Confidence and prediction intervals based on interpolated records

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Abstract

In several statistical problems, non-parametric confidence intervals for population quantiles can be constructed and their coverage probabilities can be computed exactly, but cannot in general be rendered equal to a pre-determined level. The same difficulty arises for coverage probabilities of non-parametric prediction intervals for future observations. One solution to this difficulty is to interpolate between intervals which have the closest coverage probability from above and below to the pre-determined level. In this paper, confidence intervals for population quantiles are constructed based on interpolated upper and lower records. Subsequently, prediction intervals are obtained for future upper records based on interpolated upper records. Additionally, we derive upper bounds for the coverage error of these confidence and prediction intervals. Finally, our results are applied to some real data sets. Also, a comparison via a simulation study is done with similar classical intervals obtained before.

Keywords and Phrases: Interpolated records, Quantile, Coverage probability, Coverage error
AMS Subject Classification: 62G15, 62G30.

1 Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with a continuous cumulative distribution function (cdf) F and probability density function (pdf) f. An observation X_j is defined to be an *upper record* (or *lower record*) if $X_j > X_i$ (or $X_j < X_i$), for every $i < j$. For convenience of notation, let us denote the ith upper and lower records by U_i and L_i, respectively, for $i \geq 1$. Then, the marginal pdf of the ith upper record, U_i, for $i \geq 1$, is given by (see for example, Arnold *et al.*, 1998)

$$f_{U_i}(u) = \frac{\left\{-\ln \bar{F}(u)\right\}^{i-1}}{(i-1)!} f(u), \qquad F^{-1}(0^+) < u < F^{-1}(1^-), \tag{1}$$

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where $\bar{F}(u) = 1 - F(u)$ is the survival function of X-sequence and the quantile function $F^{-1} : [0, 1] \to \mathbb{R}$ is defined by

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}, \quad y \in (0, 1),$$

and $F^{-1}(0^+) = \lim_{y \to 0^+} F^{-1}(y)$, $F^{-1}(1^-) = \lim_{y \to 1^-} F^{-1}(y)$. Furthermore, the joint pdf of U_i and U_j, $(1 \leq i < j)$, is equal to

$$f_{U_i,U_j}(u, v) = \frac{\{-\ln \bar{F}(u)\}^{i-1}}{(i-1)!} \frac{f(u)}{\bar{F}(u)} \frac{\{\ln \bar{F}(u) - \ln \bar{F}(v)\}^{j-i-1}}{(j-i-1)!} f(v), \qquad u < v. \tag{2}$$

Record data arise naturally in many practical problems. Examples include industrial stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events, and oil and mining surveys. For more details on record model, we refer the reader to Arnold *et al.* (1998), Nevzorov (2001), Gulati and Padgett (2003) and the references contained therein.

For any number p in the interval (0, 1) the pth quantile, ξ_p, of the continuous random variable X having cdf F is defined by $\xi_p = \inf\{x : F(x) \geq p\}$. Recently, several authors have discussed the construction of confidence intervals for population quantiles based on order statistics and records. Arnold *et al.* (2008, p. 183) have described how order statistics can be used to provide distribution-free confidence intervals for population quantiles and tolerance intervals. Ahmadi and Arghami (2003) obtained similar results based on record data.

In addition, as mentioned in Takada (1995) a prediction set is a set determined by the observed sample and having the property that it contains the result of a future sample with a specified probability. A prediction interval corresponds to the special case that the set is an interval. Prediction intervals are widely used for reliability and other related problems. Patel (1989) provided a review on the construction of prediction intervals. Practical problems for which such prediction intervals are appropriate can be found, for example, in Nelson (1968), Hahn (1969, 1970), Hahn and Nelson (1973) and Hall *et al.* (1975).

Now, let ξ_p be the quantile of order p of the parent distribution F. Then, Ahmadi and Arghami (2003) showed that (U_r, U_s), $1 \leq r < s$, is a two-sided confidence interval based on upper records for ξ_p with confidence coefficient

$$\pi_1(r, s; p) = P(U_r < \xi_p < U_s) = (1-p) \sum_{i=r}^{s-1} \frac{(-\ln(1-p))^i}{i!}, \tag{3}$$

which does not depend on F and is a step function of r and s. Clearly, $\pi_1(r, s; p)$ can be read from the table of Poisson probabilities. It is easy to show that $\pi_1(r, s; p) \leq p$. So, confidence intervals for ξ_p based on only upper records are not suitable for central and lower quantiles. In order to obtain suitable confidence intervals for central quantiles, Ahmadi and Arghami (2003) offered using upper and lower records jointly. To do this, they presented confidence interval of the form (L_r, U_s), $(r, s \geq 1)$, whose coverage probability is given by

$$\pi_2(r, s; p) = P(L_r < \xi_p < U_s) = p \sum_{i=0}^{r-1} \frac{(-\ln p)^i}{i!} + (1-p) \sum_{i=0}^{s-1} \frac{(-\ln(1-p))^i}{i!} - 1, \tag{4}$$

which is also a step function of r and s. Then, the possible exact levels are determined by a discrete Poisson distribution. Because of the discreteness of the Poisson distribution, the coverage probabilities cannot achieve the exact pre-fixed level. Moreover, in the context of prediction interval, let R_i be the ith future upper record from the same parent distribution F. Then, Raqab and Balakrishnan (2008) obtained a two-sided prediction interval based on upper records of the form (U_r, U_s), $1 \leq r < s$, for R_i, with coverage probability equals to

$$\pi_3(r, s; i) = P(U_r \leq R_i \leq U_s) = \sum_{j=r}^{s-1} \binom{i+j-1}{j} \frac{1}{2^{i+j}}. \tag{5}$$

Thus, $\pi_3(r, s; i)$ is a step function of r and s and can be computed simply from the binomial tables. Again, because of the discreteness of the binomial distribution, typically its values cannot be rendered equal to a pre-fixed level. The same difficulty arises for confidence intervals for population quantiles and for prediction intervals for future order statistics based on observed order statistics. For example, confidence intervals of the form $(X_{r:n}, X_{s:n})$, $(1 \leq r < s \leq n)$, cover ξ_p with probability $\pi(r, s; n, p)$ given by (see David and Nagaraja, 2003, p. 160 and Arnold *et al.*, 2008, p. 183)

$$\pi(r, s; n, p) = P(X_{r:n} \leq \xi_p \leq X_{s:n}) = \sum_{i=r}^{s-1} \binom{n}{i} p^i (1-p)^{n-i},$$

which does not depend on F and is a step function of r and s. So, the coverage probability cannot achieve the exact pre-determined level. In order to achieve a desired level more closely, different methods have been introduced. For example, Hettmansperger and Sheather (1986) obtained approximately distribution-free confidence intervals for median by interpolating the two confidence intervals which have the closest confidence coefficient from above and below to the pre-selected level. Nyblom (1992) extended their results and presented confidence intervals for an arbitrary population quantile based on non-simple interpolated order statistics. Beran and Hall (1993) used simple linear interpolated order statistics for constructing confidence intervals for population quantiles and prediction intervals for an unobserved data value. They proved that simple linear interpolation reduces the order of coverage error of both confidence and prediction intervals. It is also worth noting that, this problem was solved by some other researchers. See for example, Papadatos (1995), Hutson (1999), and Zieliński and Zieliński (2005). All these mentioned results can be applied for confidence intervals for population quantiles and prediction intervals for future records based on observed records. The main finding of the present study is solving this problem by using non-simple interpolated upper and lower records instead of upper and lower records, respectively.

The paper is organized as follows. At the beginning of Section 2, we introduce the notation used throughout the paper. Then, we construct confidence intervals for quantiles based on interpolated upper and lower records. Moreover, we derive prediction intervals for future upper records based on interpolated upper records, in Section 3. Our results are applied to some real data sets with a comparison study in Section 4. Finally, a simulation study is conducted in Section 5 for comparing our results with other existed results. We conclude with some remarks in Section 6.

2 Confidence interval for quantile

Throughout the paper, we assume that $U_1, U_2, \ldots$ and $L_1, L_2, \ldots$ are two observed sequences of upper and lower records from the X-sequence of iid random variables with cdf F and pdf f, respectively. The interpolated upper and lower records are denoted by $U_r(\lambda) = (1-\lambda)U_r + \lambda U_{r+1}$ and $L_r(\lambda) = (1-\lambda)L_r + \lambda L_{r-1}$, respectively, for $0 \le \lambda \le 1$. Moreover, let ξ_p be the quantile of order p of the parent distribution F. In the following, we are interested in confidence intervals for population quantiles with an approximate confidence coefficient $1-\alpha$.

2.1 Confidence intervals based on interpolated upper records

Suppose (U_r, U_s) is an equal-tailed confidence interval for population quantile, ξ_p, with confidence coefficient $\pi_1(r,s;p)$, where $\pi_1(\cdot,\cdot;p)$ is defined as in (3). Furthermore, assume that (U_{r+1}, U_{s-1}) is a confidence interval for ξ_p with confidence coefficient $\pi_1(r+1, s-1; p)$. We are interested in confidence interval for ξ_p with an approximate confidence coefficient $1-\alpha$, where $\pi_1(r+1, s-1; p) \le 1-\alpha \le \pi_1(r,s;p)$. Towards this end, we have the following theorem.

Theorem 1 *Let $\{X_i,\ i \ge 1\}$ be a sequence of iid continuous random variables with cdf F and pdf f and U_r be the rth upper record from the X-sequence. Moreover, let $U_r(\lambda_1) = (1-\lambda_1)U_r + \lambda_1 U_{r+1}$ and $U_{s-1}(1-\lambda_2) = \lambda_2 U_{s-1} + (1-\lambda_2)U_s$ be two interpolated upper records, for $0 \le \lambda_1 \le 1$ and $0 \le \lambda_2 \le 1$. Then, $(U_r(\lambda_1), U_{s-1}(1-\lambda_2))$ is a confidence interval for population quantile, ξ_p, which approximately has the confidence coefficient $1-\alpha$, with*

$$\lambda_1 = \lambda(\frac{\alpha}{2}, p, r, \pi_r) \qquad \text{and} \qquad \lambda_2 = 1 - \lambda(1-\frac{\alpha}{2}, p, s-1, \pi_{s-1}), \tag{6}$$

where

$$\lambda(\alpha^*, p, k, \pi_k) = \left\{1 + \frac{k(\pi_{k+1} - \alpha^*)}{\ln(1-p)(\pi_k - \alpha^*)}\right\}^{-1}, \tag{7}$$

and

$$\pi_k = (1-p)\sum_{i=0}^{k-1} \frac{(-\ln(1-p))^i}{i!}. \tag{8}$$

Proof. Before proving (6), we obtain a representation for the survival function of interpolated upper records of the form $U_k(\lambda) = (1-\lambda)U_k + \lambda U_{k+1}$, $(0 \le \lambda \le 1)$. To do this, let

$$m(x,\lambda) = -\sigma(\lambda)\ln \bar{F}(x) \qquad \text{and} \qquad \sigma(\lambda) = \frac{1}{\lambda} - 1. \tag{9}$$

Then, we have

$$P((1-\lambda)U_k + \lambda U_{k+1} < x) = \int_{-\infty}^{x}\int_{u}^{\frac{x-(1-\lambda)u}{\lambda}} f_{U_k,U_{k+1}}(u,v)dvdu.$$

Using (2), the above identity can be expressed as

$$
\begin{aligned}
&P((1-\lambda)U_k+\lambda U_{k+1}<x)\\
&=\int_{-\infty}^{x}\int_{u}^{\frac{x-(1-\lambda)u}{\lambda}}\frac{(-\ln\bar F(u))^{k-1}}{(k-1)!}\frac{f(u)}{\bar F(u)}f(v)dvdu\\
&=\int_{-\infty}^{x}\frac{(-\ln\bar F(u))^{k-1}}{(k-1)!}f(u)du-\int_{-\infty}^{x}\frac{(-\ln\bar F(u))^{k-1}}{(k-1)!}\bar F\big(\frac{x-(1-\lambda)u}{\lambda}\big)\frac{f(u)}{\bar F(u)}du\\
&=P(U_k<x)-\int_{-\infty}^{x}\frac{(-\ln\bar F(u))^{k-1}}{(k-1)!}\bar F\big(\frac{x-(1-\lambda)u}{\lambda}\big)\frac{f(u)}{\bar F(u)}du. \qquad (10)
\end{aligned}
$$

By using Taylor expansion for two functions $(k-1)\ln(-\ln\bar F(u))$ and $\ln\bar F\big(\frac{x-(1-\lambda)u}{\lambda}\big)$ at $u=x$, we find (see for example, Råde and Westergren, 2004, p. 189)

$$
(k-1)\ln(-\ln\bar F(u))=(k-1)\ln(-\ln\bar F(x))-(k-1)\frac{f(x)}{\bar F(x)\ln\bar F(x)}(u-x)+O((u-x)^2), \qquad (11)
$$

and

$$
\ln\Big(\bar F\big(\frac{x-(1-\lambda)u}{\lambda}\big)\Big)=\ln\bar F(x)+\frac{1-\lambda}{\lambda}\frac{f(x)}{\bar F(x)}(u-x)+O((u-x)^2), \qquad (12)
$$

where $h(x)\in O(g(x))$ means that, there exists $c>0$ and x_0, such that $h(x)<cg(x)$ whenever $x>x_0$. From (9), (11) and (12), we get

$$
\begin{aligned}
&(k-1)\ln(-\ln\bar F(u))+\ln\Big(\bar F\big(\frac{x-(1-\lambda)u}{\lambda}\big)\Big)\\
&=(k-1)\ln(-\ln\bar F(x))+\ln\bar F(x)-\frac{f(x)(u-x)}{\bar F(x)\ln\bar F(x)}\big(m(x,\lambda)+k-1\big)+O((u-x)^2)\\
&=(k-1)\ln(-\ln\bar F(x))+\ln\bar F(x)\\
&\quad-\big(m(x,\lambda)+k-1\big)\big\{\ln(-\ln\bar F(x))-\ln(-\ln\bar F(u))\big\}+O((u-x)^2)\\
&=-m(x,\lambda)\ln(-\ln\bar F(x))+\ln\bar F(x)+\big(m(x,\lambda)+k-1\big)\ln(-\ln\bar F(u))+O((u-x)^2), \qquad (13)
\end{aligned}
$$

where the second equality is concluded from (11). By neglecting $O((u-x)^2)$ in (13), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{x}\frac{(-\ln\bar F(u))^{k-1}}{(k-1)!}\bar F\big(\frac{x-(1-\lambda)u}{\lambda}\big)\frac{f(u)}{\bar F(u)}du\\
&\approx\frac{(-\ln\bar F(x))^{-m(x,\lambda)}}{(k-1)!}\bar F(x)\int_{-\infty}^{x}(-\ln\bar F(u))^{(m(x,\lambda)+k-1)}\frac{f(u)}{\bar F(u)}du\\
&=\frac{(-\ln\bar F(x))^{-m(x,\lambda)}}{(k-1)!}\bar F(x)\int_{0}^{-\ln\bar F(x)}t^{(m(x,\lambda)+k-1)}dt\\
&=\frac{k}{m(x,\lambda)+k}\frac{(-\ln\bar F(x))^{k}}{k!}\bar F(x). \qquad (14)
\end{aligned}
$$

Substituting (14) into (10), it follows that

$$
\begin{aligned}
P((1-\lambda)U_k + \lambda U_{k+1} < x) &\approx P(U_k < x) - \frac{k}{m(x,\lambda)+k}\frac{(-\ln \bar{F}(x))^k}{k!}\bar{F}(x) \\
&= \frac{m(x,\lambda)}{m(x,\lambda)+k}P(U_k < x) + \frac{k}{m(x,\lambda)+k}P(U_{k+1} < x),
\end{aligned}
$$

since

$$P(U_k < x) - P(U_{k+1} < x) = \frac{(-\ln \bar{F}(x))^k}{k!}\bar{F}(x).$$

Therefore, the survival function of $U_k(\lambda)$ is approximately given by

$$\bar{F}_{U_k(\lambda)}(x) \approx \frac{m(x,\lambda)}{m(x,\lambda)+k}\bar{F}_{U_k}(x) + \frac{k}{m(x,\lambda)+k}\bar{F}_{U_{k+1}}(x), \tag{15}$$

where $m(\cdot,\cdot)$ is defined as in (9).

In order to construct an equal-tailed confidence interval of the form $(U_r(\lambda_1), U_{s-1}(1-\lambda_2))$ for ξ_p with confidence coefficient $1-\alpha$, we should obtain values of λ_1 and λ_2, $(0 \le \lambda_1 \le 1,\ 0 \le \lambda_2 \le 1)$, such that

$$\frac{\alpha}{2} = P(U_r(\lambda_1) > \xi_p) \qquad \text{and} \qquad 1-\frac{\alpha}{2} = P(U_{s-1}(1-\lambda_2) > \xi_p). \tag{16}$$

We just investigate the first equation in (16) which leads to finding λ_1. By setting $x = \xi_p$ in (15) and using the first equality in (16), we immediately conclude that

$$\frac{\alpha}{2} = \frac{m(\xi_p,\lambda_1)}{m(\xi_p,\lambda_1)+r}\pi_r + \frac{r}{m(\xi_p,\lambda_1)+r}\pi_{r+1}, \tag{17}$$

where π_k and $m(\cdot,\cdot)$ are defined as in (8) and (9), respectively. Solving the equation (17) yields the result. Similarly, we can derive an expression for the second equation in (16). □

Remark 1 *From the relation* (7)*, it can be seen that* $0 \le \lambda(\alpha^*, p, k, \pi_k) \le 1$ *and therefore* $0 \le \lambda_1 \le 1$ *and* $0 \le \lambda_2 \le 1$*. Also,* $\lambda(\pi_k, p, k, \pi_k) = 0$ *and* $\lambda(\pi_{k+1}, p, k, \pi_k) = 1$*.*

As mentioned earlier, the obtained confidence interval of the form $(U_r(\lambda_1), U_{s-1}(1-\lambda_2))$ is not suitable for central quantiles (also, in this case, equal-tailed confidence intervals may not exist). In order to find confidence intervals for central quantiles, we use interpolated upper and lower records jointly. The results are presented in the next subsection.

2.2 Confidence intervals based on interpolated upper and lower records jointly

There are some situations wherein upper and lower records are observed simultaneously, just as in the case of weather data. In this case, we can use them jointly to construct confidence intervals, which are suitable for central quantiles. Suppose (L_r, U_s) is an equal-tailed confidence interval for ξ_p with confidence coefficient $\pi_2(r,s;p)$, where $\pi_2(\cdot,\cdot;p)$ is defined as in (4). Moreover, let (L_{r-1}, U_{s-1}) be a confidence interval for ξ_p with confidence coefficient $\pi_2(r-1,s-1;p)$. Theorem 2, gives a confidence interval for ξ_p based on interpolated upper and lower records with an approximate confidence coefficient $1-\alpha$, where $\pi_2(r-1,s-1;p) \le 1-\alpha \le \pi_2(r,s;p)$.

Theorem 2 *Under the assumptions of Theorem 1, let L_r be the rth lower record from the X-sequence and $L_r(\lambda_3) = (1-\lambda_3)L_r + \lambda_3 L_{r-1}$ be the interpolated lower record, for $0 \leq \lambda_3 \leq 1$. Then, $(L_r(\lambda_3), U_{s-1}(1-\lambda_2))$ is a confidence interval for population quantile, ξ_p, with an approximate confidence coefficient $1-\alpha$, with*

$$\lambda_3 = 1 - \lambda(\frac{\alpha}{2}, 1-p, r-1, \beta_{r-1}) \qquad \text{and} \qquad \lambda_2 = 1 - \lambda(1-\frac{\alpha}{2}, p, s-1, \pi_{s-1}), \tag{18}$$

where $\lambda(\cdot,\cdot,\cdot,\cdot)$ and π_k are defined as in (7) and (8), respectively, and

$$\beta_k = 1 - p\sum_{i=0}^{k-1}\frac{(-\ln p)^i}{i!}. \tag{19}$$

Proof. By proceeding as in the proof of Theorem 1, it can be shown that, the survival function of $L_r(\lambda_3)$ is approximately given by

$$\bar{F}_{L_r(\lambda_3)}(x) \approx \frac{q(x,\lambda_3)}{q(x,\lambda_3)+r-1}\bar{F}_{L_{r-1}}(x) + \frac{r-1}{q(x,\lambda_3)+r-1}\bar{F}_{L_r}(x), \tag{20}$$

where

$$q(x,\lambda_3) = -\sigma(1-\lambda_3)\ln F(x), \tag{21}$$

and $\sigma(\cdot)$ is defined as in (9). As in the proof of Theorem 1, to obtain an equal-tailed confidence interval with confidence coefficient $1-\alpha$, we should find λ_3 and λ_2, such that

$$\frac{\alpha}{2} = P(L_r(\lambda_3) > \xi_p) \qquad \text{and} \qquad 1-\frac{\alpha}{2} = P(U_{s-1}(1-\lambda_2) > \xi_p). \tag{22}$$

Setting $x = \xi_p$ in the equations (15), (20) and using (22) and after some simplifications, we immediately conclude the result. □

It is important to note that, the obtained confidence intervals are not exact. It means, their confidence coefficients approximately are equal to pre-fixed level $1-\alpha$. This approximation is concluded by using Taylor expansion which leads to finding an approximation expression for the survival function of interpolated upper and lower records (see the relation (14)). We shall now obtain upper bounds for the coverage errors of the obtained confidence intervals. For the coverage error of approximation for $P(U_r(\lambda_1) > \xi_p)$, (denoted by $ER_{U_r(\lambda_1)}$), from (14) we can write

$$ER_{U_r(\lambda_1)} = \left|\int_{-\infty}^{\xi_p}\frac{(-\ln\bar{F}(u))^{r-1}}{(r-1)!}\bar{F}\big(\frac{\xi_p-(1-\lambda)u}{\lambda}\big)\frac{f(u)}{\bar{F}(u)}du - \frac{r}{m(\xi_p,\lambda_1)+r}\frac{(-\ln(1-p))^r}{r!}(1-p)\right|. \tag{23}$$

On the other hand, for $u < \xi_p$ we have $0 \leq \bar{F}\big(\frac{\xi_p-(1-\lambda)u}{\lambda}\big) \leq \bar{F}(\xi_p) = (1-p)$. It follows that

$$0 \leq \int_{-\infty}^{\xi_p}\frac{(-\ln\bar{F}(u))^{r-1}}{(r-1)!}\bar{F}\big(\frac{\xi_p-(1-\lambda)u}{\lambda}\big)\frac{f(u)}{\bar{F}(u)}du \leq (1-p)\frac{(-\ln(1-p))^r}{r!}.$$

So, we conclude that

$$
\begin{aligned}
&-\frac{r}{m(\xi_p,\lambda_1)+r}\frac{(-\ln(1-p))^r}{r!}(1-p)\\
&\le \int_{-\infty}^{\xi_p}\frac{(-\ln\bar F(u))^{r-1}}{(r-1)!}\bar F\big(\frac{\xi_p-(1-\lambda)u}{\lambda}\big)\frac{f(u)}{\bar F(u)}du-\frac{r}{m(\xi_p,\lambda_1)+r}\frac{(-\ln(1-p))^r}{r!}(1-p)\\
&\le \frac{m(\xi_p,\lambda_1)}{m(\xi_p,\lambda_1)+r}\frac{(-\ln(1-p))^r}{r!}(1-p). \qquad (24)
\end{aligned}
$$

But, from (8), it can be easily shown that

$$\frac{(-\ln(1-p))^r}{r!}(1-p)=\pi_{r+1}-\pi_r.$$

Hence, from the above identity and relations (23) and (24), upper bounds for the coverage error of approximation for $P(U_r(\lambda_1)>\xi_p)$, (denoted by $UER_{U_r(\lambda_1)}$), is given by

$$
\begin{aligned}
UER_{U_r(\lambda_1)} &= \max\Big\{\frac{r}{m(\xi_p,\lambda_1)+r}\big(\pi_{r+1}-\pi_r\big),\frac{m(\xi_p,\lambda_1)}{m(\xi_p,\lambda_1)+r}\big(\pi_{r+1}-\pi_r\big)\Big\}\\
&\equiv UEC(r,p,\pi_r,\lambda_1),\ \text{say}, \qquad (25)
\end{aligned}
$$

where $m(\cdot,\cdot)$ is defined as in (9). In a similar way, we can show that upper bounds for the coverage error of approximations for $P(U_{s-1}(1-\lambda_2)>\xi_p)$ and $P(L_r(\lambda_3)>\xi_p)$ are equal to

$$UER_{U_{s-1}(1-\lambda_2)}=UEC(s-1,p,\pi_{s-1},1-\lambda_2), \qquad (26)$$

and

$$UER_{L_r(\lambda_3)}=UEC(r-1,1-p,\beta_{r-1},1-\lambda_3), \qquad (27)$$

respectively, since from (9) and (21), we have $m(\xi_p,\lambda)=q(\xi_{1-p},\lambda)$.

Remark 2 *It is interesting to note that, if* $p\longrightarrow 1$, *then* $UEC(k,p,\pi_k,\lambda)\longrightarrow 0$.

Remark 3 *The bounds obtained in* (25), (26) *and* (27) *are attainable if and only if* $\lambda_i=1$ *or* $\lambda_i=0$, $(i=1,2,3)$.

For symmetric distributions, we find the following result.

Remark 4 *Suppose that the cdf* F *is symmetric, say about zero, without loss of generality. Then,* $(L_r(\lambda_3),U_{s-1}(1-\lambda_2))$ *is a confidence interval for* ξ_p *with an approximate confidence coefficient* $1-\alpha$, *if and only if,* $(L_s(\lambda_2),U_{r-1}(1-\lambda_3))$ *is a confidence interval for* ξ_{1-p} *with the same confidence coefficient. In fact, in this case, we have* $U_i\overset{\mathrm{d}}{=}-L_i$ *and* $\xi_p=-\xi_{1-p}$, *where* $\overset{\mathrm{d}}{=}$ *means identical in distribution. So, we get*

$$U_{s-1}(1-\lambda_2)=(1-\lambda_2)U_s+\lambda_2U_{s-1}\overset{\mathrm{d}}{=}-\Big((1-\lambda_2)L_s+\lambda_2L_{s-1}\Big)=-L_s(\lambda_2),$$

and

$$L_r(\lambda_3) = (1-\lambda_3)L_r + \lambda_3 L_{r-1} \stackrel{\mathrm{d}}{=} -\Big((1-\lambda_3)U_r + \lambda_3 U_{r-1}\Big) = -U_{r-1}(1-\lambda_3).$$

Therefore, it can be shown that

$$\begin{aligned} P(L_r(\lambda_3) < \xi_p < U_{s-1}(1-\lambda_2)) &= P(U_{s-1}(1-\lambda_2) > \xi_p) - P(L_r(\lambda_3) > \xi_p) \\ &= P(-L_s(\lambda_2) > \xi_p) - P(-U_{r-1}(1-\lambda_3) > \xi_p) \\ &= P(L_s(\lambda_2) < -\xi_p) - P(U_{r-1}(1-\lambda_3) < -\xi_p) \\ &= P(L_s(\lambda_2) < \xi_{1-p}) - P(U_{r-1}(1-\lambda_3) < \xi_{1-p}) \\ &= P(L_s(\lambda_2) < \xi_{1-p} < U_{r-1}(1-\lambda_3)). \end{aligned}$$

Tables 1 and 2 contain confidence intervals (CI) for quantiles and their coverage probabilities (CP) for different values of p and $1-\alpha$. Moreover, Tables 1 and 2 report values of λ_2 and λ_3 based on the results obtained in Theorem 2. For validity of the theoretical conclusions, we consider the standard uniform, normal, Cauchy and Laplace distributions, which are symmetric as well as two beta distributions with parameters $(3,1)$ and $(1,3)$ which are non-symmetric. Furthermore, upper bounds for the coverage errors obtained in (26) and (27) are displayed in the two last columns of Tables 1 and 2 .

From these Tables, the following points may be observed:

- The obtained confidence intervals are approximately distribution-free. It means, the results for different distributions are quite similar.
- Furthermore, we can see that for different distributions, the maximum discrepancy between the exact and nominal values is 0.0421. Moreover, the maximum upper bound for the coverage error is 0.1034.
- The indices of upper records increase when p increases. Conversely, when p increases, the indices of lower records decrease. Also, it is seen that, for $p = 0.5$, indices of upper and lower records are the same.
- It can be seen that, for $p = 0.5$ upper bounds for the coverage errors are the same.

3 Prediction interval for future records

In the following, assume that (U_r, U_s) is an equal-tailed prediction interval for R_i with coverage probability $\pi_3(r, s; i)$, where $\pi_3(\cdot, \cdot; i)$ is defined as in (5). Furthermore, let (U_{r+1}, U_{s-1}) be a prediction interval with coverage probability $\pi_3(r+1, s-1; i)$. In order to construct prediction intervals with an approximate

Table 1: Confidence intervals for population quantiles (nominal values in parentheses) for $1-\alpha=0.9$.

				Tail probabilities			UER	
p	$(\lambda_3,\ \lambda_2)$	CI	Distribution	below the lower	above the upper	CP	$L_r(\lambda_3)$	$U_{s-1}(1-\lambda_2)$
0.1	(0.5591, 0.8948)	$(L_6,\ U_2)$	Continuous	0.0301	0.0052	0.9647	0.0341	0.0500
		$(L_5,\ U_1)$	Continuous	0.0841	0.1000	0.8159		
		Interpolated	Uniform	0.0576 (0.05)	0.0492 (0.05)	0.8934 (0.90)		
		Interpolated	Normal	0.0505 (0.05)	0.0742 (0.05)	0.8752 (0.90)		
		Interpolated	Cauchy	0.0435 (0.05)	0.0878 (0.05)	0.8686 (0.90)		
		Interpolated	Laplace	0.0484 (0.05)	0.0809 (0.05)	0.8706 (0.90)		
		Interpolated	Logistic	0.0493 (0.05)	0.0773 (0.05)	0.8735 (0.90)		
		Interpolated	$Beta(1,3)$	0.0581 (0.05)	0.0401 (0.05)	0.9017 (0.90)		
		Interpolated	$Beta(3,1)$	0.0512 (0.05)	0.0776 (0.05)	0.8712 (0.90)		
0.3	(0.3567, 0.9987)	$(L_4,\ U_3)$	Continuous	0.0341	0.0058	0.9601	0.0714	0.0442
		$(L_3,\ U_2)$	Continuous	0.1214	0.0503	0.8283		
		Interpolated	Uniform	0.0530 (0.05)	0.0502 (0.05)	0.8968 (0.90)		
		Interpolated	Normal	0.0500 (0.05)	0.0502 (0.05)	0.8999 (0.90)		
		Interpolated	Cauchy	0.0465 (0.05)	0.0500 (0.05)	0.9035 (0.90)		
		Interpolated	Laplace	0.0486 (0.05)	0.0502 (0.05)	0.9013 (0.90)		
		Interpolated	Logistic	0.0494 (0.05)	0.0502 (0.05)	0.9004 (0.90)		
		Interpolated	$Beta(1,3)$	0.0564 (0.05)	0.0501 (0.05)	0.8936 (0.90)		
		Interpolated	$Beta(3,1)$	0.0484 (0.05)	0.0502 (0.05)	0.9014 (0.90)		
0.5	(0.3177, 0.3177)	$(L_3,\ U_3)$	Continuous	0.0333	0.0333	0.9334	0.1034	0.1034
		$(L_2,\ U_2)$	Continuous	0.1534	0.1534	0.6931		
		Interpolated	Uniform	0.0488 (0.05)	0.0488 (0.05)	0.9024 (0.90)		
		Interpolated	Normal	0.0499 (0.05)	0.0499 (0.05)	0.9001 (0.90)		
		Interpolated	Cauchy	0.0522 (0.05)	0.0522 (0.05)	0.8954 (0.90)		
		Interpolated	Laplace	0.0517 (0.05)	0.0517 (0.05)	0.8965 (0.90)		
		Interpolated	Logistic	0.0502 (0.05)	0.0502 (0.05)	0.8995 (0.90)		
		Interpolated	$Beta(1,3)$	0.0546 (0.05)	0.0462 (0.05)	0.8992 (0.90)		
		Interpolated	$Beta(3,1)$	0.0462 (0.05)	0.0546 (0.05)	0.8992 (0.90)		
0.7	(0.9987, 0.3567)	$(L_3,\ U_4)$	Continuous	0.0058	0.0341	0.9601	0.0442	0.0714
		$(L_2,\ U_3)$	Continuous	0.0503	0.1214	0.8283		
		Interpolated	Uniform	0.0502 (0.05)	0.0530 (0.05)	0.8968 (0.90)		
		Interpolated	Normal	0.0503 (0.05)	0.0499 (0.05)	0.8996 (0.90)		
		Interpolated	Cauchy	0.0499 (0.05)	0.0464 (0.05)	0.9035 (0.90)		
		Interpolated	Laplace	0.0501 (0.05)	0.0485 (0.05)	0.9012 (0.90)		
		Interpolated	Logistic	0.0502 (0.05)	0.0494 (0.05)	0.9004 (0.90)		
		Interpolated	$Beta(1,3)$	0.0502 (0.05)	0.0484 (0.05)	0.9014 (0.90)		
		Interpolated	$Beta(3,1)$	0.0501 (0.05)	0.0564 (0.05)	0.8936 (0.90)		
0.9	(0.8948, 0.5591)	$(L_2,\ U_6)$	Continuous	0.0052	0.0301	0.9647	0.0500	0.0341
		$(L_1,\ U_5)$	Continuous	0.1000	0.0841	0.8159		
		Interpolated	Uniform	0.0492 (0.05)	0.0576 (0.05)	0.8934 (0.90)		
		Interpolated	Normal	0.0742 (0.05)	0.0505 (0.05)	0.8752 (0.90)		
		Interpolated	Cauchy	0.0878 (0.05)	0.0435 (0.05)	0.8686 (0.90)		
		Interpolated	Laplace	0.0809 (0.05)	0.0484 (0.05)	0.8706 (0.90)		
		Interpolated	Logistic	0.0773 (0.05)	0.0493 (0.05)	0.8735 (0.90)		
		Interpolated	$Beta(1,3)$	0.0776 (0.05)	0.0512 (0.05)	0.8712 (0.90)		
		Interpolated	$Beta(3,1)$	0.0401 (0.05)	0.0581 (0.05)	0.9017 (0.90)		

Table 2: Confidence intervals for population quantiles (nominal values in parentheses) for $1-\alpha = 0.95$.

				Tail probabilities			UER	
p	$(\lambda_3,\ \lambda_2)$	CI	Distribution	below the lower	above the upper	CP	$L_r(\lambda_3)$	$U_{s-1}(1-\lambda_2)$
0.1	(0.8881 , 0.7150)	$(L_7,\ U_2)$	Continuous	0.0094	0.0052	0.9854	0.0156	0.0750
		$(L_6,\ U_1)$	Continuous	0.0301	0.1000	0.8699		
		Interpolated	Uniform	0.0271 (0.025)	0.0182 (0.025)	0.9547 (0.95)		
		Interpolated	Normal	0.0255 (0.025)	0.0400(0.025)	0.9344 (0.95)		
		Interpolated	Cauchy	0.0215 (0.025)	0.0705 (0.025)	0.9079 (0.95)		
		Interpolated	Laplace	0.0248 (0.025)	0.0525 (0.025)	0.9226 (0.95)		
		Interpolated	Logistic	0.0251 (0.025)	0.0460 (0.025)	0.9289 (0.95)		
		Interpolated	$Beta(1,3)$	0.0272 (0.025)	0.0171 (0.025)	0.9557 (0.95)		
		Interpolated	$Beta(3,1)$	0.0259 (0.025)	0.0423 (0.025)	0.9319 (0.95)		
0.3	(0.8621, 0.8095)	$(L_5,\ U_3)$	Continuous	0.0078	0.0058	0.9863	0.0172	0.0253
		$(L_4,\ U_2)$	Continuous	0.0341	0.0503	0.9156		
		Interpolated	Uniform	0.0283 (0.025)	0.0285 (0.025)	0.9432 (0.95)		
		Interpolated	Normal	0.0263 (0.025)	0.0302 (0.025)	0.9435 (0.95)		
		Interpolated	Cauchy	0.0219 (0.025)	0.0322 (0.025)	0.9459 (0.95)		
		Interpolated	Laplace	0.0244 (0.025)	0.0323 (0.025)	0.9433 (0.95)		
		Interpolated	Logistic	0.0257 (0.025)	0.0305 (0.025)	0.9438 (0.95)		
		Interpolated	$Beta(1,3)$	0.0287 (0.025)	0.0235 (0.025)	0.9478 (0.95)		
		Interpolated	$Beta(3,1)$	0.0260 (0.025)	0.0260 (0.025)	0.9398 (0.95)		
0.5	(0.9101, 0.9101)	$(L_4,\ U_4)$	Continuous	0.0056	0.0056	0.9889	0.0194	0.0194
		$(L_3,\ U_3)$	Continuous	0.0333	0.0333	0.9334		
		Interpolated	Uniform	0.0281 (0.025)	0.0281 (0.025)	0.9439 (0.95)		
		Interpolated	Normal	0.0270 (0.025)	0.0270 (0.025)	0.9459 (0.95)		
		Interpolated	Cauchy	0.0240 (0.025)	0.0240 (0.025)	0.9518 (0.95)		
		Interpolated	Laplace	0.0251 (0.025)	0.0251 (0.025)	0.9496 (0.95)		
		Interpolated	Logistic	0.0267 (0.025)	0.0267 (0.025)	0.9467 (0.95)		
		Interpolated	$Beta(1,3)$	0.0289 (0.025)	0.0259 (0.025)	0.9452 (0.95)		
		Interpolated	$Beta(3,1)$	0.0259 (0.025)	0.0289 (0.025)	0.9452 (0.95)		
0.7	(0.8095, 0.8621)	$(L_3,\ U_5)$	Continuous	0.0058	0.0078	0.9863	0.0253	0.0172
		$(L_2,\ U_4)$	Continuous	0.0503	0.0341	0.9156		
		Interpolated	Uniform	0.0285 (0.025)	0.0283 (0.025)	0.9432 (0.95)		
		Interpolated	Normal	0.0301 (0.025)	0.0262 (0.025)	0.9435 (0.95)		
		Interpolated	Cauchy	0.0322 (0.025)	0.0218 (0.025)	0.9458 (0.95)		
		Interpolated	Laplace	0.0323 (0.025)	0.0244 (0.025)	0.9432 (0.95)		
		Interpolated	Logistic	0.0305 (0.025)	0.0257 (0.025)	0.9438 (0.95)		
		Interpolated	$Beta(1,3)$	0.0341 (0.025)	0.0260 (0.025)	0.9398 (0.95)		
		Interpolated	$Beta(3,1)$	0.0235 (0.025)	0.0287 (0.025)	0.9478 (0.95)		
0.9	(0.7150, 0.8881)	$(L_2,\ U_7)$	Continuous	0.0052	0.0094	0.9854	0.0750	0.0156
		$(L_1,\ U_6)$	Continuous	0.1000	0.0301	0.8699		
		Interpolated	Uniform	0.0182 (0.025)	0.0271 (0.025)	0.9547 (0.95)		
		Interpolated	Normal	0.0400 (0.025)	0.0255 (0.025)	0.9344 (0.95)		
		Interpolated	Cauchy	0.0705 (0.025)	0.0215 (0.025)	0.9079 (0.95)		
		Interpolated	Laplace	0.0525 (0.025)	0.0248 (0.025)	0.9226 (0.95)		
		Interpolated	Logistic	0.0460 (0.025)	0.0251 (0.025)	0.9289 (0.95)		
		Interpolated	$Beta(1,3)$	0.0423 (0.025)	0.0259 (0.025)	0.9319 (0.95)		
		Interpolated	$Beta(3,1)$	0.0171 (0.025)	0.0272 (0.025)	0.9557 (0.95)		

coverage probability $1-\alpha$, $\pi_3(r+1, s-1; i) \leq 1-\alpha \leq \pi_3(r, s; i)$, we consider two interpolated upper records as

$$U_r(\gamma_1) = (1-\gamma_1)U_r + \gamma_1 U_{r+1} \quad \text{and} \quad U_{s-1}(1-\gamma_2) = (1-\gamma_2)U_s + \gamma_2 U_{s-1}, \tag{28}$$

where $0 \leq \gamma_1 \leq 1$ and $0 \leq \gamma_2 \leq 1$. Before constructing prediction intervals, we first present the following lemma which is used for constructing prediction interval.

Lemma 1 *Under the assumptions of Theorem 1, let R_i be a future upper record. Then, we have*

$$P(U_k(\gamma) > R_i) = \left\{1 - \Big(\frac{2k}{2k+\sigma(\gamma)}\Big)^{k+i}\right\} \phi_k + \left\{\Big(\frac{2k}{2k+\sigma(\gamma)}\Big)^{k+i}\right\} \phi_{k+1}, \tag{29}$$

where

$$\phi_k = \sum_{j=0}^{k-1} \binom{i+j-1}{j} \frac{1}{2^{i+j}}. \tag{30}$$

and $\sigma(\cdot)$ is defined as in (9).

Proof. By conditional argument and using (15), we can write

$$\begin{aligned} P(U_k(\gamma) > R_i) &= \int_{-\infty}^{\infty} P(U_k(\gamma) > x) f_{R_i}(x)dx \\ &\approx \int_{-\infty}^{\infty} \left\{ \frac{m(x,\gamma)}{m(x,\gamma)+k} \bar{F}_{U_k}(x) + \frac{k}{m(x,\gamma)+k} \bar{F}_{U_{k+1}}(x) \right\} f_{R_i}(x)dx \\ &= P(U_k > R_i) + \int_{-\infty}^{\infty} \frac{k}{m(x,\gamma)+k} \frac{(-\ln \bar{F}(x))^k}{k!} \bar{F}(x) f_{R_i}(x)dx, \end{aligned} \tag{31}$$

since, it can be written

$$\bar{F}_{U_{k+1}}(x) - \bar{F}_{U_k}(x) = \frac{(-\ln \bar{F}(x))^k}{k!} \bar{F}(x).$$

On the other hand, from equations (1) and (9), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{k}{m(x,\gamma)+k} \frac{(-\ln \bar{F}(x))^k}{k!} \bar{F}(x) f_{R_i}(x)dx \\ &= \frac{1}{(k-1)!(i-1)!} \int_{-\infty}^{\infty} \frac{(-\ln \bar{F}(x))^{k+i-1}}{k - \sigma(\gamma)\ln \bar{F}(x)} \bar{F}(x) f(x)dx \\ &= \frac{1}{(k-1)!(i-1)!} \int_{0}^{\infty} \frac{t^{k+i-1}}{k+\sigma(\gamma)t} e^{-2t}dt, \end{aligned} \tag{32}$$

where $t = -\ln \bar{F}(x)$. Expanding $-\ln(k+\sigma(\gamma)t)$ at $t=0$, we get

$$-\ln(k+\sigma(\gamma)t) = -\ln k - \frac{\sigma(\gamma)}{k} t + O(t^2).$$

By neglecting $O(t^2)$ in the above equation and from (32), it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{k}{m(x,\gamma)+k}\frac{(-\ln\bar{F}(x))^k}{k!}\bar{F}(x)f_{R_i}(x)dx &\approx \frac{1}{k!(i-1)!}\int_0^{\infty}t^{k+i-1}e^{-t(2+\frac{\sigma(\gamma)}{k})}dt\\ &= \binom{k+i-1}{k}\left(\frac{k}{2k+\sigma(\gamma)}\right)^{k+i}\\ &= \Big(\phi_{k+1}-\phi_k\Big)\left(\frac{2k}{2k+\sigma(\gamma)}\right)^{k+i}. \qquad (33)\end{aligned}$$

Upon substituting (33) into (31) and after some simplifications, we arrive at the desired result. □

Finding an approximate equal-tailed prediction interval is presented in the next theorem.

Theorem 3 *Under the assumptions of Theorem 1,* $(U_r(\gamma_1), U_{s-1}(1-\gamma_2))$ *is an approximate* $100(1-\alpha)\%$ *equal-tailed prediction interval for future upper record,* R_i*, with*

$$\gamma_1 = \gamma(\frac{\alpha}{2}, r, i) \qquad \text{and} \qquad \gamma_2 = 1-\gamma(1-\frac{\alpha}{2}, s-1, i), \qquad (34)$$

where

$$\gamma(\alpha^*, k, i) = \left\{1+2k\left[\left(\frac{\phi_{k+1}-\phi_k}{\alpha^*-\phi_k}\right)^{\frac{1}{k+i}}-1\right]\right\}^{-1}, \qquad (35)$$

and ϕ_k *is defined as in* (30).

Proof. For constructing an equal-tailed prediction interval of the form $(U_r(\gamma_1), U_{s-1}(1-\gamma_2))$ for future upper record, R_i, with prediction coefficient $1-\alpha$, we should find γ_1 and γ_2, such that

$$P(U_r(\gamma_1) > R_i) = \frac{\alpha}{2} \quad \text{and} \quad P(U_{s-1}(1-\gamma_2) > R_i) = 1-\frac{\alpha}{2}. \qquad (36)$$

For deriving the desired result, it is enough to find γ such that $P(U_k(\gamma) > R_i) = \alpha^*$. By using equation (29), we find

$$\alpha^* = \left\{1-\Big(\frac{2k}{2k+\sigma(\gamma)}\Big)^{k+i}\right\}\phi_k + \left\{\Big(\frac{2k}{2k+\sigma(\gamma)}\Big)^{k+i}\right\}\phi_{k+1}.$$

It follows that

$$\sigma(\gamma) = 2k\left[\left(\frac{\phi_{k+1}-\phi_k}{\alpha^*-\phi_k}\right)^{\frac{1}{k+i}}-1\right].$$

Now, from (9), we get the result. □

Remark 5 *Notice that,* $0 \le \gamma(\alpha^*, k, i) \le 1$ *and so* $0 \le \gamma_1 \le 1$ *and* $0 \le \gamma_2 \le 1$*. Also,* $\gamma(\phi_{k+1}, k, i) = 1$ *and* $\gamma(\phi_k, k, i) = 0$*.*

Similar to Section 2, we derive upper bounds for the coverage error of the obtained prediction interval of the form $(U_r(\gamma_1), U_{s-1}(1-\gamma_2))$. First, we obtain the result for $P(U_r(\gamma_1) > R_i)$, which from (33) is equal to

$$\begin{aligned} ER_{U_r(\gamma_1)} &= \Bigg| \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{(-\ln \bar{F}(x+u))^{r-1}}{(r-1)!}\bar{F}(x-\sigma(\gamma_1)u)\frac{f(x+u)}{\bar{F}(x+u)}\frac{(-\ln \bar{F}(x))^{i-1}}{(i-1)!}f(x)dudx \\ &\qquad - \Big(\frac{2r}{2r+\sigma(\gamma_1)}\Big)^{r+i}(\phi_{r+1}-\phi_r)\Bigg|. \end{aligned} \tag{37}$$

On the other hand, we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{(-\ln \bar{F}(x+u))^{r-1}}{(r-1)!}\bar{F}(x)\frac{f(x+u)}{\bar{F}(x+u)}\frac{(-\ln \bar{F}(x))^{i-1}}{(i-1)!}f(x)dudx \\ &= \int_{-\infty}^{\infty} \frac{(-\ln \bar{F}(x))^{r+i-1}}{r!(i-1)!}\bar{F}(x)f(x)dx \\ &= \binom{r+i-1}{r}(\frac{1}{2})^{r+i} \\ &= \phi_{r+1}-\phi_r. \end{aligned} \tag{38}$$

Since $u<0$, then $0 \le \bar{F}(x-\sigma(\gamma_1)u) \le \bar{F}(x)$. So, from (38) we conclude that

$$\begin{aligned} 0 &\le \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{(-\ln \bar{F}(x+u))^{r-1}}{(r-1)!}\bar{F}(x-\sigma(\gamma_1)u)\frac{f(x+u)}{\bar{F}(x+u)}\frac{(-\ln \bar{F}(x))^{i-1}}{(i-1)!}f(x)dudx \\ &\le \phi_{r+1}-\phi_r. \end{aligned}$$

The above inequalities yield

$$\begin{aligned} &-\Big(\frac{2r}{2r+\sigma(\gamma_1)}\Big)^{r+i}(\phi_{r+1}-\phi_r) \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{(-\ln \bar{F}(x+u))^{r-1}}{(r-1)!}\bar{F}(x-\sigma(\gamma_1)u)\frac{f(x+u)}{\bar{F}(x+u)}\frac{(-\ln \bar{F}(x))^{i-1}}{(i-1)!}f(x)dudx \\ &\qquad - \Big(\frac{2r}{2r+\sigma(\gamma_1)}\Big)^{r+i}(\phi_{r+1}-\phi_r) \\ &\le \Big\{1-\Big(\frac{2r}{2r+\sigma(\gamma_1)}\Big)^{r+i}\Big\}(\phi_{r+1}-\phi_r). \end{aligned}$$

So, form relation (37) and the above inequalities, upper bounds for the coverage error of $P(U_r(\gamma_1) > R_i)$, (denoted by $UER_{U_r(\gamma_1)}$), is given by

$$\begin{aligned} UER_{U_r(\gamma_1)} &= \max\Big\{\Big(\frac{2r}{2r+\sigma(\gamma_1)}\Big)^{r+i}(\phi_{r+1}-\phi_r), \Big\{1-\Big(\frac{2r}{2r+\sigma(\gamma_1)}\Big)^{r+i}\Big\}(\phi_{r+1}-\phi_r)\Big\} \\ &\equiv UEP(r,i,\gamma_1), \text{ say.} \end{aligned} \tag{39}$$

In a similar manner, for $P(U_{s-1}(1-\gamma_2) > R_i)$, we can write

$$UER_{U_{s-1}(1-\gamma_2)} = UEP(s-1, i, 1-\gamma_2). \tag{40}$$

Remark 6 *The bounds obtained in* (39) *and* (40) *are attainable if and only if* $\gamma_i = 1$ *or* $\gamma_i = 0$, $(i = 1, 2)$.

Remark 7 *It is interesting to mention here that,* $UEP(k, i, \gamma)$ *tends to zero as* $i \longrightarrow \infty$.

Table 3: Prediction intervals for future upper records (nominal values in parentheses).

						Tail probabilities			UER	
$1-\alpha$	i	γ_1	γ_2	PI	Distribution	below the lower	above the upper	CP	$U_r(\gamma_1)$	$U_{s-1}(1-\gamma_2)$
0.90	6	0.9174	0.1071	$(U_1,\ U_{13})$	Continuous	0.0156	0.0481	0.9362	0.0345	0.0216
				$(U_2,\ U_{12})$	Continuous	0.0625	0.0717	0.8658		
				Interpolated	Uniform	0.0425 (0.05)	0.0522 (0.05)	0.9054 (0.90)		
				Interpolated	Normal	0.0533 (0.05)	0.0499 (0.05)	0.8969 (0.90)		
				Interpolated	Cauchy	0.0592 (0.05)	0.0491 (0.05)	0.8915 (0.90)		
				Interpolated	Exponential	0.0560 (0.05)	0.0551 (0.05)	0.8888 (0.90)		
	7	0.6158	0.5463	$(U_2,\ U_{15})$	Continuous	0.0352	0.0392	0.9257	0.0399	0.0108
				$(U_3,\ U_{14})$	Continuous	0.0898	0.0577	0.8525		
				Interpolated	Uniform	0.0487 (0.05)	0.0510 (0.05)	0.9003 (0.90)		
				Interpolated	Normal	0.0590 (0.05)	0.0479 (0.05)	0.8930 (0.90)		
				Interpolated	Cauchy	0.0474 (0.05)	0.0447 (0.05)	0.9077 (0.90)		
				Interpolated	Exponential	0.0619 (0.05)	0.0576 (0.05)	0.8804 (0.90)		
	8	0.9470	0.1935	$(U_2,\ U_{16})$	Continuous	0.0195	0.0466	0.9393	0.0306	0.0169
				$(U_3,\ U_{15})$	Continuous	0.0547	0.0669	0.8784		
				Interpolated	Uniform	0.0427 (0.05)	0.0529 (0.05)	0.9044 (0.90)		
				Interpolated	Normal	0.0509 (0.05)	0.0496 (0.05)	0.8994 (0.90)		
				Interpolated	Cauchy	0.0761 (0.05)	0.0483 (0.05)	0.8754 (0.90)		
				Interpolated	Exponential	0.0477 (0.05)	0.0144 (0.05)	0.9378 (0.90)		
0.95	6	0.6596	0.4118	$(U_1,\ U_{15})$	Continuous	0.0156	0.0207	0.9637	0.0375	0.0068
				$(U_2,\ U_{14})$	Continuous	0.0625	0.0318	0.9057		
				Interpolated	Uniform	0.0255 (0.025)	0.0262 (0.025)	0.9483 (0.95)		
				Interpolated	Normal	0.0337 (0.025)	0.0243 (0.025)	0.9418 (0.95)		
				Interpolated	Cauchy	0.0493 (0.025)	0.0229 (0.025)	0.9276 (0.95)		
				Interpolated	Exponential	0.0382 (0.025)	0.0242 (0.025)	0.9375 (0.95)		
	7	0.8929	0.7444	$(U_1,\ U_{17})$	Continuous	0.0078	0.0173	0.9748	0.0172	0.0077
				$(U_2,\ U_{16})$	Continuous	0.0352	0.0262	0.9386		
				Interpolated	Uniform	0.0196 (0.025)	0.0250 (0.025)	0.9554 (0.95)		
				Interpolated	Normal	0.0276 (0.025)	0.0233 (0.025)	0.9490 (0.95)		
				Interpolated	Cauchy	0.0327 (0.025)	0.0215 (0.025)	0.9457 (0.95)		
				Interpolated	Exponential	0.0296 (0.025)	0.0262 (0.025)	0.9441 (0.95)		
	8	0.5494	0.3523	$(U_2,\ U_{18})$	Continuous	0.0195	0.0216	0.9558	0.0297	0.0070
				$(U_3,\ U_{17})$	Continuous	0.0547	0.0320	0.9134		
				Interpolated	Uniform	0.0262 (0.025)	0.0260 (0.025)	0.9478 (0.95)		
				Interpolated	Normal	0.0315 (0.025)	0.0245 (0.025)	0.9438 (0.95)		
				Interpolated	Cauchy	0.0428 (0.025)	0.0234 (0.025)	0.9337 (0.95)		
				Interpolated	Exponential	0.0292 (0.025)	0.0243 (0.025)	0.9464 (0.95)		

Table 3 presents prediction intervals (PI), for future upper records and their coverage probabilities (CP), for different values of i and $1-\alpha$. Also, upper bounds for the coverage error are tabulated in the two last columns of Table 3. Values of γ_1 and γ_2 based on the results of Theorem 3 are computed in this table. In Table 3, three symmetric distributions are considered as the standard uniform, normal and Cauchy distributions. From Table 3, the following points can be drawn:

- As one would expect, by increasing values of i, indices of upper records increase.
- The obtained results for different distributions are similar. So, the obtained prediction intervals are approximately distribution-free.

- Moreover, from the results in Table 3, it can be seen that, the maximum discrepancy for different distributions between the exact and nominal coverage probabilities is 0.0224. Furthermore, the maximum upper bound for the coverage error is 0.0399.

4 Application on real data

To illustrate the inferential procedures developed in the preceding sections, and for comparing the existing methods of obtaining confidence intervals for population quantiles and prediction intervals for future upper records, we apply our technique to the following data sets in two presented examples.

Example 1. We consider the data which represent the amount of annual (Jan. 1 - Dec. 31) rainfall in inches recorded at Los Angeles Civic Center during the 100-year period from 1890 until 1989; see Arnold *et al.* (1998, p. 180). Table 4 gives the upper and lower records extracted from the data set in Arnold *et al.* (1998, p. 180). This data set was used by Ahmadi and Arghami (2003) for finding confidence intervals for quantiles. Table 5 presents the approximate equal-tailed confidence intervals based on interpolated upper and lower records and the values of λ_2 and λ_3 obtained from the results of Theorem 2.

Table 4: The upper and lower records extracted from Arnold *et al.* (1998, p. 180).

i	1	2	3	4	5	6	7	8
U_i	12.69	12.84	18.72	21.96	23.92	27.16	31.28	34.04
L_i	12.69	7.51	4.83	4.13	4.08			

Table 5: The approximate equal-tailed confidence interval for quantiles based on the data in Table 4.

p	$1-\alpha$	(r, s)	$(L_r,\ U_s)$	$(L_{r-1},\ U_{s-1})$	λ_3	λ_2	$(L_r(\lambda_3),\ U_{s-1}(1-\lambda_2))$
0.7	0.99	(4, 6)	(4.13, 27.16)	(4.83, 23.92)	0.9792	0.8351	(4.8155, 24.4543)
	0.98	(3, 5)	(4.83, 23.92)	(7.51, 21.96)	0.3686	0.2285	(5.8178, 23.4720)
	0.97	(3, 5)	(4.83, 23.92)	(7.51, 21.96)	0.5935	0.5541	(6.4206, 22.8339)
	0.96	(3, 5)	(4.83, 23.92)	(7.51, 21.96)	0.7242	0.7409	(6.7707, 22.4678)
	0.95	(3, 5)	(4.83, 23.92)	(7.51, 21.96)	0.8095	0.8621	(6.9996, 22.2303)
0.9	0.99	(3, 8)	(4.83, 34.04)	(7.51, 31.28)	0.9981	0.6224	(7.5049, 32.3223)
	0.98	(2, 7)	(7.51, 31.28)	(12.69, 27.16)	0.3372	0.0704	(9.2568, 30.9899)
	0.97	(2, 7)	(7.51, 31.28)	(12.69, 27.16)	0.5231	0.4905	(10.2198, 29.2591)
	0.96	(2, 7)	(7.51, 31.28)	(12.69, 27.16)	0.6375	0.7317	(10.8124, 28.2655)
	0.95	(2, 7)	(7.51, 31.28)	(12.69, 27.16)	0.7150	0.8881	(11.2137, 27.6209)

It is important to note that, based on this real data set and the results in Ahmadi and Arghami (2003), 96% confidence intervals for $\xi_{0.7}$ and $\xi_{0.9}$ are $(4.83, 21.96)$ and $(7.51, 27.16)$, respectively. It is clear that, our results are better in the sense of length of the intervals. So, Table 5 indicates that linear interpolation works well for confidence intervals.

Example 2. In this example, in order to use the obtained prediction intervals in Section 3, we consider the upper records of daily temperatures (in degrees Fahrenheit) recorded at the National Center of Atmospheric Research (NCAR) during the year 2005 (Earth System Research laboratory, U.S. Department of Commerce, www.cdc.noaa.gov/Boulder/data.daily.html), and the extracted upper records are tabulated in Table 6. This data set was applied by Raqab and Balakrishnan (2008) for constructing prediction intervals for upper records. Based on interpolated upper records in Table 6, approximate equal-tailed prediction intervals for future upper records were obtained with an approximate prediction coefficient $1 - \alpha = 0.90$. The results are presented in Table 7.

Table 6: Upper records from the temperature data.

n	1	2	3	4	5	6	7	8	9
Day	January 1	January 8	January 9	January 18	January 19	January 20	April 3	April 18	May 10
Records U_k	45	51	57	60	68	73	74	75	76
n	10	11	12	13	14	15	16	17	18
Day	May 16	May 19	May 20	July 6	July 7	July 8	July 16	July 19	July 21
Records U_k	78	83	90	91	94	96	97	99	101

Table 7: Approximate 90% equal-tailed prediction interval for future upper records based on data in Table 6.

i	r	s	γ_1	γ_2	$(U_r,\ U_s)$	$(U_{r+1},\ U_{s-1})$	$(U_r(\gamma_1), U_{s-1}(1-\gamma_2))$
6	1	13	0.9174	0.1071	(45, 91)	(51, 90)	(50.5000, 90.8900)
7	2	15	0.6158	0.5463	(51, 96)	(57, 94)	(54.7000, 94.9100)
8	2	16	0.9470	0.1935	(51, 97)	(57, 96)	(56.6800, 96.8100)
9	3	18	0.6954	0.6566	(57, 101)	(60, 99)	(59.0900, 99.6900)

In order to compare the presented results with those of Raqab and Balakrishnan (2008), we can see that our results are prediction intervals with shorter length. Thus, we can construct prediction intervals for future upper records with shorter length if we use interpolated upper records instead of upper records. In conclusion, linear interpolation is also beneficial for prediction intervals for future upper records.

5 Simulation study and comparisons

In this section, we have utilized a simulated set of record data in order to illustrate the procedures developed in this paper. For this purpose, we have generated upper and lower record data from the standard normal distribution. Each simulation consisted of 10000 replications. Confidence intervals for population quantiles for different values of p and $1 - \alpha$ are presented in Table 8. Similar results for the standard uniform and exponential distributions are given in Tables 9 and 10. It is evident from Tables 8, 9 and 10 that, utilizing the approach suggested in this paper leads to an improvement of coverage probability. Because this method provides confidence intervals with coverage probability approximately equals to the pre-fixed level $1 - \alpha$. The improvement is driven by using interpolated upper and lower records instead of upper and lower records, respectively. For prediction intervals for future records, simulated results for standard normal, uniform and exponential distributions are tabulated in Tables 11, 12 and 13, respectively, which reveal that in all cases, the coverage probabilities are close to the desired level $1 - \alpha$.

6 Conclusion

It this paper, a method was suggested for constructing confidence intervals for population quantiles and prediction intervals for future records based on observed record. In this method, we used interpolated upper and lower records instead of upper and lower records, respectively. Numerical results showed that this method performs well. Because the coverage probabilities were more close to $1 - \alpha$. More generally, one could obtain the results in terms of $\frac{\alpha}{k}$ and $1 - \frac{\alpha}{k}$ instead of $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$, respectively, for some $k > 0$.

Table 8: Confidence intervals for quantiles based on simulated upper and lower records from the standard normal distribution (nominal values in parentheses).

$1-\alpha$	p	λ_3	λ_2	CI	Tail probabilities below the lower limit	Tail probabilities above the upper limit	CP	Length of CI
0.90	0.5	0.3177	0.3177	$(L_3,\ U_3)$	0.0352	0.0352	0.9296	2.9577
				$(L_2,\ U_2)$	0.1563	0.1563	0.6874	1.7651
				Interpolated	0.0519 (0.05)	0.0519 (0.05)	0.8962 (0.90)	2.5788
	0.7	0.9987	0.3567	$(L_3,\ U_4)$	0.0060	0.0364	0.9576	3.4673
				$(L_2,\ U_3)$	0.0532	0.1242	0.8226	2.3903
				Interpolated	0.0530 (0.05)	0.0498 (0.05)	0.8972 (0.90)	2.6974
	0.9	0.8948	0.5591	$(L_2,\ U_6)$	0.0058	0.0301	0.9641	3.6379
				$(L_1,\ U_5)$	0.0999	0.0874	0.8127	2.3846
				Interpolated	0.0748 (0.05)	0.0528 (0.05)	0.8724 (0.90)	2.6341
0.95	0.5	0.9101	0.9101	$(L_4,\ U_4)$	0.0049	0.0049	0.9902	3.9266
				$(L_3,\ U_3)$	0.0317	0.0317	0.9366	2.9982
				Interpolated	0.0258 (0.025)	0.0258 (0.025)	0.9484 (0.95)	3.0816
	0.7	0.8095	0.8621	$(L_3,\ U_5)$	0.0070	0.0075	0.9855	3.8587
				$(L_2,\ U_4)$	0.0509	0.0360	0.9131	2.8712
				Interpolated	0.0292 (0.025)	0.0278 (0.025)	0.9430 (0.95)	3.0384
	0.9	0.7150	0.8881	$(L_2,\ U_7)$	0.0036	0.0099	0.9865	3.9441
				$(L_1,\ U_6)$	0.1025	0.0307	0.8668	2.7172
				Interpolated	0.0429 (0.025)	0.0255 (0.025)	0.9316 (0.95)	3.0112

Table 9: Confidence intervals for quantiles based on simulated upper and lower records from the standard uniform distribution (nominal values in parentheses).

$1-\alpha$	p	λ_3	λ_2	CI	Tail probabilities below the lower limit	Tail probabilities above the upper limit	CP	Length of CI
0.90	0.5	0.3177	0.3177	$(L_3,\ U_3)$	0.0349	0.0349	0.9302	0.7426
				$(L_2,\ U_2)$	0.1606	0.1606	0.6788	0.4890
				Interpolated	0.0520 (0.05)	0.0520 (0.05)	0.8960 (0.90)	0.6620
	0.7	0.9987	0.3567	$(L_3,\ U_4)$	0.0064	0.0372	0.9564	0.8088
				$(L_2,\ U_3)$	0.0490	0.1252	0.8258	0.6213
				Interpolated	0.0487 (0.05)	0.0570 (0.05)	0.8943 (0.90)	0.6622
	0.9	0.8948	0.5591	$(L_2,\ U_6)$	0.0049	0.0323	0.9628	0.7339
				$(L_1,\ U_5)$	0.0962	0.0871	0.8167	0.4706
				Interpolated	0.0486 (0.05)	0.0605 (0.05)	0.8909 (0.90)	0.5037
0.95	0.5	0.9101	0.9101	$(L_4,\ U_4)$	0.0059	0.0059	0.9882	0.8744
				$(L_3,\ U_3)$	0.0346	0.0346	0.9308	0.7493
				Interpolated	0.0302 (0.025)	0.0302 (0.025)	0.9396 (0.95)	0.7605
	0.7	0.8095	0.8621	$(L_3,\ U_5)$	0.0070	0.0076	0.9854	0.8424
				$(L_2,\ U_4)$	0.0506	0.0329	0.9165	0.6845
				Interpolated	0.0303 (0.025)	0.0272 (0.025)	0.9425 (0.95)	0.7129
	0.9	0.7150	0.8881	$(L_2,\ U_7)$	0.0048	0.0112	0.9840	0.7400
				$(L_1,\ U_6)$	0.1075	0.0330	0.8595	0.4796
				Interpolated	0.0172 (0.025)	0.0296 (0.025)	0.9532 (0.95)	0.5524

Table 10: Confidence intervals for quantiles based on simulated upper and lower records from the standard exponential distribution (nominal values in parentheses).

$1-\alpha$	p	λ_3	λ_2	CI	Tail probabilities below the lower limit	Tail probabilities above the upper limit	CP	Length of CI
0.90	0.5	0.3177	0.3177	$(L_3,\ U_3)$	0.0306	0.0306	0.9388	2.884849
				$(L_2,\ U_2)$	0.1472	0.1472	0.7056	1.668257
				Interpolated	0.0547 (0.05)	0.0413 (0.05)	0.9040 (0.90)	2.498338
	0.7	0.9987	0.3567	$(L_3,\ U_4)$	0.0053	0.0356	0.9591	3.82679
				$(L_2,\ U_3)$	0.0511	0.1245	0.8244	2.63577
				Interpolated	0.0510 (0.05)	0.0482 (0.05)	0.9008 (0.90)	3.271805
	0.9	0.8948	0.5591	$(L_2,\ U_6)$	0.0061	0.0327	0.9612	5.627995
				$(L_1,\ U_5)$	0.1055	0.0884	0.8061	3.974322
				Interpolated	0.0898 (0.05)	0.0518 (0.05)	0.8584 (0.90)	4.484135
0.95	0.5	0.9101	0.9101	$(L_4,\ U_4)$	0.0052	0.0052	0.9896	3.962896
				$(L_3,\ U_3)$	0.0298	0.0298	0.9404	2.86916
				Interpolated	0.0258 (0.025)	0.0216 (0.025)	0.9526 (0.95)	2.967487
	0.7	0.8095	0.8621	$(L_3,\ U_5)$	0.005	0.008	0.9870	4.847654
				$(L_2,\ U_4)$	0.0483	0.0309	0.9208	3.66581
				Interpolated	0.0344 (0.025)	0.0230 (0.025)	0.9426 (0.95)	3.839118
	0.9	0.7150	0.8881	$(L_2,\ U_7)$	0.0065	0.0082	0.9853	6.679745
				$(L_1,\ U_6)$	0.0968	0.0282	0.8750	5.030786
				Interpolated	0.0537 (0.025)	0.0231 (0.025)	0.9232 (0.95)	5.326247

Table 11: Prediction intervals for future upper records based on simulated upper records from the standard normal distribution (nominal values in parentheses).

$1-\alpha$	i	γ_1	γ_2	PI	Tail probabilities below the lower limit	Tail probabilities above the upper limit	CP	Length of PI
0.90	6	0.9174	0.1071	$(U_1,\ U_{13})$	0.0147	0.0507	0.9346	4.5313
				$(U_2,\ U_{12})$	0.0627	0.0739	0.8634	3.4011
				Interpolated	0.0531 (0.05)	0.0526 (0.05)	0.8943 (0.90)	3.6694
	7	0.6158	0.5463	$(U_2,\ U_{15})$	0.0348	0.0418	0.9234	4.0398
				$(U_3,\ U_{14})$	0.0893	0.0605	0.8502	3.2496
				Interpolated	0.0578 (0.05)	0.0515 (0.05)	0.8907 (0.90)	3.5670
	8	0.9470	0.1935	$(U_2,\ U_{16})$	0.0191	0.0491	0.9318	4.2172
				$(U_3,\ U_{15})$	0.0530	0.0699	0.8771	3.4338
				Interpolated	0.0481 (0.05)	0.0518 (0.05)	0.9001 (0.90)	3.6189
	9	0.6954	0.6566	$(U_3,\ U_{18})$	0.0327	0.0389	0.9284	∞
				$(U_4,\ U_{17})$	0.0721	0.0543	0.8736	∞
				Interpolated	0.0544 (0.05)	0.0479 (0.05)	0.8977 (0.90)	∞
	10	0.3981	0.3797	$(U_4,\ U_{19})$	0.0443	0.0421	0.9136	∞
				$(U_5,\ U_{18})$	0.0869	0.0618	0.8513	∞
				Interpolated	0.0560 (0.05)	0.0477 (0.05)	0.8963 (0.90)	∞
0.95	6	0.6596	0.4118	$(U_1,\ U_{15})$	0.0147	0.0208	0.9645	4.9268
				$(U_2,\ U_{14})$	0.0624	0.0303	0.9073	3.8350
				Interpolated	0.0332 (0.025)	0.0244(0.025)	0.9424 (0.95)	4.2552
	7	0.8929	0.7444	$(U_1,\ U_{17})$	0.0060	0.0169	0.9771	5.3353
				$(U_2,\ U_{16})$	0.0318	0.0256	0.9426	4.2480
				Interpolated	0.0240 (0.025)	0.0225 (0.025)	0.9535 (0.95)	4.3922
	8	0.5494	0.3523	$(U_2,\ U_{18})$	0.0190	0.0234	0.9576	∞
				$(U_3,\ U_{17})$	0.0568	0.0333	0.9099	∞
				Interpolated	0.0318 (0.025)	0.0261 (0.025)	0.9421 (0.95)	∞
	9	0.8621	0.7403	$(U_2,\ U_{20})$	0.0119	0.0157	0.9724	∞
				$(U_3,\ U_{19})$	0.0333	0.0258	0.9409	∞
				Interpolated	0.0278 (0.025)	0.0230 (0.025)	0.9492 (0.95)	∞
	10	0.5695	0.4048	$(U_3,\ U_{21})$	0.0200	0.0235	0.9565	∞
				$(U_4,\ U_{20})$	0.0474	0.0336	0.9190	∞
				Interpolated	0.0305 (0.025)	0.0263 (0.025)	0.9432 (0.95)	∞

Table 12: Prediction intervals for future upper records based on simulated upper records from the standard uniform distribution (nominal values in parentheses).

					Tail probabilities			
$1-\alpha$	i	γ_1	γ_2	PI	below the lower limit	above the upper limit	CP	Length of PI
0.90	6	0.9174	0.1071	$(U_1,\ U_{13})$	0.0152	0.0460	0.9388	0.4975
				$(U_2,\ U_{12})$	0.0611	0.0728	0.8661	0.2512
				Interpolated	0.0401 (0.05)	0.0500 (0.05)	0.9099 (0.90)	0.2717
	7	0.6158	0.5463	$(U_2,\ U_{15})$	0.0364	0.0441	0.9195	0.2496
				$(U_3,\ U_{14})$	0.0945	0.0610	0.8445	0.1232
				Interpolated	0.0506 (0.05)	0.0540 (0.05)	0.8954 (0.90)	0.1718
	8	0.9470	0.1935	$(U_2,\ U_{16})$	0.0192	0.0499	0.9309	0.2536
				$(U_3,\ U_{15})$	0.0570	0.0707	0.8723	0.1275
				Interpolated	0.0440 (0.05)	0.0557 (0.05)	0.9003 (0.90)	0.1342
	9	0.6954	0.6566	$(U_3,\ U_{18})$	0.0346	0.0369	0.9285	0.1245
				$(U_4,\ U_{17})$	0.0766	0.0531	0.8703	0.0622
				Interpolated	0.0482 (0.05)	0.0490 (0.05)	0.9028 (0.90)	0.0811
	10	0.3981	0.3797	$(U_4,\ U_{19})$	0.0461	0.0465	0.9074	0.0628
				$(U_5,\ U_{18})$	0.0914	0.0661	0.8425	0.0313
				Interpolated	0.0525 (0.05)	0.0567 (0.05)	0.8908 (0.90)	0.0503
0.95	6	0.6596	0.4118	$(U_1,\ U_{15})$	0.0169	0.0226	0.9605	0.5015
				$(U_2,\ U_{14})$	0.0645	0.0330	0.9025	0.2501
				Interpolated	0.0265 (0.025)	0.0269 (0.025)	0.9466 (0.95)	0.3357
	7	0.8929	0.7444	$(U_1,\ U_{17})$	0.0079	0.0185	0.9736	0.4991
				$(U_2,\ U_{16})$	0.0378	0.0271	0.9351	0.2496
				Interpolated	0.0202 (0.025)	0.0256 (0.025)	0.9542 (0.95)	0.2764
	8	0.5494	0.3523	$(U_2,\ U_{18})$	0.0191	0.0219	0.9590	0.2482
				$(U_3,\ U_{17})$	0.0518	0.0328	0.9154	0.1240
				Interpolated	0.0258 (0.025)	0.0268 (0.025)	0.9474 (0.95)	0.1800
	9	0.8621	0.7403	$(U_2,\ U_{20})$	0.0111	0.0172	0.9717	0.2478
				$(U_3,\ U_{19})$	0.0336	0.0264	0.9400	0.1248
				Interpolated	0.0208 (0.025)	0.0246 (0.025)	0.9546 (0.95)	0.1418
	10	0.5695	0.4048	$(U_3,\ U_{21})$	0.0186	0.0221	0.9593	0.1260
				$(U_4,\ U_{20})$	0.0440	0.0299	0.9261	0.0633
				Interpolated	0.0245 (0.025)	0.0255 (0.025)	0.9500 (0.95)	0.0903

Table 13: Prediction intervals for future upper records based on simulated upper records from the standard exponential distribution (nominal values in parentheses).

					Tail probabilities			
$1-\alpha$	i	γ_1	γ_2	PI	below the lower limit	above the upper limit	CP	Length of PI
0.90	6	0.9174	0.1071	$(U_1,\ U_{13})$	0.0166	0.0495	0.9339	11.97769
				$(U_2,\ U_{12})$	0.0645	0.0732	0.8623	9.975046
				Interpolated	0.0584 (0.05)	0.0505 (0.05)	0.8911 (0.90)	10.94984
	7	0.6158	0.5463	$(U_2,\ U_{15})$	0.0370	0.0373	0.9257	13.02129
				$(U_3,\ U_{14})$	0.0928	0.0561	0.8511	11.03152
				Interpolated	0.0643 (0.05)	0.0462 (0.05)	0.8895 (0.90)	11.8646
	8	0.9470	0.1935	$(U_2,\ U_{16})$	0.0218	0.0458	0.9324	13.96929
				$(U_3,\ U_{15})$	0.0570	0.0648	0.8782	11.95173
				Interpolated	0.0544 (0.05)	0.0489 (0.05)	0.8967 (0.90)	12.81332
	9	0.6954	0.6566	$(U_3,\ U_{18})$	0.0319	0.0398	0.9283	15.0086
				$(U_4,\ U_{17})$	0.0725	0.0580	0.8695	13.0096
				Interpolated	0.0562 (0.05)	0.0487 (0.05)	0.8951 (0.90)	13.6572
	10	0.3981	0.3797	$(U_4,\ U_{19})$	0.0479	0.0458	0.9063	∞
				$(U_5,\ U_{18})$	0.0900	0.0648	0.8452	∞
				Interpolated	0.0598 (0.05)	0.0522 (0.05)	0.8880 (0.90)	∞
0.95	6	0.6596	0.4118	$(U_1,\ U_{15})$	0.0156	0.0215	0.9629	14.04339
				$(U_2,\ U_{14})$	0.0617	0.0333	0.9050	12.03782
				Interpolated	0.0378 (0.025)	0.0256 (0.025)	0.9366 (0.95)	12.96966
	7	0.8929	0.7444	$(U_1,\ U_{17})$	0.0087	0.0177	0.9736	15.97693
				$(U_2,\ U_{16})$	0.0359	0.0271	0.9370	13.99326
				Interpolated	0.0308 (0.025)	0.0237 (0.025)	0.9455 (0.95)	14.35305
	8	0.5494	0.3523	$(U_2,\ U_{18})$	0.0194	0.0207	0.9599	∞
				$(U_3,\ U_{17})$	0.0543	0.0313	0.9144	13.96848
				Interpolated	0.0327 (0.025)	0.0230 (0.025)	0.9443 (0.95)	∞
	9	0.8621	0.7403	$(U_2,\ U_{20})$	0.0111	0.0153	0.9736	∞
				$(U_3,\ U_{19})$	0.0336	0.0237	0.9427	∞
				Interpolated	0.0282 (0.025)	0.0210 (0.025)	0.9508 (0.95)	∞
	10	0.5695	0.4048	$(U_3,\ U_{21})$	0.0180	0.0224	0.9596	∞
				$(U_4,\ U_{20})$	0.0445	0.0326	0.9229	∞
				Interpolated	0.0270 (0.025)	0.0258 (0.025)	0.9472 (0.95)	∞

Also, this method can be useful for deriving prediction and confidence intervals based on other ordered data, such as censored data.

Acknowledgements

The authors are grateful to the two anonymous referees for making many helpful comments and suggestions on an earlier version of this paper.

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