Analysis of progressive Type-II censoring in presence of competing risk data under step stress modeling

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Abstract

In this article we consider the analysis of progressively censored competing risks data obtained from a simple step-stress experiment. It is assumed that there are only two competing causes of failures at each stress level and the lifetime distribution of each one of them is one parameter exponential distribution. Based on the cumulative exposure model (CEM) assumption, the conditional maximum likelihood estimators (MLEs) of the unknown parameters can be obtained in explicit forms. Confidence intervals of the unknown parameters based on the exact distributions of the conditional MLEs and percentile bootstrap method, are constructed. Further we obtain Bayes estimates and the associated credible intervals based on a very flexible Beta-gamma prior on the unknown parameters. A simulation experiment has been performed to observe the performances of the different estimators.

Key Words and Phrases: Competing risk; progressive Type-II censoring; competing risk; beta-gamma distribution; maximum likelihood estimator; bootstrap confidence interval; Bayes credible interval.

AMS Subject Classifications: 62F10, 62F03, 62H12.

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1 Introduction

The progressive censoring scheme has received a considerable amount of attention in recent years. The progressive Type-II censoring, introduced by Herd [6], can be described as follows. Suppose n items are put on a life testing experiment at the time point zero, and m is a prefixed integer such that $m(\leq n)$. Let us further choose non negative integers $R_1, R_2, \ldots, R_m$ such that, $m + \sum_{i=1}^{m} R_i = n$. At the time of the first failure, say t_1, R_1 number of units are withdrawn from the system. Similarly, at the time of the second failure, say t_2, R_2 number of units are withdrawn from the experiment. This procedure continues till the failure of the m-th unit takes place with the remaining R_m number of units are removed from the system. It is to be noted that Type-II censoring scheme can be obtained by taking $R_1 = R_2 = \ldots = R_{m-1} = 0$ and $R_m = n-m$. For more details on progressive censoring schemes one is referred to the book by Balakrishnan and Cramer [2].

In a typical life testing experiment it may be difficult to get sufficient number of failures under normal operating conditions. Due to this reason, to carry out a statistical analysis, the experimenter artificiality changes the environment of the operating condition to get early failures. In statistical literature this is called as 'accelerated life testing' (ALT) experiment. There are different ways of conducting an ALT experiment. One of the popular methods is known as 'simple step stress life test', which is commonly called as simple SSLT experiment. In a simple SSLT experiment, the experimenter enforces a stress level at a time point in the middle of the experiment. More specifically, in a simple SSLT experiment suppose n units are placed in the experiment under the first or initial stress level S_1, then the units are subjected to another stress level say, S_2 at a predefined time point τ. The experiment stops in the second stress level when a prefixed number of failures, say m, takes place. Suppose, n_1 units have failed under first stress level and $n_2 = m - n_1$ units have failed in the second stress level before the experiment stops, then the data set becomes:

$$data = \Big\{(t_{1:n}, \dots, t_{n_1+n_2:n}) : 0 < t_{1:n} < \dots < t_{n_1:n} \le \tau < t_{n_1+1:n} < \dots < t_{n_1+n_2:n}\Big\}.$$

Note that a Type-II progressive censoring scheme can be easily incorporated to a simple SSLT experiment.

Different types of models assumptions are made in analyzing the data obtained from a simple SSLT experiment. The most common and popular model is the cumulative exposure model (CEM). The CEM was first proposed by Sedyakin [11]. Under the assumptions of CEM, the residual life of a unit at a stress level depends only on the cumulative exposure that the unit has experienced, no matter how this exposure was accumulated. Let us consider a simple SSLT experiment as described before with cumulative distribution function(CDF) of the units at the i-th stress level being $F_i(\cdot|\boldsymbol{\theta_i})$, where $\boldsymbol{\theta_i}$ is the vector of parameters associated with the i-th stress level, for $i = 1, 2$. Suppose $F(\cdot)$ denotes the CDF of an experimental unit under CEM. Then under the assumptions of CEM, $F(\cdot)$ is related to CDF under each stress level by the following set of equations.

$$F(t) = \begin{cases} F_1(t|\boldsymbol{\theta_1}), & \text{if } 0 < t < \tau, \\ F_2(t - \tau + h|\boldsymbol{\theta_2}), & \text{if } \tau < t, \\ 0, & \text{otherwise.} \end{cases}$$

where, $F_1(\tau|\boldsymbol{\theta_1}) = F_2(h|\boldsymbol{\theta_2})$. For more details on step-stress modeling, one is referred to Kundu and Ganguly [7].

In a reliability experiment, it is interesting to asses some specific risk factors in presence of other risk factors. In the statistical literature this is called the competing risk problem. In a typical competing risk problem a unit may fail due to several causes. In this case one observes time of failure and the associated cause of failure also. Let δ be an indicator variable showing the cause of failure of a unit. Then the data set is

$$data = \Big\{(z_{1:n}, \delta_1), (z_{2:n}, \delta_2), \dots, (z_{n:n}, \delta_n)\Big\}.$$

Here, $z_{i:n}$ denotes the failure time of the i-th unit. Cox [4] suggested latent failure time models in which the risks are assumed to be independent to each other. In this model, suppose n units are put in the experiment and $X_1, X_2, \ldots, X_p$ are the random variables denoting the lifetimes of p competing risks responsible for the failure of units. Then a unit fails as soon as one of the competing risks causes the unit fail. An extensive amount of work has been done under this assumption. For more details on competing risks related problems, one is referred to the book by Crowder [5].

In this article we have considered simple step stress modeling under progressive Type-II censoring in presence of competing risks. We have made the CEM assumption for both the causes. In the first stress level the distributions of the competing causes of failures are assumed to be exponential distributions with mean $\theta_{11}(>0)$ and $\theta_{21}(>0)$, respectively, whereas in the second stress level the corresponding distributional assumptions become exponential with corresponding means $\theta_{12}(>0)$ and $\theta_{22}(>0)$, respectively. It should be mentioned that the results here are close to those results obtained by Xie, Balakrishnan and Han [12] without the presence of any competing causes of failures. The proposed model is connected to Type-I progressively hybrid censored competing risks data considered by Kundu and Joarder [8] and a more general hybrid censoring model in this regard discussed by the authors in Koley and Kundu [9]. But so far no body has considered the model, under this most general set up. The exact distributions of the conditional MLEs have been obtained, and based on the exact distribution approximate confidence intervals have been constructed. Since the construction of the confidence intervals based on exact distributions of the conditional MLEs are quite challenging, we have obtained Bayes estimates and the associated credible intervals under a very general set of prior, which has not been addressed before. It is assumed that $(\theta_{11}^{-1}, \theta_{21}^{-1})$ follows a Beta-gamma distribution, and similarly, $(\theta_{12}^{-1}, \theta_{22}^{-1})$ also follows a Beta-gamma distribution. Based on this prior assumptions, we provide the Bayes estimates with respect to squared error loss function, and the associated credible intervals based on Gibbs sampling method. Extensive simulations have been performed to observe the behavior of the different estimators.

The rest of the paper is organized as follows. In Section 2 we describe model assumption and MLEs of the parameters, the distribution of the conditional MLEs are also derived in Section 2. Different types of confidence intervals namely approximate and percentile bootstrap confidence intervals are constructed in Section 3. Further we carry out the Bayesian inference in Section 4. An extensive simulation study is reported in Section 5 and finally we conclude the paper in Section 6. All the proofs of the results are given in the Appendix.

2 Model, MLEs & Distribution of Conditional MLEs

As it has been mentioned before, n, $m(< n)$ and $(R_1, \ldots, R_m)$ such that $m + \sum_{i=1}^{m} R_i = n$ are pre-fixed integers. It is assumed that an experiment starts with n units at the first stress level. The operating condition of the experiment changes to a second stressed level at a fixed time point τ. For $i \in \{1, 2, \ldots, m\}$, R_i units are removed from the remaining $(n - i - R_1 - \ldots - R_{i-1})$ units at the i-th failure time of the experiment. The experiment stops as soon as the m-th unit fails. At each stress level there are two causes of failures. For $i, j = 1, 2$, let X_{ij} denote the random variable associated with the i-th risk at the j-th stress level of the experiment. It is also assumed that the causes of failures follow Cox's latent failure time model and the distribution functions of the different causes at the two stress levels satisfy CEM assumption. One would observe the failure time points and the cause associated with each failure. The random variable associated with the failure at the first stress level is $Z_1 = \min\{X_{11}, X_{21}\}$ and similarly, $Z_2 = \min\{X_{12}, X_{22}\}$ is the random variables associated with the failure at the second stress level. Moreover, let us define a new random variable

$$Z = \begin{cases} Z_1, & \text{at the first stress level} \\ Z_2, & \text{at the second stress level.} \end{cases}$$

Thus Z is the random variable denoting the failure time of the units. Let us denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the observed data sets at the first and second stress level, respectively. Thus if D is the random variable denoting the number of failures at the first stress level then $\mathcal{D}_1$ and $\mathcal{D}_2$ are of the following form

$\mathcal{D}_1 = \{(z_{1:m:n}, R_1, \delta_{1,1}), (z_{2:m:n}, R_2, \delta_{2,1}), \ldots, (z_{d:m:n}, R_d, \delta_{d,1})\}$

$\mathcal{D}_2 = \{(z_{d+1:m:n}, R_{d+1}, \delta_{d+1,2}), (z_{d+2:m:n}, R_{d+2}, \delta_{d+2,2}), \ldots, (z_{m:m:n}, R_m, \delta_{m,2})\}$

where

$d:$ is the number of failures observed at the first stress level.

$$\delta_{i,j} = \begin{cases} 1, & \text{if the } i\text{-th failure comes from the first cause at the } j\text{-th stress level,} \\ 0, & \text{if the } i\text{-th failure comes from the second cause at the } j\text{-th stress level.} \end{cases}$$

Clearly the complete data of the experiment is $\{\mathcal{D}_1, \mathcal{D}_2\}$.

It is to be noted that, for $z_{i:m:n} < \tau$, the likelihood contribution of $(z_{i:m:n}, \delta_{i,1} = 1)$ is,

$$\begin{aligned} L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,1} = 1)) =& \frac{1}{\theta_{11}} e^{-\frac{1}{\theta_{11}} z_{i:m:n}(1+R_i)} \; e^{-\frac{1}{\theta_{21}} z_{i:m:n}(1+R_i)} \\ =& \frac{1}{\theta_{11}} e^{-(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}}) z_{i:m:n}(1+R_i)}. \end{aligned} \tag{1}$$

Similarly, the likelihood contribution of $(z_{i:m:n}, \delta_{i,1} = 0)$ is

$$L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,1} = 0)) = \frac{1}{\theta_{21}} e^{-(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}}) z_{i:m:n}(1+R_i)}. \tag{2}$$

On the other hand, for $\tau < z_{i:m:n}$, the likelihood of $(z_{i:m:n}, \delta_{i,2} = 1)$ is

$$\begin{aligned} L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,2} = 1)) =& \frac{1}{\theta_{12}} e^{-\frac{1}{\theta_{12}}(z_{i:m:n} - \tau) - \frac{\tau}{\theta_{11}}} \; e^{-\frac{1}{\theta_{22}}(z_{i:m:n} - \tau) - \frac{\tau}{\theta_{21}}} \\ =& \frac{1}{\theta_{12}} e^{-(\frac{1}{\theta_{12}} + \frac{1}{\theta_{22}})(z_{i:m:n} - \tau) - \tau(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}})}. \end{aligned} \tag{3}$$

Similarly, for $\tau < z_{i:m:n}$, the likelihood contribution of $(z_{i:m:n}, \delta_{i,2} = 0)$ is

$$L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22} | (z_{i:m:n}, \delta_{i,2} = 0)) = \frac{1}{\theta_{22}} e^{-(\frac{1}{\theta_{12}} + \frac{1}{\theta_{22}})(z_{i:m:n} - \tau) - \tau(\frac{1}{\theta_{11}} + \frac{1}{\theta_{21}})}. \tag{4}$$

Let us use the following notations:

$$W_1 = \sum_{i=1}^{D} Z_{i:m:n}\big(1 + R_i\big) + \tau\Big[\big(m - D\big) + \sum_{i=D+1}^{m} R_i\Big] \text{ and } W_2 = \sum_{i=D+1}^{m} (Z_{i:m:n} - \tau)(1 + R_i).$$

For $i, j = 1, 2$, suppose D_{ij} is the random variable denoting the number of failures occurring due to cause i in the j -th stress level. Clearly $D_{21} = D - D_{11}$ and $D_{22} = m - D - D_{12}$. The likelihood function from the expressions (1) to (4) and based on the data, can be written as,

$$L(\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22}|data) = c\Big(\frac{1}{\theta_{11}}\Big)^{d_{11}}\Big(\frac{1}{\theta_{21}}\Big)^{d-d_{11}}\Big(\frac{1}{\theta_{12}}\Big)^{d_{12}}\Big(\frac{1}{\theta_{22}}\Big)^{m-d-d_{12}} e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}}, \tag{5}$$

where, c is the normalizing constant, independent of the parameters and for $s = 1, 2$, $\frac{1}{\theta_{.s}} = \frac{1}{\theta_{1s}} + \frac{1}{\theta_{2s}}$. Here $d \in \{0, 1, \ldots, m\}$, $d_{11} \in \{0, 1, \ldots, d\}$ and $d_{12} \in \{0, 1, \ldots, m - d\}$. Derivation of c is supplied in the Appendix. For $i, j = 1, 2$, MLEs of θ_{ij} are obtained by maximizing the likelihood (or equivalently the log likelihood) function. Note that the MLE of θ_{ij} exists only if $D_{ij} > 0$ and it is given by,

$$\widehat{\theta}_{ij} = \frac{W_j}{D_{ij}}, \quad i, j = 1, 2. \tag{6}$$

Next we derive the exact distribution of the conditional MLEs of the unknown parameters. We use the following two notations. Suppose $f_G(x; a, b, c)$ is a shifted Gamma distribution with probability density function (PDF)

$$f_G(x; a, b, c) = \frac{c^b}{\Gamma(b)}(x - a)^{b-1}e^{-c(x-a)}, \; x > a, a \in (-\infty, \infty), b > 0, c > 0$$

and

$$F_G(x; a, b, c) = \int_a^x f_G(u; a, b, c)\; du.$$

Let us define an event $\mathcal{D}^* = \{D_{11} > 0, D_{21} > 0, D_{12} > 0, D_{22} > 0\}$. Note that MLEs of all the parameters exist only if the event $\mathcal{D}^*$ occurs. For $s = 1, 2$, the distribution functions of $\widehat{\theta}_{s1}$ and $\widehat{\theta}_{s2}$ are given below.

Theorem 1. *For $s = 1, 2$, the distribution function of $\widehat{\theta}_{s1}$ is given by,*

$$
\begin{aligned}
F_{\widehat{\theta}_{s1|\mathcal{D}^*}}(x) =& P(\widehat{\theta}_{s1} \leq x|\mathcal{D}^*) \\
=& \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{s1}}{\theta_{11}+\theta_{21}} \Big)^{d-i} \Big(1 - \frac{\theta_{s1}}{\theta_{11}+\theta_{21}} \Big)^{i} \Big[1 - \Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}} \Big)^{m-d} - \\
& \Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}} \Big)^{m-d} \Big] \frac{1}{\prod_{j=1}^{m-d} \sum_{p=j}^{m-d} (1+R_p)} \times \\
& \sum_{l=0}^{d} \Big[\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}} \left[l+m-d+\sum_{j=d-l+1}^{m} R_j \right]}}{\left[\prod_{j=1}^{l} \sum_{p=d-l+1}^{d-l+j} (1+R_p) \right] \left[\prod_{j=1}^{d-l} \sum_{p=j}^{d-l} (1+R_p) \right]} \times \\
& F_G\Big(x; \frac{\tau}{i} \big[l+m-d+\sum_{j=d-l+1}^{m} R_j \big], d, \frac{i}{\theta_{.1}} \Big) \Big] \Bigg].
\end{aligned}
\tag{7}
$$

Proof. See the Appendix. ∎

Corollary 1.1. *For $s = 1, 2$, the probability density function of $\widehat{\theta}_{s1}$, obtained by taking the derivative of $F_{\widehat{\theta}_{s1|\mathcal{D}^*}}(x)$, is given by,*

$$
\begin{aligned}
f_{\widehat{\theta}_{s1|\mathcal{D}^*}}(x) =& \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{i=1}^{d-1} \Bigg[\binom{d}{i} \Big(\frac{\theta_{s1}}{\theta_{11}+\theta_{21}} \Big)^{d-i} \Big(1 - \frac{\theta_{s1}}{\theta_{11}+\theta_{21}} \Big)^{i} \Big[1 - \Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}} \Big)^{m-d} - \\
& \Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}} \Big)^{m-d} \Big] \frac{1}{\prod_{j=1}^{m-d} \sum_{p=j}^{m-d} (1+R_p)} \times \\
& \sum_{l=0}^{d} \Big[\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}} \left[l+m-d+\sum_{j=d-l+1}^{m} R_j \right]}}{\left[\prod_{j=1}^{l} \sum_{p=d-l+1}^{d-l+j} (1+R_p) \right] \left[\prod_{j=1}^{d-l} \sum_{p=j}^{d-l} (1+R_p) \right]} \times \\
& f_G\Big(x; \frac{\tau}{i} \big[l+m-d+\sum_{j=d-l+1}^{m} R_j \big], d, \frac{i}{\theta_{.1}} \Big) \Big] \Bigg].
\end{aligned}
\tag{8}
$$

Theorem 2. *For $s = 1, 2$, the distribution function of $\widehat{\theta}_{s2}$ is given by,*

$$
\begin{aligned}
F_{\widehat{\theta}_{s2|\mathcal{D}^*}}(x) =& P(\widehat{\theta}_{s2} \leq x|\mathcal{D}^*) \\
=& \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2} \sum_{k=1}^{m-d-1} \Bigg[\Big[1 - \Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}} \Big)^{d} - \Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}} \Big)^{d} \Big] \binom{m-d}{k} \times \\
& \Big(1 - \frac{\theta_{s2}}{\theta_{12}+\theta_{22}} \Big)^{k} \Big(\frac{\theta_{s2}}{\theta_{12}+\theta_{22}} \Big)^{m-d-k} \frac{1}{\prod_{j=1}^{m-d} \sum_{p=j}^{m-d} (1+R_p)} \times
\end{aligned}
$$

$$\sum_{l=0}^{d} \frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{i=d-l+1}^{m} R_i\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]} F_G\Big(x;0,m-d,\frac{k}{\theta_{.2}}\Big)\Bigg]. \tag{9}$$

Proof. See the Appendix. ∎

Corollary 2.1. *For $s = 1, 2$, the probability density function of $\widehat{\theta}_{s2}$, obtained by taking the derivative of $F_{\widehat{\theta}_{s2|\mathcal{D}^*}}(x)$, is given by,*

$$\begin{aligned} f_{\widehat{\theta}_{s2|D^*}}(x) =& \frac{c}{P(\mathcal{D}^*)} \sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\Bigg[\Bigg[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^d - \Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^d\Bigg]\binom{m-d}{k}\times \\ & \Big(1-\frac{\theta_{s2}}{\theta_{12}+\theta_{22}}\Big)^k\Big(\frac{\theta_{s2}}{\theta_{12}+\theta_{22}}\Big)^{m-d-k}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\times \\ & \sum_{l=0}^{d} \frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{i=d-l+1}^{m} R_i\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]} f_G\Big(x;0,m-d,\frac{k}{\theta_{.2}}\Big)\Bigg]. \end{aligned} \tag{10}$$

3 Confidence intervals

In this section we construct approximate confidence intervals based on the exact distribution of the conditional MLEs and percentile bootstrap confidence intervals (CI) of the parameters of interest. They are discussed below.

3.1 Approximate confidence interval

For any $\alpha \in (0, 1)$, if θ_{ij}^L and θ_{ij}^U denote $100(1-\alpha)\%$ lower and upper confidence limits of θ_{ij} then they are obtained by solving for θ_{ij} the following two equations:

$$F_{\widehat{\theta}_{ij}|D^*}(\widehat{\theta}_{ij}^{obs}) = 1 - \frac{\alpha}{2}, \tag{11}$$

$$F_{\widehat{\theta}_{ij}|D^*}(\widehat{\theta}_{ij}^{obs}) = \frac{\alpha}{2}. \tag{12}$$

Here $\widehat{\theta}_{ij}^{obs}$ denotes the observed MLE of θ_{ij}. The other parameters in equations (11) and (12) are replaced by their respective MLEs. Following the approach of Balakrishnan, Cramer and Iliopoulos [3], the solutions of the above equations exist if the following two properties hold true:

Property 1:

For any $x > 0$ and $i, j = 1, 2$, the function $F_{\widehat{\theta}_{ij}|D^*}(x)$ is a monotonically decreasing function of θ_{ij}.

Property 2: For any $x > 0$, $\lim_{\theta_{ij}\to 0} F_{\widehat{\theta}_{ij}|D^*}(x) = 1$ and $\lim_{\theta_{ij}\to\infty} F_{\widehat{\theta}_{ij}|D^*}(x) = 0$.

Under the assumption that Property-1 and Property-2 hold true, we construct approximate confidence interval of the parameters. Equations (11) and (12) are non linear equations and one needs to solve them by non linear solvers namely bisection, Newton Rapshon method etc.

3.2 Percentile Bootstrap confidence interval

The other confidence interval namely percentile bootstrap confidence intervals can be constructed using the following algorithm.

Algorithm 1:

Step-1: Given the values of $n, m, R_1, R_2, \ldots, R_m$ and T, generate sample and obtain $\widehat{\theta}_{11}$, $\widehat{\theta}_{21}$, $\widehat{\theta}_{12}$, $\widehat{\theta}_{22}$ from equation (6).

Step-2: Using $n, m, R_1, R_2, \ldots, R_m, T, \widehat{\theta}_{11}, \widehat{\theta}_{21}, \widehat{\theta}_{12}, \widehat{\theta}_{22}$, generate sample and obtain values of the parameters say,$\widehat{\theta}^*_{11}, \widehat{\theta}^*_{21}, \widehat{\theta}^*_{12}, \widehat{\theta}^*_{22}$ from equation (6).

Step-3: Repeat Step-2 a large number of times say, M times and obtain $\widehat{\theta}^*_{11}, \widehat{\theta}^*_{21}, \widehat{\theta}^*_{12}, \widehat{\theta}^*_{22}$, each M times. Arrange them in increasing order to obtain $\widehat{\theta}^{*(1)}_{11} < \widehat{\theta}^{*(2)}_{11} < \ldots < \widehat{\theta}^{*(M)}_{11}$, $\widehat{\theta}^{*(1)}_{21} <$

$\widehat{\theta}_{21}^{*(2)} < \ldots < \widehat{\theta}_{21}^{*(M)}$, $\widehat{\theta}_{12}^{*(1)} < \widehat{\theta}_{12}^{*(2)} < \ldots < \widehat{\theta}_{12}^{*(M)}$ and $\widehat{\theta}_{22}^{*(1)} < \widehat{\theta}_{22}^{*(2)} < \ldots < \widehat{\theta}_{22}^{*(M)}$, respectively.

Step-4: For $\alpha \in (0,1)$, a $100(1-\alpha)\%$ percentile bootstrap confidence interval for θ_{11} is then given by, $\left(\widehat{\theta}_{11}^{*([M\frac{\alpha}{2}])}, \widehat{\theta}_{11}^{*([M(1-\frac{\alpha}{2})])}\right)$, where $[x]$ denotes the largest integer less than or equal to x. Similarly a $100(1-\alpha)\%$ percentile bootstrap confidence interval for each of θ_{21}, θ_{12} and θ_{22} is then given by, $\left(\widehat{\theta}_{21}^{*([M\frac{\alpha}{2}])}, \widehat{\theta}_{21}^{*([M(1-\frac{\alpha}{2})])}\right)$, $\left(\widehat{\theta}_{12}^{*([M\frac{\alpha}{2}])}, \widehat{\theta}_{12}^{*([M(1-\frac{\alpha}{2})])}\right)$ and $\left(\widehat{\theta}_{22}^{*([M\frac{\alpha}{2}])}, \widehat{\theta}_{22}^{*([M(1-\frac{\alpha}{2})])}\right)$, respectively.

4 Bayesian Analysis

In this section we provide the Bayes estimates of the unknown parameters under square error loss functions and the associated credible intervals based on Gibbs sampling technique. Before we proceed further, let us denote Beta-gamma distribution with parameters $b_0 > 0$, $a_0 > 0$, $a_1 > 0$, $a_2 > 0$ by $BG(b_0, a_0, a_1, a_2)$ with the following density function.

$$f(x,y) = b_0^{a_0} \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} e^{-b_0(x+y)} x^{a_1-1} y^{a_2-1} (x+y)^{a_0-a_1-a_2}, \quad x > 0, y > 0. \tag{13}$$

For further details on Beta-gamma distribution, one may see Peña and Gupta [10]. We make the re-parameterization of the parameters as $\lambda_{ij} = \frac{1}{\theta_{ij}}$, for $i,j = 1,2$. Further, we assume that the parameters of the model have the following prior distributions. For $b_{01} > 0$, $a_{01} > 0$, $a_{11} > 0$, $a_{21} > 0$, $b_{02} > 0$, $a_{02} > 0$, $a_{12} > 0$, $a_{22} > 0$,

$$(\lambda_{11}, \lambda_{21}) \sim BG(b_{01}, a_{01}, a_{11}, a_{21}), \qquad (\lambda_{12}, \lambda_{22}) \sim BG(b_{02}, a_{02}, a_{12}, a_{22}).$$

and $(\lambda_{11}, \lambda_{21})$ is independent of $(\lambda_{12}, \lambda_{22})$.

4.1 Posterior distribution

The posterior distribution of the re-parameterized parameters $(\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22})$ turn out to be in the following form.

$$\pi(\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}|data) = \pi_1(\lambda_{11}, \lambda_{21}|data) \times \pi_2(\lambda_{12}, \lambda_{22}|data), \tag{14}$$

where

$$\pi_1(\lambda_{11}, \lambda_{21}|data) = BG(b_{01} + w_1, a_{01} + d, a_{11} + d_{11}, a_{21} + d - d_{11})$$

and

$$\pi_2(\lambda_{12}, \lambda_{22}|data) = BG(b_{02} + w_2, a_{02} + m - d, a_{12} + d_{12}, a_{22} + m - d - d_{12}).$$

Under square error loss function, the Bayes estimators of θ_{ij} are obtained as

$$\widehat{\theta}_{ij} = \frac{(a_{1j} + a_{2j} + D_{1j} + D_{2j} - 1)(W_j + b_{0j})}{(a_{0j} + D_{1j} + D_{2j} - 1)(a_{ij} + D_{ij} - 1)},$$

for $i, j = 1, 2$.

4.2 Credible interval (CRI)

We now consider construction of symmetric and highest posterior density (HPD) credible intervals of the parameters. Note that, from 4.1, the joint posterior distribution of $\lambda_{11}, \lambda_{21}$ is obtained as

$$(\lambda_{11}, \lambda_{21}|data) \sim BG(b_{01} + w_1, a_{01} + d, a_{11} + d_{11}, a_{21} + d - d_{11})$$

and the joint posterior distribution of $\lambda_{12}, \lambda_{22}$ is obtained as,

$$(\lambda_{12}, \lambda_{22}|data) \sim BG(b_{02} + w_2, a_{02} + m - d, a_{12} + d_{12}, a_{22} + m - d - d_{12}).$$

We need the following lemma, to proceed further.

Lemma 1. *The random variable* (X, Y) *follows a joint distribution* $BG(b_0, a_0, a_1, a_2)$ *if and only if,*

$$\frac{X}{X+Y} \sim Beta(a_1, a_2) \qquad and \qquad X+Y \sim Gamma(a_0, b_0)$$

and they are independent.

Proof. See the Appendix. ∎

We now provide an algorithm to construct $100(1-\alpha)\%$ symmetric and HPD credible intervals of θ_{11} and θ_{21}. Similar methods can be used for θ_{12} and θ_{22} also.

Algorithm 2:

Step-1: Generate $\lambda_{11} + \lambda_{21}$ from $Gamma(a_{01} + d, b_{01} + w_1)$. Generate $\frac{\lambda_{11}}{\lambda_{11}+\lambda_{21}}$ independently from $Beta(a_{11} + d_{11}, a_{21} + d - d_{11})$.

Step-2: Obtain λ_{11} and λ_{21} from Step-1. Obtain $\theta_{11} = \frac{1}{\lambda_{11}}$ and $\theta_{21} = \frac{1}{\lambda_{21}}$.

Step-3: Repeat Step-1 and Step-2 quite a large number of times say, M, to obtain M number of θ_{11} and θ_{21}. Arrange M values of θ_{11} and θ_{21} each in increasing order as, $\theta_{11}^1 \leq \theta_{11}^2 \ldots \leq \theta_{11}^M$ and $\theta_{21}^1 \leq \theta_{21}^2 \ldots \leq \theta_{21}^M$, respectively.

Step-4: The $100(1-\alpha)\%$ symmetric and HPD credible intervals of θ_{11} are obtained as $(\theta_{11}^{[M\frac{\alpha}{2}]}, \theta_{11}^{[M(1-\frac{\alpha}{2})]})$ and $(\theta_{11}^{j^*}, \theta_{11}^{j^*+[M(1-\alpha)]})$, respectively, where, $j^* \in \{1, 2, \ldots, [M\alpha]\}$ is an integer such that, $\theta_{11}^{j^*+[M(1-\alpha)]} - \theta_{11}^{j^*} \leq \theta_{11}^{j+[M(1-\alpha)]} - \theta_{11}^{j}$ for $\forall j = 1, 2, \ldots, [M\alpha]$ and $[x]$ is the largest integer not exceeding x. In a similar way the $100(1-\alpha)\%$ symmetric and HPD credible intervals of θ_{21} are obtained.

5 Simulation results

In this section we carry out extensive simulation study to see the effectiveness of the different methods proposed so far. We replicate the process 5000 times to report average length of the approximate and percentile bootstrap confidence intervals, as well as symmetric and HPD credible intervals of the parameters at 5% and 1% levels of significance and the associated coverage percentages within brackets. We have taken $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$ as our designing parameters. For Bayesian analysis the hyper parameters are taken as $b_{01} = b_{02} = 0,\ a_{01} = a_{02} = 2,\ a_{11} = a_{21} = a_{12} = a_{22} = 1$. These values are chosen such that, they match with the corresponding MLEs of the parameters. Thus a comparison study can be made between the performance of frequentest and Bayesian methods proposed here. Different values of n and τ are considered to check the performances of the different methods under different censoring schemes. For our study we have considered three different censoring schemes and they are described as Scheme-1: $R_1 = n - m, R_2 = \ldots = R_m = 0$, Scheme-2: $R_1 = \ldots = R_{m/2-1} = 0, R_{m/2} = n - m, R_{m/2+1} = \ldots = R_m = 0$, Scheme-3: $R_1 = (n-m)/2, R_2 = \ldots = R_{m-1} = 0, R_m = (n-m)/2$. In all these cases, $m + \sum_{i=1}^{m} R_i = n$.

Table 1: Classical results for θ_{11} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Approximate CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	1	0.133	0.332	2.839 (96.02)	4.255 (99.40)	3.554 (94.20)	5.851 (98.32)
		0.6		0.126	0.297	2.534 (96.37)	3.879 (99.35)	3.149 (94.75)	5.324 (98.62)
		0.7		0.125	0.283	2.349 (95.57)	3.504 (99.57)	2.982 (93.77)	4.895 (98.12)
		0.5	2	0.134	0.321	2.626 (95.90)	3.884 (99.42)	3.270 (94.17)	5.463 (98.27)
		0.6		0.119	0.291	2.381 (95.50)	3.584 (99.40)	2.896 (94.62)	4.854 (98.55)
		0.7		0.114	0.267	2.228 (95.20)	3.334 (99.35)	2.669 (94.75)	4.481 (98.70)
		0.5	3	0.135	0.319	2.614 (95.67)	3.971 (99.40)	3.185 (95.05)	5.379 (98.67)
		0.6		0.119	0.273	2.313 (95.45)	3.507 (99.40)	2.840 (94.35)	4.801 (98.67)
		0.7		0.111	0.274	2.193 (95.07)	3.269 (99.27)	2.637 (94.30)	4.423 (98.67)
40	34	0.5	1	0.130	0.314	2.563 (95.92)	3.786 (99.47)	3.151 (93.95)	5.229 (98.40)
		0.6		0.123	0.278	2.286 (95.42)	3.380 (99.45)	2.806 (94.05)	4.703 (98.30)
		0.7		0.106	0.252	2.099 (94.82)	3.076 (99.25)	2.513 (94.42)	4.174 (98.50)
		0.5	2	0.111	0.267	2.280 (95.42)	3.400 (99.37)	2.943 (93.40)	4.902 (98.27)
		0.6		0.107	0.249	2.122 (95.02)	3.078 (99.47)	2.559 (94.62)	4.246 (98.45)
		0.7		0.101	0.255	2.048 (94.92)	2.992 (99.02)	2.310 (94.80)	3.747 (98.35)
		0.5	3	0.107	0.273	2.296 (95.62)	3.403 (99.57)	2.695 (94.25)	4.518 (98.70)
		0.6		0.107	0.245	2.104 (95.70)	3.108 (99.45)	2.449 (94.15)	4.035 (98.45)
		0.7		0.098	0.244	1.979 (94.80)	2.879 (99.30)	2.265 (94.25)	3.699 (98.90)

Table 2: Classical results for θ_{21} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Approximate CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	1	0.106	0.222	2.056 (94.82)	3.092 (99.47)	2.562 (93.55)	4.280 (98.37)
		0.6		0.097	0.197	1.857 (94.82)	2.786 (99.47)	2.206 (94.12)	3.619 (98.95)
		0.7		0.093	0.180	1.729 (94.57)	2.523 (99.05)	1.921 (95.25)	3.288 (98.67)
		0.5	2	0.086	0.196	1.864 (94.65)	2.831 (99.32)	2.178 (94.70)	3.692 (98.45)
		0.6		0.082	0.171	1.727 (95.17)	2.559 (99.32)	1.941 (94.27)	3.152 (98.82)
		0.7		0.079	0.164	1.646 (94.70)	2.391 (99.37)	1.803 (94.90)	2.887 (99.20)
		0.5	3	0.092	0.195	1.873 (94.82)	2.790 (99.37)	2.285 (94.40)	3.730 (98.85)
		0.6		0.087	0.173	1.722 (94.95)	2.508 (99.22)	2.002 (94.52)	3.318 (98.70)
		0.7		0.083	0.145	1.609 (95.37)	2.348 (99.20)	1.815 (95.50)	2.920 (98.50)
40	34	0.5	1	0.089	0.183	1.814 (94.75)	2.692 (99.25)	2.149 (94.55)	3.638 (98.37)
		0.6		0.083	0.174	1.663 (94.92)	2.413 (99.35)	1.906 (94.65)	2.975 (98.92)
		0.7		0.075	0.150	1.547 (95.07)	2.244 (99.10)	1.742 (94.70)	2.690 (98.90)
		0.5	2	0.076	0.165	1.670 (95.12)	2.462 (99.30)	1.895 (94.27)	3.113 (98.67)
		0.6		0.069	0.141	1.543 (95.15)	2.270 (99.10)	1.709 (94.60)	2.686 (98.77)
		0.7		0.067	0.134	1.473 (94.20)	2.145 (98.72)	1.607 (95.22)	2.443 (98.82)
		0.5	3	0.086	0.179	1.738 (95.67)	2.523 (99.47)	1.961 (95.02)	3.232 (98.70)
		0.6		0.081	0.154	1.577 (94.25)	2.249 (98.82)	1.780 (94.55)	2.794 (98.77)
		0.7		0.078	0.130	1.490 (95.10)	2.124 (98.97)	1.599 (95.52)	2.507 (98.75)

Table 3: Classical results for θ_{12} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Approximate CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	1	0.030	0.094	2.318 (97.70)	3.854 (99.77)	2.078 (94.95)	3.263 (99.10)
		0.6		0.031	0.092	2.755 (97.80)	4.352 (99.55)	2.187 (95.35)	3.352 (99.25)
		0.7		0.040	0.079	3.022 (98.25)	4.833 (99.65)	2.232 (95.17)	3.339 (99.15)
		0.5	2	0.039	0.096	2.709 (97.35)	4.265 (99.70)	2.178 (95.05)	3.353 (98.92)
		0.6		0.046	0.090	3.041 (97.62)	4.794 (99.65)	2.278 (95.97)	3.411 (99.25)
		0.7		0.050	0.084	3.441 (97.52)	5.115 (99.67)	2.248 (95.37)	3.314 (98.87)
		0.5	3	0.041	0.092	2.742 (97.75)	4.282 (99.65)	2.163 (95.10)	3.338 (99.32)
		0.6		0.046	0.081	3.177 (98.37)	4.852 (99.72)	2.252 (95.70)	3.387 (99.30)
		0.7		0.053	0.075	3.658 (99.22)	5.353 (99.72)	2.204 (95.37)	3.253 (98.92)
40	34	0.5	1	0.024	0.094	2.107 (97.42)	3.461 (99.77)	1.938 (94.95)	3.118 (99.02)
		0.6		0.027	0.098	2.503 (97.40)	4.069 (99.67)	2.090 (95.82)	3.272 (99.32)
		0.7		0.033	0.094	2.943 (97.47)	4.627 (99.75)	2.212 (95.75)	3.376 (99.27)
		0.5	2	0.031	0.102	2.384 (96.55)	4.003 (99.75)	2.117 (95.07)	3.309 (98.82)
		0.6		0.039	0.094	2.774 (97.12)	4.571 (99.70)	2.187 (95.95)	3.334 (99.32)
		0.7		0.047	0.085	3.256 (97.22)	4.973 (99.57)	2.235 (95.42)	3.349 (99.07)
		0.5	3	0.040	0.096	2.379 (97.62)	3.852 (99.80)	2.089 (95.37)	3.294 (99.02)
		0.6		0.041	0.099	2.977 (97.55)	4.557 (99.75)	2.185 (95.37)	3.334 (99.27)
		0.7		0.049	0.084	3.260 (97.77)	5.141 (99.72)	2.232 (95.67)	3.332 (98.97)

Table 4: Classical results for θ_{22} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Bias	MSE	Approximate CI		Bootstrap CI	
						95%	99%	95%	99%
35	30	0.5	1	0.027	0.044	1.204 (96.32)	2.117 (99.50)	1.103 (94.67)	1.798 (98.82)
		0.6		0.034	0.052	1.558 (97.00)	2.702 (99.67)	1.268 (95.35)	2.038 (98.90)
		0.7		0.037	0.051	1.863 (97.22)	3.129 (99.72)	1.387 (96.10)	2.207 (99.20)
		0.5	2	0.030	0.053	1.427 (96.62)	2.443 (99.62)	1.207 (95.42)	1.946 (98.85)
		0.6		0.032	0.052	1.725 (96.67)	2.911 (99.62)	1.339 (95.15)	2.132 (99.12)
		0.7		0.039	0.054	2.082 (97.12)	3.519 (99.75)	1.484 (95.72)	2.329 (99.02)
		0.5	3	0.027	0.048	1.525 (96.22)	2.619 (99.75)	1.219 (95.22)	1.975 (98.92)
		0.6		0.030	0.052	1.928 (96.77)	3.390 (99.60)	1.392 (95.90)	2.215 (99.35)
		0.7		0.033	0.056	2.369 (97.02)	3.943 (99.87)	1.536 (96.47)	2.391 (99.32)
40	34	0.5	1	0.024	0.039	1.038 (95.90)	1.761 (99.52)	0.999 (94.72)	1.615 (98.77)
		0.6		0.029	0.046	1.319 (96.67)	2.263 (99.65)	1.157 (94.47)	1.875 (98.85)
		0.7		0.031	0.051	1.552 (96.07)	2.764 (99.72)	1.296 (95.47)	2.074 (99.02)
		0.5	2	0.025	0.045	1.226 (96.45)	2.062 (99.30)	1.089 (95.10)	1.774 (98.60)
		0.6		0.028	0.049	1.523 (96.67)	2.706 (99.70)	1.258 (95.47)	2.033 (98.92)
		0.7		0.031	0.056	1.947 (96.70)	3.294 (99.62)	1.384 (95.70)	2.197 (99.22)
		0.5	3	0.025	0.047	1.255 (96.50)	2.135 (99.52)	1.098 (94.95)	1.787 (98.90)
		0.6		0.030	0.053	1.603 (96.45)	2.937 (99.62)	1.277 (95.55)	2.060 (98.87)
		0.7		0.032	0.051	2.053 (97.17)	3.449 (99.75)	1.440 (95.42)	2.274 (99.07)

Table 5: Bayesian results for θ_{11} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	1	2.460 (94.70)	3.872 (99.12)	2.157 (93.35)	3.391 (98.60)
		0.6		2.281 (94.40)	3.522 (98.82)	2.028 (93.52)	3.121 (98.50)
		0.7		2.112 (94.92)	3.164 (98.80)	1.905 (93.62)	2.859 (98.62)
		0.5	2	2.256 (94.62)	3.473 (98.92)	2.006 (93.30)	3.088 (98.55)
		0.6		2.095 (94.75)	3.137 (98.92)	1.889 (93.67)	2.836 (98.42)
		0.7		1.979 (94.95)	2.925 (99.02)	1.799 (94.45)	2.663 (98.67)
		0.5	3	2.261 (94.35)	3.475 (98.62)	2.009 (92.95)	3.091 (98.10)
		0.6		2.042 (94.80)	3.065 (99.12)	1.842 (93.17)	2.766 (98.77)
		0.7		1.900 (95.55)	2.797 (99.22)	1.733 (94.02)	2.554 (98.82)
40	34	0.5	1	2.262 (94.57)	3.482 (98.92)	2.014 (93.67)	3.092 (98.82)
		0.6		1.968 (94.77)	2.914 (98.90)	1.787 (93.32)	2.651 (98.47)
		0.7		1.865 (95.45)	2.736 (98.92)	1.706 (93.92)	2.507 (98.52)
		0.5	2	1.999 (95.22)	2.969 (99.07)	1.812 (94.02)	2.696 (98.82)
		0.6		1.931 (94.00)	2.841 (98.67)	1.762 (93.80)	2.596 (98.20)
		0.7		1.787 (94.82)	2.600 (98.65)	1.644 (94.05)	2.394 (98.30)
		0.5	3	2.087 (95.02)	3.121 (98.92)	1.884 (94.35)	2.821 (98.57)
		0.6		1.916 (94.72)	2.819 (99.00)	1.749 (93.80)	2.575 (98.77)
		0.7		1.820 (95.15)	2.646 (98.92)	1.674 (94.60)	2.437 (98.85)

Table 6: Bayesian results for θ_{21} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	1	1.800 (94.70)	2.711 (99.15)	1.620 (93.67)	2.440 (98.60)
		0.6		1.630 (94.67)	2.399 (98.90)	1.486 (93.67)	2.191 (98.45)
		0.7		1.508 (94.90)	2.192 (99.15)	1.387 (93.67)	2.019 (98.77)
		0.5	2	1.627 (94.37)	2.397 (99.00)	1.481 (93.10)	2.187 (98.60)
		0.6		1.544 (94.72)	2.252 (99.00)	1.418 (93.57)	2.070 (98.60)
		0.7		1.472 (95.27)	2.126 (99.00)	1.361 (94.47)	1.968 (98.87)
		0.5	3	1.643 (93.82)	2.437 (98.47)	1.492 (92.57)	2.215 (98.22)
		0.6		1.522 (94.55)	2.219 (98.72)	1.398 (93.67)	2.041 (98.30)
		0.7		1.424 (96.02)	2.050 (99.35)	1.318 (94.70)	1.900 (98.92)
40	34	0.5	1	1.597 (93.52)	2.357 (99.05)	1.347 (91.42)	2.156 (98.40)
		0.6		1.510 (94.00)	2.192 (98.82)	1.391 (93.67)	2.021 (98.25)
		0.7		1.404 (94.47)	2.013 (98.67)	1.304 (93.77)	1.872 (98.30)
		0.5	2	1.514 (95.32)	2.199 (99.07)	1.394 (94.60)	2.026 (98.80)
		0.6		1.402 (95.27)	2.013 (98.97)	1.302 (94.10)	1.870 (98.82)
		0.7		1.329 (94.20)	1.893 (98.70)	1.240 (93.37)	1.769 (98.15)
		0.5	3	1.517 (95.20)	2.205 (98.90)	1.396 (93.90)	2.030 (98.62)
		0.6		1.419 (94.85)	2.039 (98.95)	1.316 (94.45)	1.893 (98.77)
		0.7		1.346 (95.02)	1.916 (99.05)	1.256 (94.87)	1.790 (98.95)

Table 7: Bayesian results for θ_{12} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	1	1.711 (93.52)	2.967 (98.65)	1.422 (91.10)	2.459 (97.70)
		0.6		1.921 (93.77)	3.512 (98.62)	1.552 (91.05)	2.829 (97.92)
		0.7		2.186 (93.45)	4.212 (98.60)	1.717 (90.65)	3.300 (97.55)
		0.5	2	1.871 (93.20)	3.361 (98.67)	1.524 (90.67)	2.732 (97.75)
		0.6		2.130 (93.87)	4.047 (98.57)	1.686 (90.20)	3.195 (97.42)
		0.7		2.246 (93.42)	4.421 (98.17)	1.743 (89.65)	3.428 (96.80)
		0.5	3	2.084 (94.82)	3.800 (98.52)	1.685 (93.42)	3.067 (97.85)
		0.6		2.390 (94.62)	4.613 (98.62)	1.871 (91.85)	3.610 (98.18)
		0.7		2.666 (94.27)	5.448 (98.95)	2.029 (90.77)	4.144 (98.02)
40	34	0.5	1	1.597 (93.52)	2.696 (98.42)	1.347 (91.42)	2.272 (97.47)
		0.6		1.814 (94.00)	3.216 (98.72)	1.490 (91.45)	2.636 (97.90)
		0.7		2.045 (93.92)	3.828 (98.70)	1.634 (91.02)	3.050 (97.70)
		0.5	2	1.728 (94.52)	3.010 (98.82)	1.434 (92.10)	2.488 (98.20)
		0.6		2.000 (93.12)	3.714 (98.15)	1.605 (90.57)	2.960 (97.30)
		0.7		2.163 (93.20)	4.157 (98.25)	1.701 (89.52)	3.265 (96.90)
		0.5	3	1.710 (94.07)	2.996 (98.22)	1.416 (91.25)	2.469 (97.65)
		0.6		1.974 (93.50)	3.661 (98.32)	1.582 (90.07)	2.928 (97.40)
		0.7		2.291 (93.55)	4.519 (98.77)	1.777 (89.45)	3.494 (98.02)

Table 8: Bayesian results for θ_{22} with $\theta_{11} = 1.3, \theta_{21} = 1.1, \theta_{12} = 0.7, \theta_{22} = 0.5$

n	m	τ	Scheme	Symmetric CRI		HPD CRI	
				95%	99%	95%	99%
35	30	0.5	1	0.877 (93.37)	1.364 (98.35)	0.774 (91.37)	1.203 (97.67)
		0.6		0.982 (94.30)	1.590 (98.60)	0.847 (92.12)	1.370 (97.85)
		0.7		1.149 (93.85)	1.963 (98.65)	0.961 (91.27)	1.642 (97.77)
		0.5	2	0.966 (93.70)	1.555 (98.65)	0.837 (91.57)	1.344 (98.02)
		0.6		1.149 (93.67)	1.973 (98.70)	0.961 (90.92)	1.646 (97.72)
		0.7		1.282 (93.97)	2.285 (98.55)	1.051 (90.25)	1.864 (97.78)
		0.5	3	1.083 (93.67)	1.764 (98.32)	0.931 (92.87)	1.516 (98.20)
		0.6		1.277 (94.62)	2.216 (98.75)	1.062 (93.07)	1.837 (98.27)
		0.7		1.558 (94.87)	2.918 (98.85)	1.244 (93.42)	2.319 (98.57)
40	34	0.5	1	0.794 (93.60)	1.202 (98.75)	0.712 (91.40)	1.078 (98.12)
		0.6		0.912 (93.52)	1.445 (98.70)	0.798 (92.07)	1.262 (98.00)
		0.7		1.059 (93.37)	1.755 (98.52)	0.901 (90.85)	1.492 (97.55)
		0.5	2	0.869 (93.92)	1.346 (98.67)	0.767 (91.82)	1.190 (98.07)
		0.6		1.010 (94.05)	1.650 (98.72)	0.867 (91.35)	1.413 (98.00)
		0.7		1.178 (93.85)	2.026 (98.75)	0.983 (91.42)	1.688 (98.05)
		0.5	3	0.900 (94.17)	1.418 (98.82)	0.790 (92.02)	1.242 (98.10)
		0.6		1.040 (94.30)	1.724 (98.62)	0.886 (90.72)	1.466 (97.82)
		0.7		1.222 (93.47)	2.156 (98.45)	1.006 (90.67)	1.768 (97.27)

The simulation results of the parameters in different schemes are reported in Table 1 to Table 8. It is observed that average bias of a parameter decreases as the effective sample size increases. For instance, as the value of τ increases, sample size in first stress level increases and hence the average bias of the estimators of the parameters θ_{11} and θ_{21} decreases for fixed n and m in a specific scheme. Also with increasing values of n and(or) m, the average bias of the parameters decreases. On the other hand if τ increases for fixed n and m in a specific scheme, the effective sample size in the second stress level decreases and hence the bias of the estimators of the parameters θ_{12} and θ_{22} increases. The MSEs of the estimators also behave accordingly. As expected, lengths of both the approximate and bootstrap confidence intervals of the parameters decrease with increasing sample sizes. The average length of approximate confidence interval of the parameters in the first stress level namely θ_{11} and θ_{21} are smaller than their average length of bootstrap confidence interval. However, the opposite is true for the parameters in the second stress level namely θ_{12} and θ_{22}. Thus it is not evident which method of construction of confidence intervals of the parameters works better in classical analysis. However, in both the cases the corresponding coverage probabilities are matching quite well with the specified confidence coefficients. In case of Bayesian method, it is observed that the length of the credible intervals get improved with the increasing effective sample sizes of the experiment. The coverage percentages match quite close to the corresponding nominal values. Lengths of the credible intervals are smaller than the approximate and percentile bootstrap confidence intervals of the parameters with keeping the coverage percentages as close to the nominal percentages. It is evident that one can carry out the Bayesian analysis part with the prior distribution used in this case. Thus as a practitioner's point of view, if (s)he has the prior information to use the prior distribution given in (4), it is recommended to carry out the Bayesian analysis with the same choice of hyper parameters values mentioned before. Otherwise, choose the frequentest analysis and choose bootstrap method since it involves less computational cost than that of approximate confidence interval construction method.

6 Conclusion

In this article, we have discussed progressive Type-II censoring in presence of competing risks under two stress levels. We have assumed one parameter exponential distribution for each risk factor with CEM for our analysis under the two stress levels. The MLEs of the parameters are obtained and they are conditional MLEs. We have derived the distributions of the conditional MLEs of the parameters which was used to construct approximate confidence intervals of the parameters. We also have carried out the Bayesian analysis under a flexible prior distribution and obtained symmetric and HPD credible intervals. In simulation study, satisfactory outputs have come in terms of bias, MSE and coverage probabilities. It is recommended to carry out Bayesian analysis to construct credible intervals of the parameters if prior information of the set of the parameters are available, otherwise use bootstrap method to construct their confidence intervals. It is to comment that although we have carried out the analysis with exponential distribution assumption, a more general and widely applicable distribution like Weibull distribution can be used instead of that. MLEs of the parameters and derivation of their exact distributions may give serious challenges to the investigator. More work is needed along that direction.

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Appendix

Before we proceed, let us use the following notation.

$$a(d,i,k,\theta_{11},\theta_{21},\theta_{12},\theta_{22}) = \begin{cases} \binom{d}{i}\binom{m-d}{k}\left(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\right)^{i}\left(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\right)^{d-i}\left(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\right)^{k}\left(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\right)^{m-d-k}, \\ \qquad \text{if, } d \in \{0,1,\ldots,m\}, i \in \{0,1,\ldots,d\}, \text{ and } k \in \{0,1,\ldots,m-d\} \\ 0, \qquad \text{otherwise} \end{cases}$$

Derivation of constant c

The constant c in the likelihood function (5) is such that,

$$\frac{1}{c} = \sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d} a(d,d_{11},d_{12},\theta_{11},\theta_{21},\theta_{12},\theta_{22})\Big(\frac{1}{\theta_{.1}}\Big)^{d}\Big(\frac{1}{\theta_{.2}}\Big)^{(m-d)} \times$$
$$\int_{\tau}^{\infty}\cdots\int_{\tau}^{z_{d+2:m:n}}\int_{0}^{\tau}\cdots\int_{0}^{z_{2:m:n}} e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}} dz_{1:m:n}\ldots dz_{d:m:n}dz_{d+1:m:n}\ldots dz_{m:m:n} \tag{15}$$

The right hand side of the above equation is

$$=\sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d} a(d,d_{11},d_{12},\theta_{11},\theta_{21},\theta_{12},\theta_{22})\; e^{-\frac{\tau}{\theta_{.1}}(m-d+\sum\limits_{j=d+1}^{m}R_j)}\left(\frac{1}{\theta_{.1}}\right)^{d}\int_0^{\tau}\cdots\int_0^{z_{2:m:n}}\times$$

$$e^{-\frac{1}{\theta_{.1}}\sum\limits_{j=1}^{d} z_{j:m:n}(1+R_j)} dz_{1:m:n}\dots dz_{d:m:n}\left(\frac{1}{\theta_{.2}}\right)^{(m-d)}\int_{\tau}^{\infty}\cdots\int_{\tau}^{z_{d+2:m:n}} e^{-\frac{1}{\theta_{.2}}\sum\limits_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)}\times$$

$$dz_{d+1:m:n}\dots dz_{m:m:n}$$

$$=\sum_{d=0}^{m}\sum_{d_{12}=0}^{m-d}\sum_{d_{11}=0}^{d} a(d,d_{11},d_{12},\theta_{11},\theta_{21},\theta_{12},\theta_{22})\times$$

$$\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}$$

(by using Lemma 1 from Balakrishnan, Childs and Chandrasekhar [1])

$$=\sum_{d=0}^{m}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}.$$

Thus the constant c turns out to be,

$$c=\left[\sum_{d=0}^{m}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\right]^{-1}.$$

Next we prove Theorem 1 and Theorem 2. To do so, we find the conditional moment generating functions of the MLEs of the parameters. Here we provide the derivation of moment generating functions of $\widehat{\theta}_{11}$ and $\widehat{\theta}_{12}$. The same calculations can be carried out to find the conditional moment generating functions of $\widehat{\theta}_{21}$ and $\widehat{\theta}_{22}$. In these derivations, we will use Lemma 1, given by Balakrishnan, Childs and Chandrasekhar [1]. Before we proceed for the derivations, we note the conditional distribution of $Z_{1:m:n},\dots,Z_{m:m:n}$, conditioning on the random variables D, D_{11}, D_{12},

$$f_{Z_{1:m:n},\dots,Z_{m:m:n}|\{D=d,D_{11}=i,D_{12}=k\}}(z_{1:m:n},\dots,z_{m:m:n})$$
$$=\frac{c\; a(d,i,k,\theta_{11},\theta_{21},\theta_{12},\theta_{22})\left(\frac{1}{\theta_{.1}}\right)^{d}\left(\frac{1}{\theta_{.2}}\right)^{(m-d)} e^{-\frac{w_1}{\theta_{.1}}-\frac{w_2}{\theta_{.2}}}}{P(D=d,D_{11}=i,D_{12}=k)} \qquad (16)$$

Derivation of $E\left[e^{t\widehat{\theta}_{11}}|\mathcal{D}^*\right]$

$$E\left[e^{t\widehat{\theta}_{11}}|\mathcal{D}^*\right]$$

$$
\begin{aligned}
&=\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Big[E\big[e^{t\widehat{\theta}_{11}}|D=d,D_{11}=i,D_{12}=k\big]P(D=d,D_{11}=i,D_{12}=k|\mathcal{D}^*)\Big]\\
&=\frac{1}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}\Big[E\big[e^{\frac{t}{i}w_1}|D=d,D_{11}=i,D_{12}=k\big]P(D=d,D_{11}=i,D_{12}=k)\Big]\\
&=\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}a(d,i,k,\theta_{11},\theta_{21},\theta_{12},\theta_{22})\ e^{-(\frac{1}{\theta_{.1}}-\frac{t}{i})\tau\left[m-d+\sum\limits_{j=d+1}^{m}R_j\right]}\Big(\frac{1}{\theta_{.1}}\Big)^d\Big(\frac{1}{\theta_{.2}}\Big)^{(m-d)}\times\\
&\quad\Big(1-\frac{\theta_{.1}t}{i}\Big)^{-d}\int_0^{\tau}\dots\int_0^{z_{2:m:n}}\Big(\frac{1}{\theta_{.1}}-\frac{t}{i}\Big)^d e^{-(\frac{1}{\theta_{.1}}-\frac{t}{i})\sum\limits_{j=1}^{d}z_{j:m:n}(1+R_j)}dz_{1:m:n}\dots dz_{d:m:n}\times\\
&\quad\int_{\tau}^{\infty}\dots\int_{\tau}^{z_{d+2:m:n}}e^{-\frac{1}{\theta_{.2}}\sum\limits_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)}dz_{d+1:m:n}\dots dz_{m:m:n}\\
&=\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1}a(d,i,k,\theta_{11},\theta_{21},\theta_{12},\theta_{22})\Big(1-\frac{\theta_{.1}t}{i}\Big)^{-d}\frac{e^{-(\frac{1}{\theta_{.1}}-\frac{t}{i})\tau\left[m-d+\sum\limits_{j=d+1}^{m}R_j\right]}}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\times\\
&\quad\sum_{l=0}^{d}\frac{(-1)^l}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}e^{-\left(\frac{1}{\theta_{.1}}-\frac{t}{i}\right)\tau\sum_{j=d-l+1}^{d}(1+R_j)}\\
&\quad\text{(by using Lemma 1 from Balakrishnan, Childs and Chandrasekhar [1])}\\
&=\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{i=1}^{d-1}\left[\binom{d}{i}\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^i\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^{d-i}\left[1-\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^{m-d}-\Big(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\Big)^{m-d}\right]\times\right.\\
&\quad\left.\sum_{l=0}^{d}\frac{(-1)^l\ e^{-\left(\frac{1}{\theta_{.1}}-\frac{t}{i}\right)\tau\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\big[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\big]\big[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\big]}\ \frac{\Big(1-\frac{\theta_{.1}t}{i}\Big)^{-d}}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\right].
\end{aligned}
$$

Proof of Theorem 1.

The distribution of $\widehat{\theta}_{11}$ is obtained by inverting its MGF. Similarly, the distribution of $\widehat{\theta}_{21}$ is obtained by inverting its MGF and hence **T**heorem 1 is obtained.

Similarly, $E\big[e^{t\widehat{\theta}_{12}}|\mathcal{D}^*\big]$ is obtained as:

$$
\begin{aligned}
&E\big[e^{t\widehat{\theta}_{12}}|\mathcal{D}^*\big]\\
&=\frac{c}{P(\mathcal{D}^*)}\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\left[\left[1-\Big(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\Big)^d-\Big(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\Big)^d\right]\binom{m-d}{k}\Big(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\Big)^k\times\right.
\end{aligned}
$$

$$\left(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\right)^{m-d-k}\sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{i=d-l+1}^{m}R_i\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\times$$
$$\left.\frac{\left(1-\frac{t\theta_{.2}}{k}\right)^{-(m-d)}}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}\right]$$

(by using Lemma 1 from Balakrishnan, Childs and Chandrasekhar [1]).

Proof of Theorem 2

The distribution of $\widehat{\theta}_{12}$ is obtained by inverting its MGF. Similarly, the distribution of $\widehat{\theta}_{22}$ is obtained by inverting its MGF and hence **T**heorem 2 is obtained.

Derivation of $P(\mathcal{D}^*)$

$$\begin{aligned}
P(\mathcal{D}^*) =& P(D_{11}>0, D_{21}>0, D_{12}>0, D_{22}>0) \\
=& \sum_{d=2}^{m-2}\sum_{i=1}^{d-1}\sum_{k=1}^{m-d-1} P(D=d, D_{11}=i, D_{21}=k) \\
=& c\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1} a(d,i,k,\theta_{11},\theta_{21},\theta_{12},\theta_{22})\ e^{-\frac{\tau}{\theta_{.1}}\left[m-d+\sum\limits_{j=d+1}^{m}R_j\right]}\left(\frac{1}{\theta_{.1}}\right)^{d}\times \\
& \int_0^{\tau}\cdots\int_0^{z_{2:m:n}} e^{-\frac{1}{\theta_{.1}}\sum\limits_{j=1}^{d} z_{j:m:n}(1+R_j)} dz_{1:m:n}\ldots dz_{d:m:n}\left(\frac{1}{\theta_{.2}}\right)^{(m-d)}\times \\
& \int_{\tau}^{\infty}\cdots\int_{\tau}^{z_{d+2:m:n}} e^{-\frac{1}{\theta_{.2}}\sum\limits_{j=d+1}^{m}(z_{j:m:n}-\tau)(1+R_j)} dz_{d+1:m:n}\ldots dz_{m:m:n} \\
=& c\sum_{d=2}^{m-2}\sum_{k=1}^{m-d-1}\sum_{i=1}^{d-1} a(d,i,k,\theta_{11},\theta_{21},\theta_{12},\theta_{22})\times \\
& \sum_{l=0}^{d}\frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m}R_j\right]}}{\left[\prod_{j=1}^{l}\sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l}\sum_{p=j}^{d-l}(1+R_p)\right]}\frac{1}{\prod_{j=1}^{m-d}\sum_{p=j}^{m-d}(1+R_p)}
\end{aligned}$$

(by using Lemma 1 from Balakrishnan, Childs and Chandrasekhar [1])

$$=c\sum_{d=2}^{m-2}\left[\left[1-\left(\frac{\theta_{21}}{\theta_{11}+\theta_{21}}\right)^{d}-\left(\frac{\theta_{11}}{\theta_{11}+\theta_{21}}\right)^{d}\right]\left[1-\left(\frac{\theta_{22}}{\theta_{12}+\theta_{22}}\right)^{m-d}-\left(\frac{\theta_{12}}{\theta_{12}+\theta_{22}}\right)^{m-d}\right]\right.$$

$$\sum_{l=0}^{d} \frac{(-1)^l e^{-\frac{\tau}{\theta_{.1}}\left[l+m-d+\sum_{j=d-l+1}^{m} R_j\right]}}{\left[\prod_{j=1}^{l} \sum_{p=d-l+1}^{d-l+j}(1+R_p)\right]\left[\prod_{j=1}^{d-l} \sum_{p=j}^{d-l}(1+R_p)\right]} \frac{1}{\prod_{j=1}^{m-d} \sum_{p=j}^{m-d}(1+R_p)}\Bigg].$$

Proof of Lemma 1

If part: The distribution of (X, Y) is

$$f_{(X,Y)}(x,y) = b_0^{a_0} \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} e^{-b_0(x+y)} x^{a_1-1} y^{a_2-1} (x+y)^{a_0-a_1-a_2}, \; x>0, y>0$$

Define,

$$U = X+Y, \quad \text{and} \quad V = \frac{X}{X+Y}. \tag{17}$$

The Jacobian of the transformation is $-U$. Thus the distribution of (U, V) is obtained as,

$$f_{(U,V)}(u,v) = \frac{b_0^{a_0}}{\Gamma(a_0)} e^{-b_0 u} u^{a_0-1} \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} v^{a_1-1}(1-v)^{a_2-1}, \; u>0, 0<v<1.$$

Hence $U \sim Gamma(a_0, b_0)$ and $V \sim Beta(a_1, a_2)$.

Only if part: The joint distribution of U and V, where U and V are same as defined in (17) is,

$$f_{(U,V)}(u,v) = \frac{b_0^{a_0}}{\Gamma(a_0)} e^{-b_0 u} u^{a_0-1} \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} v^{a_1-1}(1-v)^{a_2-1}, \; u>0, 0<v<1.$$

Thus $X = UV$ and $Y = U(1-V)$ and the Jacobian of the transformation is $-\frac{1}{U}$. Hence the distribution of (X, Y) is obtained as,

$$f_{(X,Y)}(x,y) = b_0^{a_0} \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} e^{-b_0(x+y)} x^{a_1-1} y^{a_2-1} (x+y)^{a_0-a_1-a_2}, \; x>0, y>0.$$

Hence $(X, Y) \sim BG(b_0, a_0, a_1, a_2)$.

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