Inferences on stress-strength reliability from weighted Lindley distributions

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Abstract

This paper deals with the estimation of the stress-strength parameter $R = P(Y < X)$, when X and Y are two independent weighted Lindley random variables with a common shape parameter. The MLEs can be obtained by maximizing the profile log-likelihood function in one dimension. The asymptotic distribution of the MLEs are also obtained, and they have been used to construct the asymptotic confidence interval of R. Bootstrap confidence intervals are also proposed. Monte Carlo simulations are performed to verify the effectiveness of the different estimation methods, and data analysis has been performed for illustrative purposes.

Keywords Weighted Lindley distribution; Maximum likelihood estimator; Asymptotic distribution; Bootstrap confidence intervals.

Mathematics Subject Classification Primary 62F10, 62F12; Secondary 62F40, 62F15.

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1 Introduction

Lindley (1958) originally proposed a lifetime distribution which has the following probability density function (PDF)

$$f_1(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \qquad x > 0, \quad \theta > 0. \tag{1}$$

The distribution with PDF (1) is well known as the Lindley distribution. This distribution was proposed in the context of Bayesian statistics, and it is used as a counter example of fudicial statistics. Ghitany et al. (2008) discussed different properties of the Lindley distribution. It is known that the shape of the PDF of the Lindley distribution is either unimodal or a decreasing function, and the hazard function is always an increasing function.

Although, Lindley distribution has some interesting properties, due to presence of only one parameter, it may not be very useful in practice. Due to this reason, Ghitany et al. (2011) proposed a two-parameter extension of the Lindley distribution, and named it as the weighted Lindley (WL) distribution. The WL distribution is more flexible than the Lindley distribution, and Lindley distribution can be obtained as a special case of the WL distribution. The WL distribution can be defined as follows. A random variable X is said to have a WL distribution if it has the PDF

$$f(x) = \frac{\theta^{c+1}}{(\theta+c)\ \Gamma(c)}\ x^{c-1}(1+x)e^{-\theta x}, \qquad x > 0, \quad c, \theta > 0. \tag{2}$$

From now on a WL distribution with the parameters c and θ, and PDF (2) will be denoted by WL(c, θ). The WL distribution has several interesting properties. A brief description of the WL distribution, and some of its new properties are discussed in Section 2.

The main aim of this paper is to consider the inference on the stress-strength parameter $R = P(Y < X)$, where $X \sim$ WL(c, θ_1), $Y \sim$ WL(c, θ_2) and they are independently distributed. Here the notation $\sim$ means 'follows' or has the distribution. The estimation of the stress-strength parameter arises quite naturally in the area of system reliability. For example, if X is the strength of a component which is subject to stress Y, then R is a measure of system performance.

In this paper we consider the case when all the three parameters c, θ_1, θ_2 are unknown. The MLEs of the unknown parameters cannot be obtained in explicit forms. The asymptotic distribution of the MLEs of the unknown parameters and also of R are obtained. The asymptotic distribution has been used to construct asymptotic confidence interval of the stress strength parameter. Since the asymptotic variance of the MLE of R is quite involved, we have proposed to use parametric and non-parametric Bootstrap confidence intervals also. Different estimation methods are compared using Monte Carlo simulations and data analysis has been performed for illustrative purposes.

It may be mentioned that the estimation of the stress-strength parameter has received considerable attention in the statistical literature starting with the pioneering work of Birnbaum (1956), where the author has provided an interesting connection between the classical Mann-Whitney statistic and the stress-strength model. Since then extensive work has been done on developing the inference procedure on R for different distributions, both from the Bayesian and frequentist points of view. The monograph by Kotz et al. (2003) provided an excellent review on this topic till that time. For some of the recent references the readers are referred to the articles by Kundu and Gupta (2005, 2006), Gupta and Li (2006), Kim and Chung (2006), Krishnomoorthy et al. (2007), Jiang and Wong (2008), Gupta and Peng (2009), Gupta et al. (2010, 2012, 2013), Al-Mutairi et al. (2011, 2013), Saraçoğlu et al. (2012), Ghitany et al. (2015), and the references cited therein.

The rest of the paper is organized as follows. In Section 2, we briefly discuss about the WL distribution, and discuss some of its properties. In Section 3, we discuss the MLE of R and derive its asymptotic properties when all the parameters are unknown. Different bootstrap confidence intervals are discussed in Section 4. Monte Carlo simulation results are provided in Section 5 and the data analysis has been presented in Section 6. Finally, we conclude the paper in Section 7.

2 Weighted Lindley Distribution

A random variable X is said to have a WL distribution with parameters c and θ, if it has the PDF (2). The corresponding survival function (SF) takes the following form:

$$S(x) = \frac{(\theta+c)\ \Gamma(c,\theta x) + (\theta x)^c\ e^{-\theta x}}{(\theta+c)\ \Gamma(c)}, \qquad x > 0, \quad c, \theta > 0, \tag{3}$$

where

$$\Gamma(a,z) = \int_z^\infty y^{a-1} e^{-y} dy, \qquad a > 0, z \geq 0,$$

is the upper incomplete gamma function and $\Gamma(a) = \Gamma(a,0)$ is the usual gamma function. It has been observed by Ghitany et al. (2011) that the PDF of the WL distribution is either decreasing, unimodal or decreasing-increasing-decreasing shape depending on the values of c and θ. The hazard function of the WL distribution is either bathtub shaped or an increasing function depending only on the value of c.

It is clear from (3) that the inversion of the SF cannot be performed analytically. The following mixture representation is useful in generating random samples and also deriving some other properties for the WL distribution. The PDF (2) of WL distribution can be written as

$$f(x) = p f_{GA}(x; c, \theta) + (1-p) f_{GA}(x; c+1, \theta). \tag{4}$$

Here $p = \theta/(\theta + c)$, and

$$f_{GA}(x; c+j-1, \theta) = \frac{\theta^{c+j-1}}{\Gamma(c+j-1)} x^{c+j-2} e^{-\theta x}; \qquad x > 0, \quad c, \theta > 0, j = 1, 2,$$

is the PDF of the gamma distribution with the shape parameter $c+j-1$ and scale parameter θ. Since the generation from a gamma distribution can be performed quite efficiently, see for example Kundu and Gupta (2007), the generation from a WL distribution becomes quite straight forward. We further have the following result regarding likelihood ratio ordering.

THEOREM 1: Let $X \sim \text{WL}(c_1, \theta_1)$ and $Y \sim \text{WL}(c_2, \theta_2)$. If $c_1 \geq c_2$ and $\theta_1 \leq \theta_2$, then X is larger than Y in the sense of likelihood ratio order, written as $X \geq_{lr} Y$.

PROOF: From the definition of the likelihood ratio,

$$\frac{f(x; c_1, \theta_1)}{f(x; c_2, \theta_2)} = \frac{(c_2 + \theta_2)\, \Gamma(c_2)\, \theta_1^{c_1 - 1}}{(c_1 + \theta_1)\, \Gamma(c_1)\, \theta_2^{c_2 - 1}} x^{c_1 - c_2} e^{-(\theta_1 - \theta_2)\, x}$$

increases in x, since $c_1 \geq c_2$ and $\theta_1 \leq \theta_2$. Therefore the result follows. ∎

3 Maximum Likelihood Estimator of R

In this section first we provide the expression of R, and then provide the maximum likelihood estimator of R when all the parameters are unknown.

3.1 Expression of R

To derive the expression of R, we will be using the following notation:

$$\int_0^\infty \Gamma(a, x)\, x^{s-1}\, e^{-\beta x} dx = \frac{\Gamma(a+s)}{s(1+\beta)^{a+s}}\ {}_2F_1(a+s, 1; s+1; \frac{\beta}{1+\beta}), \qquad s > 0, \quad \beta > -1.$$

where

$${}_2F_1(a_1, a_2; b; z) = {}_2F_1(a_2, a_1; b; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n\, (a_2)_n}{(b)_n} \frac{z^n}{n!}, \qquad |z| < 1, \quad (a)_0 = 1,\ (a)_n = \prod_{k=0}^{n-1} (a+k),$$

is the hyper-geometric function. We assume $b \neq 0, -1, -2, \ldots$ to prevent the denominators from vanishing, see Erdélyi et al. (1954), p. 325, formula (16).

Let $X \sim \mathrm{WL}(c, \theta_1)$ and $Y \sim \mathrm{WL}(c, \theta_2)$ be independent random variables. Then

$$\begin{aligned}
R &= P(X > Y) \\
&= \int_0^\infty P(X > Y | Y = y) \ . \ f_Y(y) \ dy \\
&= \int_0^\infty S_X(y) \ . \ f_Y(y) \ dy \\
&= \frac{\theta_2^{c+1}}{(\theta_1 + c)(\theta_2 + c) \ \Gamma^2(c)} \Big\{ (\theta_1 + c) \int_0^\infty \Gamma(c, \theta_1 y) \ . \ (y^{c-1} + y^c) e^{-\theta_2 y} \ dy \\
&\qquad + \theta_1^c \int_0^\infty (y^{2c-1} + y^{2c}) e^{-(\theta_1 + \theta_2) y} \ dy \Big\} \\
&= \frac{\theta_2^{c+1}}{(\theta_1 + c)(\theta_2 + c) \ \Gamma^2(c)} \Big\{ \frac{(\theta_1 + c)}{\theta_1^{c+1}} \int_0^\infty \Gamma(c, t) \ . \ (\theta_1 t^{c-1} + t^c) e^{-(\theta_2/\theta_1) t} \ dt \\
&\qquad + \frac{\theta_1^c \ \Gamma(2c)}{(\theta_1 + \theta_2)^{2c+1}} (\theta_1 + \theta_2 + 2c) \Big\} \\
&= \frac{\theta_1^c \ \theta_2^{c+1} \Gamma(2c)}{(\theta_1 + c)(\theta_2 + c)(\theta_1 + \theta_2)^{2c+1} \ \Gamma^2(c)} \Big\{ \frac{(\theta_1 + c)(\theta_1 + \theta_2)}{c} \ {}_2F_1(2c, 1; c+1; \frac{\theta_2}{\theta_1 + \theta_2}) \\
&\qquad + \frac{2c(\theta_1 + c)}{c+1} \ {}_2F_1(2c+1, 1; c+2; \frac{\theta_2}{\theta_1 + \theta_2}) + \theta_1 + \theta_2 + 2c \Big\} \\
&= \frac{\theta_1^c \ \theta_2^{c+1} \Gamma(2c)}{(\theta_2 + c)(\theta_1 + \theta_2)^{2c+1} \ \Gamma^2(c)} \Big\{ \frac{\theta_1 + \theta_2}{c} \ {}_2F_1(2c, 1; c+1; \frac{\theta_2}{\theta_1 + \theta_2}) \\
&\qquad + \frac{2c}{c+1} \ {}_2F_1(2c+1, 1; c+2; \frac{\theta_2}{\theta_1 + \theta_2}) + \frac{\theta_2 + c}{\theta_1 + c} + 1 \Big\}
\end{aligned}$$

Remarks.

(i) When $\theta_1 = \theta_2$, $R = 0.5$. This, of course, is expected since, in this case, X and Y are i.i.d. and there is an equal chance that X is bigger than Y.

3.2 Maximum Likelihood Estimator of R

Suppose $x_1, x_2, \ldots, x_{n_1}$ is a random sample of size n_1 from the $\mathrm{WL}(c, \theta_1)$ and $y_1, y_2, \ldots, y_{n_2}$ is an independent random sample of size n_2 from the $\mathrm{WL}(c, \theta_2)$. The log-likelihood function

$\ell \equiv \ell(c, \theta_1, \theta_2) = \ell(\boldsymbol{\theta})$ based on the two independent random samples is given by

$$
\begin{aligned}
\ell &= \sum_{i=1}^{n_1} \ln[f_X(x_i)] + \sum_{j=1}^{n_2} \ln[f_Y(y_j)] \\
&= n_1[(c+1)\ln(\theta_1) - \ln(\theta_1 + c) - \ln(\Gamma(c))] + \sum_{i=1}^{n_1} \ln(1+x_i) + (c-1)\sum_{i=1}^{n_1} \ln(x_i) - n_1\theta_1\bar{x} \\
&+ n_2[(c+1)\ln(\theta_2) - \ln(\theta_2 + c) - \ln(\Gamma(c))] + \sum_{j=1}^{n_2} \ln(1+y_j) + (c-1)\sum_{j=1}^{n_2} \ln(y_j) - n_2\theta_2\bar{y},
\end{aligned}
$$

where $\bar{x}$ and $\bar{y}$ are the sample means of $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$, respectively. The maximum likelihood estimator (MLE) $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is the solutions of the non-linear equations:

$$
\begin{aligned}
\frac{\partial \ell}{\partial c} &= n_1\left[\ln(\theta_1) - \frac{1}{\theta_1 + c}\right] + n_2\left[\ln(\theta_2) - \frac{1}{\theta_2 + c}\right] - (n_1+n_2)\psi(c) \\
&\quad + \sum_{i=1}^{n_1} \ln(x_i) + \sum_{j=1}^{n_2} \ln(y_i) = 0, \\
\frac{\partial \ell}{\partial \theta_1} &= n_1 \frac{c(\theta_1 + c + 1)}{\theta_1(\theta_1 + c)} - n_1\bar{x} = 0, \\
\frac{\partial \ell}{\partial \theta_2} &= n_2 \frac{c(\theta_2 + c + 1)}{\theta_2(\theta_2 + c)} - n_2\bar{y} = 0,
\end{aligned}
$$

where $\psi(c) = \dfrac{d}{dc}\ln(\Gamma(c))$ is the digamma function. It is clear that the MLEs of the unknown parameters cannot be obtained in explicit forms. The MLEs are as follows:

$$
\begin{aligned}
\widehat{\theta_1} \equiv \widehat{\theta_1}(\widehat{c}) &= \frac{-\widehat{c}(\bar{x}-1) + \sqrt{\widehat{c}^2(\bar{x}-1)^2 + 4\widehat{c}(\widehat{c}+1)\bar{x}}}{2\bar{x}} \\
\widehat{\theta_2} \equiv \widehat{\theta_2}(\widehat{c}) &= \frac{-\widehat{c}(\bar{y}-1) + \sqrt{\widehat{c}^2(\bar{y}-1)^2 + 4\widehat{c}(\widehat{c}+1)\bar{y}}}{2\bar{y}},
\end{aligned}
$$

where $\widehat{c}$ is the solution of the non-linear equation:

$$
n_1\left[\ln(\widehat{\theta_1}(\widehat{c})) - \frac{1}{\widehat{\theta_1}(\widehat{c}) + \widehat{c}}\right] + n_2\left[\ln(\widehat{\theta_2}(\widehat{c})) - \frac{1}{\widehat{\theta_2}(\widehat{c}) + \widehat{c}}\right] - (n_1+n_2)\psi(\widehat{c}) + \sum_{i=1}^{n_1} \ln(x_i) + \sum_{j=1}^{n_2} \ln(y_i) = 0.
$$

Some iterative procedure needs to be used to solve the above non-linear equation.

Now we provide the elements of the expected Fisher information matrix, which will be needed to construct the asymptotic confidence intervals of the unknown parameters. We will use the following notations.

$$p_1 = \lim_{n_1, n_2 \to \infty} \frac{n_1}{n_1 + n_2}, \qquad p_2 = \lim_{n_1, n_2 \to \infty} \frac{n_2}{n_1 + n_2}$$

and

$$\mathbf{H}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\partial^2 l}{\partial c^2} & \frac{\partial^2 l}{\partial c \partial \theta_1} & \frac{\partial^2 l}{\partial c \partial \theta_2} \\ \frac{\partial^2 l}{\partial \theta_1 \partial c} & \frac{\partial^2 l}{\partial \theta_1^2} & \frac{\partial^2 l}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 l}{\partial \theta_2 \partial c} & \frac{\partial^2 l}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 l}{\partial \theta_2^2} \end{bmatrix}.$$

The elements of the symmetric expected Fisher information matrix of (c, θ_1, θ_2) is given by

$$\mathbf{I}(\boldsymbol{\theta}) = [I_{ij}(\boldsymbol{\theta})] = \lim_{n_1, n_2 \to \infty} E\left[\frac{-1}{n_1 + n_2} \mathbf{H}(\boldsymbol{\theta})\right].$$

In this case

$$\begin{aligned} I_{11}(\boldsymbol{\theta}) &= \psi'(c) - \frac{p_1}{(\theta_1 + c)^2} - \frac{p_2}{(\theta_2 + c)^2}, \\ I_{22}(\boldsymbol{\theta}) &= p_1 \Big[\frac{c+1}{\theta_1^2} - \frac{1}{(\theta_1 + c)^2}\Big], \\ I_{33}(\boldsymbol{\theta}) &= p_2 \Big[\frac{c+1}{\theta_2^2} - \frac{1}{(\theta_2 + c)^2}\Big], \\ I_{12}(\boldsymbol{\theta}) &= -p_1 \Big[\frac{1}{\theta_1} + \frac{1}{(\theta_1 + c)^2}\Big], \\ I_{13}(\boldsymbol{\theta}) &= -p_2 \Big[\frac{1}{\theta_2} + \frac{1}{(\theta_2 + c)^2}\Big], \\ I_{23}(\boldsymbol{\theta}) &= 0, \end{aligned}$$

where $\psi'(c) = \frac{d}{dc}\psi(c)$ is the trigamma function. Since WL family satisfies all the regularity conditions of the consistency and asymptotic normality of the MLEs, see for example Lehmann and Casella (1998), pp. 461-463, we have the following result.

THEOREM 2: The asymptotic distribution of the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ satisfies

$$\sqrt{n_1 + n_2}\, (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} N_3(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta})),$$

where $\xrightarrow{d}$ denotes convergence in distribution and $\mathbf{I}^{-1}(\boldsymbol{\theta})$ is the inverse of the matrix $\mathbf{I}(\boldsymbol{\theta})$.

The asymptotic variance-covariance matrix of the MLE $\widehat{\boldsymbol{\theta}}$ is given by

$$\frac{1}{n_1+n_2}\,\mathbf{I}^{-1}(\boldsymbol{\theta}) = \begin{pmatrix} Var(\hat{c}) & Cov(\hat{c},\hat{\theta}_1) & Cov(\hat{c},\hat{\theta}_2) \\ Cov(\hat{c},\hat{\theta}_1) & Var(\hat{\theta}_1) & Cov(\hat{\theta}_1,\hat{\theta}_2) \\ Cov(\hat{c},\hat{\theta}_2) & Cov(\hat{\theta}_1,\hat{\theta}_2) & Var(\hat{\theta}_2) \end{pmatrix}.$$

We need the following integral representation for further development, see for example Andrews (1998), p. 364:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad |zt|<1, \quad c>b>0,$$

$${}_2F_1(2c,1;c+1;z) = c\int_0^1 (1-t)^{c-1}\,(1-zt)^{-2c}\,dt$$

$$\begin{aligned} J_1(c,z) &= \frac{\partial}{\partial c}\left\{\frac{1}{c}\;{}_2F_1(2c,1;c+1;z)\right\} \\ &= \int_0^1 (1-t)^{c-1}\,(1-zt)^{-2c}\,\ln\left(\frac{1-t}{(1-zt)^2}\right)\,dt \end{aligned}$$

$${}_2F_1(2c+1,1;c+2;z) = (c+1)\int_0^1 (1-t)^{c}\,(1-zt)^{-2c-1}\,dt$$

$$\begin{aligned} J_2(c,z) &= \frac{\partial}{\partial c}\left\{\frac{1}{c+1}\;{}_2F_1(2c+1,1;c+2;z)\right\} \\ &= \int_0^1 (1-t)^{c}\,(1-zt)^{-2c-1}\,\ln\left(\frac{1-t}{(1-zt)^2}\right)\,dt. \end{aligned}$$

In order to establish the asymptotic normality of R, we further define

$$\mathbf{d}(\boldsymbol{\theta}) = \Big(\frac{\partial R}{\partial c},\frac{\partial R}{\partial \theta_1},\frac{\partial R}{\partial \theta_2}\Big)^T = \big(d_1,d_2,d_3\big)^T,$$

where T is the transpose operation and

$$
\begin{aligned}
d_1 &= \frac{\theta_1^c\ \theta_2^{c+1}\Gamma(2c)}{(\theta_2+c)(\theta_1+\theta_2)^{2c+1}\ \Gamma^2(c)}\Big\{(\theta_1+\theta_2)\ J_1(c,\frac{\theta_2}{\theta_1+\theta_2})+2c\ J_2(c,\frac{\theta_2}{\theta_1+\theta_2}) \\
&\qquad + \frac{2}{c+1}\ {}_2F_1(2c+1,1;c+2;\frac{\theta_2}{\theta_1+\theta_2})+\frac{\theta_1-\theta_2}{(\theta_1+c)^2}\Big\} \\
&\qquad + R\left\{\ln\left(\frac{\theta_1\theta_2}{(\theta_1+\theta_2)^2}\right)+2\psi(2c)-2\psi(c)-\frac{1}{\theta_2+c}\right\}
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= d_2(\theta_1,\theta_2) \\
&= -\frac{\Gamma(c+\frac{1}{2})\Big\{4c^3+(\theta_1+1)(\theta_1+\theta_2)^2+c^2(2+8\theta_1+4\theta_2)+c[5\theta_1^2+\theta_2(\theta_2+2)+\theta_1(4+6\theta_2)]\Big\}}{2^{1-2c}\ \theta_1^{-c}\ \theta_2^{-c-1}\ \sqrt{\pi}\ \Gamma(c)(\theta_1+c)^2(\theta_2+c)(\theta_1+\theta_2)^{2c+2}},
\end{aligned}
$$

$$
d_3 = -\ d_2(\theta_2,\theta_1).
$$

Therefore, using Theorem 2 and delta method, we obtain the asymptotic distribution of $\widehat{R}$, the MLE of R, as

$$
\sqrt{n_1+n_2}\ (\hat{R}-R) \xrightarrow{d} N\Big(0,\mathbf{d}^T(\boldsymbol{\theta})\ \mathbf{I}^{-1}(\boldsymbol{\theta})\ \mathbf{d}(\boldsymbol{\theta})\Big). \tag{5}
$$

From (5) the asymptotic variance of $\widehat{R}$ is obtained as

$$
\begin{aligned}
Var(\widehat{R}) &= \frac{1}{n_1+n_2}\ \mathbf{d}^T(\boldsymbol{\theta})\ \mathbf{I}^{-1}(\boldsymbol{\theta})\ \mathbf{d}(\boldsymbol{\theta}) \\
&= d_1^2\ Var(\widehat{c})+d_2^2\ Var(\widehat{\theta}_1)+d_3^2\ Var(\widehat{\theta}_2)+2\ d_1\ d_2\ Cov(\widehat{c},\widehat{\theta}_1) \\
&\qquad +2\ d_1\ d_3\ Cov(\widehat{c},\widehat{\theta_2})+2\ d_2\ d_3\ Cov(\widehat{\theta_1},\widehat{\theta_2}).
\end{aligned}
$$

Hence, using (5), an asymptotic $100(1-\alpha)\%$ confidence interval for R can be obtained as

$$
\widehat{R} \mp z_{\frac{\alpha}{2}}\ \sqrt{\widehat{Var}(\widehat{R})},
$$

where $z_{\frac{\alpha}{2}}$ is the upper $\alpha/2$ quantile of the standard normal distribution. Since, the expression of asymptotic variance of $\widehat{R}$ is quite involved, we consider different bootstrap confidence intervals of R.

4 Bootstrap Confidence Intervals

In this section we provide different parametric and non-parametric bootstrap confidence intervals of R. It is assumed that we have independent random samples $x_1, \cdots, x_{n_1}$ and $y_1, \cdots, y_{n_2}$ obtained from WL(c, θ_1) and WL(c, θ_2), respectively. First, we propose to use the following method to generate non-parametric bootstrap samples, as suggested by Efron and Tibshirani (1998), from the given random samples.

Algorithm: (Non-parametric bootstrap sampling)

- Step 1: Generate independent bootstrap samples $x_1^*, \cdots, x_{n_1}^*$ and $y_1^*, \cdots, y_{n_2}^*$ taken with replacement from the given samples $x_1, \cdots, x_{n_1}$ and $y_1, \cdots, y_{n_2}$, respectively. Based on the bootstrap samples, compute the MLE $(\hat{c}^*, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$ of (c, θ_1, θ_2) as well as $\widehat{R}^* = R(\hat{c}, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$ of R.
- Step 2: Repeat Step 1, B times to obtain a set of bootstrap estimates of R, say $\{\widehat{R}_j^*, j = 1, \cdots, B\}$.

Using the above bootstrap sample values of R, we obtain three different bootstrap confidence interval of R. The ordered $\widehat{R}_j^*$ for $j = 1, \cdots B$ will be denoted as:

$$\widehat{R}^{*(1)} < \cdots < \widehat{R}^{*(B)}.$$

(i) *Percentile bootstrap (p-boot) confidence interval*:

Let $\widehat{R}^{*(\tau)}$ be the τ percentile of $\{\widehat{R}_j^*, j = 1, 2, \ldots, B\}$, i.e. $\widehat{R}^{*(\tau)}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I(\widehat{R}_j^* \le \widehat{R}^{*(\tau)}) = \tau, \qquad 0 < \tau < 1,$$

where $I(\cdot)$ is the indicator function.

A $100(1-\alpha)\%$ p-boot confidence interval of R is given by

$$(\widehat{R}^{*(\alpha/2)}, \widehat{R}^{*(1-\alpha/2)}).$$

(ii) *Student's t bootstrap (t-boot) confidence interval*:

Let $\overline{\widehat{R}^{*}}$ and $se(\widehat{R}^{*})$ be the sample mean and sample standard deviation of $\{\widehat{R}_j^{*}, j = 1, 2, \ldots, B\}$, where $\widehat{R}_j^{*}$ is the MLE of R for the jth bootstrap sample.

Also, let $\widehat{t}^{*(\tau)}$ be the τ percentile of $\{\frac{\widehat{R}_j^{*}-\overline{\widehat{R}^{*}}}{se(\widehat{R}^{*})}, j = 1, 2, \ldots, B\}$, i.e. $\widehat{t}^{*(\tau)}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I(\frac{\widehat{R}_j^{*}-\overline{\widehat{R}^{*}}}{se(\widehat{R}^{*})} \leq \widehat{t}^{*(\tau)}) = \tau, \qquad 0 < \tau < 1.$$

A $100(1-\alpha)\%$ t-boot confidence interval of R is given by

$$\widehat{R} \pm \widehat{t}^{*(\alpha/2)}\ se(\widehat{R}^{*}).$$

(iii) *Bias-corrected and accelerated bootstrap (BC_a-boot) confidence interval*:

Let $z^{(\tau)}$ and $\hat{z}_0$, respectively, be such that $z^{(\tau)} = \Phi^{-1}(\tau)$ and

$$\hat{z}_0 = \Phi^{-1}\Big(\frac{1}{B}\sum_{j=1}^{B} I(\widehat{R}_j^{*} \leq \widehat{R})\Big),$$

where $\Phi^{-1}(.)$ is the inverse CDF of the standard normal distribution. The value $\hat{z}_0$ is called bias-correction. Also, let

$$\hat{a} = \frac{\sum_{i=1}^{n}\ \Big(\widehat{R}_{(.)} - \widehat{R}_{(i)}\Big)^3}{6\Big[\ \sum_{i=1}^{n}\ \Big(\widehat{R}_{(.)} - \widehat{R}_{(i)}\Big)^2\ \Big]^{3/2}}$$

where $\widehat{R}_{(i)}$ is the MLE of R based of $(n-1)$ observations after excluding the ith observation and $\widehat{R}_{(\cdot)} = \frac{1}{n}\sum_{i=1}^{n}\widehat{R}_{(i)}$. The value $\hat{a}$ is called acceleration factor.

A $100(1-\alpha)\%$ BC_a-boot confidence interval of R is given by

$$(\widehat{R}^{*(\nu_1)}, \widehat{R}^{*(\nu_2)}),$$

where

$$\nu_1 = \Phi\Big(\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha/2)}}{1 - \hat{a}(\hat{z}_0 + z^{(\alpha/2)})}\Big), \qquad \nu_2 = \Phi\Big(\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha/2)}}{1 - \hat{a}(\hat{z}_0 + z^{(1-\alpha/2)})}\Big).$$

Now we provide the following method to generate parametric bootstrap samples from the given random samples, and they can be used to construct different parametric bootstrap confidence intervals.

Algorithm: (Parametric bootstrap sampling)

- Step 1: Compute the MLEs of c, θ_1 and θ_2 from the given random samples, say $\widehat{c}$, $\widehat{\theta}_1$ and $\widehat{\theta}_2$, respectively. Generate independent bootstrap samples $x_1^*, \cdots, x_{n_1}^*$ and $y_1^*, \cdots, y_{n_2}^*$, from WL$(\widehat{c}, \widehat{\theta}_1)$ and WL$(\widehat{c}, \widehat{\theta}_2)$, respectively. Based on the bootstrap samples, compute the MLE $(\hat{c}^*, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$ of (c, θ_1, θ_2) as well as $\widehat{R}^* = R(\hat{c}, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$ of R.
- Step 2: Repeat Step 1, B times to obtain a set of bootstrap estimates of R, say $\{\widehat{R}_j^*, j = 1, \cdots, B\}$.

Using the above bootstrap samples of R we can obtain three different parametric bootstrap confidence intervals of R similar to the non-parametric ones. It may be mentioned that all the bootstrap confidence intervals can be obtained even in the logit scale also, and we have presented those results in Section 5.

5 Monte Carlo Simulations

In this section, we present some Monte Carlo simulation results mainly to compare the performances of different methods for different sample sizes and for different parameter values. We mainly investigate the performance of the point and interval estimation of the reliability $R = P(X > Y)$ based on maximum likelihood procedure when all the parameters are unknown. Specifically, we investigate the bias and mean square error (MSE) of the simulated MLEs. Also, we investigate the coverage probability and the length of the simulated 95% confidence intervals based on maximum likelihood method and different bootstrap procedures presented in Section 4.

For this purpose, we have generated $3,000$ samples with $B = 1,000$ bootstrap samples from each of independent WL(c, θ_1) and WL(c, θ_2) distributions where (c, θ_1, θ_2): (0.75, 2.5, 1), (2, 1, 1) and (4, 2, 5). These parameter values correspond to $R = 0.26, 0.5$ and 0.91, respectively. We have taken different sample sizes namely (n_1, n_2): (20,20), (20,30), (20,50), (30,20), (30,30),(30,50), (50,20), (50,30), (50,50).

In Tables 1 - 3, we report the average biases and the mean squared errors of the estimates of R based on MLEs, parametric bootstrap and non-parametric bootstrap methods. In Tables 4 - 6, we provide the coverage percentages and the average lengths of the confidence intervals of R based on MLE of R and using different bootstrap methods.

Some of the points are quite clear from these experiments. Even for small sample sizes the performances of the MLEs are quite satisfactory in terms of biases and MSEs. It is observed that when $n_1 = n_2 = n$, and increases, the bias and MSEs decrease. It verifies the consistency property of the MLE of R. For fixed $n_1(n_2)$ as $n_2(n_1)$ increases, the MSEs decrease. Comparing the average confidence interval lengths and coverage percentages, it is observed that the performance of the confidence intervals of R, based on the asymptotic distribution of MLE is quite satisfactory. It maintains the nominal coverage percentages even for small sample sizes. Among the different bootstrap confidence intervals the biased corrected parametric bootstrap confidence intervals perform the best in terms of the coverage percentages. In most of the cases considered here, the confidence intervals based on MLE performs better than the biased corrected bootstrap confidence intervals in terms of shorter average confidence intervals.

Table 1: Average bias (mean squared error) of different estimators of $R = P(X > Y)$ when $c = 0.75$, $\theta_1 = 2.5$ and $\theta_2 = 1$.

n_1, n_2	MLE	Parametric Boot	Non-Parametric Boot
20, 20	-0.0045 (0.0046)	-0.0075 (0.0047)	-0.0072 (0.0047)
20, 30	-0.0030 (0.0037)	-0.0084 (0.0038)	-0.0064 (0.0037)
20, 50	-0.0059 (0.0029)	-0.0081 (0.0029)	-0.0096 (0.0030)
30, 20	-0.0017 (0.0037)	-0.0017 (0.0039)	-0.0024 (0.0037)
30, 30	-0.0023 (0.0029)	-0.0046 (0.0031)	-0.0042 (0.0029)
30, 50	-0.0046 (0.0023)	-0.0053 (0.0023)	-0.0071 (0.0024)
50, 20	0.0011 (0.0031)	0.0049 (0.0033)	0.0027 (0.0032)
50, 30	0.0009 (0.0023)	-0.0002 (0.0023)	0.0008 (0.0023)
50, 50	-0.0004 (0.0018)	-0.0016 (0.0017)	-0.0016 (0.0018)

Table 2: Average bias (mean squared error) of different estimators of $R = P(X > Y)$ when $c = 2$, $\theta_1 = 1$ and $\theta_2 = 1$.

n_1, n_2	MLE	Parametric Boot	Non-Parametric Boot
20, 20	0.0001 (0.0078)	0.0008 (0.0079)	0.0000 (0.0080)
20, 30	-0.0042 (0.0065)	-0.0007 (0.0065)	-0.0054 (0.0067)
20, 50	-0.0021 (0.0055)	-0.0060 (0.0051)	-0.0044 (0.0056)
30, 20	-0.0013 (0.0064)	0.0041 (0.0067)	-0.0001 (0.0065)
30, 30	-0.0012 (0.0050)	-0.0014 (0.0052)	-0.0012 (0.0052)
30, 50	-0.0024 (0.0040)	0.0005 (0.0041)	-0.0035 (0.0040)
50, 20	0.0046 (0.0052)	0.0050 (0.0056)	0.0069 (0.0053)
50, 30	0.0008 (0.0040)	0.0015 (0.0042)	0.0018 (0.0040)
50, 50	0.0016 (0.0030)	0.0003 (0.0030)	0.0016 (0.0030)

Table 3: Average bias (mean squared error) of different estimators of $R = P(X > Y)$ when $c = 4$, $\theta_1 = 2$ and $\theta_2 = 5$.

n_1, n_2	MLE	Parametric Boot	Non-Parametric Boot
20, 20	0.0026 (0.0016)	0.0037 (0.0015)	0.0040 (0.0016)
20, 30	0.0008 (0.0013)	0.0020 (0.0012)	0.0015 (0.0013)
20, 50	0.0004 (0.0010)	-0.0009 (0.0011)	0.0001 (0.0010)
30, 20	0.0019 (0.0014)	0.0036 (0.0013)	0.0033 (0.0013)
30, 30	0.0014 (0.0011)	0.0025 (0.0010)	0.0024 (0.0011)
30, 50	0.0001 (0.0009)	0.0010 (0.0009)	0.0004 (0.0008)
50, 20	0.0010 (0.0010)	0.0019 (0.0011)	0.0021 (0.0010)
50, 30	0.0019 (0.0008)	0.0028 (0.0008)	0.0029 (0.0008)
50, 50	0.0013 (0.0006)	0.0009 (0.0007)	0.0020 (0.0006)

Table 4: Coverage probability (average confidence interval length) of different estimators of $R = P(X > Y)$ when $c = 0.75$, $\theta_1 = 2.5$ and $\theta_2 = 1$.

n_1, n_2	MLE	Parametric Bootstrap			Non-parametric Bootstrap		
		p-boot	t-boot	BC_a-boot	p-boot	t-boot	BC_a-boot
20, 20	0.9400 (0.2473)	0.9263 (0.2546)	0.9827 (0.3050)	0.9543 (0.2635)	0.9127 (0.2496)	0.9757 (0.3026)	0.9387 (0.2566)
20, 30	0.9427 (0.2265)	0.9290 (0.2299)	0.9823 (0.2764)	0.9493 (0.2379)	0.9140 (0.2252)	0.9700 (0.2739)	0.9370 (0.2310)
20, 50	0.9483 (0.2048)	0.9330 (0.2075)	0.9827 (0.2474)	0.9523 (0.2142)	0.9050 (0.2003)	0.9710 (0.2432)	0.9320 (0.2057)
30, 20	0.9507 (0.2277)	0.9317 (0.2348)	0.9763 (0.2648)	0.9500 (0.2397)	0.9293 (0.2312)	0.9707 (0.2624)	0.9523 (0.2347)
30, 30	0.9487 (0.2044)	0.9277 (0.2086)	0.9760 (0.2392)	0.9543 (0.2129)	0.9260 (0.2048)	0.9723 (0.2358)	0.9497 (0.2085)
30, 50	0.9390 (0.1815)	0.9350 (0.1845)	0.9763 (0.2128)	0.9507 (0.1884)	0.9193 (0.1805)	0.9707 (0.2101)	0.9380 (0.1840)
50, 20	0.9490 (0.2090)	0.9373 (0.2163)	0.9583 (0.2288)	0.9487 (0.2174)	0.9333 (0.2113)	0.9573 (0.2234)	0.9447 (0.2119)
50, 30	0.9530 (0.1842)	0.9463 (0.1879)	0.9717 (0.2036)	0.9547 (0.1897)	0.9400 (0.1855)	0.9653 (0.2017)	0.9513 (0.1868)
50, 50	0.9433 (0.1599)	0.9430 (0.1621)	0.9720 (0.1792)	0.9527 (0.1640)	0.9323 (0.1597)	0.9640 (0.1770)	0.9417 (0.1614)

Table 5: Coverage probability (average confidence interval length) of different estimators of $R = P(X > Y)$ when $c = 2$, $\theta_1 = 1$ and $\theta_2 = 1$.

n_1, n_2	MLE	Parametric Bootstrap			Non-parametric Bootstrap		
		p-boot	t-boot	BC_a-boot	p-boot	t-boot	BC_a-boot
20, 20	0.9413 (0.3172)	0.9383 (0.3361)	0.9630 (0.3391)	0.9567 (0.3380)	0.9303 (0.3355)	0.9510 (0.3398)	0.9497 (0.3365)
20, 30	0.9450 (0.2918)	0.9400 (0.3060)	0.9650 (0.3135)	0.9607 (0.3071)	0.9343 (0.3035)	0.9583 (0.3122)	0.9503 (0.3041)
20, 50	0.9413 (0.2690)	0.9507 (0.2791)	0.9687 (0.2904)	0.9590 (0.2795)	0.9310 (0.2756)	0.9543 (0.2880)	0.9410 (0.2759)
30, 20	0.9480 (0.2920)	0.9293 (0.3059)	0.9440 (0.3031)	0.9510 (0.3073)	0.9323 (0.3040)	0.9497 (0.3012)	0.9523 (0.3049)
30, 30	0.9457 (0.2635)	0.9450 (0.2740)	0.9593 (0.2760)	0.9577 (0.2749)	0.9383 (0.2724)	0.9527 (0.2743)	0.9493 (0.2728)
30, 50	0.9513 (0.2373)	0.9450 (0.2444)	0.9607 (0.2496)	0.9517 (0.2447)	0.9453 (0.2432)	0.9607 (0.2492)	0.9543 (0.2433)
50, 20	0.9480 (0.2693)	0.9347 (0.2786)	0.9413 (0.2713)	0.9503 (0.2797)	0.9323 (0.2747)	0.9400 (0.2661)	0.9487 (0.2755)
50, 30	0.9527 (0.2372)	0.9433 (0.2448)	0.9517 (0.2422)	0.9557 (0.2455)	0.9413 (0.2423)	0.9487 (0.2394)	0.9550 (0.2427)
50, 50	0.9510 (0.2069)	0.9400 (0.2121)	0.9527 (0.2132)	0.9480 (0.2124)	0.9387 (0.2107)	0.9507 (0.2117)	0.9483 (0.2109)

Table 6: Coverage probability (average confidence interval length) of different estimators of $R = P(X > Y)$ when $c = 4$, $\theta_1 = 2$ and $\theta_2 = 5$.

n_1, n_2	MLE	Parametric Bootstrap			Non-parametric Bootstrap		
		p-boot	t-boot	BC_a-boot	p-boot	t-boot	BC_a-boot
20, 20	0.9570 (0.1641)	0.9303 (0.1473)	0.8943 (0.1305)	0.9580 (0.1731)	0.9010 (0.1426)	0.8677 (0.1254)	0.9393 (0.1646)
20, 30	0.9547 (0.1490)	0.9263 (0.1368)	0.8960 (0.1226)	0.9660 (0.1543)	0.9037 (0.1346)	0.8713 (0.1200)	0.9447 (0.1507)
20, 50	0.9560 (0.1325)	0.9347 (0.1262)	0.9123 (0.1155)	0.9530 (0.1363)	0.9317 (0.1235)	0.9040 (0.1124)	0.9500 (0.1330)
30, 20	0.9527 (0.1485)	0.9140 (0.1344)	0.8833 (0.1190)	0.9477 (0.1546)	0.9023 (0.1305)	0.8660 (0.1147)	0.9393 (0.1478)
30, 30	0.9477 (0.1331)	0.9263 (0.1235)	0.8987 (0.1100)	0.9547 (0.1377)	0.9147 (0.1208)	0.8833 (0.1071)	0.9350 (0.1337)
30, 50	0.9543 (0.1179)	0.9330 (0.1113)	0.9057 (0.1007)	0.9520 (0.1200)	0.9297 (0.1102)	0.8963 (0.0994)	0.9463 (0.1183)
50, 20	0.9527 (0.1327)	0.9230 (0.1227)	0.8887 (0.1094)	0.9450 (0.1370)	0.9153 (0.1191)	0.8817 (0.1054)	0.9377 (0.1319)
50, 30	0.9563 (0.1170)	0.9323 (0.1097)	0.9020 (0.0980)	0.9500 (0.1203)	0.9193 (0.1077)	0.8893 (0.0955)	0.9463 (0.1175)
50, 50	0.9553 (0.1020)	0.9360 (0.0981)	0.9060 (0.0884)	0.9483 (0.1047)	0.9317 (0.0960)	0.9040 (0.0862)	0.9473 (0.1023)

6 Data Analysis

In this section we consider two data sets and describe all the details for illustrative purposes. The two data sets were originally reported by Bader and Priest (1982), on failure stresses (in GPa) of single carbon fibers of lengths 20 mm and 50 mm, respectively. We present the data sets below.

Length 20 mm: X ($n_1 = 69$): 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Length 50 mm: Y ($n_2 = 65$): 1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.84, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.27, 2.272, 2.28, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.39, 2.41, 2.43, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.62, 2.633, 2.67, 2.682, 2.699, 2.705, 2.735, 2.785, 3.02, 3.042, 3.116, 3.174.

Before progressing further first we perform some preliminary data analysis. Figure 1 shows the empirical scaled total time on test (TTT)-transform for each considered data set where

$$T(r/n) = \frac{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}}{\sum_{i=1}^{n} x_{i:n}}, \qquad r = 1, 2, \ldots, n,$$

where $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ are the order statistics from a random sample $x_1, x_2, \ldots, x_n$. For more details on the empirical scaled total time on test (TTT)-transform and its relation to the behavior of the hazard rate function, we refer the reader to the seminal paper by Barlow and Campo (1975). Inspection of Figure 1 shows concave behavior above the diagonal line, indicating that each of the considered data sets is drawn from a population with an increasing hazard rate. Therefore, WL can be used to analyze the two data sets.

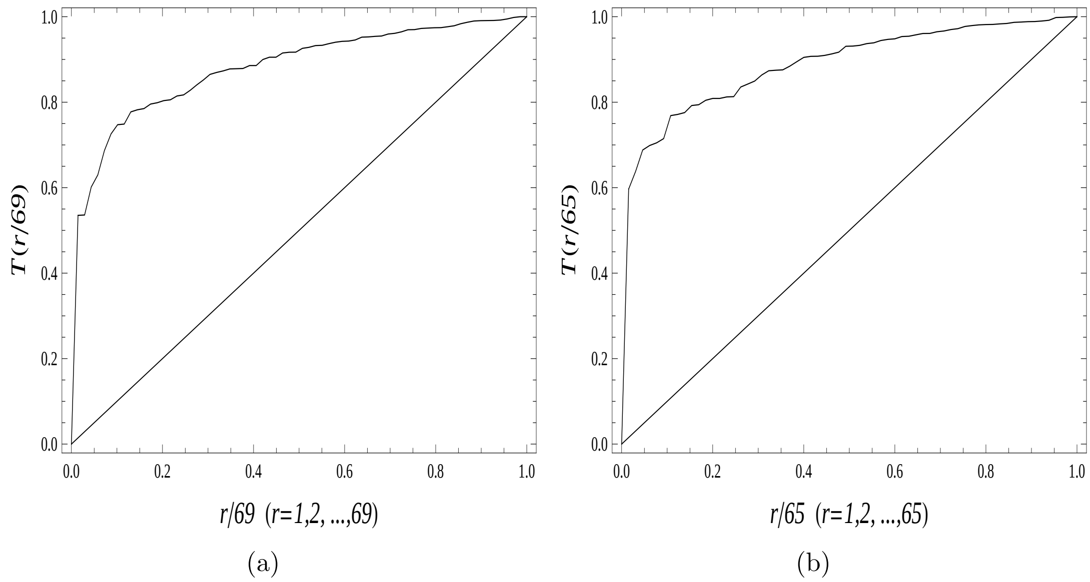

Figure 1: Scaled TTT plots: (a) Length 20 mm data (b) Length 50 mm data.

First it is assumed that $X \sim \text{WL}(c_1, \theta_1)$ and $Y \sim \text{WL}(c_2, \theta_2)$. The MLEs of the unknown parameters are as follows: $\widehat{c}_1 = 22.8930$, $\widehat{\theta}_1 = 8.2277$, $\widehat{c}_2 = 28.0882$, $\widehat{\theta}_2 = 11.3519$, and the associated log-likelihood value is $L_1 = -85.0880$. Second suppose that $X \sim \text{WL}(c, \theta_1)$ and $Y \sim \text{WL}(c, \theta_2)$. Then, the MLEs of the unknown parameters are as follows: $\widehat{c} = 25.1549$, $\widehat{\theta}_1 = 10.5491$, $\widehat{\theta}_2 = 11.5153$, and the associated log-likelihood value is $L_0 = -85.4284$. We perform the following testing of hypothesis;

$$H_0 : c_1 = c_2, \quad vs. \quad H_1 : c_1 \neq c_2,$$

and in this case $-2(L_0 - L_1) = 0.6808$. Hence, the null hypothesis cannot be rejected. Therefore, in this case the assumption of $c_1 = c_2$ is justified.

To compute the MLEs of c, θ_1 and θ_2, we use the profile likelihood method. Figure 2 provides the profile log-likelihood of c. It indicates that it has a unique maximum. In Table 7 we provide the MLEs of the parameters c, θ_1, θ_2 as well as the Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) goodness-of-fit tests. The table shows that both the tests accept the null hypothesis that each data set is drawn from WL distribution. Such conclusions are also supported by various diagnostic plots in Figures 3 and 4. Based on the MLEs $\hat{c}, \hat{\theta}_1, \hat{\theta}_2$, the point estimate of R is 0.6235 and the 95% confidence interval of R is $(0.5164, 0.7307)$ with confidence interval length 0.2143. MLE, parametric and non-parametric bootstrap estimates of R are provided in Table 8.

Table 7: MLEs and K-S and A-D goodness-of-fit tests.

Plane	MLEs	K-S statistic	p-value	A-D statistic	p-value
Length 20 mm	$\hat{c} = 25.1549$ $\hat{\theta}_1 = 10.5491$	0.0559	0.9823	0.3309	0.9130
Length 50 mm	$\hat{c} = 25.1549$ $\hat{\theta}_2 = 11.5153$	0.0723	0.8857	0.3311	0.9128

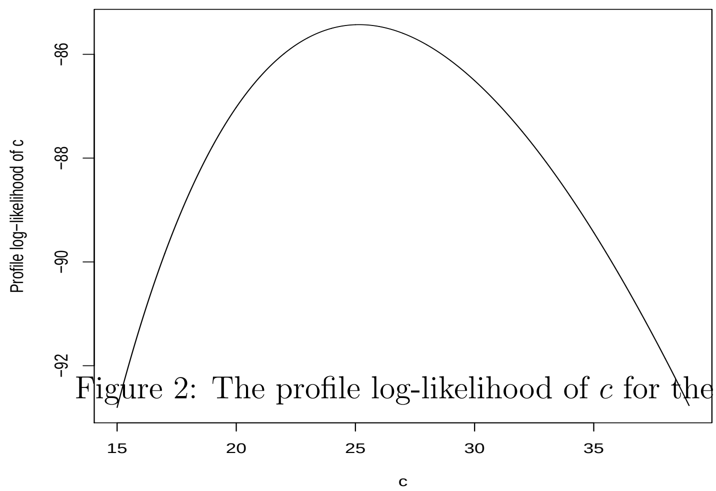

Figure 2: The profile log-likelihood of c for the given data.

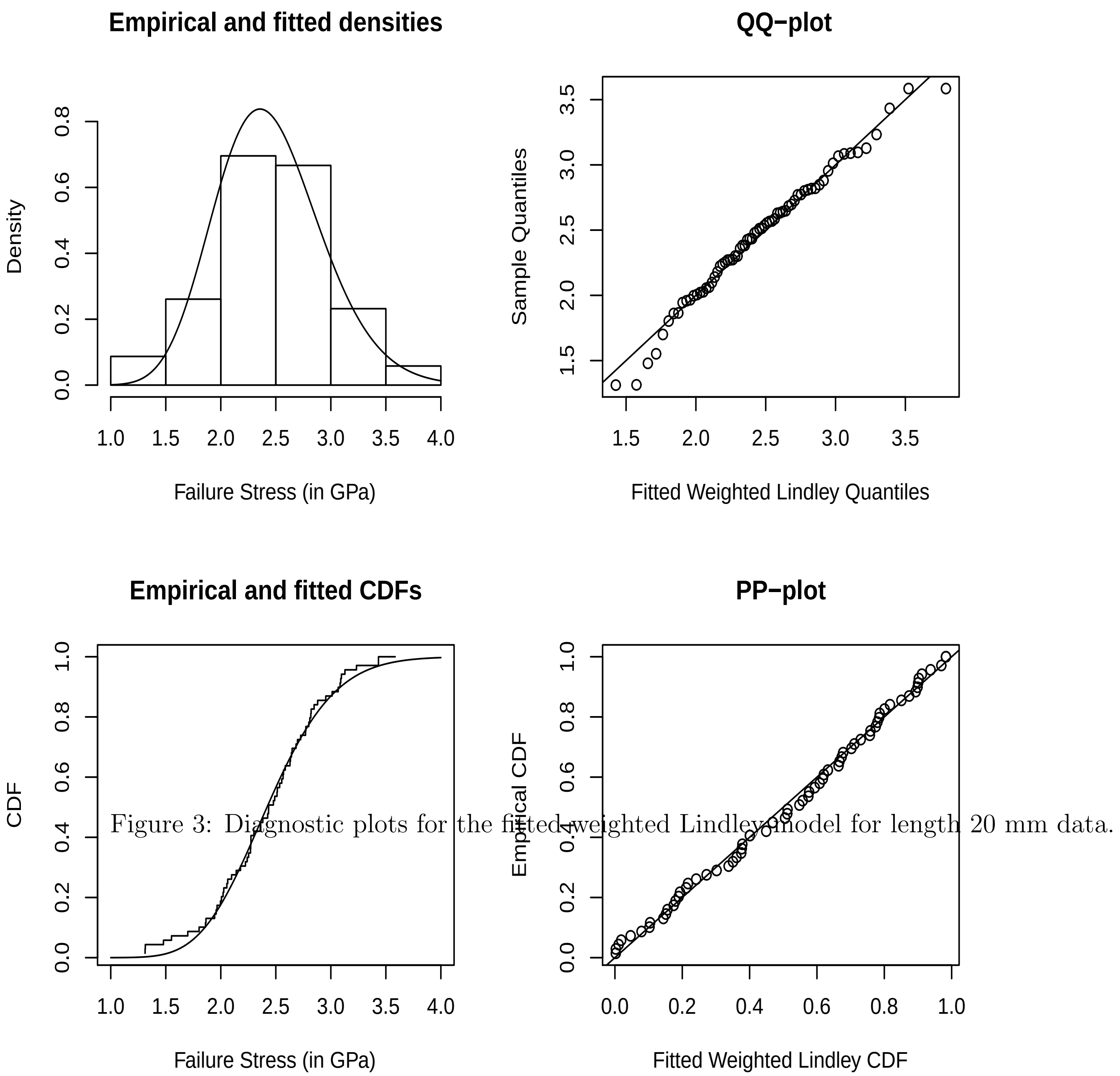

Figure 3: Diagnostic plots for the fitted weighted Lindley model for length 20 mm data.

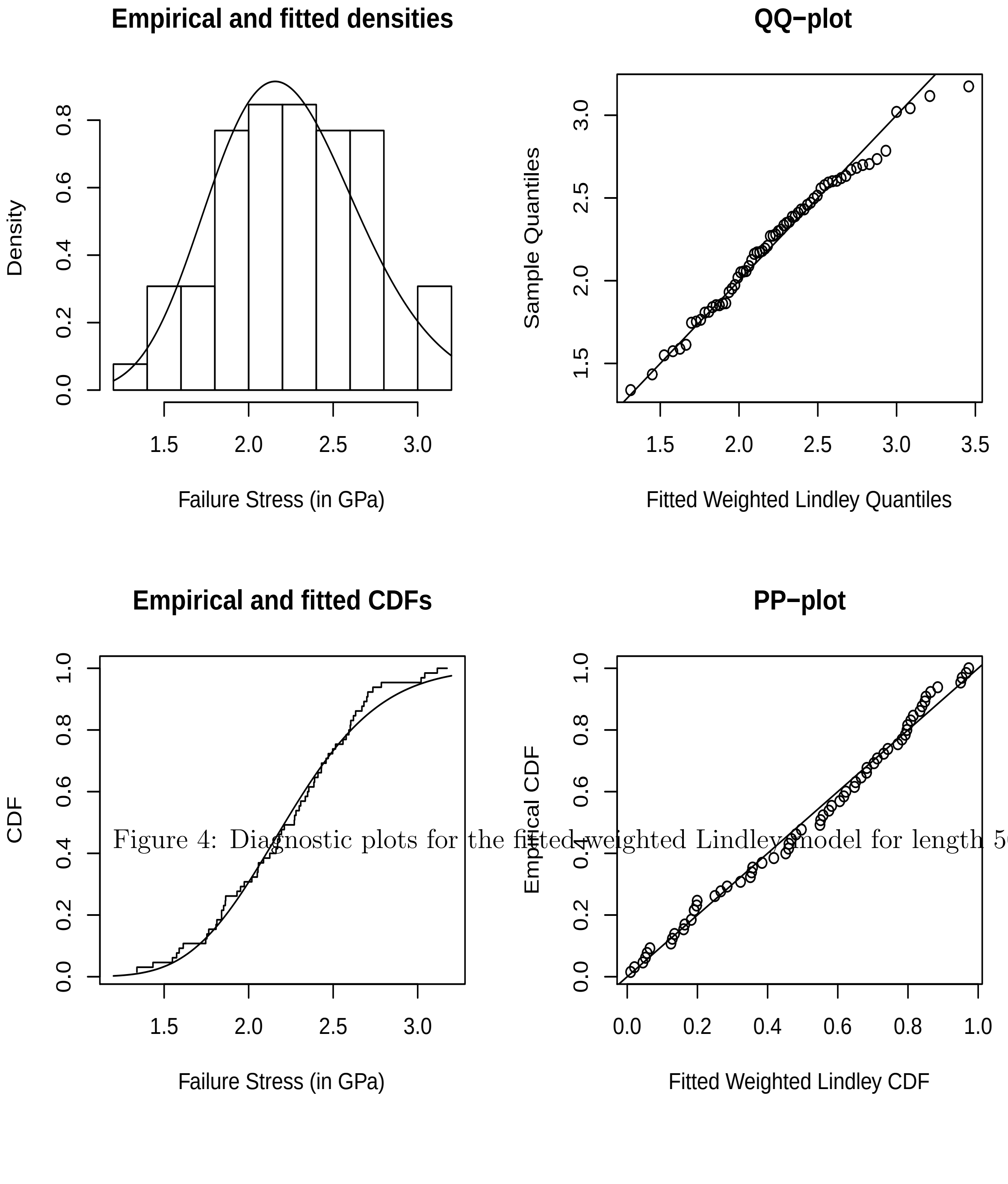

Figure 4: Diagnostic plots for the fitted weighted Lindley model for length 50 mm data.

Table 8: Statistical inference of R for the different estimation methods.

Estimation Method	$\hat{R}$	95% C.I. of R	confidence interval length
MLE	0.6235	(0.5286, 0.7099)	0.1813
Parametric Bootstrap			
p-boot	0.6232	(0.5265, 0.7154)	0.1889
t-boot	0.6232	(0.5265, 0.7116)	0.1851
BC_a-boot	0.6232	(0.5218, 0.7140)	0.1922
Non-Parametric Bootstrap			
p-boot	0.6251	(0.5364, 0.7081)	0.1717
t-boot	0.6251	(0.5364, 0.7033)	0.1669
BC_a-boot	0.6251	(0.5328, 0.7030)	0.1702

7 Concluding Remarks

In this paper we consider the statistical inference of $R = P(X > Y)$, where X and Y are independent weighted Lindley random variables with a common shape parameter. This probability is a measure of discrimination between two groups and has been studied quite extensively under different conditions by various authors. We investigate Maximum likelihood, p-, t-, and BC_a-bootstrapping estimation methods (point and interval) of R and their performances are examined by extensive simulations. It is observed that the maximum likelihood method provides very satisfactory results both for point and interval estimation of R. An example is provided to illustrate these results. It is hoped that our investigation will be useful for researchers dealing with the kind of data considered in this paper.

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