + ... + r_m}, i = 1, ..., m$$

$$\nu_{i:m:n} = log[-log(1 - \alpha_{i:m:n})]$$

$$\alpha_i = e^{\nu_{i:m:n}}(1 - \nu_{i:m:n}) \text{ and } \beta_i = e^{\nu_{i:m:n}} \geq 0,$$

then the approximate maximum estimators of the Extreme value parameters under progressive Type-II censoring are given by

$$\tilde{\mu} = K + L\tilde{\sigma},$$

where

$$K = \frac{\sum_{i=1}^{m}(r_i + 1)\beta_i y_i}{\sum_{i=1}^{m}(r_i + 1)\beta_i}, \quad L = \frac{\sum_{i=1}^{m}(r_i + 1)\alpha_i - m}{\sum_{i=1}^{m}(r_i + 1)\beta_i}, \quad \tilde{\sigma} = \frac{A + \sqrt{A^2 + 4mB}}{2m},$$

and

$$A = \sum_{i=1}^{m}[(r_i + 1)\alpha_i - 1](y_i - K), \quad B = \sum_{i=1}^{m}(r_i + 1)\beta_i(y_i - K)^2 \geq 0.$$

Using the relationship between $T$ and $Y$, we can substitute $Y = \log(T)$ in the previous equations and hence get the approximate maximum estimators of the IW parameters under progressive Type-II censoring as follows

$$\tilde{\alpha} = \exp(\tilde{\mu}) \quad \text{and} \quad \tilde{\beta} = \frac{1}{\tilde{\sigma}}.$$

# 3 Bayesian Inference

## 3.1 Prior Assumptions

Observe that if the shape parameter $\alpha$ is known then the scale parameter $\lambda$ has a conjugate prior, which is a gamma prior. When both the parameters are unknown clearly they do not have conjugate priors. So we consider the following priors on $\lambda$ and $\alpha$. $\pi_1(\lambda)$ has a gamma prior with the scale parameter $a$ and shape parameter $b$, Gamma$(a, b)$, *i.e.* it has the PDF

$$\pi_1(\lambda) = \frac{a^b}{\Gamma(b)}\lambda^{b-1}e^{-a\lambda}, \alpha > 0, \lambda > 0, a > 0, b > 0.$$

It is further assumed that $\alpha$ has a non-informative prior $\pi_2(\alpha)$, namely

$$\pi(\alpha) = \frac{1}{\alpha}, \quad \alpha > 0,$$

and consider the priors of $\lambda$ and $\alpha$ are independent. Therefore, the joint prior distribution on $\alpha$ and $\lambda$ can be written as

$$\pi(\lambda, \alpha) = \frac{a^b}{\alpha\Gamma(b)}\lambda^{b-1}e^{-a\lambda}, \alpha > 0, \lambda > 0, a > 0, b > 0. \tag{3.1}$$

## 3.2 Posterior Analysis

Based on the observed sample $\mathcal{D}$ and using the prior assumption as stated above, the joint posterior density function of $\alpha$ and $\lambda$ can be obtained as

$$\pi(\lambda, \alpha|\mathcal{D}) \propto \frac{\pi(\lambda, \alpha)L(\alpha, \lambda)}{\int_0^\infty \int_0^\infty \pi(\lambda, \alpha)L(\alpha, \lambda)d\alpha d\lambda},$$

where $L(\alpha, \lambda)$ is same as defined in (2.8). Hence

$$\pi(\lambda, \alpha|\mathcal{D}) = \frac{\alpha^{m-1}\lambda^{m+b-1}}{k}e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})}\prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i}, \tag{3.2}$$

where the normalizing constant

$$k = \int_0^\infty \int_0^\infty \alpha^{m-1}\lambda^{m+b-1}e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})}\prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i}d\alpha d\lambda.$$

Now, we derive the Bayes estimators and generalized Bayes estimator, respectively, for the unknown parameter $\alpha$ and $\lambda$ under the square error loss function and Linex loss function.

If $\mu$ is the parameter to be estimated by an estimator $\tilde{\mu}$ then the square error loss function is defined as:

$$L_s(\mu, \tilde{\mu}) = (\tilde{\mu} - \mu)^2,$$

and the Linex loss function is defined by

$$L_N(\mu, \tilde{\mu}) = a(e^{c(\tilde{\mu}-\mu)} - c(\tilde{\mu} - \mu) - 1), a > 0, c \neq 0,$$

where $a$ and $c$ are shape and scale parameters of the loss function $L_N$. For detail exposition on the loss function $L_N$ one may refer to the paper by Zellner (1986). Without loss of generality, we take $a$ to be 1. It is well known that under the loss function $L_s$ the Bayes estimator of $\mu$ is the posterior mean of $\mu$. However, in case of the loss function $L_N$ the Bayes estimator $\tilde{\mu}$ of $\mu$ is given by the expression

$$\widetilde{\mu} = -\frac{1}{c}\log\{E^{(\mu|\mathcal{D})}(e^{-c\mu}|\mathcal{D})\},$$

here expectation is taken with respect to the posterior distribution of $\mu$. Thus, under the loss function $L_s$ the Bayes estimator of $\lambda$ and the generalized Bayes estimator of $\alpha$ are, respectively,

$$\begin{aligned}\widetilde{\lambda}_s = E(\lambda|\mathcal{D}) &= \frac{1}{k}\int_0^\infty \int_0^\infty \alpha^{m-1}\lambda^{m+b} e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})} \\ &\times \prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i} d\alpha d\lambda,\end{aligned}$$

and

$$\begin{aligned}\tilde{\alpha}_s &= E(\alpha|\mathcal{D}) = \frac{1}{k}\int_0^\infty \int_0^\infty \alpha^{m}\lambda^{m+b-1} e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})} \\ &\times \prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i} d\alpha d\lambda.\end{aligned}$$

Under Linex loss function $L_N$, the Bayes estimator of $\lambda$ is

$$\widetilde{\lambda}_N = -\frac{1}{c}\log\{E^{(\lambda|\mathcal{D})}(e^{-c\lambda}|\mathcal{D})\},$$

where

$$\begin{aligned}E^{(\lambda|\mathcal{D})}(e^{-c\lambda}|\mathcal{D}) &= \frac{1}{k}\int_0^\infty \int_0^\infty \alpha^{m-1}\lambda^{m+b-1} e^{-\lambda(c+a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})} \\ &\times \prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i} d\alpha d\lambda,\end{aligned}$$

and the generalized Bayes estimator of $\alpha$ is

$$\tilde{\alpha}_N = -\frac{1}{c}\log\{E^{(\alpha|\mathcal{D})}(e^{-c\alpha}|\mathcal{D})\},$$

where

$$\begin{aligned} E^{(\alpha|\mathcal{D})}(e^{-c\alpha}|\mathcal{D}) &= \frac{1}{k}\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{m+b-1}e^{-c\alpha}e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})} \\ &\times \prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1-e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i}d\alpha d\lambda. \end{aligned}$$

Proceeding as above, the Bayes estimator of $R(t)$ for the loss function $L_s$ is obtained as

$$\begin{aligned} \tilde{R}_s(t) &= E(R(t)|\mathcal{D}) = \frac{1}{k}\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{m+b-1}e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})}(1-e^{-\lambda t^{-\alpha}}) \\ &\times \prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1-e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i}d\alpha d\lambda. \end{aligned}$$

Under loss function $L_N$ it is given by

$$\tilde{R}_N(t) = -\frac{1}{c}\log\{E(e^{-cR(t)}|\mathcal{D})\}, \quad c \neq 0,$$

where

$$\begin{aligned} E(e^{-cR(t)}|\mathcal{D}) &= \frac{1}{k}\int_0^\infty\int_0^\infty \alpha^{m-1}\lambda^{m+b-1}e^{-c[1-e^{-\lambda t^{-\alpha}}]}e^{-\lambda(a+\sum_{i=1}^m x_{i:m:n}^{-\alpha})} \\ &\times \prod_{i=1}^m x_{i:m:n}^{-(\alpha+1)}[1-e^{-\lambda x_{i:m:n}^{-\alpha}}]^{R_i}d\alpha d\lambda. \end{aligned}$$

In the next section all estimators considered in this section are obtained using the well known approximation method.

## 3.3 Lindley Approximation

In the previous section, we have obtained Bayes and generalized Bayes estimators of $\lambda$ and $\alpha$ under various loss functions such as squared error and Linex. Note that these estimators are of the form of ratio of two integrals which can not be simplified in to a closed form. However, using the approach developed by Lindley (1980), one can approximate these Bayes estimators in to a form containing no integrals. This method provides a simplified form of Bayes estimator which is easy to use in practice.

Consider the ratio of integral $I(X)$, where

$$I(X) = \frac{\int_{(\lambda,\alpha)} U(\lambda, \alpha) e^{L(\lambda,\alpha)+\rho(\lambda,\alpha)} d(\lambda, \alpha)}{\int_{(\lambda,\alpha)} e^{L(\lambda,\alpha)+\rho(\lambda,\alpha)} d(\lambda, \alpha)},$$

where $U(\lambda, \alpha)$ is function of $\lambda$ and $\alpha$ only and $L(\lambda, \alpha)$ is the log-likelihood and $\rho(\lambda, \alpha) = \log \pi(\lambda, \alpha)$. Let $(\hat{\lambda}, \hat{\alpha})$ denote the MLE of $(\lambda, \alpha)$. For sufficiently large sample size $n$, using the approach developed in Lindley (1980), the ratio of integral $I(X)$ as defined in (5.1) can be written as

$$\begin{aligned} I(X) &= U(\hat{\lambda}, \hat{\alpha}) \\ &+ \frac{1}{2}\Big[(\hat{U}_{\lambda\lambda} + 2\hat{U}_{\lambda}\hat{\rho}_{\lambda})\hat{\sigma}_{\lambda\lambda} + (\hat{U}_{\alpha\lambda} + 2\hat{U}_{\alpha}\hat{\rho}_{\lambda})\hat{\sigma}_{\alpha\lambda} + (\hat{U}_{\lambda\alpha} + 2\hat{U}_{\lambda}\hat{\rho}_{\alpha})\hat{\sigma}_{\lambda\alpha} \\ &+ (\hat{U}_{\alpha\alpha} + 2\hat{U}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha}\Big] \\ &+ \frac{1}{2}\Big[(\hat{U}_{\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{U}_{\alpha}\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) \\ &+ (\hat{U}_{\lambda}\hat{\sigma}_{\alpha\lambda} + \hat{U}_{\alpha}\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha} + \hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})\Big], \end{aligned}$$

where $U_{\lambda\lambda}$ denoted the second derivative of the function $U(\lambda, \alpha)$ with respect to $\lambda$ and $\hat{U}_{\lambda\lambda}$ represents the same expression evaluated at $\lambda = \hat{\lambda}$ and $\alpha = \hat{\alpha}$. All other quantities appearing in the above expression of $I(X)$ have the following representations:

$$\begin{aligned} \hat{\sigma}_{\lambda\lambda} &= -\frac{1}{\hat{L}_{\lambda\lambda}}, \\ \hat{L}_{\lambda\lambda} &= \frac{-m}{\lambda^2} - \sum_{i=1}^{m} \frac{r_i (x_{i:m:n}^{-\alpha})^2 e^{-\lambda x_{i:m:n}^{-\alpha}}}{[1 - e^{-\lambda x_{i:m:n}^{-\alpha}}]^2} \\ \hat{\sigma}_{\alpha\alpha} &= -\frac{1}{\hat{L}_{\alpha\alpha}}, \\ \hat{L}_{\alpha\alpha} &= -\frac{m}{\hat{\alpha}^2} - \hat{\lambda} \sum_{i=1}^{m} \{ x_{i:m:n}^{-\hat{\alpha}} (\log x_{i:m:n})^2 \\ &+ \frac{r_i x_{i:m:n}^{-\hat{\alpha}} (\log x_{i:m:n})^2 e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}} (\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}} + e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}} - 1)}{[1 - e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}}]^2} \} \\ \hat{\sigma}_{\alpha\lambda} &= \hat{\sigma}_{\lambda\alpha} = 0 \\ \hat{L}_{\lambda\lambda\lambda} &= \frac{\partial^3 \log L}{\partial \lambda^3} = \frac{2m}{\hat{\lambda}^3} - \sum_{i=1}^{m} \frac{R_i x_{i:m:n}^{-3\hat{\alpha}} e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}} (3e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}} - 1)}{[1 - e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}}]^3} \\ \hat{L}_{\alpha\lambda\lambda} &= \frac{\partial^3 \log L}{\partial \alpha \partial \lambda^2} = \sum_{i=1}^{m} \frac{R_i x_{i:m:n}^{-2\hat{\alpha}} e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}} \log x_{i:m:n}}{[1 - e^{-\hat{\lambda} x_{i:m:n}^{-\hat{\alpha}}}]^3} \end{aligned}$$

$$
\begin{aligned}
& \times \quad \{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} - 2e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} - \hat{\lambda}x_{i:m:n}^{-\hat{\alpha}} + 2\} \\
\hat{L}_{\alpha\alpha\lambda} &= \frac{\partial^3 \log L}{\partial\alpha^2\partial\lambda} = -\sum_{i=1}^{m} x_{i:m:n}^{-\hat{\alpha}} \log x_{i:m:n}^2 + \sum_{i=1}^{m} \frac{R_i x_{i:m:n}^{-\hat{\alpha}} e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} \log x_{i:m:n}^2}{[1-e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}}]^3} \\
& \times \quad \{(2\hat{\lambda}^2 x_{i:m:n}^{-2\hat{\alpha}} + \hat{\lambda}x_{i:m:n}^{-\hat{\alpha}} + 3e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} - 4)e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} - \hat{\lambda}x_{i:m:n}^{-\hat{\alpha}} + 1\} \\
\hat{L}_{\alpha\alpha\alpha} &= \frac{\partial^3 \log L}{\partial\alpha^3} = \frac{2m}{\hat{\alpha}^3} + \hat{\lambda}\sum_{i=1}^{m} x_{i:m:n}^{-\hat{\alpha}}(\log x_{i:m:n})^3 - \hat{\lambda}\sum_{i=1}^{m} \frac{R_i x_{i:m:n}^{-\hat{\alpha}} e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} (\log x_{i:m:n})^3}{[1-e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}}]^3} \\
& \times \quad \{(2\hat{\lambda}^2 x_{i:m:n}^{-2\hat{\alpha}} + \hat{\lambda}x_{i:m:n}^{-\hat{\alpha}} + 3e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} - 4)e^{-\hat{\lambda}x_{i:m:n}^{-\hat{\alpha}}} - \hat{\lambda}x_{i:m:n}^{-\hat{\alpha}} + 1\} \\
\hat{\rho}(\lambda,\alpha) &= (b-1)\log(\hat{\lambda}) - a\hat{\lambda} - \log(\hat{\alpha}), \\
\hat{\rho}_\lambda &= \frac{(b-1)}{\hat{\lambda}} - a, \hat{\rho}_\alpha = -\frac{1}{\hat{\alpha}}
\end{aligned}
$$

Now the Bayes estimate of $\lambda$ under loss function $L_s$ is obtained as

$$
U(\lambda,\alpha) = \lambda,\ U_\lambda = 1, U_{\lambda\lambda} = U_\alpha = U_{\alpha\alpha} = U_{\alpha\lambda} = U_{\lambda\alpha} = 0,
$$
$$
E(\lambda|\mathcal{D}) = \hat{\lambda} + \hat{\sigma}_{\lambda\lambda}\left( (\frac{b-1}{\hat{\lambda}} - a) + \frac{1}{2}(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha}) \right).
$$

Also the generalized Bayes estimates of $\alpha$ under loss function $L_s$ is obtained as

$$
U(\lambda,\alpha) = \alpha, U_\alpha = 1, U_{\alpha\alpha} = U_\lambda = U_{\lambda\lambda} = U_{\alpha\lambda} = U_{\lambda\alpha} = 0,
$$
$$
E(\alpha|\mathcal{D}) = \hat{\alpha} + \hat{\sigma}_{\alpha\alpha}[-\frac{1}{\hat{\alpha}} + \frac{1}{2}(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})].
$$

Next, the Bayes estimates of $\lambda$ against the loss function $L_N$ is obtained as

$$
\begin{aligned}
U(\lambda,\alpha) &= e^{-c\hat{\lambda}}, U_\lambda = -ce^{-c\hat{\lambda}}, U_{\lambda\lambda} = c^2 e^{-c\hat{\lambda}}, U_\alpha = U_{\alpha\alpha} = U_{\alpha\lambda} = U_{\lambda\alpha} = 0, \\
E(e^{-c\hat{\lambda}}|\mathcal{D}) &= e^{-c\hat{\lambda}} + \frac{1}{2}\hat{\sigma}_{\lambda\lambda}[c^2 e^{-c\hat{\lambda}} - ce^{-c\hat{\lambda}}(2(\frac{b-1}{\hat{\lambda}} - a) + \hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} \\
&+ \hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha})].
\end{aligned}
$$

So,

$$
\tilde{\lambda}_N = -\frac{1}{c} log\{E^{(\lambda|\mathcal{D})}(e^{-c\lambda}|\mathcal{D})\},
$$

Finally, the generalized Bayes estimates of $\alpha$ is given by

$$
\begin{aligned}
U(\lambda,\alpha) &= e^{-c\hat{\alpha}}, U_\alpha = -ce^{-c\hat{\alpha}}, U_{\alpha\alpha} = c^2 e^{-c\hat{\alpha}}, U_\lambda = U_{\lambda\lambda} = U_{\alpha\lambda} = U_{\lambda\alpha} = 0, \\
E(e^{-c\hat{\alpha}}|\mathcal{D}) &= e^{-c\hat{\alpha}} + \frac{1}{2}\hat{\sigma}_{\alpha\alpha}[c^2 e^{-c\hat{\alpha}} - ce^{-c\hat{\alpha}}(2(\frac{b-1}{\hat{\lambda}} - a) + \hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda} \\
&+ \hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})].
\end{aligned}
$$

Hence,

$$\tilde{\alpha}_N = -\frac{1}{c} log\{E^{(\hat{\alpha}|\mathcal{D})}(e^{-c\hat{\alpha}}|\mathcal{D})\}.$$

Proceeding similarly the Bayes estimates for the reliability function can be evaluated under both loss functions.

## 3.4 HPD Credible Intervals

In the previous section we consider the Bayes estimates of the unknown parameters using Lindley's approximation. Although, Lindley's approximation provides approximate Bayes estimates of the unknown parameters, it cannot be used to construct highest posterior density (HPD) credible intervals of the unknown parameters. We propose to use the importance sampling technique similarly as Kundu and Pradhan (2009) to compute simulation consistent Bayes estimates of the unknown parameters, and also to construct associated HPD credible intervals.

Under the prior assumptions mentioned in Section 3.1, the posterior density function of the unknown parameters has been provided in (3.2). Note that $\pi(\lambda, \alpha|\mathcal{D})$ can be written as

$$\pi(\lambda, \alpha|\mathcal{D}) \propto f_1(\lambda|\alpha, \mathcal{D}) f_2(\alpha|\mathcal{D}) h(\alpha, \lambda), \tag{3.3}$$

where $f_1(\lambda|\alpha, \mathcal{D})$ is a gamma density function with the shape parameter $m + b$, and scale parameter $(a + \sum_{i=1}^{m} x_{i:m:n}^{-\alpha})$, *i.e.*,

$$f_1(\lambda|\alpha, \mathcal{D}) = \frac{(a + \sum_{i=1}^{m} x_{i:m:n}^{-\alpha})^{m+b}}{\Gamma(m+b)} \lambda^{m+b-1} e^{-\lambda(a+\sum_{i=1}^{m} x_{i:m:n}^{-\alpha})},$$

and

$$f_2(\alpha|\mathcal{D}) = K \frac{\alpha^{m-1} \prod_{i=1}^{m} x_{i:m:n}^{-\alpha}}{(a + \sum_{i=1}^{m} x_{i:m:n}^{-\alpha})^{m+b}}.$$

Here $K$ is the normalizing constant, such that $\int_0^\infty f_2(\alpha|\mathcal{D}) d\alpha = 1$, and

$$h(\alpha, \lambda) = \prod_{i=1}^{m} \left(1 - e^{-\lambda x_{i:m:n}^{-\alpha}}\right)^{R_i}.$$

Let us denote the right hand side of (3.3) as $\pi_N(\lambda, \alpha|\mathcal{D})$, and it is clear that $\pi(\lambda, \alpha|\mathcal{D})$ and $\pi_N(\lambda, \alpha|\mathcal{D})$ differ only by the proportionality constant. The Bayes estimate of any function of $\alpha$ and $\lambda$ say $g(\alpha, \lambda)$ can be obtained as

$$\widehat{g}_B(\alpha, \lambda) = \frac{\int_0^\infty \int_0^\infty g(\alpha, \lambda) \pi_N(\lambda, \alpha|\mathcal{D}) d\alpha d\lambda}{\int_0^\infty \int_0^\infty \pi_N(\lambda, \alpha|\mathcal{D}) d\alpha d\lambda}. \tag{3.4}$$

It is clear from (3.4) that to approximate $\widehat{g}_B(\alpha,\lambda)$ using the importance sampling technique it is not needed to compute the normalizing constant. Note that to use the importance sampling procedure we need to generate samples from $f_1(\lambda|\alpha,\mathcal{D})$ and $f_2(\alpha|\mathcal{D})$. It is straight forward to generate samples from $f_1(\lambda|\alpha,\mathcal{D})$, as it is a gamma density function. Using Lemma 2 of Kundu (2008), it can be shown that the $f_2(\alpha|\mathcal{D})$ is log-concave. Therefore, using the method of Devroye (1984), it is possible to generate samples from $f_2(\alpha|\mathcal{D})$.

We propose the following algorithm along the line of Kundu and Pradhan (2009) to compute the Bayes estimate of $g(\alpha,\lambda)$ and also to construct the associated HPD credible interval.

Step 1: Generate

$$\alpha \sim f_2(\alpha|\mathcal{D}) \quad \text{and} \quad \lambda|\alpha \sim f_1(\lambda|\alpha,\mathcal{D}).$$

Step 2: Repeat this procedure to obtain $(\alpha_1,\lambda_1),\ldots,(\alpha_M,\lambda_M)$

Step 3: A simulation consistent estimate of (3.4) can be obtained as

$$\frac{\sum_{i=1}^M g(\alpha_i,\lambda_i)h(\alpha_i\lambda_i)}{\sum_{i=1}^M h(\alpha_i,\lambda_i)}.$$

Now to compute the HPD credible interval of $g(\alpha,\lambda)$, we propose the following procedure. Suppose for $0<p<1$, $g_p$ is such that $P(g(\alpha,\lambda)\le g_p|\mathcal{D})=p$. For a given $p$, first we provide a method to estimate $g_p$, and that can be used to construct HPD credible intervals of $g(\alpha,\lambda)$.

Let us denote for simplicity, $g_i=g(\alpha_i,\lambda_i); i=1,\ldots,M$, and suppose

$$w_i = \frac{h(\alpha_i,\lambda_i)}{\sum_{i=1}^M h(\alpha_i,\lambda_i)};\ i=1,\ldots,M.$$

Arrange $\{(g_1,w_1),\ldots,(g_M,w_M)\}$, as $\{(g_{(1)},w_{[1]}),\ldots,(g_{(M)},w_{[M]})\}$ where $g_{(1)}<\ldots<g_{(M)}$, and $w_{[i]}$ is associated with $g_{(i)}$ for $i=1,\ldots,M$. Then a simulation consistent Bayes estimate of $g_p$ is $\widehat{g}_p=g_{(M_p)}$, where $M_p$ is the integer satisfying

$$\sum_{i=1}^{M_p} w_{[i]} \le p < \sum_{i=1}^{M_p+1} w_{[i]}.$$

Now using the above procedure a $100(1-\gamma)\%$ credible interval of $g(\alpha,\lambda)$ can be obtained as $(\widehat{g}_\delta,\widehat{g}_{\delta+1-\gamma})$, for $\delta=w_{[1]},w_{[1]}+w_{[2]},\ldots,\sum_{i=1}^{M_{1-\gamma}} w_{[i]}$. Therefore, a $100(1-\gamma)\%$ HPD credible interval of $g(\alpha,\lambda)$ can be obtained as $(\widehat{g}_{\delta^*},\widehat{g}_{\delta^*+1-\gamma})$, where $\delta^*$ is

$$\widehat{g}_{\delta^*+1-\gamma}-\widehat{g}_{\delta^*} \le \widehat{g}_{\delta+1-\gamma}-\widehat{g}_{\delta}, \quad \text{for all} \quad \delta.$$

# 4 Monte Carlo Simulations and Data Analysis

## 4.1 Simulations

In this section, we compare the performance of all the estimates proposed in the previous sections, in terms of their mean squared errors. In Tables 1 and 2, the estimated risk of estimates $\hat{\alpha}$, $\hat{\lambda}$, $\tilde{\alpha}_s$, $\tilde{\lambda}_s$, $\tilde{\alpha}_N$ and $\tilde{\lambda}_N$ are tabulated for various choices of $\alpha, \lambda, m$ and $n$. All these results are based on 1000 replications. In Tables 3 and 4 the estimated risk of the estimates $\hat{R}_{MLE}(t), \tilde{R}_s(t)$ and $\tilde{R}_N(t)$ of reliability function $R(t)$ are tabulated for different choices of $t$. Again the estimated risk values are simulated by using Monte Carlo technique. From the results of Tables 1 to 4, the following conclusions can be made:

It is clear from the Tables 1 and 2 that the performance of Bayes estimate $\widetilde{\alpha}_N$ of $\alpha$ is better than that of its MLE $\widehat{\alpha}$. It is also observed that the generalized Bayes estimates $\widetilde{\lambda}_s$ and $\widetilde{\lambda}_N$ of $\lambda$ performs worse than its MLE $\widehat{\lambda}$ in terms of their estimated risk values. The risk performance of the Bayes estimates shows that in most cases the performance of $\widetilde{\alpha}_N$ under the loss function $L_N$ is better than $\widetilde{\alpha}_s$ under the loss function $L_s$, while the performance of $\widetilde{\lambda}_N$ under the loss function $L_N$ is worse than $\widetilde{\lambda}_s$ under the loss function $L_s$.

In term of consistency, it is clear from Table 1 and 2 that the consistency is incomparable since the risk function is depending on the scheme, but in cases where $R_i's = 0$ the consistency is clear for different $n$. It is also observed that as the values of $\alpha$ and $\lambda$ increase, the performances of $\tilde{\alpha}_s$ and $\tilde{\alpha}_N$ become worse, and the corresponding performances of $\tilde{\lambda}_s$ and $\tilde{\lambda}_N$ become better, in terms of their risk values.

From the risk values tabulated in Tables 3 and 4, it is observed that the performance of Bayes estimate $\tilde{R}_s(t)$ of the reliability function $R(t)$ is better than that of its MLE $\hat{R}_{MLE}(t)$ and the Bayes estimate $\tilde{R}_N(t)$, and that $\tilde{R}_N(t)$ is worse than $\hat{R}_{MLE}(t)$, in terms of their risk values.

Table 1: The estimated risk values of all estimates for different choices of $n$ and $m$
$\alpha = 2, \lambda = 0.25, a = 1.1, b = 0.275, c = 1$

| $n$ | $m$ | *Scheme* | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | (0,2,1,0*2,(2,1)*2,1) | 0.007612 | 0.014687 | 0.007142 | 0.000924 | 0.000954 | 0.001659 |
| | 15 | (2,2,0*12,1) | 0.004057 | 0.007477 | 0.003816 | 0.000539 | 0.000551 | 0.000891 |
| | 20 | (0*20) | 0.003220 | 0.036017 | 0.002708 | 0.003011 | 0.003410 | 0.003312 |
| 30 | 10 | ((2,0,4)*3,2) | 0.001767 | 0.003757 | 0.001660 | 0.000377 | 0.000384 | 0.000486 |
| | 15 | (1*15) | 0.000175 | 0.003567 | 0.000150 | 0.000145 | 0.000155 | 0.000399 |
| | 20 | ((0,1)*10) | 0.000740 | 0.003705 | 0.000691 | 0.000218 | 0.000226 | 0.000429 |
| | 30 | (0*30) | 0.005900 | 0.007710 | 0.005717 | 0.000647 | 0.000652 | 0.000933 |
| 50 | 10 | ((4,8)*3,4,0,0,0) | 0.005838 | 0.007651 | 0.005684 | 0.000889 | 0.000894 | 0.001013 |
| | 15 | ((0,3,4,0,4,3)*2,0,3,4) | 0.002195 | 0.010979 | 0.002082 | 0.000618 | 0.000632 | 0.001228 |
| | 20 | (3*10,0*10) | 0.000293 | 0.007137 | 0.000260 | 0.000245 | 0.000256 | 0.000773 |
| | 30 | ((0,2)*10,0*20) | 0.003547 | 0.025489 | 0.003413 | 0.000135 | 0.000137 | 0.002808 |
| | 40 | (0*30,1*10) | 0.002977 | 0.014689 | 0.002888 | 0.000562 | 0.000577 | 0.001551 |
| | 50 | (0*50) | 0.00111 | 0.003781 | 0.001083 | 0.000199 | 0.000201 | 0.000419 |

Table 2: The estimated risk values of all estimates for different choices of $n$ and $m$
$\alpha = 3, \lambda = 0.5, a = 1.5, b = 0.75, c = 1$

| $n$ | $m$ | *Scheme* | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ |
|---|---|---|---|---|---|---|---|---|
| 20 | 10 | (0,2,1,0*2,(2,1)*2,1) | 0.030138 | 0.044216 | 0.027934 | 0.000441 | 0.000455 | 0.000511 |
| | 15 | (2,2,0*12,1) | 0.019518 | 0.017894 | 0.018351 | 0.000213 | 0.000215 | 0.000236 |
| | 20 | (0*20) | 0.021618 | 0.062249 | 0.019809 | 0.000437 | 0.000438 | 0.000543 |
| 30 | 10 | ((2,0,4)*3,2) | 0.015738 | 0.017883 | 0.014660 | 0.000187 | 0.000192 | 0.000216 |
| | 15 | (1*15) | 0.003810 | 0.017702 | 0.003458 | 0.000105 | 0.000108 | 0.000135 |
| | 20 | ((0,1)*10) | 0.014840 | 0.035662 | 0.013869 | 0.000273 | 0.000332 | 0.000332 |
| | 30 | (0*30) | 0.014875 | 0.017858 | 0.013838 | 0.000228 | 0.000228 | 0.000269 |
| 50 | 10 | ((4,8)*3,4,0,0,0) | 0.004878 | 0.008936 | 0.004373 | 0.000127 | 0.000129 | 0.000136 |
| | 15 | ((0,3,4,0,4,3)*2,0,3,4) | 0.015310 | 0.043862 | 0.014658 | 0.000350 | 0.000357 | 0.000424 |
| | 20 | (3*10,0*10) | 0.006635 | 0.017813 | 0.006299 | 0.000219 | 0.000221 | 0.000240 |
| | 30 | ((0,2)*10,0*20) | 0.031784 | 0.035786 | 0.030894 | 0.000409 | 0.000411 | 0.000447 |
| | 40 | (0*30,1*10) | 0.007480 | 0.017880 | 0.007087 | 0.000068 | 0.000070 | 0.000107 |
| | 50 | (0*50) | 0.002642 | 0.017773 | 0.002512 | 0.000111 | 0.000112 | 0.000144 |

Table 3: The estimated risk values of all estimates for different choices of $n$ and $m$
$\alpha = 2, \lambda = 0.25, a = 1.1, b = 0.275, c = 1$

| | | | $R(0.5)$ | | | $R(1)$ | | |
|---|---|---|---|---|---|---|---|---|
| n | m | scheme | $\hat{R}_{MLE}(t)$ | $\tilde{R}_s(t)$ | $\tilde{R}_N(t)$ | $\hat{R}_{MLE}(t)$ | $\tilde{R}_s(t)$ | $\tilde{R}_N(t)$ |
| 20 | 10 | (0,2,1,0*2,(2,1)*2,1) | 0.000283 | 0.000247 | 0.000365 | 0.000464 | 0.000424 | 0.000878 |
| | 15 | (2,2,0*12,1) | 0.001167 | 0.000817 | 0.001009 | 0.001029 | 0.001007 | 0.001738 |
| | 20 | (0*20) | 0.002105 | 0.004297 | 0.154180 | 0.005045 | 0.000503 | 0.002867 |
| 30 | 10 | ((2,0,4)*3,2) | 0.001218 | 0.001095 | 0.001999 | 0.000263 | 0.000245 | 0.004344 |
| | 15 | (1*15) | 0.003044 | 0.002776 | 0.002970 | 0.000152 | 0.000143 | 0.007118 |
| | 20 | ((0,1)*10) | 0.000218 | 0.000202 | 0.000405 | 0.000435 | 0.000414 | 0.000879 |
| | 30 | (0*30) | 0.000483 | 0.000178 | 0.000503 | 0.001 | 0.001 | 0.000135 |
| 50 | 10 | ((4,8)*3,4,0,0,0) | 0.001109 | 0.001024 | 0.002089 | 0.000664 | 0.000631 | 0.004096 |
| | 15 | ((0,3,4,0,4,3)*2,0,3,4) | 0.007704 | 0.007209 | 0.010016 | 0.000510 | 0.000487 | 0.021722 |
| | 20 | (3*10,0*10) | 0.000964 | 0.000876 | 0.000149 | 0.001 | 0.000999 | 0.000135 |
| | 30 | ((0,2)*10,0*20) | 0.002838 | 0.002712 | 0.003488 | 0.000141 | 0.000137 | 0.007893 |
| | 40 | (0*30,1*10) | 0.000542 | 0.000876 | 0.000406 | 0.001 | 0.001 | 0.000135 |
| | 50 | (0*50) | 0.000353 | 0.000342 | 0.000308 | 0.000133 | 0.000132 | 0.000795 |

Table 4: The estimated risk values of all estimates for different choices of $n$ and $m$
$\alpha = 3, \lambda = 0.5, a = 1.5, b = 0.75, c = 1$

| | | | $R(0.5)$ | | | $R(1)$ | | |
|---|---|---|---|---|---|---|---|---|
| n | m | scheme | $\hat{R}_{MLE}(t)$ | $\tilde{R}_s(t)$ | $\tilde{R}_N(t)$ | $\hat{R}_{MLE}(t)$ | $\tilde{R}_s(t)$ | $\tilde{R}_N(t)$ |
| 20 | 10 | (0,2,1,0*2,(2,1)*2,1) | 0.000401 | 0.000361 | 0.000872 | 0.000321 | 0.000303 | 0.001597 |
| | 15 | (2,2,0*12,1) | 0.001313 | 0.001110 | 0.001869 | 0.001073 | 0.001072 | 0.002391 |
| | 20 | (0*20) | 0.006024 | 0.005906 | 0.008079 | 0.000999 | 0.000988 | 0.001065 |
| 30 | 10 | ((2,0,4)*3,2) | 0.006540 | 0.005996 | 0.008889 | 0.00442 | 0.000418 | 0.019229 |
| | 15 | (1*15) | 0.000284 | 0.000264 | 0.000354 | 0.000289 | 0.000278 | 0.000716 |
| | 20 | ((0,1)*10) | 0.002006 | 0.001893 | 0.002476 | 0.000853 | 0.000832 | 0.005726 |
| | 30 | (0*30) | 0.004186 | 0.004067 | 0.004511 | 0.000110 | 0.000110 | 0.000810 |
| 50 | 10 | ((4,8)*3,4,0,0,0) | 0.000980 | 0.000950 | 0.001344 | 0.000288 | 0.000282 | 0.001956 |
| | 15 | ((0,3,4,0,4,3)*2,0,3,4) | 0.003791 | 0.003593 | 0.004579 | 0.000308 | 0.000298 | 0.009944 |
| | 20 | (3*10,0*10) | 0.000339 | 0.000327 | 0.000889 | 0.000291 | 0.000289 | 0.001852 |
| | 30 | ((0,2)*10,0*20) | 0.000246 | 0.000238 | 0.000375 | 0.000171 | 0.000169 | 0.000771 |
| | 40 | (0*30,1*10) | 0.000213 | 0.000207 | 0.000401 | 0.000158 | 0.000156 | 0.000779 |
| | 50 | (0*50) | 0.001120 | 0.000623 | 0.001779 | 0.000201 | 0.000200 | 0.000198 |

## 4.2 Data Analysis

Example 1:

We generate three different progressively type-II censored samples from the IW($\alpha = 3, \lambda = 0.5$) distribution of sample size $n = 50$ and $m = 30$, namely: $r_1$=(0*29,20), $r_2$=(0*27,5*2,10) and $r_3$=(0*26,5*4), we obtained the following data sets

| $n$ | $m$ | Scheme | Simulated data |
|---|---|---|---|
| 50 | 30 | (0*29,20) | 0.4819231, 0.5323240, 0.5494268, 0.5859643, 0.5868108, 0.5961870, 0.6106631, 0.6260398, 0.6289551, 0.6306613, 0.6914318, 0.7260768, 0.7373865, 0.7381686, 0.7610778, 0.7611810, 0.7711611, 0.7726352, 0.7792885, 0.7803440, 0.7889365, 0.7905907, 0.7935983, 0.7955530, 0.7960173, 0.8059086, 0.8288102, 0.8341882, 0.8424389, 0.8478130 |
| 50 | 30 | (0*27,5*2,10) | 0.4528989, 0.5233849, 0.5771970, 0.5971890, 0.6216006, 0.6283978, 0.6509377, 0.6555913, 0.6693977, 0.6843925, 0.6932899, 0.6981735, 0.7077837, 0.7185488, 0.7371240, 0.7409210, 0.7479689, 0.7579143, 0.7691301, 0.7872442, 0.7986704, 0.8021387, 0.8050323, 0.8052003, 0.8202584, 0.8375498, 0.8377457, 0.8419965, 0.8637634, 0.8664239 |
| 50 | 30 | (0*26,5*4) | 0.5061006, 0.5698172, 0.5776521, 0.5817129, 0.5875164, 0.5928514, 0.6053401, 0.6254125, 0.6294302, 0.6483531, 0.6697083, 0.6739602, 0.6778033, 0.6787557, 0.6990153, 0.7159138, 0.7198770, 0.7224557, 0.7528722, 0.7537209, 0.7593391, 0.7665029, 0.7788091, 0.8091189, 0.8097992, 0.8149457, 0.8153077, 0.8356457, 0.8402571, 0.8861024 |

The corresponding estimates are summarized as follows:

| $n$ | $m$ | scheme | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ |
|---|---|---|---|---|---|---|---|---|
| 50 | 30 | (0*29,20) | 3.405503 | 3.374701 | 3.039060 | 0.288318 | 0.284208 | 0.753233 |
| | | (0*27,5*2,10) | 3.326512 | 3.294541 | 3.042312 | 0.331219 | 0.326268 | 0.722401 |
| | | (0*26,5*4) | 3.220688 | 2.989171 | 3.057058 | 0.4017295 | 0.394935 | 0.674800 |

The associated 95% HPD credible intervals are as follows:

| $n$ | $m$ | scheme | $\alpha$ | $\lambda$ |
|---|---|---|---|---|
| 50 | 30 | (0*29,20) | (2.1327, 4.5321) | (0.1523, 0.5234) |
| | | (0*27,5*2,10) | (2.1054, 4.5523) | (0.1235, 0.5199) |
| | | (0*26,5*4) | (2.1167, 4.5881) | (0.1397, 0.5343) |

From the above table, we can see that, for the shape parameter $\alpha$, the Bayes estimate under the Linex loss function $\tilde{\alpha}_N$ is better in terms of the Bayesian criterion, than the MLE $\hat{\alpha}$ and the Bayes estimate under the squared error loss function $\tilde{\alpha}_s$. For the scale parameter $\lambda$, the MLE $\hat{\lambda}$ is marginally better "according to the Bayesian criterion" than the Bayes estimate under the squared error loss function $\tilde{\lambda}_s$ and significantly better than the Bayes estimate under the Linex loss function $\tilde{\lambda}_N$.

Example 2:

In this example, we consider a real life data set and illustrate the methods proposed in the previous sections. The data set is given by Dumonceaux and Antle (1973), and it represents the maximum flood levels (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as:

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379,0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

It may be mentioned that Maswadah (2003) performed the goodness of fit test of the IW distribution to the above flood data set, and found that the IW fits the data very well. We generate artificially progressively Type-II censored data from the above data set, and compute the estimates of the different unknown parameters and the associated confidence and credible intervals. The results are reported below.

It is clear that the estimates of the scale parameters are quite close in all the cases. In case of the shape parameter, the MLE and the Bayes estimates based on squared error loss function are quite close, but the Bayes estimates based on Linex loss function are slightly different.

| $n$ | $m$ | scheme | censored data |
|---|---|---|---|
| 20 | 20 | (0*20) | 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265 |
| 20 | 15 | (0*14,5) | 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494 |
| 20 | 15 | (0*10,1*5) | 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.418, 0.494, 0.338, 0.484 |
| 20 | 10 | (0*9,10) | 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, |

The corresponding estimates are summarized as follows:

| $n$ | $m$ | scheme | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ |
|---|---|---|---|---|---|---|---|---|
| 20 | 20 | (0*20) | 4.3139 | 4.2849 | 3.8073 | 0.0119 | 0.0121 | 0.0121 |
| 20 | 15 | (0*14,5) | 3.3685 | 3.3313 | 3.0958 | 0.0465 | 0.0459 | 0.0459 |
| 20 | 15 | (0*10,1*5) | 4.0717 | 4.0348 | 3.6266 | 0.0192 | 0.0191 | 0.0191 |
| 20 | 10 | (0*9,10) | 5.2895 | 5.2348 | 4.2586 | 0.0059 | 0.0059 | 0.0059 |

The associated 95% HPD credible intervals are as follows:

| $n$ | $m$ | scheme | $\alpha$ | $\lambda$ |
|---|---|---|---|---|
| 20 | 20 | (0*20) | (2.1053, 6.2111) | (0.0048, 0.0235) |
| 20 | 15 | (0*14,5) | (2.0123, 6.2786) | (0.0037, 0.0270) |
| 20 | 15 | (0*10,1*5) | (2.0128, 6.2867) | (0.0031, 0.0267) |
| 20 | 10 | (0*9,10) | (1.9954, 7.2134) | (0.0025, 0.0289) |

# 5 Optimal Progressive Censoring Scheme

## 5.1 Estimating of Quantiles:

In a reliability context, we often may be interested in estimating the $p$th quintile of the population. The MLE of the $p$th quintile is given by

$$\hat{Q}_p = \hat{\mu} + \hat{\sigma} F^{-1}(p). \tag{5.1}$$

Where $F^{-1}(p)$ is the inverse cdf of the standard inverse Weibull distribution, that is ,

$$F^{-1}(p) = (\frac{-1}{\hat{\lambda}} \ln(p))^{-\frac{1}{\hat{\alpha}}} \tag{5.2}$$

In this case optimality can be simply defined in terms of minimizing the variance of the estimate of the $p$th quantile or, equivalently,

$$V(\hat{Q}_p) = V_{11}(\hat{\theta}) + [F^{-1}(p)]^2 V_{22}(\hat{\theta}) + 2F^{-1}(1-p) V_{12}(\hat{\theta}). \tag{5.3}$$

In the finite sample situation, we can list all possible choices of censoring plans and compute the corresponding objective functions, then determine the optimal censoring plan through an extensive search.
We illustrate this approach of optimality in the following example.

## 5.2 Illustrative Example:

Using the previous example of Dumonceaux and Antle (1973) (Example 2), we illustrate how we can find the corresponding optimal censoring plans subject to the optimality criterion described earlier.
The following four Tables 5-8 present comparisons of different optimal censoring schemes by estimating the variance of the 5th and 95th quantiles as an optimality criterion used by Ng *et al.* (2004) for $n = 20$ and $m = 15, 10$. It is observed that, the results based on negative values of $c$ are smaller than the ones based on positive value of $c$. It is clear from these values that the optimal censoring plan is the one with the minimum quantiles variance.

Table 5: Comparison of different censoring plans for $n = 20$ and $m = 15$
$a = 1.5, b = 0.75, c = 1$

| scheme | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ | $V(\hat{Q}_{.95})$ | $V(\hat{Q}_{.05})$ |
|---|---|---|---|---|---|---|---|---|
| (0*14,5) | 3.3686 | 3.3338 | 3.3590 | 0.0465 | 0.0461 | 0.0465 | 0.5199 | 0.5562 |
| (0*13,1,4) | 3.8696 | 3.8381 | 3.8701 | 0.0258 | 0.0257 | 0.0255 | 0.6571 | 0.3473 |
| (0*13,2,3) | 4.1868 | 4.1578 | 4.1711 | 0.0166 | 0.0167 | 0.0164 | 0.8706 | 0.6029 |
| (0*12,1,1,3) | 3.9283 | 3.8982 | 3.9020 | 0.0227 | 0.0227 | 0.0224 | 0.7142 | 0.4132 |
| (1*12,1,2,2) | 3.6556 | 3.6230 | 3.6565 | 0.0330 | 0.0329 | 0.0328 | 0.5878 | 0.2335 |
| (0*12,1,3,1) | 3.3848 | 3.3506 | 3.3754 | 0.0462 | 0.0459 | 0.0461 | 0.5168 | 0.0926 |
| (0*12,3,1,1) | 3.7835 | 3.7520 | 3.7915 | 0.0273 | 0.0272 | 0.0271 | 0.6470 | 0.3208 |
| (0*10,1*5) | 4.0717 | 4.0397 | 4.0690 | 0.0192 | 0.0191 | 0.0190 | 0.7518 | 0.4854 |
| (0*8,1,0*5,4) | 3.8237 | 3.7922 | 3.8401 | 0.0293 | 0.0292 | 0.0290 | 0.6040 | 0.2835 |
| (0*8,1,0*3,2,0,2) | 4.0316 | 4.0011 | 4.0277 | 0.0215 | 0.0214 | 0.0212 | 0.7276 | 0.4422 |

Table 6: Comparison of different censoring plans for $n = 20$ and $m = 10$
$a = 1.5, b = 0.75, c = 1$

| scheme | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ | $V(\hat{Q}_{.95})$ | $V(\hat{Q}_{.05})$ |
|---|---|---|---|---|---|---|---|---|
| (0*8,5,5) | 3.4919 | 3.4576 | 3.4890 | 0.0576 | 0.0569 | 0.0576 | 0.8264 | 0.0455 |
| (0*8,1,9) | 4.3783 | 4.3538 | 4.3775 | 0.0119 | 0.0122 | 0.0114 | 1.5354 | 1.1788 |
| (0*8,3,7) | 4.0761 | 4.0456 | 4.0675 | 0.0301 | 0.0299 | 0.0299 | 1.0589 | 0.5093 |
| (0*7,1,2,7) | 3.8934 | 3.8625 | 3.8832 | 0.0332 | 0.0329 | 0.0330 | 0.9957 | 0.4138 |
| (1*7,1,4,5) | 3.3879 | 3.3539 | 3.3875 | 0.0577 | 0.0571 | 0.0576 | 0.8013 | 0.0240 |
| (0*6,1,1,1,7) | 3.3271 | 3.2956 | 3.3290 | 0.0517 | 0.0511 | 0.0516 | 0.7449 | 0.0586 |
| (0*6,2,2,3,3) | 4.3956 | 4.3682 | 4.3787 | 0.0170 | 0.0169 | 0.0167 | 1.6141 | 1.1225 |
| (0*6,5,0*2,5) | 3.5050 | 3.4706 | 3.4899 | 0.0580 | 0.0573 | 0.0580 | 0.8175 | 0.0428 |
| (0*4,1,0*4,9) | 4.7072 | 4.6856 | 4.6989 | 0.0073 | 0.0076 | 0.0070 | 4.0238 | 3.4425 |
| (0*2,2,0*5,4,4) | 3.9532 | 3.9233 | 3.9447 | 0.0237 | 0.0238 | 0.0233 | 0.9861 | 0.5462 |

Table 7: Comparison of different censoring plans for $n = 20$ and $m = 15$
$a = 1.5, b = 0.75, c = -1$

| scheme | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ | $V(\hat{Q}_{.95})$ | $V(\hat{Q}_{.05})$ |
|---|---|---|---|---|---|---|---|---|
| (0*14,5) | 3.3686 | 3.3338 | 3.0624 | 0.0465 | 0.0461 | 0.0461 | 0.5199 | 0.5562 |
| (0*13,1,4) | 3.8696 | 3.8381 | 3.2965 | 0.0258 | 0.0257 | 0.0257 | 0.6571 | 0.3473 |
| (0*13,2,3) | 4.1868 | 4.1578 | 3.1554 | 0.0166 | 0.0167 | 0.0167 | 0.8706 | 0.6029 |
| (0*12,1,1,3) | 3.9283 | 3.8982 | 3.2654 | 0.0227 | 0.0227 | 0.0227 | 0.7142 | 0.4132 |
| (1*12,1,2,2) | 3.6556 | 3.6230 | 3.2235 | 0.0330 | 0.0329 | 0.0329 | 0.5878 | 0.2335 |
| (0*12,1,3,1) | 3.3848 | 3.3506 | 3.0767 | 0.0462 | 0.0459 | 0.0459 | 0.5168 | 0.0926 |
| (0*12,3,1,1) | 3.7835 | 3.7520 | 3.2556 | 0.0273 | 0.0272 | 0.0272 | 0.6470 | 0.3208 |
| (0*10,1*5) | 4.0717 | 4.0397 | 3.2138 | 0.0192 | 0.0191 | 0.0191 | 0.7518 | 0.4854 |
| (0*8,1,0*5,4) | 3.8237 | 3.7922 | 3.3164 | 0.0293 | 0.0292 | 0.0293 | 0.6040 | 0.2835 |
| (0*8,1,0*3,2,0,2) | 4.0316 | 4.0011 | 3.3051 | 0.0215 | 0.0214 | 0.0214 | 0.7276 | 0.4422 |

Table 8: Comparison of different censoring plans for $n = 20$ and $m = 10$
$a = 1.5, b = 0.75, c = -1$

| scheme | $\hat{\alpha}$ | $\tilde{\alpha}_s$ | $\tilde{\alpha}_N$ | $\hat{\lambda}$ | $\tilde{\lambda}_s$ | $\tilde{\lambda}_N$ | $V(\hat{Q}_{.95})$ | $V(\hat{Q}_{.05})$ |
|---|---|---|---|---|---|---|---|---|
| (0*8,5,5) | 3.4919 | 3.4576 | 3.1788 | 0.0576 | 0.0569 | 0.0569 | 0.8264 | 0.0455 |
| (0*8,1,9) | 4.3783 | 4.3538 | 3.9909 | 0.0119 | 0.0122 | 0.0122 | 1.5354 | 1.1788 |
| (0*8,3,7) | 4.0761 | 4.0456 | 3.4668 | 0.0301 | 0.0299 | 0.0299 | 1.0589 | 0.5093 |
| (0*7,1,2,7) | 3.8934 | 3.8625 | 3.3678 | 0.0332 | 0.0329 | 0.0330 | 0.9957 | 0.4138 |
| (1*7,1,4,5) | 3.3879 | 3.3539 | 3.0830 | 0.0577 | 0.0571 | 0.0571 | 0.8013 | 0.0240 |
| (0*6,1,1,1,7) | 3.3271 | 3.2956 | 3.0327 | 0.0517 | 0.0511 | 0.0511 | 0.7449 | 0.0586 |
| (0*6,2,2,3,3) | 4.3956 | 4.3682 | 3.0836 | 0.0170 | 0.0169 | 0.0170 | 1.6141 | 1.1225 |
| (0*6,5,0*2,5) | 3.5050 | 3.4706 | 3.1907 | 0.0580 | 0.0573 | 0.0574 | 0.8175 | 0.0428 |
| (0*4,1,0*4,9) | 4.7072 | 4.6856 | 4.0136 | 0.0073 | 0.0076 | 0.0076 | 4.0238 | 3.4425 |
| (0*2,2,0*5,4,4) | 3.9532 | 3.9233 | 3.1533 | 0.0237 | 0.0238 | 0.0238 | 0.9861 | 0.5462 |

# 6 Conclusions

In this paper the statistical inference of the unknown parameters of a two-parameter IW distribution under Type-II progressively censoring has been considered. Both the classical and Bayesian inference of the unknown parameters are provided. It is observed that the MLEs of the unknown parameters cannot be obtained in closed form, hence approximate MLEs have been proposed, which can be obtained in explicit forms. The performance of the MLEs and the approximate MLEs are very close to each other. We consider the Bayes estimates of the unknown parameters based on different loss functions, and it is observed that they cannot be obtained in explicit forms, hence Lindley's approximation has been incorporated. Gibbs sampling technique has been used to compute HPD credible intervals, and it is observed that the implementation of the Gibbs sampling procedure is very simple in this case. In this paper although we have mainly considered Type-II progressive censoring case, the same method can be extended for other censoring schemes also. More work is needed along that direction.

## Acknowledgements:

The authors would like to thank two unknown referees for their valuable suggestion, and the editor-in-chief Prof. R.G. Krutchkoff for his encouragement.

# References

Ali Mousa, M.A. and Jaheen, Z.f. (2002). Statistical inference for the Burr model base on the progressively censored data, *Computational Mathematical Application*, **43**, 1441-1449.

Balakrishnan, N. and Aggarwalla, R. (2000). *Progressive Censoring: Theory, Methods and Applications*, Birkhäuser, Boston.

Balakrishnan, N.,Kannan, N., Lin, C. T., and Wu, S.J.S. (2004). Inference for the extreme value distribution under progressively Type-II censoring, *Journal of Statistical Computational and Simulation*, **74**, 25-45.

Calabria, R. and Pulcini, G. (1989). Confidence limits for reliability and tolerance limits in the inverse Weibull distribution, *Reliability Engineering and System Safety*, **24**, 77-85.

Calabria, R. and Pulcini, G. (1990). On the maximum likelihood and least-squares estimation in the Inverse Weibull distributions , *Statistica Applicata*, **2(1)**, 53-66.

Calabria, R. and Pulcini, G. (1992). Bayes probability intervals in a load-strength model, *Communications in Statistics-Theory and Method*, **21**, 3393-3405.

Calabria, R. and Pulcini, G. (1994). Bayes 2-sample prediction for the inverse Weibull distribution, *Communications in Statistics-Theory and Method*, **23(6)**, 1811-1824.

Devroye, L. (1984), A simple algorithm for generating random variables with log-concave density, *Computing*, **33**, 247 - 257.

Jiang . R, Ji. P . and Xiao .X (2003). Aging property of unimodal failure rate models , *Reliability Engineering and System Safety*, **79**, 113-116.

Keller, A.Z. and Kanath, A.R.R., 1982. Alternative reliability models for mechanical systems, Third International Conference on Reliability and Maintainability, 18-21 October, Toulse, France.

Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring, *Technometrics*, **50,2**, 144-154.

Kundu, D. and Howlader, H.(2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data, *Computational Statistics and Data Analysis*, **54**, 1547-1558.

Kundu, D. and Pradhan, B. (2009). Bayesian inference and life testing plans for generalized exponential distribution, *Science in China Series A: Mathematics*, **52**, 1373 - 1388.

Lawless, J.F. (1982). *Statistical Models & Methods For Lifetime Data*, John Wiley & Sons, New York.

Lindley, D.V.(1980). Approximate Bayesian Method, *Trabajos de Estadistica*, **31**, 223-237.

Mahmoud, M.A., Sultan, K.S. and Amer, S.M.(2003). Order statistics from the inverse Weibull distribution and characterizations, *Metron*, **LXI n.3**, 389-401.

Maswadah, M.(2003). Conditional confidence interval estimation for the inverse Weibull distribution based on censored generalized order statistics, *Journal of Statistical Computation and Simulation*, **73,12**, 887-898.

Ng, H. K. T., Chan, P. S. and Balakrishnan, N.(2004). Optimal progressive censoring plans for the Weibull distribution, *Technometrics*, **46,4**, 470-481.

Tierney, T. and Kadane, J.B.(1986). Accurate approximations for posterior moments and marginal densities , *Journal of American Statistical Association*, **81**, 82-86.

Tiku, M., Tan, W. and Balakrishnan, N. (1986). *Robust Inference*, Marcel Dekker, New York.

Zellner, A.(1986). Bayesian estimation and prediction using asymetric loss function, *Journal of American Statistical Association*, **81**, 446-451.