A Simple Step-Stress Model for Lehmann family of distributions

Ayan Pal, Debashis Samanta, Sharmishtha Mitra and Debasis Kundu

Abstract In this article, we consider a flexible simple step-stress model for the Lehmann family of distributions, also known as the exponentiated distributions, when the data are Type-II censored. At each stress level, we assume that the lifetime distribution of the experimental units follows a member of the Lehmann family of distributions with different shape and scale parameters. The distribution under each stress level is connected through a failure rate based step-stress accelerated life testing (SSALT) model. We obtain the maximum likelihood estimators (MLEs) of the unknown model parameters. It is observed that the MLEs of the unknown parameters do not always exist and whenever they exist, they are not in closed form. However, the failure rate based SSALT model assumption simplifies the inference problem to a significant extent. It is not possible to obtain the exact distribution of the MLEs, and hence, we have constructed the asymptotic confidence intervals (CIs) based on the observed Fisher information matrix. We have also obtained the bootstrap CIs for model parameters. Extensive simulation study is carried out when the lifetime distribution is a two-parameter generalized exponential(GE) distribution, an important member of the Lehmann family. A real data set has been analyzed assuming the lifetimes follow a few important members of the Lehmann family for illustration purposes.

Ayan Pal
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur-208016, India, e-mail: ayanpal@iitk.ac.in/ayan.pal33@gmail.com

Debashis Samanta
Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal, Pin 700156, India, e-mail: deba007.stat@gmail.com

Sharmishtha Mitra
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur-208016, India, e-mail: smitra@iitk.ac.in

Debasis Kundu
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur-208016, India, e-mail: kundu@iitk.ac.in

1 Introduction

Industrial products nowadays are highly reliable due to advancement in science and technology, and one of the major recent challenges in reliability analysis is to conduct their life testing experiments. Mean time to failure is quite high in industries like VLSI(very large scale integrated) electronic devices, computer equipments, missiles, automobile parts, etc. Performing life tests under normal operating condition (NOC) often turns out to be impractical, expensive and time intensive.

Censoring is a well known statistical technique to truncate the life testing experiment in a well planned manner before all the items fail. But censoring of a life testing experiment under NOC will not resolve the issue of insufficient number of failures for proper statistical analysis. To address this problem, accelerated life testing (ALT) experiment has been introduced which ensures a faster rate of failure. The step-stress accelerated life testing (SSALT) experiment is a special class of the ALT experiment in which the experimenter has the flexibility to perform the experiment under one or more stress levels. In a multiple step-stress model set up, n identical units are placed on a life testing experiment at an initial stress level s_1 and then the stress level is gradually increased to $s_2 < s_3 < \ldots < s_{m+1}$ at pre-fixed times $\tau_1 < \tau_2 < \ldots < \tau_m$, respectively. If $m = 1$, the corresponding model is called a simple step-stress model. One way to truncate the life testing experiment is to fix some positive integer $1 \leq r \leq n$, and then the experiment is allowed to stop as soon as the $r-$th failure occurs. This is the usual Type-II censoring case and total time of experimental duration here is random. The successive failure times thus recorded may then be extrapolated to estimate the failure time distribution under NOC. If there are only two stress levels, the model is called a simple step-stress model. To analyze the failure time data from any SSALT experiment, we need a model that relates the distributions under different stress levels. The most popular in the literature is the cumulative exposure model (CEM), introduced by [20] and later generalized by [4] and [17]. Here, if $F_1(.)$ and $F_2(.)$ are the cumulative distribution functions (CDFs) of lifetimes under the constant stress levels s_1 and s_2, respectively, the CDF of the lifetime of the experimental unit under the CEM is given by

$$F_{CEM}(t)= \begin{cases} F_1(t) & \text{if} \quad 0 < t \leq \tau \\ F_2(t + \tau - \tau^*) & \text{if} \quad \tau < t < \infty. \end{cases} \tag{1}$$

Here, τ^* is the solution of the equation $F_2(\tau^*) = F_1(\tau)$ and τ is the stress changing time. Another widely used model is the proportional hazards model (PHM) introduced by [6]. It describes the impact of the covariates on the lifetime distribution. Later, as a variation of Cox's PHM, [5] introduced the tampered failure rate model (TFRM). The assumption here is the effect of increasing the stress level from s_1 to s_2 is equivalent to multiply the initial failure rate function at stress level s_1 by an unknown factor $\alpha > 0$. Therefore, the hazard function(HF) of the lifetime of the experimental unit under the TFRM is given by

$$h_{TFRM}(t) = \begin{cases} h_1(t) & \text{if} \quad 0 < t \leq \tau \\ \alpha h_1(t) & \text{if} \quad \tau < t < \infty, \end{cases} \tag{2}$$

where $h_1(.)$ is the HF at the stress level s_1. It may be mentioned that [15] used the TFRM for the multiple SSALT model ($m \geq 2$) when the lifetime distribution across the stress levels is Weibull with common shape parameter and different scale parameters. Under similar set up, [14] derived the MLEs of the common shape parameter and different scale parameters using a log-link model. Their model is well known as the Khamis-Higgins model (KHM).

In this chapter, we work with a flexible failure rate based SSALT model with pre-fixed but arbitrarily chosen failure rates at different stress levels. If s_1 and s_2 are the two stress levels and τ is the stress changing time point, it is assumed that the hazard rate of the distribution under the step-stress pattern is as follows:

$$h(t) = \begin{cases} h_1(t) & \text{if} \quad 0 < t \leq \tau \\ h_2(t) & \text{if} \quad \tau < t < \infty, \end{cases} \tag{3}$$

where $h_i(t)$ is the HF corresponding to the CDF $F_i(t)$, $i = 1, 2.$ On simplifying, one can obtain the distribution function corresponding to $h(t)$ as follows :

$$F(t) = \begin{cases} F_1(t) & \text{if} \quad 0 < t \leq \tau \\ 1 - \dfrac{1 - F_1(\tau)}{1 - F_2(\tau)}(1 - F_2(t)) & \text{if} \quad \tau < t < \infty. \end{cases} \tag{4}$$

In fact, the CEM, TFRM and the failure rate based SSALT model coincide when the underlying distributions at the two stress levels follow exponential distribution. The flexibility of this model allows us to assume difference in both shape and scale parameters of the underlying failure distribution in the different stress levels. This assumption will take prior information about the failure situations under different stress levels into account. For the probabilistic interpretation, flexibility and application of this model, readers are referred to [12, 13].

A family of distributions is said to belong to Lehmann family if the CDF is given by:

$$F^*(t; \alpha, \lambda) = \begin{cases} [G_0(t; \lambda)]^\alpha, & \text{if} \quad t > 0, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

Here $G_0(t; \lambda)$ is some baseline absolutely continuous distribution assumed to be completely specified except for the unknown parameter λ and depending on $G_0(.)$, $\alpha > 0$ and $\lambda > 0$ can be shape, scale or location parameters. In general, a standard form of the baseline distribution is assumed to take the following form.

$$G_0(t; \lambda) = 1 - e^{-\lambda Q(t)}, \tag{6}$$

where $Q(t)$ is a strictly increasing function, differentiable on $(0, \infty)$ with $Q(0) = 0$ and $Q(\infty) = \infty$. The Lehmann family of distributions discussed in this chapter is also known as the exponentiated distributions. This model is quite flexible in reliability analysis in the sense that one can obtain the various well-known lifetime distributions as special cases for different choices of $Q(.)$. For example,

- $Q(t) = t$ gives the exponentiated (generalized) exponential(GE) distribution [See [10]].
- $Q(t) = t^2$ leads to the generalized Rayleigh(GR) distribution. It is also known as Burr type-X distribution [See [22]].
- $Q(t) = \ln(1 + t)$ gives the exponentiated Pareto (EP) distribution [See [9, 21]].
- $Q(t) = \frac{a}{\lambda}t + \frac{b}{2\lambda}t^2$ leads to the generalized failure rate distribution with parameters $a,\ b,\ \alpha$ and will be denoted by $GFRD(a, b, \alpha)$ [See [19]]

Not many authors focus on inferential procedures in SSALT set up, when the lifetime distribution is a member or belongs to the Lehmann family of distributions. It may be mentioned that [1] considered the inference of parameters of a GE distribution for simple SSALT model for Type-I censored data based on the CEM assumptions. The lifetime distribution at each of the two stress levels is assumed to have the common shape parameter and the only difference lies in the scale parameters across the stress levels. [7] considered the problem of maximum likelihood estimation for a simple step-stress accelerated GE distribution with Type II censored data based on the CEM assumptions and keeping shape parameters fixed at both the stress levels. [11] considered the maximum likelihood estimation of the GE distribution parameters and the acceleration factor under step-stress partially accelerated life testing (SSPALT) when the data are Type-II censored and the underlying model is the tampered random variable model (TRVM). Recently [18] provided an order restricted inference of the multiple step-stress model when the lifetime distribution at the different stress levels is GE distribution with common shape parameter and different scale parameters. However, a flexible SSALT experiment with difference in both the shape and scale parameters is yet to be addressed.

The main intent of this chapter is to consider the likelihood inference of a simple step-stress model for the Lehmann family of distributions based on Type-II censoring under failure rate based SSALT model assumptions.

It is assumed that the lifetime distribution of the experimental units at each of the stress levels belongs to the Lehmann family with difference in shape and scale parameters. In particular, the CDF, probability density function (PDF) and the HF of the lifetime distribution at the i−th stress level for $i = 1, 2$ are given by

$$F_i^*(t) = \begin{cases} [G_0(t;\lambda_i)]^{\alpha_i}, & \text{if} \quad t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$f_i^*(t) = \begin{cases} \alpha_i [G_0(t;\lambda_i)]^{\alpha_i - 1} [g_0(t;\lambda_i)] & \text{if} \quad t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and

$$h_i^*(t)=\begin{cases}\dfrac{\alpha_i[G_0(t;\lambda_i)]^{\alpha_i-1}g_0(t;\lambda_i)}{1-[G_0(t;\lambda_i)]^{\alpha_i}}, & \text{if} \quad t>0\\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

The existence of MLEs of the unknown parameters depends on the number of failures at the two stress levels s_1 and s_2. However, given that they exist, the MLEs cannot be obtained in closed form, but can be obtained by solving a four dimensional optimization problem. The assumption of the failure rate based SSALT model simplifies the optimization problem in terms of dimension reduction. The MLEs can then be obtained by solving a one-dimensional and a two-dimensional optimization problems. In the complete sample ($r = n$) case, the optimization problem gets even more simplified. The MLEs can then be obtained by solving two one - dimensional optimization problems.

It is not possible to obtain the exact distributions of the MLEs, as they are not in closed forms. We suggest to use the observed Fisher information matrix to construct the asymptotic CIs of the unknown model parameters assuming the asymptotic normality of the MLEs. The parametric bootstrap CIs are also proposed as an alternative as it is easy to implement in practice.

The rest of the chapter is organized as follows. In Section 2, we provide the model description, likelihood function and MLEs of the unknown parameters. For illustration purpose, some specific results for the GE distribution are demonstrated. The construction of both asymptotic and bootstrap CIs is discussed in Section 3. To see the effectiveness of the proposed methods, an extensive simulation experiment is carried out in Section 4 for different sample sizes, censoring times and stress changing time points. We have analyzed a real life data set for illustrative purpose, assuming that the lifetime distribution follows different important members of the Lehmann family. Finally we have concluded the chapter in Section 5.

2 Model description, likelihood function and MLEs

2.1 Model description

We consider a simple step-stress model with two stress levels s_1 and s_2 under Type-II censoring scheme. Initially n identical units are placed on the life testing experiment at the stress level s_1. The stress level is changed to a higher level s_2 at the pre-fixed time τ ($0 < \tau < \infty$) and the experiment terminates as soon as the r-th failure occurs (r is a prefixed integer less than or equal to n). Let n_i be the number of units that fail at stress level s_i ($i = 1, 2$). With this notation, we observe the following ordered failure time data:

$$\mathcal{D} = \Big\{t_{1:n} < \ldots < t_{n_1:n} < \tau < t_{n_1+1:n} < \ldots < t_{r:n}\Big\}, \tag{10}$$

where $r = n_1 + n_2$.

Suppose, the lifetime distributions of the experimental units at stress levels s_1 and s_2 belong to the Lehmann family of distributions with difference in both the shape and scale parameters. To relate the cumulative distribution functions(CDFs) of lifetime distributions at two consecutive stress levels to the CDF of the lifetime under the used conditions, we follow the failure rate based SSALT model assumptions. Under the assumption of the failure rate based SSALT model to analyze the failure time data, the HF $h(t)$, the corresponding CDF $G(t)$ and the associated PDF $g(t)$ of the lifetime of an experimental unit are respectively given by

$$h(t) = \begin{cases} \dfrac{\alpha_1[G_0(t;\lambda_1)]^{\alpha_1-1}g_0(t;\lambda_1)}{1-[G_0(t;\lambda_1)]^{\alpha_1}}, & \text{if} \quad 0 < t \le \tau \\ \\ \dfrac{\alpha_2[G_0(t;\lambda_2)]^{\alpha_2-1}g_0(t;\lambda_2)}{1-[G_0(t;\lambda_2)]^{\alpha_2}} & \text{if} \quad \tau < t < \infty, \end{cases}$$

$$G(t) = \begin{cases} [G_0(t;\lambda_1)]^{\alpha_1}, & \text{if} \quad 0 < t \le \tau \\ 1 - \dfrac{\left\{1-[G_0(\tau;\lambda_1)]^{\alpha_1}\right\}}{\left\{1-[G_0(\tau;\lambda_2)]^{\alpha_2}\right\}}\left\{1-[G_0(t;\lambda_2)]^{\alpha_2}\right\} & \text{if} \quad \tau < t < \infty, \end{cases}$$

$$g(t) = \begin{cases} \alpha_1[G_0(t;\lambda_1)]^{\alpha_1-1}[g_0(t;\lambda_1)] & \text{if} \quad 0 < t \le \tau \\ \alpha_2\dfrac{\left\{1-[G_0(\tau;\lambda_1)]^{\alpha_1}\right\}}{\left\{1-[G_0(\tau;\lambda_2)]^{\alpha_2}\right\}}[G_0(t;\lambda_2)]^{\alpha_2-1}[g_0(t;\lambda_2)] & \text{if} \quad \tau < t < \infty. \end{cases}$$

2.2 Likelihood Function and MLEs

2.2.1 Type-II Censoring case

In this subsection, we consider the likelihood function based on the observed Type-II censored data in (10) and obtain the MLEs of the unknown parameters $\alpha_1, \lambda_1, \alpha_2$ and λ_2.

If $T_{1:n} < \ldots < T_{r:n}$ denote the ordered Type-II censored sample from any absolutely continuous CDF $F_T(.)$, PDF $f_T(.)$, then the likelihood function of this censored sample [see [3]] can be written as

$$L(\boldsymbol{\theta} \mid \text{Data}) = \frac{n!}{(n-r)!}\left\{\prod_{k=1}^{r} f_T(t_{k:n})\right\}\{1-F_T(t_{r:n})\}^{n-r}, \quad 0 < t_{1:n} < \ldots < t_{r:n} < \infty, \tag{11}$$

where $\boldsymbol{\theta}$ is the vector of model parameters.

Let $\boldsymbol{\theta} = (\alpha_1,\ \lambda_1,\ \alpha_2,\ \lambda_2)$ be the set of unknown model parameters of interest. Based on the observed Type-II censored data in (10) of failure time from the Lehmann family of distributions with difference in both the shape and scale parameters at each of the two stress levels and assuming a failure rate based simple SSALT model, we obtain the likelihood function $L^{II}(\boldsymbol{\theta} \mid \mathcal{D})$ as

$$\begin{aligned} L^{\mathrm{II}}(\boldsymbol{\theta} \mid \mathcal{D}) = \frac{n!}{(n-r)!}\,\alpha_1^{n_1}\,\alpha_2^{n_2} \prod_{k=1}^{n_1}[G_0(t_{k:n};\lambda_1)]^{\alpha_1-1} \times \\ \prod_{k=n_1+1}^{r}[G_0(t_{k:n};\lambda_2)]^{\alpha_2-1} \prod_{k=1}^{n_1}[g_0(t_{k:n};\lambda_1)] \prod_{k=n_1+1}^{r}[g_0(t_{k:n};\lambda_2)] \times \\ \left\{1-[G_0(t_{r:n};\lambda_2)]^{\alpha_2}\right\}^{n-r}\left[\frac{1-[G_0(\tau;\lambda_1)]^{\alpha_1}}{1-[G_0(\tau;\lambda_2)]^{\alpha_2}}\right]^{n-n_1}, \\ 0 < t_{1:n} < \ldots < t_{n_1:n} < \tau < t_{n_1+1:n} < \ldots < t_{r:n} < \infty. \end{aligned} \tag{12}$$

The MLE of $\boldsymbol{\theta}$, say $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}_1,\ \widehat{\lambda}_1,\ ,\widehat{\alpha}_2,\ \widehat{\lambda}_2)$ can be obtained by maximizing (12) over the region $\Theta = (0,\infty)\times(0,\infty)\times(0,\infty)\times(0,\infty)$. The associated log-likelihood function $l^{\mathrm{II}}(\boldsymbol{\theta} \mid \mathcal{D})$ of the observed data without additive constants is given by

$$l^{\mathrm{II}}(\boldsymbol{\theta} \mid \mathcal{D}) = g_1(\alpha_1,\lambda_1) + g_2(\alpha_2,\lambda_2), \tag{13}$$

where

$$\begin{aligned} g_1(\alpha_1,\lambda_1) = {} & n_1 \ln\alpha_1 + (\alpha_1-1)\sum_{k=1}^{n_1}\ln G_0(t_{k:n};\lambda_1) + \sum_{k=1}^{n_1}\ln g_0(t_{k:n};\lambda_1) \\ & +(n-n_1)\ln\{1-[G_0(\tau;\lambda_1)]^{\alpha_1}\}, \end{aligned} \tag{14}$$

$$\begin{aligned} g_2(\alpha_2,\lambda_2) = {} & n_2 \ln\alpha_2 + (\alpha_2-1)\sum_{k=n_1+1}^{r}\ln G_0(t_{k:n};\lambda_2) \\ & + \sum_{k=n_1+1}^{r}\ln g_0(t_{k:n};\lambda_2) + (n-r)\ln\{1-[G_0(t_{r:n};\lambda_2)]^{\alpha_2}\} \\ & -(n-n_1)\ln\{1-[G_0(\tau;\lambda_2)]^{\alpha_2}\}. \end{aligned} \tag{15}$$

Hence, $\widehat{\boldsymbol{\theta}}$ can be obtained by maximizing the log-likelihood function (13) over the region Θ. The log-likelihood function (13) can be written as the sum of two terms $g_1(\alpha_1,\lambda_1)$ and $g_2(\alpha_2,\lambda_2)$. Differentiating the log-likelihood function (13) with respect to $\alpha_1, \lambda_1, \alpha_2$ and λ_2 respectively and equating them to zero, the four normal equations are obtained (see Appendix Section 6.1.1). $\widehat{\boldsymbol{\theta}}$ can directly be obtained by solving the four normal equations. In this context, it is to note that the assumption of the failure rate based SSALT model assumption yields the simple representation of the log-likelihood function. It indicates that the separate maximization of the constituent functions $g_1(\alpha_1,\lambda_1)$ and $g_2(\alpha_2,\lambda_2)$, is sufficient to obtain $(\widehat{\alpha}_1,\ \widehat{\lambda}_1)$, $(\widehat{\alpha}_2,\ \widehat{\lambda}_2)$, and hence $\widehat{\boldsymbol{\theta}}$, provided the log-likelihood function (13) is unimodal. Additionally, from the normal equations associated with $g_1(\alpha_1, \lambda_1)$, it can be easily seen that $\widehat{\alpha}_1(\lambda_1)$ maximizes $g_1(\alpha_1, \lambda_1)$ for a given λ_1, where $\widehat{\alpha}_1(\lambda_1)$ is

$$\frac{\ln[G_0(\tau;\lambda_1)] \sum_{k=1}^{n_1} \frac{n_o(t_{k:n};\lambda_1)}{g_0(t_{k:n};\lambda_1)} - \frac{n_1 m_0(\tau;\lambda_1)}{G_0(\tau;\lambda_1)} - \ln[G_0(\tau;\lambda_1)] \sum_{k=1}^{n_1} \frac{m_0(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)}}{\frac{m_0(\tau;\lambda_1)}{G_0(\tau;\lambda_1)} - \ln[G_0(\tau;\lambda_1)] \sum_{k=1}^{n_1} \frac{m_0(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)}}. \tag{16}$$

For details of the calculation and expressions for $m_0(.;\lambda_1)$ and $n_0(.;\lambda_1)$, the readers are referred to Appendix Section 6.1.1. This provides an extra edge over solving the usual four dimensional optimization problem in the sense that we are now maximizing a single one dimensional nonlinear function for estimating λ_1 and a single two dimensional nonlinear function $g_2(\alpha_2, \lambda_2)$ for estimating α_2 and λ_2. Note that once we obtain $\widehat{\lambda}_1$, by maximizing $g_1(\widehat{\alpha}_1(\lambda_1), \lambda_1)$, we can obtain $\widehat{\alpha}_1 = \widehat{\alpha}_1(\widehat{\lambda}_1)$. Next, we address the complete sample ($r = n$) scenario, a particular case of Type-II censoring. In fact, inference becomes simplified to a large extent in the complete sample case.

2.2.2 Complete Sample ($r = n$) case.

The likelihood function $L^c(\boldsymbol{\theta} \mid \mathcal{D})$ of the observed complete data is given by

$$\begin{aligned} L^c(\boldsymbol{\theta} \mid \mathcal{D}) = n!\, \alpha_1^{n_1}\, \alpha_2^{n_2} \prod_{k=1}^{n_1} [G_0(t_{k:n};\lambda_1)]^{\alpha_1 - 1} \prod_{k=n_1+1}^{n} [G_0(t_{k:n};\lambda_2)]^{\alpha_2 - 1} \times \\ \prod_{k=1}^{n_1} [g_0(t_{k:n};\lambda_1)] \prod_{k=n_1+1}^{n} [g_0(t_{k:n};\lambda_2)] \left[\frac{1 - [G_0(\tau;\lambda_1)]^{\alpha_1}}{1 - [G_0(\tau;\lambda_2)]^{\alpha_2}}\right]^{n-n_1}. \end{aligned} \tag{17}$$

The MLE of $\boldsymbol{\theta}$, say $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}_1, \widehat{\lambda}_1, \widehat{\alpha}_2, \widehat{\lambda}_2)$ can be obtained by maximizing (17) over the region $\Theta = (0,\infty) \times (0,\infty) \times (0,\infty) \times (0,\infty)$. The associated log-likelihood function $l^c(\boldsymbol{\theta} \mid \mathcal{D})$ of the observed complete data without additive constants is given by

$$l^c(\boldsymbol{\theta} \mid \mathcal{D}) = m_1(\alpha_1, \lambda_1) + m_2(\alpha_2, \lambda_2), \tag{18}$$

where

$$m_1(\alpha_1, \lambda_1) = n_1 \ln \alpha_1 + (\alpha_1 - 1) \sum_{k=1}^{n_1} \ln G_0(t_{k:n}; \lambda_1) + \sum_{k=1}^{n_1} \ln g_0(t_{k:n}; \lambda_1) + (n - n_1) \ln\{1 - [G_0(\tau; \lambda_1)]^{\alpha_1}\}. \tag{19}$$

$$m_2(\alpha_2, \lambda_2) = n_2 \ln \alpha_2 + (\alpha_2 - 1) \sum_{k=n_1+1}^{r} \ln G_0(t_{k:n}; \lambda_2) + \sum_{k=n_1+1}^{r} \ln g_0(t_{k:n}; \lambda_2) - (n - n_1) \ln\{1 - [G_0(\tau; \lambda_2)]^{\alpha_2}\}. \tag{20}$$

Hence, $\widehat{\boldsymbol{\theta}}$ can be obtained by maximizing the log-likelihood function (18) over the region Θ. In this case, also the log-likelihood function (18) can be written as the sum of two terms $m_1(\alpha_1, \lambda_1)$ and $m_2(\alpha_2, \lambda_2)$. Differentiating the log-likelihood function (18) with respect to $\alpha_1, \lambda_1, \alpha_2$ and λ_2 respectively and equating them to zero, the four normal equations are obtained (see Appendix 6.1.2). It is to note that $m_1(\alpha_1, \lambda_1) = g_1(\alpha_1, \lambda_1)$. Hence, from the normal equations associated with $g_1(\alpha_1, \lambda_1)$, it is obvious that $\widehat{\alpha}_1(\lambda_1)$ maximizes $m_1(\alpha_1, \lambda_1)$ for a given λ_1, where $\widehat{\alpha}_1(\lambda_1)$ is given by the equation (16). Once we obtain $\widehat{\lambda}_1$, by maximizing $m_1(\widehat{\alpha}_1(\lambda_1), \lambda_1)$, we can obtain $\widehat{\alpha}_1 = \widehat{\alpha}_1(\widehat{\lambda}_1)$. Unlike the Type-II censoring case, the inference associated with $m_2(\alpha_2, \lambda_2)$ is much simplified for obtaining $\widehat{\alpha}_2$ and $\widehat{\lambda}_2$ in the complete sample case.

From the normal equations associated with $m_2(\alpha_2, \lambda_2)$ and proceeding along the same lines as in Appendix 6.1.1, it can be easily seen that $\widehat{\alpha}_2(\lambda_2)$ maximizes $m_2(\alpha_2, \lambda_2)$ for a given λ_2, where $\widehat{\alpha}_2(\lambda_2)$ is

$$\frac{\ln[G_0(\tau; \lambda_2)] \sum\limits_{k=n_1+1}^{n} \dfrac{n_0(t_{k:n}; \lambda_2)}{g_0(t_{k:n}; \lambda_2)} - \dfrac{n_1 m_0(\tau; \lambda_2)}{G_0(\tau; \lambda_2)} - \ln[G_0(\tau; \lambda_2)] \sum\limits_{k=n_1+1}^{n} \dfrac{m_0(t_{k:n}; \lambda_2)}{G_0(t_{k:n}; \lambda_2)}}{\dfrac{m_0(\tau; \lambda_2)}{G_0(\tau; \lambda_2)} - \ln[G_0(\tau; \lambda_2)] \sum\limits_{k=n_1+1}^{n} \dfrac{m_0(t_{k:n}; \lambda_2)}{G_0(t_{k:n}; \lambda_2)}}. \tag{21}$$

For details of the calculation and expressions for $m_0(.; \lambda_2)$ and $n_0(.; \lambda_2)$, the readers are referred to Appendix 6.1.1. Note that once we obtain $\widehat{\lambda}_2$, by maximizing $m_2(\widehat{\alpha}_2(\lambda_2), \lambda_2)$, we can obtain $\widehat{\alpha}_2 = \widehat{\alpha}_2(\widehat{\lambda}_2)$. The four dimensional optimization problem now thus boils down to maximizing two one dimensional nonlinear functions, one for each one of the scale parameters.

Remarks.

1. The MLEs of $\alpha_1, \lambda_1, \alpha_2$, and λ_2 exist only

when $\{2 \leq n_1, n_2 \leq r - 2,\ r \geq 4\}$.

2. In case of equality of shape or equality in scale parameters, dimension of the optimization problem cannot be reduced. We need to perform a three dimensional optimization problem by using some numerical routines.

Special Case: Generalized Exponential (GE) Distribution
$Q(t) = t$ in (6) gives rise to the two-parameter GE distribution with shape parameter α and scale parameter λ in (5). This distribution was first considered by [10] as an alternative to the well known gamma or Weibull distribution. It has received considerable amount of attention in recent years. Interested readers are referred to a survey on this distribution by [16]; and a recent monograph by [2].

Based on the observed Type-II censored data in (10) and assuming a failure rate based simple SSALT model, we obtain the log-likelihood function of $\boldsymbol{\theta}$ as follows.

$$l_{GE}(\boldsymbol{\theta} \mid \mathcal{D}) = \quad g_1(\alpha_1, \lambda_1) + g_2(\alpha_2, \lambda_2),$$

where

$$\begin{aligned} g_1(\alpha_1, \lambda_1) = \ln n! - \ln(n-r)! + n_1 \ln \alpha_1 + n_1 \ln \lambda_1 - \lambda_1 \sum_{k=1}^{n_1} t_{k:n} + \\ (\alpha_1 - 1) \sum_{k=1}^{n_1} \ln(1 - e^{-\lambda_1 t_{k:n}}) + \\ (n - n_1) \ln\{1 - (1 - e^{-\lambda_1 \tau})^{\alpha_1}\}, \end{aligned} \tag{22}$$

$$\begin{aligned} g_2(\alpha_2, \lambda_2) = n_2 \ln \alpha_2 + n_2 \ln \lambda_2 - (n - n_1) \ln\{1 - (1 - e^{-\lambda_2 \tau})^{\alpha_2}\} \\ +(\alpha_2 - 1) \sum_{k=n_1+1}^{r} \ln(1 - e^{-\lambda_2 t_{k:n}}) - \lambda_2 \sum_{k=n_1+1}^{r} t_{k:n} + \\ (n - r) \ln\{1 - (1 - e^{-\lambda_2 t_{r:n}})^{\alpha_2}\}. \end{aligned} \tag{23}$$

Differentiating the log-likelihood function $l_{GE}(\boldsymbol{\theta} \mid \mathcal{D})$ with respect to $\alpha_1, \lambda_1, \alpha_2$ and λ_2 respectively, the normal equations are obtained in Section 6.2.1. From the normal equations associated with $g_1(\alpha_1, \lambda_1)$ in (22), it can be easily seen that $\widehat{\alpha}_1(\lambda_1)$ maximizes $g_1(\alpha_1, \lambda_1)$ for a given λ_1, where

$$\widehat{\alpha}_1(\lambda_1) = \frac{\dfrac{n_1}{\lambda_1} \ln(1 - e^{-\lambda_1 \tau}) - \dfrac{n_1 \tau e^{-\lambda_1 \tau}}{1 - e^{-\lambda_1 \tau}} - \ln(1 - e^{-\lambda_1 \tau}) \sum_{k=1}^{n_1} \dfrac{t_{k:n}}{1 - e^{-\lambda_1 t_{k:n}}}}{\dfrac{\tau e^{-\lambda_1 \tau}}{1 - e^{-\lambda_1 \tau}} \sum_{k=1}^{n_1} \ln(1 - e^{-\lambda_1 t_{k:n}}) - \ln(1 - e^{-\lambda_1 \tau}) \sum_{k=1}^{n_1} \dfrac{t_{k:n} e^{-\lambda_1 t_{k:n}}}{1 - e^{-\lambda_1 t_{k:n}}}}. \tag{24}$$

Once we obtain $\widehat{\lambda}_1$, by maximizing $g_1(\widehat{\alpha}_1(\lambda_1), \lambda_1)$, we can obtain $\widehat{\alpha}_1 = \widehat{\alpha}_1(\widehat{\lambda}_1)$.
Remarks.
1. In Type -II Censoring case, we need to maximize $g_2(\alpha_2, \lambda_2)$ to obtain $\widehat{\alpha}_2$ and $\widehat{\lambda}_2$.
2. Additionally, in the complete sample ($r = n$) case, maximization of $g_2(\alpha_2, \lambda_2)$ becomes much simpler in the sense that from the normal equations associated with $g_2(\alpha_2, \lambda_2)$ in (23), it can be easily seen that $\widehat{\alpha}_2(\lambda_2)$ maximizes $g_2(\alpha_2, \lambda_2)$ for a given λ_2, where

$$\widehat{\alpha}_2(\lambda_2) = \frac{\dfrac{n_2}{\lambda_2}\ln(1-e^{-\lambda_2\tau}) - \dfrac{n_2\tau e^{-\lambda_2\tau}}{1-e^{-\lambda_2\tau}} - \ln(1-e^{-\lambda_2\tau})\sum\limits_{k=n_1+1}^{n}\dfrac{t_{k:n}}{1-e^{-\lambda_2 t_{k:n}}}}{\dfrac{\tau e^{-\lambda_2\tau}}{1-e^{-\lambda_2\tau}}\sum\limits_{k=n_1+1}^{n}\ln(1-e^{-\lambda_2 t_{k:n}}) - \ln(1-e^{-\lambda_2\tau})\sum\limits_{k=n_1+1}^{n}\dfrac{t_{k:n}e^{-\lambda_2 t_{k:n}}}{1-e^{-\lambda_2 t_{k:n}}}}. \tag{25}$$

Once we obtain $\widehat{\lambda}_2$, by maximizing $g_2(\widehat{\alpha}_2(\lambda_2), \lambda_2)$, we can obtain $\widehat{\alpha}_2 = \widehat{\alpha}_2(\widehat{\lambda}_2)$.

3 Interval estimation

In this section, we present two different methods for construction of CIs of the unknown parameters $\alpha_1, \lambda_1, \alpha_2$ and λ_2. Since the closed forms of the MLEs do not exist, we cannot obtain the exact CIs of the unknown parameters. First, we provide asymptotic CIs assuming the asymptotic normality of the MLEs and then the parametric bootstrap CIs.

3.1 Asymptotic Confidence Intervals

Here, we present a method which assumes asymptotic normality of the MLEs to obtain the CIs for $\alpha_1, \lambda_1, \alpha_2$, and λ_2, using the observed Fisher information matrix. This method is useful for its computational simplicity and provides good coverage probabilities (close to the nominal value) for large sample sizes.

At first, explicit expressions for elements of the Fisher information matrix $I(\boldsymbol{\theta})$ need to be obtained. Then, the $100(1-\gamma)\%$ asymptotic CIs for α_1, λ_1, α_2, and λ_2 are, respectively

$$(\widehat{\alpha}_1 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}),\ (\widehat{\lambda}_1 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}),\ (\widehat{\alpha}_2 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}),\ (\widehat{\lambda}_2 \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{44}}),$$

where z_q is the $q-$th upper percentile of a standard normal distribution, V_{ij} is the $(i, j)-$ th element of the inverse of the Fisher information matrix $I(\boldsymbol{\theta})$.

3.2 Bootstrap Confidence Intervals

In this subsection, we obtain the parametric bootstrap CIs for $\alpha_1, \lambda_1, \alpha_2$ and λ_2. The following algorithm can be employed to construct the parametric bootstrap CIs.
Algorithm 1:
Step 1: For given n, r and τ, the MLEs of $(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$, say $(\widehat{\alpha}_1, \widehat{\lambda}_1, \widehat{\alpha}_2, \widehat{\lambda}_2)$, are computed based on the original sample $t = (t_1, \ldots, t_{n_1}, t_{n_1+1}, \ldots, t_r)$.
Step 2: To generate the bootstrap sample from the proposed model, first generate n observations from $U(0,1)$ distribution and sort them. Suppose, the ordered observations are $u_{1:n} < u_{2:n} < \ldots < u_{n:n}$.

Step 3: Find $n_1 = \max\{k : u_{k:n} \leq G(\tau) \leq u_{k+1:n}\}$, where $G(\tau) = \{1 - e^{-\widehat{\lambda}_1 Q(\tau)}\}^{\widehat{\alpha}_1}$.
Step 4: For $1 \leq i \leq n_1$, find t_i^* by solving $u_{i:n} = [G_0(t_i^*; \widehat{\lambda}_1)]^{\widehat{\alpha}_1}$ and for $n_1 + 1 \leq i \leq r$, find t_i^* by solving $u_{i:n} = 1 - \frac{\left\{1-[G_0(\tau;\widehat{\lambda}_1)]^{\widehat{\alpha}_1}\right\}}{\left\{1-[G_0(\tau;\widehat{\lambda}_2)]^{\widehat{\alpha}_2}\right\}}\left\{1 - [G_0(t_i^*; \widehat{\lambda}_2)]^{\widehat{\alpha}_2}\right\}$.
Step 5: Based on $n,\ r,\ \tau$ and the bootstrap sample $\{t_1^*, t_2^*, \ldots, t_{n_1}^*, t_{n_1+1}^*, \ldots, t_r^*\}$, the MLEs of $\alpha_1, \lambda_1, \alpha_2,$ and λ_2 are computed, say $(\widehat{\alpha}_1^{(1)}, \widehat{\lambda}_1^{(1)}, \widehat{\alpha}_2^{(1)}, \widehat{\lambda}_2^{(1)})$.
Step 6: Suppose $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4) = (\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ and $\widehat{\boldsymbol{\delta}}^{(i)} = (\widehat{\delta}_1^{(i)}, \widehat{\delta}_2^{(i)}, \widehat{\delta}_3^{(i)}, \widehat{\delta}_4^{(i)})$ $= (\widehat{\alpha}_1^{(i)}, \widehat{\lambda}_1^{(i)}, \widehat{\alpha}_2^{(i)}, \widehat{\lambda}_2^{(i)})$. Repeat steps 2 – 5, B times to obtain B sets of MLE of $\boldsymbol{\delta}$, say $\widehat{\boldsymbol{\delta}}^{(i)}$; $i = 1, 2, \ldots, B$.
Step 7: Arrange $\widehat{\delta}_j^{(1)}, \widehat{\delta}_j^{(2)}, \ldots, \widehat{\delta}_j^{(B)}$ in ascending order and denote the ordered MLEs as $\widehat{\delta}_j^{[1]} < \widehat{\delta}_j^{[2]} < \ldots < \widehat{\delta}_j^{[B]}$; $j = 1, 2, 3, 4$.
A two sided $100(1-\alpha)\%$ bootstrap confidence interval of δ_j is then given by $(\widehat{\delta}_j^{[\frac{\alpha}{2}B]}, \widehat{\delta}_j^{[(1-\frac{\alpha}{2})B]})$ where $[x]$ denotes the largest integer less than or equal to x.

The performance of all these confidence intervals are evaluated through an extensive simulation study in Section 4.

4 Simulation Studies and Data Analysis

4.1 Simulation Studies

In this section, we perform extensive Monte Carlo simulation study to evaluate the performance of the proposed parameter estimation method in a simple step-stress set up. In the simulation study, we consider an important member of the Lehmann family - the GE distribution as the failure time distribution with difference in both the shape and scale parameters across the two stress levels. For analysis purpose, we have taken different sample sizes (n) ranging from moderate to large (n = 40, 60, 80, 100), two different values of r ($r = 0.8n$ and $r = n$). Associated with each of these choices of (n, r), we have considered different values of stress-changing time points (τ = 0.45, 0.50, 0.55). The parameter values are set as $\alpha_1 = 3.0,\ \lambda_1 = 1.0,\ \alpha_2 = 2.0$, and $\lambda_2 = 1.5$. The average biases and the associated mean squared errors (MSEs) of the MLEs are provided in Table 1.

For interval estimation, we resort to 95% asymptotic CIs and 95% parametric bootstrap CIs. The average lengths(ALs) and the associated coverage probabilities (CPs) of the two different CIs are reported in Table 2 and Table 3 respectively. For computing the asymptotic CIs, the elements of the Fisher information matrix $I^{\mathrm{GE}}(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ are provided in Section 6.3.1. All the results are based on 10000 replications and for bootstrap CIs we have taken B=15000.

Some of the observations are quite obvious from the above results. It can be observed from the biases in Table 1 that for fixed n and r, as τ increases, the estimates of α_1 and λ_1 approach towards the true values of α_1 and λ_1 and the corresponding MSEs decrease. On the other hand, as expected, one observes exactly the opposite behavior for the estimates of α_2 and λ_2. Again for fixed n, as r increases we can see the improved behavior of the estimates and MSEs of α_2 and λ_2. For fixed (r, τ), as the sample size grows, the MLEs approach the true values and the corresponding MSEs decrease. It indicates the asymptotic consistency property of the MLEs. The performance of both types of CIs are quite satisfactory in terms of CP. For fixed n and r, as τ increases, the average lengths of α_1 and λ_1 decrease, while under same conditions, the average lengths of α_2 and λ_2 increase. As the sample size increases, the average lengths of all the model parameters decrease, which is also quite expected.

Table 1: Biases and MSEs of MLEs of model parameters.

			α_1		λ_1		α_2		λ_2	
n	r	τ	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
40	32	0.45	0.3364	1.8161	0.6881	0.8821	0.5698	2.5607	0.0904	0.1631
		0.50	0.2974	1.7611	0.5199	0.6048	0.6046	2.6813	0.0838	0.1714
		0.55	0.2884	1.7503	0.3981	0.4416	0.6178	2.7907	0.0767	0.1825
	40	0.45	0.3492	1.8072	0.6950	0.8885	0.4577	2.1413	0.0698	0.1033
		0.50	0.3063	1.7540	0.5242	0.6062	0.4952	2.2976	0.0681	0.1093
		0.55	0.3011	1.7650	0.4034	0.4483	0.5150	2.4648	0.0604	0.1095
60	48	0.45	0.3004	1.7216	0.4288	0.5312	0.4852	2.1975	0.0771	0.1302
		0.50	0.2643	1.6982	0.2949	0.3776	0.5103	2.3723	0.0722	0.1207
		0.55	0.2699	1.6887	0.2118	0.2942	0.5274	2.4894	0.0635	0.1217
	60	0.45	0.2734	1.6581	0.4184	0.5130	0.3825	1.7558	0.0602	0.0836
		0.50	0.2516	1.6376	0.2927	0.3695	0.4240	1.9755	0.0583	0.0847
		0.55	0.2210	1.6105	0.1925	0.2807	0.4320	2.1076	0.0516	0.0891
80	64	0.45	0.2606	1.6678	0.2821	0.3935	0.4099	1.8933	0.0619	0.1023
		0.50	0.2410	1.5942	0.1741	0.2883	0.4519	2.1073	0.0629	0.1030
		0.55	0.2325	1.5960	0.1143	0.2388	0.4848	2.2681	0.0607	0.0967
	80	0.45	0.2585	1.6335	0.2812	0.3882	0.2711	1.3723	0.0388	0.0668
		0.50	0.2302	1.6258	0.1780	0.2956	0.3177	1.5895	0.0397	0.0668
		0.55	0.2186	1.5247	0.1127	0.2325	0.3844	1.8279	0.0470	0.0670
100	80	0.45	0.2424	1.6052	0.1968	0.3290	0.3473	1.6198	0.0549	0.0935
		0.50	0.2196	1.5334	0.1175	0.2570	0.3990	1.8790	0.0544	0.0898
		0.55	0.2206	1.4764	0.0753	0.2129	0.4428	2.0723	0.0561	0.0848
	100	0.45	0.2315	1.5871	0.1937	0.3263	0.2379	1.1450	0.0348	0.0566
		0.50	0.2518	1.5701	0.1304	0.2626	0.2665	1.3618	0.0375	0.0569
		0.55	0.2381	1.5151	0.0793	0.2170	0.3118	1.5766	0.0375	0.0570

Table 2: CP and AL of 95 % asymptotic CIs of the model parameters.

			α_1		λ_1		α_2		λ_2	
n	r	τ	CP	AL	CP	AL	CP	AL	CP	AL
40	32	0.45	0.9549	8.3843	0.9980	4.0943	0.9997	8.8531	0.9955	2.1452
		0.50	0.9503	8.2463	0.9969	3.6549	0.9997	9.9223	0.9962	2.2166
		0.55	0.9496	8.1102	0.9942	3.2972	0.9998	9.9766	0.9974	2.2837
	40	0.45	0.9558	8.4811	0.9980	4.1150	0.9949	6.9413	0.9862	1.5422
		0.50	0.9570	8.2816	0.9961	3.6664	0.9991	7.8038	0.9886	1.5818
		0.55	0.9772	7.9878	0.9927	3.2748	0.9998	8.6274	0.9906	1.6057
60	48	0.45	0.9494	7.8394	0.9906	3.5575	0.9964	6.9265	0.9934	1.6912
		0.50	0.9465	7.6278	0.9839	3.1531	0.9988	7.8597	0.9963	1.7511
		0.55	0.9500	7.3855	0.9804	2.8270	0.9990	8.7875	0.9963	1.8054
	60	0.45	0.9491	7.7510	0.9912	3.5374	0.9822	5.5080	0.9834	1.2324
		0.50	0.9477	7.6035	0.9838	3.1512	0.9921	6.1893	0.9873	1.2606
		0.55	0.9393	7.3490	0.9772	2.8164	0.9973	6.8647	0.9874	1.2907
80	64	0.45	0.9468	7.3980	0.9798	3.2116	0.9899	5.8899	0.9899	1.4351
		0.50	0.9448	7.1114	0.9709	2.8329	0.9968	6.6501	0.9956	1.4868
		0.55	0.9467	6.8140	0.9665	2.5261	0.9995	7.5039	0.9966	1.5437
	80	0.45	0.9469	7.3705	0.9801	3.2015	0.9687	4.6845	0.9686	1.0560
		0.50	0.9472	7.1696	0.9731	2.8496	0.9817	5.2523	0.9819	1.0811
		0.55	0.9506	6.8342	0.9666	2.5317	0.9894	5.8523	0.9868	1.1048
100	80	0.45	0.9431	7.0682	0.9677	2.9701	0.9808	5.2058	0.9868	1.2756
		0.50	0.9455	6.6814	0.9609	2.6011	0.9918	5.8950	0.9941	1.3208
		0.55	0.9467	6.3581	0.9547	2.3077	0.9967	6.6584	0.9948	1.3736
	100	0.45	0.9484	7.0957	0.9702	2.9746	0.9664	4.1607	0.9646	0.9394
		0.50	0.9455	6.6820	0.9632	2.6017	0.9777	4.6742	0.9755	0.9626
		0.55	0.9458	6.3185	0.9568	2.3027	0.9812	5.1905	0.9795	0.9860

Table 3: CP and AL of 95 % bootstrap CIs of the model parameters.

			α_1		λ_1		α_2		λ_2	
n	r	τ	CP	AL	CP	AL	CP	AL	CP	AL
40	32	0.45	0.9993	5.6187	0.8156	2.7256	0.9976	6.7509	0.9843	1.6206
		0.50	0.9994	5.6028	0.8628	2.5029	0.9928	6.9105	0.9827	1.6209
		0.55	0.9996	5.5967	0.8700	2.2507	0.9996	7.0273	0.9906	1.6227
	40	0.45	0.9990	5.6346	0.8176	2.7304	0.9776	6.1875	0.9593	1.3525
		0.50	0.9996	5.6229	0.8330	2.4722	0.9923	6.4338	0.9700	1.3533
		0.55	0.9996	5.6046	0.8800	2.2555	0.9963	6.6588	0.9763	1.3675
60	48	0.45	0.9986	5.5270	0.8850	2.4634	0.9733	6.1632	0.9686	1.4139
		0.50	0.9993	5.5029	0.9400	2.2253	0.9876	6.4390	0.9820	1.4141
		0.55	0.9973	5.4828	0.9773	2.0406	0.9976	6.6909	0.9870	1.4210
	60	0.45	0.9993	5.5175	0.8810	2.4599	0.9536	5.4065	0.9490	1.1519
		0.50	0.9990	5.5080	0.9356	2.2284	0.9686	5.7482	0.9566	1.1545
		0.55	0.9996	5.4862	0.9763	2.0467	0.9786	6.0299	0.9636	1.1548
80	64	0.45	0.9996	5.4548	0.9486	2.2963	0.9696	5.6035	0.9636	1.2732
		0.50	0.9900	5.4083	0.9830	2.0855	0.9783	5.9800	0.9670	1.2742
		0.55	0.9976	5.3813	0.9900	1.9152	0.9866	6.3047	0.9763	1.2767
	80	0.45	0.9883	5.4617	0.9503	2.3050	0.9543	4.7643	0.9436	1.0108
		0.50	0.9890	5.4203	0.9810	2.0881	0.9560	5.2115	0.9510	1.0143
		0.55	0.9866	5.3690	0.9856	1.9179	0.9646	5.5305	0.9493	1.0800
100	80	0.45	0.9890	5.3960	0.9813	2.1937	0.9583	5.1345	0.9560	1.1724
		0.50	0.9883	5.3445	0.9893	1.9904	0.9650	5.5649	0.9586	1.1722
		0.55	0.9843	5.2703	0.9890	1.8300	0.9750	5.9379	0.9680	1.1726
	100	0.45	0.9890	5.3952	0.9826	2.1945	0.9560	4.2940	0.9530	0.9216
		0.50	0.9850	5.3325	0.9800	1.9885	0.9620	4.6512	0.9526	0.9217
		0.55	0.9833	5.2506	0.9866	1.8265	0.9613	5.0611	0.9453	0.9260

4.2 Data Analysis

4.2.1 Fish data set

In this section, we consider a real life step-stress fish data set obtained from [8]. A sample of 15 fish swum at initial flow rate 15 cm/sec. The time at which a fish could not maintain its position is recorded as its failure time. To ensure an early failure, the stress level was increased (flow rate by 5 cm/sec) at time 110, 130, 150 and 170 minutes, respectively. The observed failure time data is presented in Table 4 below.

Table 4: Fish Data Set

Stress level	Failure times
s_1	91.00, 93.00, 94.00, 98.20
s_2	115.81, 116.00, 116.50, 117.25, 126.75, 127.50
s_3	No failures
s_4	154.33, 159.50, 164.00
s_5	184.14, 188.33

There are five stress levels and number of failures at each stress level is 4, 6, 0, 3 and 2, respectively. For our analysis purpose, we consider it as a simple step-stress data merging the first two stress levels to a single stress level and the remaining stress levels to another single stress level. For computational purpose, we have subtracted 80 from each data points, divide them by 150 and then analyze the data with $n = r = 15$, $\tau = 0.33$. Here, we consider the complete sample for analysis purpose and stop at the $15-$ th failure to facilitate the Type-II censoring.

As a choice of the failure time distribution, we consider three members (GE, GR and EP) of the Lehmann family of distributions. All these three distributions are fitted to the Fish data set assuming difference in both the shape and scale parameters at each of the two stress levels. MLEs of the model parameters are obtained by solving one dimensional optimization problems. The MLEs of the model parameters, the Kolmogorov-Smirnov (K-S) distances between the fitted and the empirical distribution functions and the p-values for the K-S test are presented in Table 5. 90%, 95%, and 99%, bootstrap CIs are presented in Table 6.

Table 5: MLEs, K-S statistics and the corresponding p-values of the Fish data set

Model	MLEs		K-S Statistics	p-value
GE	$\widehat{\alpha}_1$	2.8304	0.1833	0.6437
	$\widehat{\lambda}_1$	5.8775		
	$\widehat{\alpha}_2$	2264.2688		
	$\widehat{\lambda}_2$	13.8730		
GR	$\widehat{\alpha}_1$	0.9459	0.1639	0.7776
	$\widehat{\lambda}_1$	9.2678		
	$\widehat{\alpha}_2$	35.1339		
	$\widehat{\lambda}_2$	11.3299		
EP	$\widehat{\alpha}_1$	3.3031	0.1901	0.6022
	$\widehat{\lambda}_1$	7.2645		
	$\widehat{\alpha}_2$	17876.6484		
	$\widehat{\lambda}_2$	22.1745		

Table 6: Bootstrap CIs of parameters based on the Fish data set

		α_1		λ_1		α_2		λ_2	
Model	Level	LL	UL	LL	UL	LL	UL	LL	UL
GE	90%	1.4493	8.5005	2.9325	9.7810	976.3842	4129.729	11.7081	15.4846
	95%	1.2838	10.7785	2.4991	10.6449	938.3585	4315.795	11.4094	15.8299
	99%	1.0355	16.1782	1.6474	12.1417	907.2052	4460.666	10.7977	16.5033
GR	90%	0.5371	1.6997	3.1702	13.1903	12.3428	88.2046	7.6982	15.2679
	95%	0.4878	1.9145	2.3662	13.5989	11.1726	93.8493	7.2067	16.0640
	99%	0.4097	2.4103	1.5100	13.9127	10.2514	98.6554	6.4699	17.8533
EP	90%	1.6466	10.1209	3.7740	11.8878	12572.91	28660.54	20.2098	24.1174
	95%	1.4745	12.6201	3.2021	12.7611	12284.68	29350.65	19.7901	24.4845
	99%	1.1784	17.2071	2.0931	14.5229	12052.65	29883.83	18.9644	25.2057

Examining the K-S statistics and the p-values, it is observed that although all the three distributions fit the data well, the GR distribution has a better fit than the other two distributions. The plot of the empirical distribution function along with the fitted distribution function based on the GR fit is provided in Figure 1.

In Figure 2 and Figure 3, we have provided the plots for the profile likelihood functions of λ_1 and λ_2 respectively. It is observed that both the functions are unimodal functions of the respective parameters.

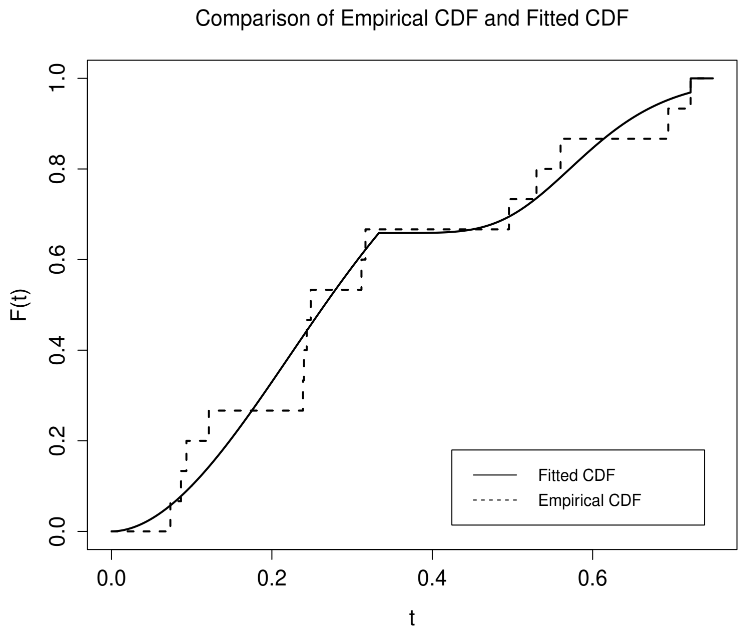


Fig. 1: Plot of empirical and fitted CDFs of Fish data set assuming GR distribution.

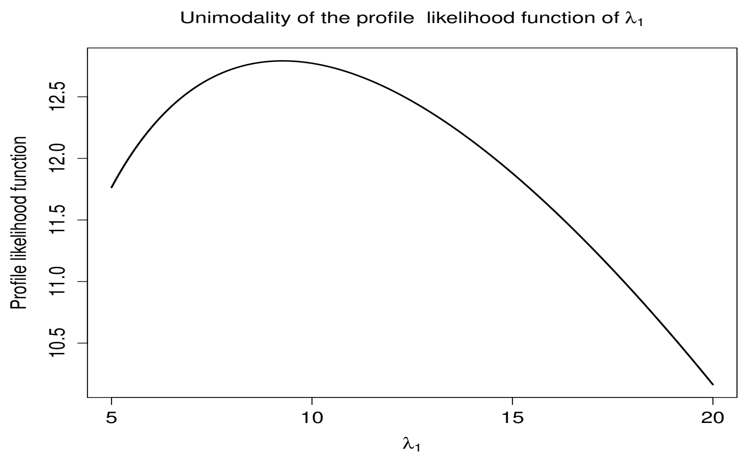


Fig. 2: Unimodality of the profile log-likelihood function of λ_1 for Fish data set assuming GR distribution.

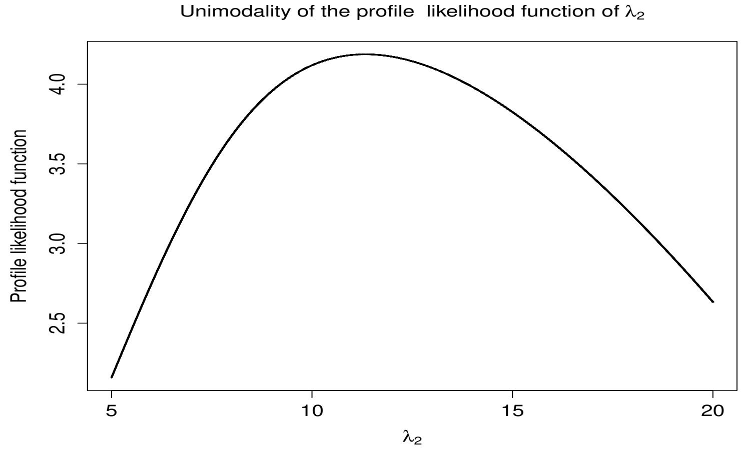


Fig. 3: Unimodality of the profile log-likelihood function of λ_2 for Fish data set assuming GR distribution.

One may be interested to know whether we can analyze the same data assuming equality of the shape or equality of the scale parameters. For this purpose, we perform the corresponding likelihood ratio based tests. First, we want to test the following hypothesis.

A. $H_o : \alpha_1 = \alpha_2$ **vs** $H_1 : \alpha_1 \neq \alpha_2$.

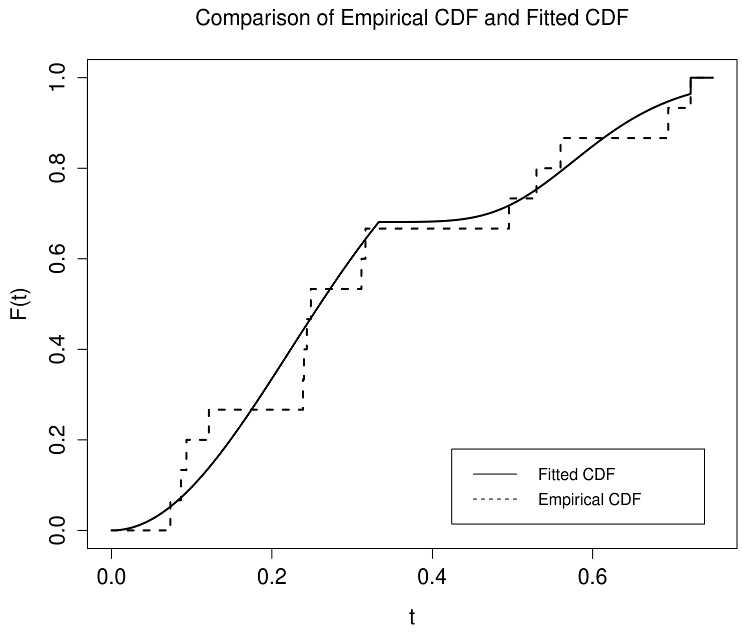
Under H_o, the MLEs are $\widetilde{\alpha} = 1.0367$, $\widetilde{\lambda}_1 = 10.0857$, $\widetilde{\lambda}_2 = 3.9102$. The likelihood ratio test (LRT) statistic and $-2lnLRT$ are obtained as 0.0896 and 4.8245, respectively and the corresponding p value is 0.0280. Thus, we reject the null hypothesis in this case at 5% level of significance.

Now we consider the following testing problem.

B. $H_o : \lambda_1 = \lambda_2$ **vs** $H_1 : \lambda_1 \neq \lambda_2$.

Under H_o, the MLEs are : $\widetilde{\alpha}_1 = 1.0125$, $\widetilde{\alpha}_2 = 26.6399$, $\widetilde{\lambda} = 10.3797$. The likelihood ratio test (LRT) statistic and $-2lnLRT$ are obtained as 0.9490 and 0.1045, respectively. The corresponding p value is 0.7464. Thus, in this case, we accept the null hypothesis.

The two testing procedures lead us to assume difference in shape parameters and equality in scale parameters and carry on the appropriate analysis assuming the GR distribution. Assuming different shape parameters and common scale parameters, the MLEs of the model parameters are obtained as $\widehat{\alpha}_1 = 1.0125$, $\widehat{\alpha}_2 = 26.6399$, $\widehat{\lambda} = 10.3797$. As a measure of the goodness of fit we have computed the K-S statistics and the associated p-value. The K-S distance and the associated p-value are obtained as 0.1753 and 0.7023, respectively. It indicates a good fit of the given data. The plot of the empirical v/s the fitted CDFs is shown in Figure 4. 90%, 95%, and 99% asymptotic and bootstrap CIs of the model parameters are given in Table 7. The elements of the associated Fisher Information matrix $I_{\text{sc}}^{\text{GR}}(\alpha_1, \lambda, \alpha_2)$ are provided in Section 6.3.2.



(a)

Fig. 4: Plot of empirical and fitted CDFs of Fish data set assuming GR distribution with different shape parameters and common scale parameter.

Table 7: Asymptotic and Bootstrap CIs of Fish data Set.

		α_1		λ		α_2	
CI	Level	LL	UL	LL	UL	LL	UL
	90%	0.4807	1.5442	5.1565	15.6028	0	72.0466
Asymptotic	95%	0.3788	4.1556	16.6038	1.6461	0	80.7481
	99%	0.1796	1.8453	2.1995	18.5500	0	97.7535
	90%	0.6674	1.8023	6.8463	14.1895	8.3102	62.6906
Bootstrap	95%	0.6159	2.0243	6.2858	14.9762	6.7643	66.2360
	99%	0.5307	2.5515	5.3487	16.6555	4.3802	69.2758

5 Conclusion

In this chapter, we have considered the likelihood inference of a simple-step stress model under Type-II censoring, when the lifetime distribution at each of the stress levels belongs to the Lehmann family of distributions. To relate the distributions at the two stress levels, a failure rate based SSALT model is proposed. Further, all the underlying two parameters (especially the shape parameter) of the lifetime distribution are allowed to vary across the stress levels, which makes this model a flexible one. In case of Type-II censoring, the MLEs of the model parameters can be obtained by solving a one-dimensional and a two-dimensional optimization problems. Again, in the complete sample case, inference becomes much simplified in terms of dimension reduction, as expected. The MLEs can then be obtained by solving two one-dimensional optimization problems. Due to absence of closed form solutions of the likelihood equations, the asymptotic and the bootstrap CIs have been obtained here. Performance of the proposed model for analyzing time to failure data has been evaluated through an extensive simulation study considering GE distribution, an important distribution of the Lehmann family. From the simulation study, it has been observed that the estimators are consistent and the CPs of the different CIs are close to the nominal values. A real data set is analyzed for illustrative purpose. It has been observed that different members of the general family the model fit the data set quite well.

Though we have only considered the Type-II censoring here, the data set from other censoring schemes can also be analyzed in a similar way. Again, not restricting to a simple step-stress model, the analysis can be easily be extended to a multiple step-stress scenario. In this work, we have only considered the classical inference of the proposed model. However, it would be interesting to work on the Bayesian analysis of the proposed model. The order restricted inference can also be considered for this model. More work is needed along these directions.

Acknowledgements The authors would like to thank the reviewers for their constructive comments which had helped to improve the manuscript significantly.

6 Appendix

6.1 Lehmann Family of Distributions

6.1.1 Normal Equations for the Type-II Censoring case

The normal equations associated with the log-likelihood function (13) are given by

$$\frac{\partial l^{II}}{\partial \alpha_1} = \frac{n_1}{\alpha_1} + \sum_{k=1}^{n_1} \ln G_0(t_{k:n};\lambda_1) - \frac{(n-n_1)\{G_0(\tau;\lambda_1)\}^{\alpha_1}\ln G_0(\tau;\lambda_1)}{1-\{G_0(\tau;\lambda_1)\}^{\alpha_1}} = 0, \quad (26)$$

$$\begin{aligned}\frac{\partial l^{II}}{\partial \lambda_1} =& (\alpha_1 - 1)\sum_{k=1}^{n_1} \frac{m_0(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)} - \frac{(n-n_1)\alpha_1\{G_0(\tau;\lambda_1)\}^{\alpha_1-1} m_0(\tau;\lambda_1)}{1-\{G_0(\tau;\lambda_1)\}^{\alpha_1}} \\ &+ \sum_{k=1}^{n_1} \frac{n_0(t_{k:n};\lambda_1)}{g_0(t_{k:n};\lambda_1)} = 0, \end{aligned} \quad (27)$$

$$\begin{aligned}\frac{\partial l^{II}}{\partial \alpha_2} =& \frac{n_2}{\alpha_2} - \sum_{k=n_1+1}^{r} \ln G_0(t_{k:n};\lambda_1) + \frac{(n-n_1)\{G_0(\tau;\lambda_2)\}^{\alpha_2}\ln G_0(\tau;\lambda_2)}{1-\{G_0(\tau;\lambda_2)\}^{\alpha_2}} \\ &- \frac{(n-r)\{G_0(t_{r:n};\lambda_2)\}^{\alpha_2}\ln G_0(t_{r:n};\lambda_2)}{1-\{G_0(t_{r:n};\lambda_2)\}^{\alpha_2}} = 0, \end{aligned} \quad (28)$$

$$\begin{aligned}\frac{\partial l^{II}}{\partial \lambda_2} =& (\alpha_2 - 1)\sum_{k=n_1+1}^{r} \frac{m_0(t_{k:n};\lambda_2)}{G_0(t_{k:n};\lambda_2)} + \frac{(n-n_1)\alpha_2\{G_0(\tau;\lambda_2)\}^{\alpha_2-1} m_0(\tau;\lambda_2)}{1-\{G_0(\tau;\lambda_2)\}^{\alpha_2}} \\ &+ \sum_{k=n_1+1}^{r} \frac{n_0(t_{k:n};\lambda_2)}{g_0(t_{k:n};\lambda_2)} - \frac{(n-r)\alpha_2\{G_0(t_{r:n};\lambda_2)\}^{\alpha_2-1} m_0(t_{r:n};\lambda_2)}{1-\{G_0(t_{r:n};\lambda_2)\}^{\alpha_2}} = 0. \end{aligned} \quad (29)$$

where

$$m_0(.;\lambda_1) = \frac{\partial}{\partial \lambda_1} G_0(.;\lambda_1), \quad n_0(.;\lambda_1) = \frac{\partial}{\partial \lambda_1} g_0(.;\lambda_1),$$
$$m_0(.;\lambda_2) = \frac{\partial}{\partial \lambda_2} G_0(.;\lambda_2), \quad n_0(.;\lambda_2) = \frac{\partial}{\partial \lambda_2} g_0(.;\lambda_2).$$

Now multiplying (26) by $\frac{\alpha_1 m_0(\tau;\lambda_1)}{G_0(\tau;\lambda_1)}$ and (27) by $\ln G_0(\tau;\lambda_1)$, respectively, we have

$$\begin{aligned}&\frac{n_1 m_0(\tau;\lambda_1)}{G_0(\tau;\lambda_1)} - \frac{(n-n_1)\alpha_1\{G_0(\tau;\lambda_1)\}^{\alpha_1-1} m_0(\tau;\lambda_1)\ln G_0(\tau;\lambda_1)}{1-\{G_0(\tau;\lambda_1)\}^{\alpha_1}} \\ &\qquad + \frac{\alpha_1 m_0(\tau;\lambda_1)}{G_0(\tau;\lambda_1)} \sum_{k=1}^{n_1} \ln G_0(t_{k:n};\lambda_1) = 0, \end{aligned} \quad (30)$$

$$-\frac{(n-n_1)\alpha_1\{G_0(\tau;\lambda_1)\}^{\alpha_1-1}m_0(\tau;\lambda_1)\ln G_0(\tau;\lambda_1)}{1-\{G_0(\tau;\lambda_1)\}^{\alpha_1}}+\ln G_0(\tau;\lambda_1)\sum_{k=1}^{n_1}\frac{n_0(t_{k:n};\lambda_1)}{g_0(t_{k:n};\lambda_1)}$$
$$+(\alpha_1-1)\ln G_0(\tau;\lambda_1)\sum_{k=1}^{n_1}\frac{m_0(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)}=0. \quad (31)$$

Subtracting (31) from (30), and after little simplification, finally we establish the following relation and $\widehat{\alpha}_1(\lambda_1)$ is

$$\frac{\ln[G_0(\tau;\lambda_1)]\sum_{k=1}^{n_1}\frac{n_o(t_{k:n};\lambda_1)}{g_0(t_{k:n};\lambda_1)}-\frac{n_1 m_o(\tau;\lambda_1)}{G_0(\tau;\lambda_1)}-\ln[G_0(\tau;\lambda_1)]\sum_{k=1}^{n_1}\frac{m_o(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)}}{\frac{m_o(\tau;\lambda_1)}{G_0(\tau;\lambda_1)}-\ln[G_0(\tau;\lambda_1)]\sum_{k=1}^{n_1}\frac{m_o(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)}} \quad (32)$$

6.1.2 Normal equations for the Complete Sample case

$$\frac{\partial l^c}{\partial\alpha_1}=\frac{n_1}{\alpha_1}+\sum_{k=1}^{n_1}\ln G_0(t_{k:n};\lambda_1)-\frac{(n-n_1)\{G_0(\tau;\lambda_1)\}^{\alpha_1}\ln G_0(\tau;\lambda_1)}{1-\{G_0(\tau;\lambda_1)\}^{\alpha_1}}=0,$$

$$\frac{\partial l^c}{\partial\lambda_1}=(\alpha_1-1)\sum_{k=1}^{n_1}\frac{m_o(t_{k:n};\lambda_1)}{G_0(t_{k:n};\lambda_1)}-\frac{(n-n_1)\alpha_1\{G_0(\tau;\lambda_1)\}^{\alpha_1-1}m_o(\tau;\lambda_1)}{1-\{G_0(\tau;\lambda_1)\}^{\alpha_1}}$$
$$+\sum_{k=1}^{n_1}\frac{n_o(t_{k:n};\lambda_1)}{g_0(t_{k:n};\lambda_1)}=0,$$

$$\frac{\partial l^c}{\partial\alpha_2}=\frac{n_2}{\alpha_2}-\sum_{k=n_1+1}^{r}\ln G_0(t_{k:n};\lambda_1)+\frac{(n-n_1)\{G_0(\tau;\lambda_2)\}^{\alpha_2}\ln G_0(\tau;\lambda_2)}{1-\{G_0(\tau;\lambda_2)\}^{\alpha_2}}=0,$$

$$\frac{\partial l^c}{\partial\lambda_2}=(\alpha_2-1)\sum_{k=n_1+1}^{r}\frac{m_o(t_{k:n};\lambda_2)}{G_0(t_{k:n};\lambda_2)}+\frac{(n-n_1)\alpha_2\{G_0(\tau;\lambda_2)\}^{\alpha_2-1}m_o(\tau;\lambda_2)}{1-\{G_0(\tau;\lambda_2)\}^{\alpha_2}}$$
$$+\sum_{k=n_1+1}^{r}\frac{n_o(t_{k:n};\lambda_2)}{g_0(t_{k:n};\lambda_2)}=0$$

6.2 Special Case : GE Distribution

6.2.1 Normal equations for the Type-II Censoring Case

$$\frac{\partial l_{GE}}{\partial \alpha_1} = \frac{n_1}{\alpha_1} + \sum_{k=1}^{n_1} \ln(1 - e^{-\lambda_1 t_{k:n}}) - A(\alpha_1, \lambda_1) = 0, \tag{33}$$

$$\frac{\partial l_{GE}}{\partial \lambda_1} = \frac{n_1}{\lambda_1} - \sum_{k=1}^{n_1} t_{k:n} + (\alpha_1 - 1) \sum_{k=1}^{n_1} \frac{t_{k:n} e^{-\lambda_1 t_{k:n}}}{1 - e^{-\lambda_1 t_{k:n}}} - B(\alpha_1, \lambda_1) = 0, \tag{34}$$

$$\frac{\partial l_{GE}}{\partial \alpha_2} = \frac{n_2}{\alpha_2} + \sum_{k=n_1+1}^{n_1+n_2} \ln(1 - e^{-\lambda_2 t_{k:n}}) + C_1(\alpha_2, \lambda_2) - C_2(\alpha_2, \lambda_2) = 0, \tag{35}$$

$$\begin{aligned} \frac{\partial l_{GE}}{\partial \lambda_2} = \frac{n_2}{\lambda_2} - \sum_{k=n_1+1}^{n_1+n_2} t_{k:n} + (\alpha_2 - 1) \sum_{k=n_1+1}^{n_1+n_2} \frac{t_{k:n} e^{-\lambda_2 t_{k:n}}}{1 - e^{-\lambda_2 t_{k:n}}} + D_1(\alpha_2, \lambda_2) \\ - D_2(\alpha_2, \lambda_2) = 0, \end{aligned} \tag{36}$$

where

$$\begin{aligned} A(\alpha_1, \lambda_1) &= \frac{(n - n_1)(1 - e^{-\lambda_1 \tau})^{\alpha_1}}{1 - (1 - e^{-\lambda_1 \tau})^{\alpha_1}} \ln(1 - e^{-\lambda_1 \tau}), \\ B(\alpha_1, \lambda_1) &= \frac{(n - n_1)\alpha_1 \tau e^{-\lambda_1 \tau}(1 - e^{-\lambda_1 \tau})^{\alpha_1 - 1}}{1 - (1 - e^{-\lambda_1 \tau})^{\alpha_1}}, \\ C_1(\alpha_2, \lambda_2) &= \frac{(n - n_1)(1 - e^{-\lambda_2 \tau})^{\alpha_2}}{1 - (1 - e^{-\lambda_2 \tau})^{\alpha_2}} \ln(1 - e^{-\lambda_2 \tau}), \\ C_2(\alpha_2, \lambda_2) &= \frac{(n - r)(1 - e^{-\lambda_2 t_{r:n}})^{\alpha_2}}{1 - (1 - e^{-\lambda_2 t_{r:n}})^{\alpha_2}} \ln(1 - e^{-\lambda_2 t_{r:n}}), \\ D_1(\alpha_2, \lambda_2) &= \frac{(n - n_1)\alpha_2 \tau (1 - e^{-\lambda_2 \tau})^{\alpha_2 - 1} e^{-\lambda_2 \tau}}{1 - (1 - e^{-\lambda_2 \tau})^{\alpha_2}}, \\ D_2(\alpha_2, \lambda_2) &= \frac{(n - r)\alpha_2 t_{r:n} (1 - e^{-\lambda_2 t_{r:n}})^{\alpha_2 - 1} e^{-\lambda_2 t_{r:n}}}{1 - (1 - e^{-\lambda_2 t_{r:n}})^{\alpha_2}}. \end{aligned}$$

6.3 Elements of the Fisher Information Matrix

6.3.1 GE Distribution

The Fisher information matrix $I^{\text{GE}}(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$ can be expressed using two block diagonal matrices viz, $I_1^{\text{GE}}(\alpha_1, \lambda_1)$ and $I_2^{\text{GE}}(\alpha_2, \lambda_2)$. Thus, we have

$$I^{\text{GE}}(\alpha_1,\lambda_1,\alpha_2,\lambda_2)=\begin{bmatrix} I_1^{\text{GE}}(\alpha_1,\lambda_1) & \mathbf{0}\\ \mathbf{0} & I_2^{\text{GE}}(\alpha_2,\lambda_2)\end{bmatrix}$$

The elements of $I_1^{\text{GE}}(\alpha_1,\lambda_1)=\begin{bmatrix} -\dfrac{\partial^2 l^{\text{GE}}}{\partial\alpha_1^2} & -\dfrac{\partial^2 l^{\text{GE}}}{\partial\alpha_1\partial\lambda_1}\\ -\dfrac{\partial^2 l^{\text{GE}}}{\partial\alpha_1\partial\lambda_1} & -\dfrac{\partial^2 l^{\text{GE}}}{\partial\lambda_1^2}\end{bmatrix}$ are

$$\frac{\partial^2 l^{\text{GE}}}{\partial\alpha_1^2}=-\Big(\frac{n_1}{\alpha_1^2}+\frac{(n-n_1)(1-e^{-\lambda_1\tau})^{\alpha_1}\{\ln[1-e^{-\lambda_1\tau}]\}^2}{\{1-(1-e^{-\lambda_1\tau})^{\alpha_1}\}^2}\Big),$$

$$\frac{\partial^2 l^{\text{GE}}}{\partial\alpha_1\partial\lambda_1}=\sum_{k=1}^{n_1}\frac{t_{k:n}e^{-\lambda_1 t_{k:n}}}{1-e^{-\lambda_1 t_{k:n}}}-(n-n_1)\tau e^{-\lambda_1\tau}\varrho_1(\alpha_1,\lambda_1),$$

$$\frac{\partial^2 l^{\text{GE}}}{\partial\lambda_1^2}=-\Big(\frac{n_1}{\lambda_1^2}-(\alpha_1-1)\psi_1(\lambda_1)+(n-n_1)\alpha_1\tau\kappa_1(\alpha_1,\lambda_1,\tau)\Big),$$

where

$$\varrho_1(\alpha_1,\lambda_1)=\frac{\{1-r(\lambda_1)^{\alpha_1}\}\left[r(\lambda_1)^{\alpha_1}\{\alpha_1\ln(r(\lambda_1)+1\}\right]+\alpha_1 r(\lambda_1)^{2\alpha_1}\ln(r(\lambda_1)}{r(\lambda_1)\{1-r(\lambda_1)^{\alpha_1}\}^2},$$

$$\psi_1(\lambda_1)=-\sum_{k=1}^{n_1}\frac{t_{k:n}^2 e^{-\lambda_1 t_{k:n}}}{(1-e^{-\lambda_1 t_{k:n}})^2},$$

$$\kappa_1(\alpha_1,\lambda_1)=\frac{\{1-r(\lambda_1)^{\alpha_1}\}\left[r(\lambda_1)^{\alpha_1-2}\tau e^{-\lambda_1\tau}\{\alpha_1 e^{-\lambda_1\tau}-1\}\right]+\alpha_1\tau e^{-2\lambda_1\tau}r(\lambda_1)^{2\alpha_1-2}}{\{1-r(\lambda_1)^{\alpha_1}\}^2},$$

$$r_1(\lambda_1)=(1-e^{-\lambda_1\tau}).$$

The elements of $I_2^{\text{GE}}(\alpha_2,\lambda_2)=\begin{bmatrix} -\dfrac{\partial^2 l^{\text{GE}}}{\partial\alpha_2^2} & -\dfrac{\partial^2 l^{\text{GE}}}{\partial\alpha_2\partial\lambda_2}\\ -\dfrac{\partial^2 l^{\text{GE}}}{\partial\alpha_1\partial\lambda_1} & -\dfrac{\partial^2 l^{\text{GE}}}{\partial\lambda_2^2}\end{bmatrix}$ are

$$\frac{\partial^2 l^{\text{GE}}}{\partial\alpha_2^2}=-\Big(\frac{n_2}{\alpha_2^2}-(n-n_1)\beta_1(\alpha_2,\lambda_2)+(n-r)\eta_1(\alpha_2,\lambda_2)\Big),$$

$$\frac{\partial^2 l^{\text{GE}}}{\partial\alpha_2\partial\lambda_2}=\sum_{k=n_1+1}^{n_1+n_2}\frac{t_{k:n}e^{-\lambda_2 t_{k:n}}}{1-e^{-\lambda_2 t_{k:n}}}+(n-n_1)\xi_1(\alpha_2,\lambda_2)-(n-r)\Upsilon_1(\alpha_2,\lambda_2),$$

$$\frac{\partial^2 l^{\mathrm{GE}}}{\partial \lambda_2^2} = -\Big(\frac{n_2}{\lambda_2^2} - (n-n_1)\sigma_1(\alpha_2,\lambda_2) - (\alpha_2-1)\delta_1(\lambda_2) + (n-r)\zeta_1(\alpha_2,\lambda_2)\Big),$$

where

$$\beta_1(\alpha_2,\lambda_2) = \frac{(1-e^{-\lambda_2\tau})^{\alpha_2}\{\ln[1-e^{-\lambda_2\tau}]\}^2}{\{1-(1-e^{-\lambda_2\tau})^{\alpha_2}\}^2},$$

$$\eta_1(\alpha_2,\lambda_2) = \frac{(1-e^{-\lambda_2 t_{r:n}})^{\alpha_2}\{\ln[1-e^{-\lambda_2 t_{r:n}}]\}^2}{\{1-(1-e^{-\lambda_2 t_{r:n}})^{\alpha_2}\}^2},$$

$$\xi_1(\alpha_2,\lambda_2) = \tau e^{-\lambda_2\tau}\frac{\{1-q_1(\lambda_2)^{\alpha_2}\}\big[q_1(\lambda_2)^{\alpha_2}\{\alpha_2\ln(q_1(\lambda_2)+1\}\big]}{q_1(\lambda_2)\{1-q_1(\lambda_2)^{\alpha_2}\}^2},$$

$$+\,\tau e^{-\lambda_2\tau}\frac{\alpha_2 q_1(\lambda_2)^{2\alpha_2}\ln(q_1(\lambda_2)}{q_1(\lambda_2)\{1-q_1(\lambda_2)^{\alpha_2}\}^2},$$

$$\Upsilon_1(\alpha_2,\lambda_2) = t_{r:n}e^{-\lambda_2 t_{r:n}}\frac{\{1-s_1(\lambda_2)^{\alpha_2}\}\big[s_1(\lambda_2)^{\alpha_2}\{\alpha_2\ln(s_1(\lambda_2)+1\}\big]}{s_1(\lambda_2)\{1-s_1(\lambda_2)^{\alpha_2}\}^2},$$

$$+\,t_{r:n}e^{-\lambda_2 t_{r:n}}\frac{\alpha_2 s_1(\lambda_2)^{2\alpha_2}\ln(s_1(\lambda_2)}{s_1(\lambda_2)\{1-s_1(\lambda_2)^{\alpha_2}\}^2},$$

$$\sigma_1(\alpha_2,\lambda_2) = \alpha_2\tau\frac{\{1-q_1(\lambda_2)^{\alpha_2}\}\big[q_1(\lambda_2)^{\alpha_2-2}\tau e^{-\lambda_2\tau}\{\alpha_2 e^{-\lambda_2\tau}-1\}\big]}{\{1-q_1(\lambda_2)^{\alpha_2}\}^2},$$

$$+\,\alpha_2\tau\frac{\alpha_2\tau e^{-2\lambda_2\tau}q_1(\lambda_2)^{2\alpha_2-2}}{\{1-q_1(\lambda_2)^{\alpha_2}\}^2},$$

$$\delta(\lambda_2) = -\sum_{k=n_1+1}^{n_1+n_2}\frac{t_{k:n}^2 e^{-\lambda_2 t_{k:n}}}{(1-e^{-\lambda_2 t_{k:n}})^2}\Big],$$

$$q_1(\lambda_2) = (1-e^{-\lambda_2\tau}),\ s_1(\lambda_2) = (1-e^{-\lambda_2 t_{r:n}}).$$

$$\zeta_1(\alpha_2, \lambda_2) = \alpha_2 t_{r:n} \frac{\{1 - s_1(\lambda_2)^{\alpha_2}\}\left[s_1(\lambda_2)^{\alpha_2 - 2} t_{r:n} e^{-\lambda_2 t_{r:n}} \{\alpha_2 e^{-\lambda_2 t_{r:n}} - 1\}\right]}{\{1 - s_1(\lambda_2)^{\alpha_2}\}^2},$$

$$+ \alpha_2 t_{r:n} \frac{\alpha_2 t_{r:n} e^{-2\lambda_2 t_{r:n}} s_1(\lambda_2)^{2\alpha_2 - 2}}{\{1 - s_1(\lambda_2)^{\alpha_2}\}^2},$$

6.3.2 GR Distribution with different shape and common scale parameter

Let the Fisher information matrix associated with the parameters $\alpha_1,\ \lambda,\ \alpha_2$ respectively be

$$I_{\text{sc}}^{\text{GR}}(\alpha_1, \lambda, \alpha_2) = \begin{bmatrix} -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1^2} & -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1 \partial \lambda} & -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1 \partial \alpha_2} \\ -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1 \partial \lambda} & -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \lambda^2} & -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_2 \partial \lambda} \\ -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1 \partial \alpha_2} & -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_2 \partial \lambda} & -\dfrac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_2^2} \end{bmatrix}$$

The corresponding elements are

$$\frac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1^2} = -\Big(\frac{n_1}{\alpha_1^2} + \frac{(n - n_1)(1 - e^{-\lambda\tau^2})^{\alpha_1}\{\ln[1 - e^{-\lambda\tau^2}]\}^2}{\{1 - (1 - e^{-\lambda\tau^2})^{\alpha_1}\}^2}\Big),$$

$$\frac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1 \partial \lambda} = \sum_{k=1}^{n_1} \frac{t_{k:n}^2 e^{-\lambda t_{k:n}^2}}{1 - e^{-\lambda t_{k:n}^2}} - (n - n_1)\varrho_3(\alpha_1, \lambda), \quad \frac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_1 \partial \alpha_2} = 0,$$

$$\frac{\partial^2 l^{\text{GR}}}{\partial \lambda^2} = -\Big(\frac{n}{\lambda^2} - (\alpha_1 - 1)\psi_3(\lambda) + (n - n_1)\kappa_3(\alpha_1, \lambda) - (n - n_1)\sigma_3(\alpha_2, \lambda) - (\alpha_2 - 1)\delta_3(\lambda)\Big),$$

$$\frac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_2 \partial \lambda} = \sum_{k=n_1+1}^{n_1+n_2} \frac{t_{k:n}^2 e^{-\lambda t_{k:n}^2}}{1 - e^{-\lambda t_{k:n}^2}} + (n - n_1)\xi_3(\alpha_2, \lambda),$$

$$\frac{\partial^2 l_{\text{sc}}^{\text{GR}}}{\partial \alpha_2^2} = -\Big(\frac{n_2}{\alpha_2^2} + \frac{(n-n_1)(1-e^{-\lambda\tau^2})^{\alpha_2}\{\ln[1-e^{-\lambda\tau^2}]\}^2}{\{1-(1-e^{-\lambda\tau^2})^{\alpha_2}\}^2}\Big),$$

where

$$\begin{aligned}\varrho_3(\alpha_1,\lambda) &= \tau^2 e^{-\lambda\tau^2}\frac{\{1-r_3(\lambda)^{\alpha_1}\}\big[r_3(\lambda)^{\alpha_1}\{\alpha_1\ln(r_3(\lambda)+1\}\big]}{r(\lambda)\{1-r_3(\lambda)^{\alpha_1}\}^2},\\ &+ \tau^2 e^{-\lambda\tau^2}\frac{\alpha_1 r_3(\lambda)^{2\alpha_1}\ln(r_3(\lambda)}{r(\lambda)\{1-r_3(\lambda)^{\alpha_1}\}^2},\end{aligned}$$

$$\psi_3(\lambda) = -\sum_{k=1}^{n_1}\frac{t_{k:n}^4 e^{-\lambda t_{k:n}^2}}{(1-e^{-\lambda t_{k:n}^2})^2},$$

$$\begin{aligned}\kappa_3(\alpha_1,\lambda_1) &= \alpha_1\tau^2\frac{\{1-r_3(\lambda)^{\alpha_1}\}\big[r_3(\lambda)^{\alpha_1-2}\tau^2 e^{-\lambda\tau^2}\{\alpha_1 e^{-\lambda\tau^2}-1\}\big]}{\{1-r_3(\lambda)^{\alpha_1}\}^2},\\ &+ \alpha_1\tau^2\frac{\alpha_1\tau^2 e^{-2\lambda\tau^2} r_3(\lambda)^{2\alpha_1-2}}{\{1-r_3(\lambda)^{\alpha_1}\}^2},\end{aligned}$$

$$\begin{aligned}\sigma_3(\alpha_2,\lambda) &= \alpha_2\tau^2\frac{\{1-r_3(\lambda)^{\alpha_2}\}\big[r_3(\lambda)^{\alpha_2-2}\tau^2 e^{-\lambda\tau^2}\{\alpha_2 e^{-\lambda\tau^2}-1\}\big]}{\{1-r_3(\lambda)^{\alpha_2}\}^2},\\ &+ \alpha_2\tau^2\frac{\alpha_2\tau^2 e^{-2\lambda\tau^2} r_3(\lambda)^{2\alpha_2-2}}{\{1-r_3(\lambda)^{\alpha_2}\}^2},\end{aligned}$$

$$\delta_3(\lambda) = -\sum_{k=n_1+1}^{n_1+n_2}\frac{t_{k:n}^4 e^{-\lambda t_{k:n}^2}}{(1-e^{-\lambda t_{k:n}^2})^2}\Big],$$

$$\begin{aligned}\xi_3(\alpha_2,\lambda) &= \tau^2 e^{-\lambda\tau^2}\frac{\{1-r_3(\lambda)^{\alpha_2}\}\big[r_3(\lambda)^{\alpha_2}\{\alpha_2\ln(r_3(\lambda)+1\}\big]}{r_3(\lambda)\{1-r_3(\lambda)^{\alpha_2}\}^2},\\ &+ \tau^2 e^{-\lambda\tau^2}\frac{\alpha_2 r_3(\lambda)^{2\alpha_2}\ln(r_3(\lambda)}{r_3(\lambda)\{1-r_3(\lambda)^{\alpha_2}\}^2},\end{aligned}$$

$$r_3(\lambda) = (1-e^{-\lambda\tau^2}).$$

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