On Type-II Progressively Hybrid Censoring

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Abstract

Progressive Type-II censoring scheme has become quite popular for the last few years. One of the major drawbacks of the progressive censoring scheme is that the length of the experiment can be very large if the items are highly reliable. Because of that, recently Kundu and Joarder (*Computational Statistics and Data Analysis*, 50, 2509-2528, 2006) introduced the Type-II progressively hybrid censored scheme and analyzed the data assuming that the lifetimes of the items are exponentially distributed. This article presents the analysis of the Type-II progressively hybrid censored data when the lifetime distributions of the items follow Weibull distributions. Maximum likelihood estimators and approximate maximum likelihood estimators are developed for estimating the unknown parameters. Asymptotic confidence intervals based on maximum likelihood estimators and approximate maximum likelihood estimators are proposed. Different methods have been compared using Monte Carlo simulations. One real data set has been analyzed for illustrative purposes.

Keywords: Maximum likelihood estimators; Approximate maximum likelihood estimators; Type-I censoring; Type-II censoring; Monte Carlo simulation.

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1 Introduction

The Type-II progressive censoring scheme has become very popular recently. It can be described as follows. Consider n units in a study and suppose $m < n$ is fixed before the experiment. Moreover, m other integers, $R_1, \ldots, R_m$ are also fixed before hand so that $R_1 + \ldots + R_m + m = n$. At the time of the first failure, say $Y_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure, say $Y_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Finally, at the time of the $m - th$ failure, say $Y_{m:m:n}$, the rest of the R_m units are removed. Extensive work has been done on this particular scheme during the last ten years. See for example the book by Balakrishnan and Aggarwala [3] and also the review article by Balakrishnan [2] for an exhaustive list of references on this particular topic.

Unfortunately the major problem about the Type-II progressive censoring scheme is that the time length of the experiment can be very large. Because of that problem, Kundu and Joarder [9] introduced a new censoring scheme named as Type-II progressively hybrid censoring scheme, which ensures that the length of the experiment can not exceed a prespecified time point T. The detailed description and the advantages of the Type-II progressively hybrid censoring can be obtained in Kundu and Joarder [9], see also Childs, Chandrasekar and Balakrishnan [5]. In Kundu and Joarder [9] and also in Childs, Chandrasekar and Balakrishnan [5], the authors assumed the lifetime distributions of the items to be exponential.

Since the exponential distribution has its limitations, in this article we consider the Type-II progressively hybrid censored lifetime data, when the lifetime follows two-parameter Weibull distribution. We provide the maximum likelihood estimators (MLEs) of the unknown parameters. It is observed that the MLEs can not be obtained in explicit forms. They can be obtained by solving a non-linear equation and we propose a simple iterative scheme to solve the non-linear equation. We also suggest approximate maximum likelihood estimators (AMLEs), which have explicit expressions. It is not possible to compute the exact distributions of the MLEs, and we use the asymptotic distribution to construct confidence intervals. Monte Carlo simulations are used to compare different methods and one data analysis is performed for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we describe the model. The MLEs and AMLE of the unknown parameter are provided in Section 3 and Section 4 respectively. Numerical results are presented in Section 5. One real data set has been analyzed in Section 6 and finally we conclude the paper in Section 7.

2 Model Description and Notations

Suppose the lifetime random variable Y has a Weibull distribution with the shape and scale parameters as α and λ respectively, *i.e.*, the probability density function (PDF) of Y is;

$$f_Y(y;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-(\frac{y}{\lambda})^{\alpha}}; \qquad y > 0, \tag{1}$$

where $\alpha > 0$, $\lambda > 0$ are the natural parameter space. If the random variable Y has the density function (1), then $X = \ln Y$ has the extreme value distribution with PDF;

$$f_X(x;\mu,\sigma) = \frac{1}{\sigma} e^{\left(\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}}\right)}; \qquad -\infty < x < \infty, \tag{2}$$

where $\mu = \ln\lambda$, $\sigma = \frac{1}{\alpha}$. The density function as described by (2) is known as the density function of an extreme value distribution with location and scale parameter as μ and σ respectively. Models (1) and (2) are equivalent models in the sense, the procedure developed under one model can be easily used for the other model. Although, they are equivalent models, sometimes it is easier to work with the model (2) than the model (1), because in the model (2), the two parameters μ and σ appear as location and scale parameters. For $\mu = 0$

and $\sigma = 1$, the model (2) is known as the standard extreme value distribution and it has the following PDF;

$$f_Z(z; 0, 1) = e^{z-e^z}; \qquad -\infty < z < \infty. \tag{3}$$

Now we describe the data available under the Type-II progressively hybrid censoring scheme. Note that under this Type-II progressively hybrid censoring scheme, it is assumed that n identical items are put on a test and the lifetime distributions of the n items are denoted by $Y_1, \ldots, Y_n$. The integer $m < n$ is pre-fixed and also $R_1, \ldots, R_m$ are m pre-fixed integers satisfying $R_1 + \ldots + R_m + m = n$. T is a pre-fixed time point. At the time of first failure $Y_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly at the time of the second failure $Y_{2:m:n}$, R_2 of the remaining units are removed and so on. If the $m - th$ failure $Y_{m:m:n}$ occurs before the time point T, the experiment stops at the time point $Y_{m:m:n}$. On the other hand suppose the $m - th$ failure does not occur before time point T and only J failures occur before the time point T, where $0 \leq J < m$, then at the time point T all the remaining R_J^* units are removed and the experiment terminates at the time point T. Note that $R_J^* = n-(R_1+\ldots+R_J)-J$. We denote the two cases as Case I and Case II respectively and call this censoring scheme as the Type-II progressively hybrid censoring scheme. See Kundu and Joarder [9] for details.

Therefore, in presence of Type-II progressively hybrid censoring scheme, we have one of the following types of observations;

$$\text{Case I:} \qquad \{Y_{1:m:n}, \ldots, Y_{m:m:n}\}; \quad \text{if} \quad Y_{m:m:n} < T, \qquad \text{or} \tag{4}$$

$$\text{Case II:} \qquad \{Y_{1:m:n}, \ldots, Y_{J:m:n}\}; \quad \text{if} \quad Y_{J:m:n} < T < Y_{J+1:m:n}. \tag{5}$$

For Case II, it may be mentioned that although we do not observe $Y_{J+1:m:n}$, but $Y_{J:m:n} < T < Y_{J+1:m:n}$ means that the J-th failure took place before T and no failure took place between $Y_{J:m:n}$ and T *i.e.* $Y_{J+1:m:n}, \ldots, Y_{m:m:n}$ are not observed.

The conventional Type-I progressive censoring scheme needs the pre-specification of $R_1, \ldots, R_m$ and also $T_1, \ldots, T_m$, see Cohen [6, 7] for details. The choices of $T_1, \ldots, T_m$ are not trivial. For the conventional Type-II progressive censoring scheme the experimental time is unbounded. In our proposed censoring scheme, the choice of T depends how much maximum experimental time the experimenter can afford to continue. Moreover, the experimental time is bounded.

3 Maximum Likelihood Estimator

In this section we provide the maximum likelihood estimators of the unknown parameters. Based on the observed data, the likelihood function for Case I is

$$l(\alpha, \lambda) = K_1 \left(\frac{\alpha}{\lambda}\right)^m \prod_{i=1}^{m} \left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha-1} e^{-\left[\sum_{i=1}^{m}(1+R_i)\left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha}\right]}, \tag{6}$$

and for Case II, it is

$$l(\alpha, \lambda) = K_2 \left(\frac{\alpha}{\lambda}\right)^J \prod_{i=1}^{J} \left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha-1} e^{-\left[\sum_{i=1}^{J}(1+R_i)\left(\frac{y_{i:m:n}}{\lambda}\right)^{\alpha} + R_J^*\left(\frac{T}{\lambda}\right)^{\alpha}\right]} \quad \text{if} \quad J > 0, \tag{7}$$

$$= e^{-n\left(\frac{T}{\lambda}\right)^{\alpha}} \quad \text{if} \quad J = 0, \tag{8}$$

where $K_1 = \prod_{i=1}^{m}\left[n - \sum_{k=1}^{i-1}(1+R_k)\right]$ and $K_2 = \prod_{i=1}^{J}\left[n - \sum_{k=1}^{i-1}(1+R_k)\right]$ both are constant.

The logarithm of (6) and (7), can be written (without the constant terms) as

$$L(\alpha, \lambda) = d(\ln\alpha - \ln\lambda) + (\alpha - 1)\left[\sum_{i=1}^{d} \ln y_{i:m:n} - d\ln\lambda\right] - \frac{1}{\lambda^{\alpha}} W. \tag{9}$$

Here $d = m$, $W(\alpha) = \sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}$ and $d = J$, $W(\alpha) = \sum_{i=1}^{J}(1+R_i)y_{i:m:n}^{\alpha} + R_J^* T^{\alpha}$ for Case -I and Case-II respectively. It is assumed that $d > 0$, otherwise the MLEs do not exist. Taking derivatives with respect to α and λ of (9) and equating them to zero we obtain;

$$\frac{\partial L(\alpha, \lambda)}{\partial \lambda} = -\frac{d\alpha}{\lambda} + \frac{\alpha}{\lambda^{\alpha+1}} W(\alpha) = 0, \tag{10}$$

$$\frac{\partial L(\alpha, \lambda)}{\partial \alpha} = \frac{d}{\alpha} + \sum_{i=1}^{d} \ln y_{i:m:n} - d\ln\lambda - \frac{1}{\lambda^{\alpha}} V(\alpha) + \frac{1}{\lambda^{\alpha}} W(\alpha)\ln\lambda = 0. \tag{11}$$

Here $V(\alpha) = \sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n}$ and $V(\alpha) = \sum_{i=1}^{J}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n} + R_J^{*}T^{\alpha}\ln T$ for Case -I and Case-II respectively. Note that

$$\lambda^{\alpha} = \frac{W(\alpha)}{d} = u(\alpha) \quad \text{(say)} \tag{12}$$

and the MLE of α can be obtained by solving

$$\alpha = h(\alpha), \tag{13}$$

where

$$h(\alpha) = \frac{d}{-\sum_{i=1}^{d}\ln y_{i:m:n} + \frac{1}{u(\alpha)}W}.$$

We propose a simple iterative scheme to obtain the MLE of α from (13). Start with an initial guess of α, say $\alpha^{(0)}$, obtain $\alpha^{(1)} = h(\alpha^{(0)})$ and proceeding in this way obtain $\alpha^{(n+1)} = h(\alpha^{(n)})$. Stop the iterative procedure, when $\left|\alpha^{(n+1)} - \alpha^{(n)}\right| < \epsilon$, some pre-assigned tolerance limit. Once the MLE of α is obtained the MLE of λ can be easily obtained from (12). Since the MLE's when they exist, are not in compact forms, we propose the following approximate MLE's and then have explicit expressions.

4 Approximate Maximum Likelihood Estimators

Let us use the following notations; $x_{i:m:n} = \ln y_{i:m:n}$ and $S = \ln T$. Therefore, the likelihood equation of the observed data $x_{i:m:n}$ for Case-I is

$$l(\mu,\sigma) = \frac{1}{\sigma^m}\prod_{i=1}^{m}\left[n - \sum_{k=1}^{i-1}(1+R_k)\right] g(z_{i:m:n})\left(\bar{G}(z_{i:m:n})\right)^{R_i}, \tag{14}$$

and for Case II, it is

$$l(\mu,\sigma) = \frac{1}{\sigma^J}\prod_{i=1}^{J}\left[n - \sum_{k=1}^{i-1}(1+R_k)\right] g(z_{i:m:n})\left(\bar{G}(z_{i:m:n})\right)^{R_i}\left(\bar{G}(V)\right)^{R_J^{*}} \tag{15}$$

where $z_{i:m:n} = \dfrac{(x_{i:m:n} - \mu)}{\sigma}$, $V = \frac{S-\mu}{\sigma}$, $g(x) = e^{x-e^x}$, $\bar{G}(x) = e^{-e^x}$, $\mu = ln\lambda$ and $\sigma = \dfrac{1}{\alpha}$. Ignoring the constant term, we obtain the following log-likelihood from (15);

$$L(\mu,\sigma) = \ln\left[l(\mu,\sigma)\right] \quad = \quad -m\ln\sigma + \sum_{i=1}^{m}\ln\left(g(z_{i:m:n})\right) + \sum_{i=1}^{m} R_i \ln\left(\bar{G}(z_{i:m:n})\right). \qquad (16)$$

From (16) we obtain the following approximate MLE's of μ and σ (see Appendix - 1),

$$\tilde{\mu} = \frac{(c_1 - c_2 - m)\tilde{\sigma} + d_1}{c_1}, \quad \tilde{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \qquad (17)$$

where $c_1 = \sum_{i=1}^{m} D_i e^{\mu_i}$, $c_2 = \sum_{i=1}^{m} D_i \mu_i e^{\mu_i}$, $d_1 = \sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i}$, $d_2 = \sum_{i=1}^{m} D_i X^2_{i:m:n} e^{\mu_i}$, $d_3 = \sum_{i=1}^{m} D_i \mu_i X_{i:m:n} e^{\mu_i}$, $A = mc_1$, $B = c_1(d_3 + m\bar{X}) - d_1(c_2 + m)$, $C = d_1^2 - c_1 d_2$, $\mu_i = G^{-1}(p_i) = \ln(-\ln q_i)$, $p_i = \frac{i}{n+1}$, $q_i = 1 - p_i$ and $D_i = 1 + R_i$ for $i = 1, \ldots, m$. Now, for Case-II, ignoring the constant term, we obtain the log-likelihood as

$$L(\mu,\sigma) = \ln\left[l(\mu,\sigma)\right] = -J\ln\sigma + \sum_{i=1}^{J}\ln\left(g(z_{i:m:n})\right) + \sum_{i=1}^{J} R_i \ln\left(\bar{G}(z_{i:m:n})\right) + R_J^* \ln \bar{G}(V). \quad (18)$$

In this case the approximate MLE's are (see Appendix - 2),

$$\tilde{\mu} = \frac{(c_1^{'} - c_2^{'} - J)\tilde{\sigma} + d_1^{'}}{c_1^{'}}, \quad \tilde{\sigma} = \frac{-B^{'} + \sqrt{{B^{'}}^2 - 4A^{'}C^{'}}}{2A^{'}}, \qquad (19)$$

where $c_1^{'} = \sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}$, $c_2^{'} = \sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*}$, $d_1^{'} = \sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*}$, $d_2^{'} = \sum_{i=1}^{J} D_i X^2_{i:m:n} e^{\mu_i} + R_J^* S^2 e^{\mu_J^*}$, $d_3^{'} = \sum_{i=1}^{J} D_i \mu_i X_{i:m:n} e^{\mu_i} + R_J^* \mu_J^* S e^{\mu_J^*}$, $A^{'} = Jc_1^{'}$, $B^{'} = c_1^{'}(d_3^{'} + J\bar{X}) - d_1^{'}(c_2^{'} + J)$, $C^{'} = {d_1^{'}}^2 - c_1^{'} d_2^{'}$. Here μ_i and D_i are same as before for $i = 1, \ldots, J$, $\mu_J^* = G^{-1}(p_J^*) = \ln(-\ln q_J^*)$, $p_J^* = (p_J + p_{J+1})/2$ and $q_J^* = 1 - p_J^*$.

5 Numerical Results and Discussions

Since the performance of the different methods can not be compared theoretically, we present Monte Carlo simulation in this section to compare the performances of the different methods proposed in the previous sections for different parameter values and for different sampling schemes. The term *different sampling schemes* means different sets of R_i's and for different T values. We mainly compare the performances of the MLEs and AMLEs estimators of the unknown parameters, in terms of their biases and mean squared errors (MSEs) for different censoring schemes. We also compare the average lengths of the asymptotic confidence intervals and their coverage percentages. All the computations are performed in the Pentium IV processor using FORTRAN-77 program. In all cases we use the random deviate generator RAN2 proposed in Press et al. [13]. Since λ is the scale parameter, we have taken in all cases $\lambda = 1$ with out loss of generality. For simulation purposes, we present the results when T is of the form $T^{\frac{1}{\alpha}}$. The reason to choose T in that form is the following: if $\hat{\alpha}$ represents the MLE or AMLE of α, then the distribution of $\frac{\hat{\alpha}}{\alpha}$ becomes independent of α in that case for $\lambda = 1$. For that purpose we report the result only for $\alpha = 1$ without loss of generality. But these results can be used for any other α also.

Before progressing further, first we describe how we generate Type-II progressively hybrid censored data for a given set n, m, $R_1, \ldots, R_m$ and T. We use the following transformation for exponential distribution suggested in Balakrishnan and Aggarwala [3].

$$
\begin{aligned}
Z_1 &= nE_{1:m:n} \\
Z_2 &= (n - R_1 - 1)(E_{2:m:n} - E_{1:m:n}) \\
&\vdots \\
Z_m &= (n - R_1 - \ldots - R_{m-1} - m + 1)(E_{m:m:n} - E_{m-1:m:n}).
\end{aligned}
\tag{20}
$$

It is known that if E_i's are $i.i.d$ standard exponential, then the spacings Z_i's are also $i.i.d$ standard exponential random variables. From (20) it follows that

$$
\begin{aligned}
E_{1:m:n} &= \frac{1}{n} Z_1 \\
E_{2:m:n} &= \frac{1}{n - R_1 - 1} Z_2 + \frac{1}{n} Z_1
\end{aligned}
$$

$$\vdots$$

$$E_{m:m:n} = \frac{1}{n - R_1 - \ldots - R_{m-1} - m + 1} Z_m + \ldots + \frac{1}{n} Z_1. \tag{21}$$

Using (21) and parameter α and λ, Type-II progressively hybrid censored data for Weibull distribution can be easily generated. For a given n, m, $R_1, \ldots, R_m$, we generate $Y_{1:m:n}, \ldots, Y_{m:m:n}$. If $Y_{m:m:n} < T$, then we have Case I and the corresponding sample is $\{(Y_{1:m:n}, R_1), \ldots, (Y_{m:m:n}, R_m)\}$. If $Y_{m:m:n} > T$, then we have Case II and we find J, such that $Y_{J:m:n} < T < Y_{J+1:m:n}$. The corresponding Type-II hybrid censored sample is $\{(Y_{1:m:n}, R_1), \ldots, (Y_{J:m:n}, R_J)\}$ and R_J^*, where R_J^* is same as defined before.

We consider different n, m and T. We have used two different sampling schemes, namely: Scheme 1: $R_1 = \ldots = R_{m-1} = 0$ and $R_m = n - m$. Scheme 2: $R_1 = \ldots = R_{m-1} = 1$ and $R_m = n - 2m + 1$. Note that Scheme 1 is the usual Type-II censoring scheme and Scheme 2, is a typical progressive censoring scheme. In each case we compute the MLEs and AMLEs estimates of the unknown parameters. We compute the 95% asymptotic condence intervals based on MLEs and replacing the MLEs by AMLEs. We replicate the process 1000 times and report the average estimates, the MSEs, the average confidence lengths and coverage percentages. The results are reported in the following Tables 1 - 8.

From Table 1-4, the following observations are made for MLE and from Table 5-8 the corresponding observations for AMLE. For fixed n as m increases the biases and the MSEs decrease for both α and λ as expected. But for fixed m as n increases it may not be true. It shows that the effective sample size (m) plays an important role than the actual sample size (n). It is also observed that the MLEs for schemes 1 and 2 behave quite similarly in terms of biases and MSEs unless both n and m are small. The performances in terms of biases and MSEs improve as T increases. Similar results are observed for AMLEs also.

Now we compare different confidence intervals in terms of their average lengths and coverage probabilities. In general it is observed that both the methods work well even for small n and m. For both the methods, it is observed that the average confidence lengths decrease as n increases for fixed m or the other way. For both MLE and AMLE methods, scheme 1 and scheme 2 behave very similarly although the confidence intervals for scheme 1 are usually slightly shorter than scheme 2.

6 Data Analysis

Kundu and Joarder [9] analyzed the following two data sets using exponential distributions. They were obtained from Lawless [11]. They are presented below.

Example 1: In this case $n = 36$ and we take $m = 10$, $T = 2600$, $R_1 = R_2 = \ldots = R_9 = 2$, $R_{10} = 8$. Thus the Type II progressively hybrid censored sample is : 11, 35, 49, 170, 329, 958, 1925, 2223, 2400, 2568. From the above sample data, we obtain $D = m = 10$ which yields $\hat{\alpha}$ and $\hat{\lambda}$ based on Maximum Likelihood Estimates (MLEs) and Approximate Maximum Likelihood Estimates (AMLEs) are

$$(\hat{\alpha} = 6.29773 \times 10^{-1}, \hat{\lambda} = 8113.80176), \quad (\tilde{\alpha} = 6.33116 \times 10^{-1}, \tilde{\lambda} = 6511.83036)$$

respectively. Using the above estimates we obtain the 95% asymptotic confidence interval for α and λ based on Maximum Likelihood Estimates (MLEs) and Approximate Maximum Likelihood Estimates (AMLEs) which are

$$(6.29773 \times 10^{-1}, 6.29882 \times 10^{-1}), (8113.40869, 8114.19482)$$

and

$$(6.33116 \times 10^{-1}, 6.33176 \times 10^{-1}), (6511.4344, 6512.2264)$$

respectively.

EXAMPLE 2: Now consider $m = 10$ and $T = 2000$ and R_i's are same as before. In this case the progressively hybrid censored sample obtained as : 11, 35, 49, 170, 329, 958, 1925, and $D = J = 7$. The MLE and AMLEs of α and λ are

$$(\hat{\alpha} = 4.77441 \times 10^{-1}, \hat{\lambda} = 25148.8613) \quad (\tilde{\alpha} = 4.77589 \times 10^{-1}, \tilde{\lambda} = 23092.3759)$$

respectively. From the above estimates we obtain again the 95% asymptotic confidence interval for α and λ based on MLEs and AMLEs and they are

$$(4.77383 \times 10^{-1}, 4.77499 \times 10^{-1}), \qquad (25148.5078, 25149.2148)$$

and

$$(4.77529 \times 10^{-1}, 4.77649 \times 10^{-1}), \qquad (23092.0219, 23092.7299)$$

respectively. In both the cases it is clear that if we want to test $H_0 : \alpha = 1$, then it will be rejected. Therefore, it implies that Weibull distribution should be used rather than exponential distribution, in this case.

7 Conclusions

In this paper we discuss the the Type-II progressively hybrid censored data for the two parameters Weibull distribution. It is observed that the maximum likelihood estimator of the shape parameter can be obtained by using an iterative procedure. The proposed approximate maximum likelihood estimators of the shape and scale parameters can be obtained in explicit forms. Although we could not construct the exact confidence intervals but it is observed that the asymptotic confidence intervals work reasonably well at least for MLEs. Although we have used the frequentest approach but Bayes estimates and credible intervals also can be obtained under suitable priors along the same line as Kundu [10]. Work is in progress and it will be reported elsewhere.

Appendix - 1

For case-I, taking derivatives with respect to μ and σ of $L(\mu, \sigma)$ defined in (16), gives

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{i=1}^{m} R_i \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{m} \frac{g^{'}(z_{i:m:n})}{g(z_{i:m:n})}\right] = 0 \tag{22}$$

$$\frac{\partial L(\mu,\sigma)}{\partial \sigma} = \frac{1}{\sigma}\left[\sum_{i=1}^{m} R_i z_{i:m:n} \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{m} z_{i:m:n}\frac{g^{'}(z_{i:m:n})}{g(z_{i:m:n})} - m\right] = 0 \tag{23}$$

Clearly, (22) and (23) do not have explicit analytical solutions. We consider a first-order Taylor approximation to $g^{'}(z_{i:m:n})/g(z_{i:m:n})$ and $g(z_{i:m:n})/\bar{G}(z_{i:m:n})$ by expanding around the actual mean μ_i of the standardized order statistic $Z_{i:m:n}$, where $\mu_i = G^{-1}(p_i) = ln(-lnq_i)$, $p_i = \frac{i}{n+1}$, $q_i = 1 - p_i$ for $i = 1, \ldots, m$, similar as Balakrishnan and Varadan [4], see for reasoning David [8] or Arnold and Balakrishnan [1]. Otherwise, the necessary procedures for obtaining μ_i, $i = 1, \ldots, m$, were made available by Mann [12] and Thomas and Wilson [14]. Note that for $i = 1, \ldots, m$

$$\frac{g^{'}(z_{i:m:n})}{g(z_{i:m:n})} \approx \alpha_i - \beta_i z_{i:m:n} \tag{24}$$

$$\frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} \approx 1 - \alpha_i + \beta_i z_{i:m:n} \tag{25}$$

where

$$\alpha_i = \frac{g^{'}(\mu_i)}{g(\mu_i)} - \mu_i\left[\frac{g^{''}(\mu_i)}{g^{'}(\mu_i)} - \left(\frac{g^{'}(\mu_i)}{g(\mu_i)}\right)^2\right] = 1 + lnq_i(1 - ln(-lnq_i)),$$

$$\beta_i = \left[-\frac{g^{''}(\mu_i)}{g^{'}(\mu_i)} + \left(\frac{g^{'}(\mu_i)}{g(\mu_i)}\right)^2\right] = -lnq_i$$

Using the approximation (24) and (25) in (22) and (23), we obtain

$$\left[\sum_{i=1}^{m} D_i e^{\mu_i} - \sum_{i=1}^{m} D_i \mu_i e^{\mu_i} - m\right]\sigma + \sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i} - \mu \sum_{i=1}^{m} D_i e^{\mu_i} = 0 \tag{26}$$

and

$$\left[m\sum_{i=1}^{m} D_i e^{\mu_i}\right]\sigma^2 + \left[\sum_{i=1}^{m} D_i e^{\mu_i}\left(\sum_{i=1}^{m} D_i \mu_i X_{i:m:n} e^{\mu_i} + m\bar{X}\right)\right]\sigma + \left[\sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i}\right]^2$$

$$-\left[\sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i}\left(\sum_{i=1}^{m} D_i \mu_i e^{\mu_i}+m\right)\right]\sigma-\left[\sum_{i=1}^{m} D_i e^{\mu_i}\right]\left[\sum_{i=1}^{m} D_i X_{i:m:n}^2 e^{\mu_i}\right]=0 \quad (27)$$

The above two equations (26) and (27) can be written as

$$(c_1 - c_2 - m)\sigma + d_1 - \mu c_1 = 0 \quad (28)$$

$$A\sigma^2 + B\sigma + C = 0 \quad (29)$$

where $c_1 = \sum_{i=1}^{m} D_i e^{\mu_i}$, $c_2 = \sum_{i=1}^{m} D_i \mu_i e^{\mu_i}$, $d_1 = \sum_{i=1}^{m} D_i X_{i:m:n} e^{\mu_i}$, $d_2 = \sum_{i=1}^{m} D_i X_{i:m:n}^2 e^{\mu_i}$, $d_3 = \sum_{i=1}^{m} D_i \mu_i X_{i:m:n} e^{\mu_i}$, $A = mc_1$, $B = c_1(d_3 + m\bar{X}) - d_1(c_2 + m)$, $C = d_1^2 - c_1 d_2$ and $D_i = 1 + R_i$, for $i = 1, \ldots, m$. The solution to the preceding equations yields the approximate MLE's are

$$\hat{\mu} = \frac{(c_1 - c_2 - m)\hat{\sigma} + d_1}{c_1} \quad (30)$$

$$\hat{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad (31)$$

Here we consider only positive root of σ. It is easily seen that these approximate estimators are equivalent but not unbiased. Unfortunately, it is not possible to compute the exact bias of $\hat{\mu}$ and $\hat{\sigma}$ theoretically because of intractability encountered in finding the expectation of $\sqrt{B^2 - 4AC}$.

Appendix - 2

For case-II, taking derivatives with respect to μ and σ of $L(\mu, \sigma)$ defined in (18), gives (similarly as Case-I)

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \frac{1}{\sigma}\left[\sum_{i=1}^{J} R_i \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{J} \frac{g^{'}(z_{i:m:n})}{g(z_{i:m:n})} + R_J^* \frac{g(V)}{\bar{G}(V)}\right] = 0 \quad (32)$$

$$\frac{\partial L(\mu,\sigma)}{\partial \sigma} = \frac{1}{\sigma}\left[\sum_{i=1}^{J} R_i z_{i:m:n} \frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})} - \sum_{i=1}^{J} z_{i:m:n} \frac{g^{'}(z_{i:m:n})}{g(z_{i:m:n})} + R_J^* V \frac{g(V)}{\bar{G}(V)} - J\right] = 0 \quad (33)$$

Here we again consider the first-order Taylor approximation to $g'(z_{i:m:n})/g(z_{i:m:n})$ and $g(z_{i:m:n})/\bar{G}(z_{i:m:n})$ by expanding around the actual mean μ_i of the standardized order statistic $Z_{i:m:n}$, where μ_i's are defined in Appendix - 1. Here we also expand $g(V)/\bar{G}(V)$ in Taylor series around the point μ_J^*, where $\mu_J^* = G^{-1}(p_J^*) = ln(-lnq_J^*)$, $p_J^* = (p_J + p_{J+1})/2$ and $q_J^* = 1 - p_J^*$. Note that

$$\begin{aligned} \frac{g'(V)}{g(V)} &\approx \alpha_J^* - \beta_J^* V && (34) \\ \frac{g(V)}{\bar{G}(V)} &\approx 1 - \alpha_J^* + \beta_J^* V && (35) \end{aligned}$$

where

$$\begin{aligned} \alpha_J^* &= \frac{g'(\mu_J^*)}{g(\mu_J^*)} - \mu_J^* \left[\frac{g''(\mu_J^*)}{g'(\mu_J^*)} - \left(\frac{g'(\mu_J^*)}{g(\mu_J^*)} \right)^2 \right] = 1 + lnq_J^*(1 - ln(-lnq_J^*)), \\ \beta_J^* &= \left[-\frac{g''(\mu_J^*)}{g'(\mu_J^*)} + \left(\frac{g'(\mu_J^*)}{g(\mu_J^*)} \right)^2 \right] = -lnq_J^* \end{aligned}$$

Using the approximation (24), (25), (34) and (35) in (32) and (33), we obtain

$$\left[\left(\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*} \right) - \left(\sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*} \right) - J \right] \sigma$$
$$+ \left[\sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*} \right] - \mu \left[\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*} \right] = 0 \qquad (36)$$

and

$$\left[J \left(\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*} \right) \right] \sigma^2 + \left[\left(\sum_{i=1}^{J} D_i e^{\mu_i} + + R_J^* e^{\mu_J^*} \right) \left(\sum_{i=1}^{J} D_i \mu_i X_{i:m:n} e^{\mu_i} + R_J^* \mu_J^* S e^{\mu_J^*} + J\bar{X} \right) \right] \sigma$$
$$- \left[\left(\sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*} \right) \left(\sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*} + J \right) \right] \sigma + \left[\sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*} \right]^2$$
$$- \left[\sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*} \right] \left[\sum_{i=1}^{m} D_i X_{i:m:n}^2 e^{\mu_i} + R_J^* S^2 e^{\mu_J^*} \right] = 0 \qquad (37)$$

The above two equations (36) and (37) can be written as

$$\begin{aligned} (c_1' - c_2' - J)\sigma + d_1' - \mu c_1' &= 0 && (38) \\ A'\sigma^2 + B'\sigma + C' &= 0 && (39) \end{aligned}$$

where $c_1^{'} = \sum_{i=1}^{J} D_i e^{\mu_i} + R_J^* e^{\mu_J^*}$, $c_2^{'} = \sum_{i=1}^{J} D_i \mu_i e^{\mu_i} + R_J^* \mu_J^* e^{\mu_J^*}$, $d_1^{'} = \sum_{i=1}^{J} D_i X_{i:m:n} e^{\mu_i} + R_J^* S e^{\mu_J^*}$, $d_2^{'} = \sum_{i=1}^{J} D_i X_{i:m:n}^2 e^{\mu_i} + R_J^* S^2 e^{\mu_J^*}$, $d_3^{'} = \sum_{i=1}^{J} D_i \mu_i X_{i:m:n} e^{\mu_i} + R_J^* \mu_J^* S e^{\mu_J^*}$, $A^{'} = J c_1^{'}$, $B^{'} = c_1^{'}(d_3^{'} + J\bar{X}) - d_1^{'}(c_2^{'} + J)$, $C^{'} = {d_1^{'}}^2 - c_1^{'} d_2^{'}$ and $D_i = 1 + R_i$, for $i = 1, \ldots, J$. The solution to the preceding equations yields the approximate MLE's are

$$\hat{\mu} = \frac{(c_1^{'} - c_2^{'} - J)\hat{\sigma} + d_1^{'}}{c_1^{'}} \tag{40}$$

$$\hat{\sigma} = \frac{-B' + \sqrt{{B'}^2 - 4A'C'}}{2A'} \tag{41}$$

Here we consider only positive root of σ. It is easily seen that these approximate estimators are equivalent but not unbiased. Unfortunately, it is not possible to compute the exact bias of $\hat{\mu}$ and $\hat{\sigma}$ theoretically because of intractability encountered in finding the expectation of $\sqrt{{B'}^2 - 4A'C'}$.

References

[1] Arnold, B. C. and Balakrishnan, N.(1989), *Relations, Bounds and Approximations for Order Statistics*, Lecture Notes in Statistics - 53, Springer Verlag, New York.

[2] Balakrishnan, N. (2007), "Progressive Censoring; an Apprisal (with discussions)", (to appear in *TEST*).

[3] Balakrishnan, N. and Aggarwala, R. (2000), *Progressive Censoring: Theory, Methods and Applications*, Boston: Birkhäuser.

[4] Balakrishnan, N. and Varadan, J. (1991), "Approximate MLEs for the Location and Scale Parameters of the Extreme Value Distribution with Censoring", *IEEE Transactions on Reliability*, vol. 40, 146 - 151.

[5] Childs, A., Chandrasekhar, B., Balakrishnan, N. (2007), "Exact likelihood inference for an exponential parameter under progressively hybrid schemes", *Statistical*

Models and Methods for Biomedical and Technical Systems, Eds. F. Vonta, M. Nikulin, N. Limnios and C. Huber-Carol, Birkhäuser, Boston.

[6] Cohen, A.C. (1963), "Progressively censored samples in life testing", *Technometrics*, vol. 5, 327 - 329.

[7] Cohen, A.C. (1966), "Life-testing and early failure", *Technometrics*, vol. 8, 539 - 549.

[8] David, H. A. (1981), *Order Statistics*, Second Edition, John Wiley and Sons, New York.

[9] Kundu, D. and Joarder, A. (2006), "Analysis of Type-Ii progressively hybrid censored data", *Computational Statistics and Data Analysis*, vol. 50, 2509 - 2528.

[10] Kundu, D. (2007), "On hybrid censored Weibull distribution", *Journal of Statistical Planning and Inference*, vol. 137, 2127 - 2142.

[11] Lawless, J.F. (1982), *Statistical Models and Methods for Lifetime Data*, Wiley, New York.

[12] Mann, N. R. (1971), "Best linear invariant estimation for Weibull parameters under progressive censoring", *Technometrics*, Vol. 13, 521-533.

[13] Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T. (1991), *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge, U.K.

[14] Thomas, D. R. and Wilson, W. M. (1972), "Linear order statistics estimation for the two-parameter Weibull and extreme value distributions from Type-II progressively censored samples", *Technometrics*, vol. 14, 679-691.

Table 1: MLE Estimate for T = 0.75 .

N, M		Scheme - 1	Scheme - 2
30,15	α	1.0968(0.0862),1.2913(94.5)	1.0751(0.0838),1.1898(93.5)
	λ	1.0358(0.1611),1.6019(89.1)	1.0760(0.2937),1.6015(88.8)
40,20	α	1.0898(0.0623),1.0099(96.6)	1.0750(0.0626),1.0167(94.9)
	λ	1.0111(0.0934),1.2453(92.3)	1.0413(0.1321),1.3662(90.7)
60,20	α	1.1046(0.0644),1.1701(92.3)	1.0916(0.0554),1.0255(94.7)
	λ	0.9777(0.0962),1.6693(88.5)	0.9842(0.0902),1.4432(91.2)
60,30	α	1.0473(0.0342),0.7386(96.5)	1.0385(0.0364),0.7681(95.1)
	λ	1.0109(0.0653),0.9055(92.7)	1.0350(0.0962),1.0315(90.9)
80,30	α	1.0566(0.0344),0.7918(95.6)	1.0435(0.0302),0.7074(96.1)
	λ	0.9913(0.0633),1.0782(92.5)	1.0081(0.0731),0.9630(92.6)
80,40	α	1.0401(0.0252),0.6275(97.3)	1.0301(0.0269),0.6501(95.6)
	λ	1.0060(0.0449),0.7670(93.2)	1.0261(0.0614),0.8732(91.7)
100,40	α	1.0471(0.0256),0.6620(97.4)	1.0323(0.0219),0.5932(96.4)
	λ	0.9904(0.0406),0.8781(93.4)	1.0096(0.0465),0.7985(93.8)
100,50	α	1.0369(0.0209),0.5544(96.2)	1.0281(0.0232),0.5811(95.6)
	λ	0.9996(0.0292),0.6760(93.6)	1.0185(0.0418),0.7800(93.0)

Table 2: MLE Estimate for T = 1.00.

N, M		Scheme - 1	Scheme - 2
30,15	α	1.1102(0.0841),1.2367(96.0)	1.0719(0.0730),1.0287(95.6)
	λ	0.9982(0.1171),1.5080(92.1)	1.0383(0.1397),1.3333(91.2)
40,20	α	1.0983(0.0600),0.9891(97.7)	1.0704(0.0518),0.8833(96.4)
	λ	0.9864(0.0629),1.2035(93.7)	1.0179(0.0817),1.1445(92.1)
60,20	α	1.1046(0.0644),1.1701(92.3)	1.0933(0.0550),1.0249(95.2)
	λ	0.9781(0.0982),1.6692(88.5)	0.9776(0.0793),1.4394(91.6)
60,30	α	1.0539(0.0329),0.7320(97.0)	1.0358(0.0291),0.6855(95.9)
	λ	0.9945(0.0510),0.8876(94.2)	1.0157(0.0616),0.8892(92.3)
80,30	α	1.0567(0.0344),0.7918(95.7)	1.0487(0.0291),0.7049(96.9)
	λ	0.9906(0.0605),1.0781(92.5)	0.9926(0.0553),0.9508(93.9)
80,40	α	1.0456(0.0246),0.6214(97.8)	0.0313(0.0225),0.5879(97.0)
	λ	0.9927(0.0331),0.7531(94.1)	1.0110(0.0429),0.7624(92.2)
100,40	α	1.0473(0.0255),0.6621(97.4)	1.0396(0.0211),0.5788(97.4)
	λ	0.9895(0.0385),0.8781(93.4)	0.9936(0.0364),0.7655(94.0)
100,50	α	1.0397(0.0205),0.5493(96.9)	1.0252(0.0190),0.5216(94.7)
	λ	0.9927(0.0243),0.6653(94.0)	1.0120(0.0301),0.6773(93.5)

Table 3: MLE Estimate for T = 1.50

N, M		Scheme - 1	Scheme - 2
30,15	α	1.1130(0.0833),1.2367(96.3)	1.0727(0.0630),0.9343(95.8)
	λ	0.9857(0.0820),1.5075(92.7)	1.0196(0.1079),1.2004(92.7)
40,20	α	1.0992(0.0599),0.9886(97.8)	1.0682(0.0430),0.7962(97.5)
	λ	0.9841(0.0600),1.2025(93.6)	1.0025(0.0593),1.0237(94.4)
60,20	α	1.1046(0.0644),1.1701(92.3)	1.0932(0.0550),1.0248(95.2)
	λ	0.9781(0.0982),1.6692(88.5)	0.9779(0.0807),1.4394(91.6)
60,30	α	1.0544(0.0327),0.7320(97.2)	1.0366(0.0259),0.6251(94.9)
	λ	0.9920(0.0451),0.8875(94.2)	1.0054(0.0498),0.8042(93.0)
80,30	α	1.0567(0.0344),0.7918(95.7)	1.0492(0.0288),0.7051(97.0)
	λ	0.9906(0.0605),1.0781(92.5)	0.9900(0.0503),0.9508(93.8)
80,40	α	1.0458(0.0245),0.6215(97.8)	1.0308(0.0192),0.5357(96.8)
	λ	0.9919(0.0312),0.7531(94.1)	1.0031(0.0319),0.6896(93.6)
100,40	α	1.0473(0.0255),0.6621(97.4)	1.0407(0.0209),0.5785(97.7)
	λ	0.9895(0.0385),0.8781(93.4)	0.9901(0.0322),0.7645(94.0)
100,50	α	1.0397(0.0205),0.5492(96.9)	1.0277(0.0156),0.4768(94.7)
	λ	0.9928(0.0243),0.6652(94.0)	1.0008(0.0231),0.6138(94.7)

Table 4: MLE Estimate for T = 2.00

N, M		Scheme - I	Scheme - II
30,15	α	1.1130(0.0833),1.2367(96.3)	1.0754(0.0602),0.9106(96.1)
	λ	0.9857(0.0820),1.5075(92.7)	1.0045(0.0882),1.1750(92.6)
40,20	α	1.0992(0.0599),0.9885(97.8)	1.0695(0.0423),0.7720(95.8)
	λ	0.9841(0.0602),1.2025(93.6)	0.9966(0.0538),0.9983(94.6)
60,20	α	1.1046(0.0644),1.1701(92.3)	1.0932(0.0550),1.0248(95.2)
	λ	0.9781(0.0982),1.6692(88.5)	0.9779(0.0807),1.4394(91.6)
60,30	α	1.0544(0.0327),0.7320(97.2)	1.0379(0.0248),0.6054(95.6)
	λ	0.9920(0.0451),0.8875(94.2)	1.0004(0.0433),0.7836(94.2)
80,30	α	1.0567(0.0344),0.7918(95.7)	1.0492(0.0288),0.7051(97.0)
	λ	0.9906(0.0605),1.0781(92.5)	0.9900(0.0503),0.9508(93.8)
80,40	α	1.0458(0.0245),0.6215(97.8)	1.0321(0.0176),0.5179(96.6)
	λ	0.9919(0.0312),0.7531(94.1)	0.9986(0.0283),0.6709(94.1)
100,40	α	1.0473(0.0255),0.6621(97.4)	1.0407(0.0209),0.5785(97.7)
	λ	0.9895(0.0385),0.8781(93.4)	0.9901(0.0322),0.7645(94.0)
100,50	α	1.0397(0.0205),0.5492(96.9)	1.0286(0.0149),0.4608(94.6)
	λ	0.9928(0.0243),0.6652(94.0)	0.9986(0.0219),0.5969(94.1)

Table 5: Approximate MLE Estimate for T = 0.75

N, M		Scheme - 1	Scheme - 2
30,15	α	1.0873(0.0847),1.2073(94.2)	1.0814(0.0889),1.2070(93.3)
	λ	1.0354(0.1640),1.4941(89.1)	1.0104(0.3033),1.5970(88.6)
40,20	α	1.0832(0.0615),0.9924(96.2)	1.0837(0.0659),1.0651(95.4)
	λ	1.0103(0.0941),1.2224(92.2)	0.9752(0.1333),1.4107(91.3)
60,20	α	1.0998(0.0638),1.1226(92.1)	1.0915(0.0554),1.0236(94.5)
	λ	0.9792(0.0968),1.6005(88.4)	0.9435(0.0823),1.4270(92.8)
60,30	α	1.0432(0.0340),0.7349(96.5)	1.0486(0.0386),0.7959(95.5)
	λ	1.0102(0.0655),0.9005(92.7)	0.9679(1.0962),1.0530(92.1)
80,30	α	1.0533(0.0342),0.7870(95.5)	1.0492(0.0308),0.7288(96.4)
	λ	0.9920(0.0635),1.0714(92.5)	0.9520(0.0688),0.9797(94.7)
80,40	α	1.0372(0.0251),0.6253(97.1)	1.0409(0.0284),0.6735(96.3)
	λ	1.0054(0.0450),0.7640(93.2)	0.9588(0.0620),0.8914(92.2)
100,40	α	1.0447(0.0255),0.6593(97.2)	1.0417(0.0227),0.6140(97.4)
	λ	0.9906(0.0407),0.8743(93.4)	0.9463(0.0453),0.8151(94.8)
100,50	α	1.0346(0.0208),0.5529(96.2)	1.0392(0.0243),0.6029(96.4)
	λ	0.9991(0.0292),0.6739(93.6)	0.9512(0.0421),0.7976(93.9)

Table 6: Approximate MLE Estimate for T = 1.00.

N, M		Scheme - I	Scheme - II
30,15	α	1.1003(0.0827),1.1683(95.3)	1.0921(0.0811),1.0824(96.1)
	λ	0.9968(0.1177),1.4208(92.0)	0.9378(0.1395),1.3736(92.3)
40,20	α	1.0916(0.0592),0.9731(97.4)	1.0936(0.0582),0.9316(97.0)
	λ	0.9851(0.0628),1.1827(93.8)	0.9175(0.0809),1.1822(94.3)
60,20	α	1.0998(0.0638),1.1226(92.1)	1.0933(0.0550),1.0232(94.9)
	λ	0.9797(0.0989),1.6004(88.4)	0.9359(0.0703),1.4234(93.4)
60,30	α	1.0497(0.0327),0.7283(97.0)	1.0586(0.0326),0.7217(96.7)
	λ	0.9936(0.0510),0.8827(94.2)	0.9150(0.0608),0.9169(92.7)
80,30	α	1.0534(0.0342),0.7870(95.6)	1.0555(0.0296),0.7275(97.0)
	λ	0.9912(0.0607),1.0712(92.5)	0.9309(0.0476),0.9685(96.6)
80,40	α	1.0426(0.0244),0.6193(97.7)	1.0546(0.0251),0.6189(97.2)
	λ	0.9921(0.0330),0.7502(94.2)	0.9102(0.0429),0.7863(92.2)
100,40	α	1.0448(0.0254),0.6594(97.2)	1.0518(0.0218),0.6009(98.0)
	λ	0.9897(0.0385),0.8743(93.4)	0.9183(0.0311),0.7825(95.7)
100,50	α	1.0374(0.0204),0.5478(96.8)	1.0484(0.0211),0.5489(95.6)
	λ	0.9922(0.0243),0.6633(94.0)	0.9112(0.0299),0.6984(93.7)

Table 7: Approximate MLE Estimate for T = 1.50

N, M		Scheme - 1	Scheme - 2
30,15	α	1.1030(0.0819),1.1681(95.7)	1.1153(0.0735),1.0048(96.1)
	λ	0.9841(0.0820),1.4202(92.7)	0.8709(0.0978),1.2553(93.6)
40,20	α	1.0925(0.0592),0.9726(97.6)	1.1158(0.0519),0.8603(96.9)
	λ	0.9827(0.0598),1.1818(93.7)	0.8541(0.0524),1.0754(95.5)
60,20	α	1.0998(0.0638),1.1226(92.1)	1.0932(0.0550),1.0231(94.9)
	λ	0.9797(0.0989),1.6004(88.4)	0.9361(0.0712),1.4233(93.4)
60,30	α	1.0502(0.0325),0.7284(97.1)	1.0832(0.0313),0.6753(95.1)
	λ	0.9910(0.0450),0.8826(94.2)	0.8563(0.0443),0.8445(92.8)
80,30	α	1.0534(0.0342),0.7870(95.6)	1.0560(0.0293),0.7277(97.2)
	λ	0.9912(0.0607),1.0712(92.5)	0.9280(0.0424),0.9684(96.6)
80,40	α	1.0428(0.0244),0.6193(97.7)	1.0778(0.0232),0.5787(95.8)
	λ	0.9912(0.0311),0.7502(94.2)	0.8540(0.0287),0.7242(92.0)
100,40	α	1.0448(0.0254),0.6594(97.2)	1.0532(0.0215),0.6009(98.3)
	λ	0.9897(0.0385),0.8743(93.4)	0.9136(0.0262),0.7816(96.8)
100,50	α	1.0374(0.0204),0.5477(96.8)	1.0748(0.0188),0.5152(94.4)
	λ	0.9922(0.0243),0.6632(94.0)	0.8514(0.0205),0.6447(91.6)

Table 8: Approximate MLE Estimate for T = 2.00

N, M		Scheme - 1	Scheme - 2
30,15	α	1.1030(0.0819),1.1681(95.7)	1.1337(0.0710),0.9924(96.1)
	λ	0.9841(0.0820),1.4202(92.7)	0.8327(0.0690),1.2439(95.0)
40,20	α	1.0925(0.0592),0.9726(97.6)	1.1326(0.0512),0.8454(96.0)
	λ	0.9828(0.0601),1.1818(93.7)	0.8245(0.0414),1.0610(96.1)
60,20	α	1.0998(0.0638),1.1226(92.1)	1.0932(0.0550),1.0231(94.9)
	λ	0.9797(0.0989),1.6004(88.4)	0.9361(0.0712),1.4233(93.4)
60,30	α	1.0502(0.0325),0.7284(97.1)	1.0990(0.0302),0.6630(94.8)
	λ	0.9910(0.0450),0.8826(94.2)	0.8269(0.0336),0.8319(92.5)
80,30	α	1.0534(0.0342),0.7870(95.6)	1.0560(0.0293),0.7277(97.2)
	λ	0.9912(0.0607),1.0712(92.5)	0.9280(0.0424),0.9684(96.6)
80,40	α	1.0428(0.0244),0.6193(97.7)	1.0946(0.0217),0.5678(94.6)
	λ	0.9912(0.0311),0.7502(94.2)	0.8247(0.0222),0.7129(90.8)
100,40	α	1.0448(0.0254),0.6594(97.2)	1.0532(0.0215),0.6009(98.3)
	λ	0.9897(0.0385),0.8743(93.4)	0.9136(0.0262),0.7816(96.8)
100,50	α	1.0374(0.0204),0.5477(96.8)	1.0916(0.0185),0.5053(92.4)
	λ	0.9922(0.0243),0.6632(94.0)	0.8245(0.0170),0.6346(89.4)