Bayesian Order Restricted Inference of A Weibull Multi-step Step-Stress Model

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Abstract

Standard life testing experiments are not appropriate for testing highly reliable items as they often turn out to be time consuming and expensive under normal operating conditions. Under such scenario, accelerated life test (ALT) is often employed to obtain failure time data. In this paper, we consider a multi-step step-stress ALT (SSALT) model when the data are Type-II censored. The lifetime distribution of the experimental units at each stress level is assumed to follow a two-parameter Weibull distribution. Further, the distributions under each of the stress levels are connected through a failure rate based multi-step SSALT model. In a step-stress experiment, the mean lifetime of the experimental units is expected to shorten with elevation in stress levels. Taking this into account, the main aim of this paper is to develop the order restricted inference of the model parameters in a multi-step step-stress set up in a Bayesian approach. Although for the given shape parameter, the order restricted maximum likelihood estimators (MLEs) of the model parameters can be obtained explicitly, they are in complicated forms. Additionally, the exact joint distribution of the MLEs and hence the exact CIs cannot be obtained. Under such scenario, Bayesian choice seems to be a natural one. Prior information about the model parameters is incorporated through a flexible multivariate prior distribution and it preserves the ordering of the mean lifetimes in the Bayes estimates(BEs). We propose to use the importance sampling algorithm to obtain the BEs and the associated credible intervals(CRIs). An extensive simulation

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study has been carried out and our approach is illustrated with two real datasets. A novel optimal plan is proposed to determine the stress changing time points based on the sum of the posterior variances.

Key Words: Weibull distribution; step-stress model; failure rate based model; Bayes estimator; credible interval; optimal stress-changing time.

1 Introduction

Industrial life testing experiments often turn out to be a challenging task for reliability practitioners. With the unprecedented progress of science and technology over the last few decades, quality of products has improved a lot and hence, products with high reliability are quite common nowadays. However, under NOCs, the implementation of life testing experiments for such products require huge cost and excessive time. Further, it fails to meet the requirements of reliability evaluation. Under such scenarios, ALT experiments are usually undertaken to speed up the failure process, thus saving time and cost.

A special case of the ALT experiment is the SSALT experiment, wherein the condition is made extreme at given time points or after specified number of failures. A general set up of a fixed time point multi-step SSALT experiment is as follows. Suppose, n items are exposed to an initial stress level s_1. At fixed time points $\tau_1,\ \tau_2, \ldots, \tau_m$, the stress levels are increased to $s_2,\ s_3, \ldots, s_{m+1}$, respectively. In this process, distributional assumptions are usually made for the lifetime of the units corresponding to each stress level and the failure times are recorded in an ascending order. Inference on the lifetime distributions under each stress level is performed first under the accelerated set up and then the results are translated back to predict the reliability characteristics under normal conditions. Some of the key references on different ALT models are available in Nelson [26], Bagdonavicius and Nikulin [4] and the references cited therein.

The two most common models to assess the impact of stress levels on the lifetime distribution are the cumulative exposure model (CEM) proposed by Sedyakin [33] and the tampered failure rate model (TFRM) introduced by Bhattacharyya and Soejoeti [9]. Similar to TFRM, Khamis-Higgins model (KHM) was proposed by Khamis and Higgins [17] when lifetimes at the different stress levels follow Weibull distribution with a common shape parameter and different scale parameters. Under CEM, the remaining lifetime of an experimental unit depends only on the cumulative exposure accumulated and the current stress level, irrespective of how it has been actually accumulated, while under TFRM, the change of stress level has a multiplicative effect on the subsequent hazard rate. If $h_i(.)$ is the failure rate corresponding to the absolutely continuous CDF $F_i(.)$ with PDF $f_i(.)$, of the failure time distribution at the $i-$th stress level, then $h_i(.)$ is given by

$$h_i(t) = \frac{f_i(t)}{1 - F_i(t)}; \;\; i = 1, 2, \ldots, m+1.$$

Recently, Kateri and Kamps [15] introduced the failure rate-based SSALT model. It states that the overall hazard function $h(t)$ under the step-stress pattern for $i = 1, 2, \ldots, m+1$, is expressed as

$$h(t)= \begin{cases} h_1(t) & \text{if} \quad 0 < t \le \tau_1 \\ h_i(t) & \text{if} \quad \tau_{i-1} < t \le \tau_i. \end{cases}$$

Hence, the compound CDF $G(.)$ for $i = 1, 2, \ldots, m+1$, can be expressed as

$$G(t)= \begin{cases} F_1(t) & \text{if} \quad 0 < t \le \tau_1 \\ 1 - \Big\{ \prod_{j=1}^{i-1} \frac{1 - F_j(\tau_j)}{1 - F_{j+1}(\tau_j)} \Big\} (1 - F_i(t)) & \text{if} \quad \tau_{i-1} < t \le \tau_i. \end{cases}$$

In a life testing experiment, experimenters often come across units with high reliability. Though, reduction in duration of a life testing experiment under a SSALT set up is obvious, yet it is not controlled. The control of the total experimental duration is equally crucial and can be achieved by imposing different censoring schemes. The experimenter can fix either a) its duration $\tau > \tau_m$ or b) the total number of observed failures ($R = r$, $1 \leq r \leq n$). In case a), the corresponding sample is Type-I censored, where the number of failures ($R \geq 1$) upto time τ, is random. In case b), the corresponding sample is Type-II censored and the experiment stops as soon as the $r-$th failure occurs.

A two-parameter Weibull (Weib) distribution with the shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ has the following cumulative distribution function (CDF), probability density function (PDF) and hazard function (HF), respectively,

$$F(t; \alpha, \lambda) = 1 - e^{-\lambda t^{\alpha}}, \qquad t > 0, \tag{1}$$

$$f(t; \alpha, \lambda) = \alpha \lambda t^{\alpha - 1} e^{-\lambda t^{\alpha}}, \qquad t > 0, \tag{2}$$

$$H(t; \alpha, \lambda) = \alpha \lambda t^{\alpha - 1}, \qquad t > 0. \tag{3}$$

Because of the various shapes of the PDF and its convenient representation of the survival function, the Weibull distribution has been used very effectively for analyzing lifetime data, especially for censored data, which is very common in most life testing experiments. From now on, a Weibull distribution with the shape parameter α and the scale parameter λ will be denoted by Weib(α, λ). Note that Weib$(1, \lambda)$ is the standard exponential distribution with mean $\frac{1}{\lambda}$.

The CEM is extensively studied in the literature when the lifetime distribution is exponential, see, for example, Bagdonavicius [3], Nelson [26], and the recent exhaustive review article by Balakrishnan [7]. Analysis of the CEM was a dressed by Komori [18], when the lifetimes of the experimental units follow the Weibull distribution. Under the same distributional assumptions, Bai and Kim [5] and Kateri and Balakrishnan [14] discussed the inferential aspects of step-stress model under Type-I and Type-II censoring schemes, respectively.

The main intent of a SSALT experiment is to reduce the lifetime of the experimental units by increasing the stress level. Therefore, it is very natural to make the assumption that the expected lifetime of the experimental units is lower at the higher stress level. However, most of the contributions on parametric inference in SSALT experiments ignore this assumption. **?**] first incorporated this information in a multi-step SSALT and considered the maximum likelihood estimation of the model parameters assuming that the lifetime of the experimental units follows exponential distribution. Recently Pal et al. [27] considered the classical order restricted inference of the model parameters under similar set up when the failure time distribution at the $i-$th stress level is Weib(α, λ_i), $i = 1, 2, \ldots, m+1$. Although for fixed α, the order restricted MLEs of the scale parameters can be obtained explicitly and the MLE of the shape parameter needs to be obtained numerically, the estimates are in complicated form. Additionally, the exact joint distribution of the MLEs and hence the exact CIs cannot be obtained in case the order restriction is present. Under such scenarios, Bayesian inference becomes a natural alternative. Bayesian inference for the analysis of SSALT has been addressed by many authors. Dorp et al. [12] considered the exponential Type-I censored case, where the transformed failure rates at different stress levels are modeled with a generalized Dirichlet prior. Dorp and Mazzuchi [11] developed the Bayesian framework for CEM and Weibull failure times. Sha and Pan [34] considered the Bayesian analysis for the Weibull proportional hazard (PH) model in multi-step SSALT set up where the scale parameters are related to the stress levels through a log-linear function. A "conjugate-like " prior is constructed and a Markov chain Monte Carlo (MCMC) algorithm with adaptive rejection sampling technique is used for posterior inference.

Recently Samanta et al. [29] and Samanta and Kundu [31] also considered the similar problem mentioned in Balakrishnan et al. [8]. They developed the order restricted classical inference of the model parameters using a reparametrization technique and the corresponding Bayesian inference under the squared error loss function based on importance sampling technique in case of exponential and generalized exponential distributions respectively.

This work aims in developing the statistical analysis of a multi-step SSALT model from a Bayesian viewpoint when the lifetime distribution of the experimental units under $i-$th stress level is Weib(α, λ_i), $i = 1, 2, \ldots, m+1$, and it satisfies the failure rate based SSALT model assumptions. Assumption of common shape parameter in Weibull distribution is quite common (see for example Mondal and Kundu [25], Samanta et al. [30]) and it makes the problem mathematically more tractable. However, different shape parameters may also be incorporated at different stress levels to make the model more flexible, although it has not been attempted here. Order restricted Bayesian inference of the model parameters is developed using a very flexible multivariate prior assumption which retains the natural ordering in the mean lifetimes ($\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1}$) without assuming any functional structure. The BEs of the unknown model parameters are derived and the associated CRIs are constructed using the importance sampling algorithm . Extensive simulation study is carried out for both the censored and the complete sample cases to see the effectiveness of the model. In addition, we provide the analysis of two real data sets for illustrative purposes.

The SSALT being a quite common practice in product life testing, reduces the time duration between product design and product release time. It has hence a direct impact on product performance and reliability improvement. Undoubtedly, an important related aspect is thus the optimal planning of the SSALT subject to certain optimization criteria. Keeping in mind the order restriction on the mean failure times, we propose a novel optimal plan to choose the optimal hold time points $\tau_1, \ldots, \tau_m$, at which the stress levels are altered. The optimization criterion here is defined as a constrained minimization of sum of the expected posterior variances of the BEs of the model parameters. The proposed method is simple to implement even when stress changes are frequent, or in other words, the number of stress levels (m) is large. Since, the posterior variances of the BEs of the unknown model parameters are not in closed form, we resort to the Lindley's approximation (See Lindley [21] for details). We present the optimal stress-changing time points for different prior assumptions and sample sizes in Section 5.

Rest of the paper is organized as follows. In Section 2, we provide the model assumption and the likelihood function constructed on the basis of the available data. In Section 3.1, we state the prior assumptions on the model parameters keeping in mind the order restriction. The proposed methodology to obtain the BEs and the associated CRIs is discussed in Section

3.2. Numerical exercise and the analysis of two real life data sets are presented in Section 4. In Section 5, the optimal planning of a multi-step SSALT is considered. Finally, we conclude the paper in Section 6.

2 Model Assumptions and the likelihood function

We assume that n identical units are placed on a life testing experiment, and subjected to an initial stress level s_1 at the time point 0. At pre- fixed time points $\tau_1, \tau_2, \ldots, \tau_m$, the stress level increases to $s_2, s_3, \ldots, s_{m+1}$, respectively. Finally, the experiment terminates as soon as the $r-$th $(1 \leq r \leq n)$ unit fails. The Type-II censored failure time data thus obtained from this multi-step SSALT experiment, is given by

$$\mathcal{D} = \{t_{1:n} < \ldots < t_{\bar{n}_1:n} < \tau_1 < t_{\bar{n}_1+1:n} < \ldots < t_{\bar{n}_2:n} < \tau_2 < \ldots < \tau_m < t_{\bar{n}_m+1:n} < \ldots < t_{r:n}\}. \tag{4}$$

Here n_k is the number of failures under stress level s_k $(k = 1, \ldots, m+1)$ and $\bar{n}_j = \sum_{i=1}^{j} n_i$ is the total number of failures upto the $j-$ th stress level. Further, it is assumed that the lifetime distribution of the experimental units under the stress level s_k follows Weib(α, λ_k). Hence, for $\alpha > 0$, $\lambda_k > 0$ and $t > 0$, the CDF and the HF of the lifetime distribution at the $k-$ th stress level is given by

$$F_k(t) = \left(1 - e^{-\lambda_k t^\alpha}\right), \; h_k(t) = \alpha \lambda_k t^{\alpha-1}, \qquad k = 1, \ldots, m+1.$$

If s_j ; $j = 1, 2, \ldots, m+1$ are the $m+1$ stress levels and τ_i ; $i = 1, 2, \ldots, m$ are the pre-determined time points at which the stress level changes from s_i to s_{i+1}, the overall HF based on the failure rate based SSALT model assumptions under the step-stress pattern is given by

$$h(t)=\begin{cases}\lambda_1 h_o(t) & \text{if} \quad 0<t\le\tau_1,\\ \lambda_k h_o(t) & \text{if} \quad \tau_{k-1}<t<\tau_k,; \; k=2,3,\ldots,m,\\ \lambda_{m+1} h_o(t) & \text{if} \quad \tau_m<t<\infty,\end{cases}$$

where $h_o(t)$ is the HF corresponding to the baseline CDF $F_o(t)=1-e^{-t^{\alpha}}$. The flexibility of this model can be very useful in multi-step SSALT experiments. Some recent references for more insight and detailed interpretations of this model can be found in Kateri and Kamps [15, 16].

We denote $\boldsymbol{\theta}=(\alpha,\,\lambda_1,\,\lambda_2,\,\ldots,\lambda_{m+1})$, the set of model parameters to be estimated. Using the one-to-one correspondence between the HF and the CDF, the overall CDF and the associated PDF are given by

$$F(t)=\begin{cases}0 & \text{if } t<0,\\ 1-e^{-\lambda_1 t^{\alpha}} & \text{if } 0<t\le\tau_1,\\ 1-e^{-\sum\limits_{j=1}^{k-1}(\lambda_j-\lambda_{j+1})\tau_j^{\alpha}-\lambda_k t^{\alpha}} & \text{if } \tau_{k-1}<t\le\tau_k \;;\; k=2,3,\ldots,m,\\ 1-e^{-\sum\limits_{j=1}^{m}(\lambda_j-\lambda_{j+1})\tau_j^{\alpha}-\lambda_{m+1} t^{\alpha}} & \text{if } \tau_m<t<\infty,\end{cases}$$

$$f(t)=\begin{cases}\alpha\lambda_1 t^{\alpha-1}e^{-\lambda_1 t^{\alpha}} & \text{if } 0<t\le\tau_1,\\ e^{-\sum\limits_{j=1}^{k-1}(\lambda_j-\lambda_{j+1})\tau_j^{\alpha}}\alpha\lambda_k t^{\alpha-1}e^{-\lambda_k t^{\alpha}} & \text{if } \tau_{k-1}<t\le\tau_k \;;\; k=2,3,\ldots,m,\\ e^{-\sum\limits_{j=1}^{m}(\lambda_j-\lambda_{j+1})\tau_j^{\alpha}}\alpha\lambda_{m+1} t^{\alpha-1}e^{-\lambda_{m+1} t^{\alpha}} & \text{if } \tau_m<t<\infty,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, based on the censored data (4), the likelihood function can be written as

$$L(\boldsymbol{\theta}|\mathcal{D}) \;\propto\; \alpha^r\lambda_1^{n_1}\lambda_2^{n_2}\ldots\lambda_{m+1}^{r-\bar{n}_m}\Big(\prod_{i=1}^{r} t_{i:n}^{\alpha-1}\Big)e^{-(\lambda_1 D_1(\alpha)+\lambda_2 D_2(\alpha)+\ldots+\lambda_{m+1}D_{m+1}(\alpha))}, \tag{5}$$

where for $j = 1, 2, \ldots, m+1$,

$$D_j(\alpha) \;= \Big[\sum_{i=\overline{n}_{j-1}+1}^{\overline{n}_j} t_{i:n}^{\alpha} + (n - \overline{n}_j)\tau_j^{\alpha} - (n - \overline{n}_{j-1})\tau_{j-1}^{\alpha} \Big], \;\; \overline{n}_0 = \tau_0 = 0, \;\; \bar{n}_{m+1} = r.$$

3 Bayesian inference

In this section, we consider the order restricted Bayesian inference of the unknown model parameters. Before proceeding further, we would like to introduce the following notations which will be used in the subsequent sections. A gamma random variable with the shape parameter $\widetilde{\alpha} > 0$ and scale parameter $\widetilde{\lambda} > 0$ has the PDF

$$f_{GA}(x; \widetilde{\alpha}, \widetilde{\lambda}) = \frac{\widetilde{\lambda}^{\widetilde{\alpha}}}{\Gamma(\widetilde{\alpha})} x^{\widetilde{\alpha}-1} e^{-\widetilde{\lambda}x}; \quad x > 0.$$

It will be denoted by $\text{GA}(\widetilde{\alpha}, \widetilde{\lambda})$. A Dirichlet random vector with the parameters $a_i > 0$, $i = 1, \ldots, m+1$, has the PDF

$$f_{DIR}(\mathbf{x}; \theta) = \frac{\Gamma(a_1 + a_2 + \ldots + a_{m+1})}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_{m+1})} \Big\{ \prod_{i=1}^{m} x_i^{a_i - 1} \Big\} \Big(1 - \sum_{i=1}^{m} x_i \Big)^{a_{m+1}-1}; \quad 0 < \sum_{i=1}^{m} x_i < 1,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_{m+1})$. It will be denoted by $\text{DIR}(\boldsymbol{a})$. The following prior assumptions are made on the common shape parameter α and on the scale parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{m+1})$, considering the order restriction on λ_i's. Following the ideas of Pena and Gupta [28], it is assumed that

$$\begin{aligned} \lambda &= \sum_{i=1}^{m+1} \lambda_i \sim GA(a_0, b_0) \\ \mathbf{p} &= \Big(\frac{\lambda_1}{\sum_{i=1}^{m+1} \lambda_i}, \frac{\lambda_2}{\sum_{i=1}^{m+1} \lambda_i}, \ldots, \frac{\lambda_m}{\sum_{i=1}^{m+1} \lambda_i} \Big) \sim DIR(\boldsymbol{a}) \end{aligned} \tag{6}$$

with $a_0 > 0$, $b_0 > 0$, $a_i > 0$, $i = 1, \ldots, m+1$, and they are independently distributed. The joint PDF of $\boldsymbol{\lambda}$ can be obtained as follows

$$\begin{aligned} \pi(\boldsymbol{\lambda}|a_o, b_o, \boldsymbol{a}) &= \frac{\Gamma(a_1 + a_2 + \ldots + a_{m+1})}{\Gamma(a_o)} (b_o\lambda)^{a_o - a_1 - a_2 - \ldots - a_{m+1}} \times \\ & \left(\frac{b_o^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1 - 1} e^{-b_o\lambda_1} \right) \times \left(\frac{b_o^{a_2}}{\Gamma(a_2)} \lambda_2^{a_2 - 1} e^{-b_o\lambda_2} \right) \times \\ & \vdots \\ & \times \left(\frac{b_o^{a_{m+1}}}{\Gamma(a_{m+1})} \lambda_{m+1}^{a_{m+1} - 1} e^{-b_o\lambda_{m+1}} \right). \end{aligned} \tag{7}$$

It is known as the Dirichlet- Gamma PDF, and will be denoted by $\text{DG}(a_o, b_o, \boldsymbol{a})$. It may be noted that $\text{DG}(a_o, b_o, \boldsymbol{a})$ is a very flexible multivariate distribution with support over the unit m- simplex. The Dirichlet-Gamma prior can be used quite flexibly to model the scale parameters. It can assume different shapes depending on the values of the hyperparameters. The following result gives the mean and variance of the Dirichlet- Gamma PDF.

Result 1. If $\boldsymbol{\lambda} \sim$ DG $(a_o, b_o, \boldsymbol{a})$, then for $i = 1, 2, \ldots, m+1$,

$$E(\lambda_i) = \frac{a_o a_i}{b_o \sum_{i=0}^{m+1} a_i}, \text{ and } V(\lambda_i) = \frac{a_o a_i}{b_o^2 \sum_{i=0}^{m+1} a_i} \left\{ \frac{(a_o + 1)(a_i + 1)}{(\sum_{i=0}^{m+1} a_i + 1)} - \frac{a_o a_i}{\sum_{i=0}^{m+1} a_i} \right\}.$$

Proof: See Appendix.

It is quite simple to generate a sample from a Dirichlet-Gamma distribution using the condition that $\boldsymbol{\lambda} \sim \text{DG}(a_o, b_o, \boldsymbol{a})$, if and only if $\sum_{i=1}^{m+1} \lambda_i$ has a gamma distribution and the vector of proportions $\left(\frac{\lambda_1}{\sum_{i=1}^{m+1} \lambda_i}, \frac{\lambda_2}{\sum_{i=1}^{m+1} \lambda_i}, \ldots, \frac{\lambda_m}{\sum_{i=1}^{m+1} \lambda_i} \right)$ has a Dirichlet distribution, and they are independently distributed. The following steps are followed to generate a sample from $\text{DG}(a_o, b_o, \boldsymbol{a})$.

Step 1. Generate X from $\text{GA}(a_o, b_o)$.

Step 2. Generate $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$ from $\text{DIR}(\boldsymbol{a})$.

To generate from $\text{DIR}(\boldsymbol{a})$, we provide below the method of Devroye [10].

- Generate $Z_1, Z_2, \ldots, Z_{m+1}$ independent gamma random variables such that $Z_k \sim \text{GA}(a_k, 1)$, $k = 1, 2, \ldots, m+1$.

- Define $Z = \sum_{i=1}^{m} Z_i$ and $Y_i = \frac{Z_i}{Z}$, $i = 1, 2, \ldots, m$. Then $(Y_1, Y_2, \ldots, Y_m) \sim \text{DIR}(\boldsymbol{a})$,

STEP 3. Now obtain $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{m+1})$ by setting $\lambda_i = Y_i X$, $i = 1, 2, \ldots, m$ and $\lambda_{m+1} = (1 - Y_1 - Y_2 - \ldots - Y_m)X$.

For $m = 1$, the corresponding analogue is the Beta-Gamma distribution, see, for example Kundu and Pradhan [19].

3.1 Prior assumptions : order restricted

Keeping in mind the order restriction on the scale parameters $(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1})$, the following prior assumption is made on $(\lambda_1, \lambda_2, \ldots, \lambda_{m+1})$.

$$\pi_{\text{ord}}(\boldsymbol{\lambda}|a_o, b_o, \boldsymbol{a}) \propto \frac{\Gamma(a_1 + a_2 + \ldots + a_{m+1})}{\Gamma(a_o)\Gamma(a_1) \ldots \Gamma(a_{m+1})} {b_o}^{a_o} \lambda^{a_o - \sum_{i=1}^{m+1} a_i} e^{-b_o \lambda} \times \sum_{\mathcal{P}} \left(\lambda_{i_1}^{a_1 - 1} \lambda_{i_2}^{a_2 - 1} \ldots \lambda_{i_{m+1}}^{a_{m+1} - 1} \right); \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m+1} < \infty. \tag{8}$$

Note that (8) is the ordered Dirichlet-Gamma PDF and we will denote it by $\text{ODG}(a_o, b_o, \boldsymbol{a})$. Here, $\mathcal{P}$ denotes the set of all the $(m+1)!$ permutations of $\{1, 2, \ldots, m+1\}$. It is quite straightforward to generate a sample from an ordered Dirichlet-Gamma distribution. At first, we generate a sample from a Dirichlet-Gamma distribution and then by sorting, we obtain a random sample from an ordered Dirichlet-Gamma distribution.

3.2 Bayes estimates and credible intervals

In this subsection, we provide BEs of the unknown model parameters and the associated CRIs. Although we have considered the squared error loss function, any other loss function can easily be incorporated. It is assumed that α, $\boldsymbol{\lambda}$ have prior distributions $\pi_{\text{shape}}(.)$, $\pi_{\text{ord}}(.)$, respectively. For $a > 0$, $b > 0$, $a_i > 0$, $i = 1, 2, \ldots, m+1$, the priors for the order restricted case are

$$\pi_{\text{shape}}(\alpha|a, b) \sim \text{GA(a,b)}, \;\; \pi_{\text{ord}}(\boldsymbol{\lambda}|a_o, b_o, \boldsymbol{a}) \sim \text{ODG}(a_o, b_o, \boldsymbol{a}),$$

and they are assumed to be independently distributed. Hence, the joint posterior density function of $\boldsymbol{\theta}$ for $\alpha > 0$, $0 < \lambda_1 \leq \lambda_2 \leq \ldots < \lambda_{m+1}$, can be written as

$$\begin{aligned}\pi(\boldsymbol{\theta}|\mathcal{D}) \;\propto\; & \lambda^{a_o - a_1 - a_2 - \ldots - a_{m+1}} \times \prod_{j=1}^{m+1} \left(\lambda_j^{n_j} e^{-\lambda_j (b_o + D_j(\alpha))} \right) \times \\ & \sum_{(i_1, i_2, \ldots, i_{m+1}) \in \mathcal{P}} \left(\lambda_{i_1}^{a_1 - 1} \lambda_{i_2}^{a_2 - 1} \ldots \lambda_{i_{m+1}}^{a_{m+1} - 1} \right) \times \alpha^{r + a - 1} e^{-b\alpha} \prod_{i=1}^{r} t_{i:n}^{\alpha}. \end{aligned} \tag{9}$$

The BE of some parametric function of $\boldsymbol{\theta}$, say $h(\boldsymbol{\theta})$, under the squared error loss function is the posterior expectation of $h(\boldsymbol{\theta})$ and it is given by

$$\widehat{h}(\boldsymbol{\theta}) = E_{\pi(\boldsymbol{\theta}|\mathcal{D})} \left(h(\boldsymbol{\theta}) \right) = \int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty} h(\boldsymbol{\theta}) \pi(\boldsymbol{\theta}|\mathcal{D}) d\alpha d\lambda_1 \ldots d\lambda_{m+1}, \tag{10}$$

provided the expectation exists. Due to absence of explicit form of (10), we propose importance sampling technique to compute the BE and the associated CRI. We rewrite the posterior density in (9) as follows.

$$\begin{aligned}\pi(\boldsymbol{\theta}|\mathcal{D}) \;\propto\; & \lambda^{a_o - a_1 - a_2 - \ldots - a_{m+1}} \times \prod_{j=1}^{m+1} \left(\lambda_j^{n_j - J} e^{-\lambda_j (D_j(\alpha)) - S(\alpha)} \right) \times \\ & \sum_{\mathcal{P}} \left(\lambda_{i_1}^{a_1 + J - 1} \lambda_{i_2}^{a_2 + J - 1} \ldots \lambda_{i_{m+1}}^{a_{m+1} + J - 1} \right) \times e^{-\lambda (b_o + S(\alpha))} \times \\ & \alpha^{r + a - 1} e^{-\alpha (b - \sum\limits_{i=1}^{r} \log t_{i:n})}, \end{aligned}$$

where $\lambda = \sum_{j=1}^{m+1} \lambda_j$, $J = \min\{n_1, n_2, \ldots, n_{m+1}\}$, and $S(\alpha) = \min\{D_1(\alpha), D_2(\alpha), \ldots, D_{m+1}(\alpha)\}$. The posterior density function in this case can be written as

$$\pi(\alpha, \boldsymbol{\lambda}|\mathcal{D}) \;\propto\; \pi_1^*(\boldsymbol{\lambda}|\mathcal{D}, \alpha) \times \pi_2^*(\alpha|\mathcal{D}) \times \mathrm{g}(\alpha, \boldsymbol{\lambda}|\mathcal{D}), \tag{11}$$

where

$$\pi_1^*(\boldsymbol{\lambda}|\mathcal{D},\alpha) \sim \text{ODG}(a_o+(m+1)J, b_o+S(\alpha), a_1+J, a_2+J, \ldots, a_{m+1}+J), \quad (12)$$

$$\pi_2^*(\alpha|\mathcal{D}) \propto \alpha^{r+a-1} e^{-\alpha(b-\sum\limits_{i=1}^{r}\ln t_{i:n})}, \quad (13)$$

$$g(\boldsymbol{\theta}|\mathcal{D}) = \frac{\prod\limits_{j=1}^{m+1}[\lambda_j^{n_j-J} e^{-\lambda_j(D_j(\alpha)-S(\alpha))}]}{[b_o+S(\alpha)]^{a_o+(m+1)J}}. \quad (14)$$

Now for $\pi_2^*(\alpha|\mathcal{D})$ to be a proper density, we require $b > \sum\limits_{i=1}^{r} \log t_{i:n}$. Since, $\pi_2^*(\alpha|\mathcal{D})$ is a well known log-concave density and generation from $\pi_1^*(\boldsymbol{\lambda}|\mathcal{D},\alpha)$ is quite convenient to perform, we are able to use the following importance sampling algorithm to compute the BE and the associated CRI of $h(\boldsymbol{\theta})$.

Algorithm 1

Step 1. Generate α from $\pi_2^*(\alpha|\mathcal{D})$.

Step 2. For a given α, generate $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}$ from $\pi_1^*(\boldsymbol{\lambda}|\mathcal{D},\alpha)$.

Step 3. Repeat the procedure M times to generate $(\alpha_1, \lambda_{11}, \lambda_{12}, \ldots, \lambda_{1m+1})$, $\ldots, (\alpha_M, \lambda_{M1}, \lambda_{M2}, \ldots, \lambda_{Mm+1})$.

Step 4. To obtain BE of $h(\boldsymbol{\theta})$, compute $(h_1, h_2, \ldots, h_M)$, where $h_i = h(\boldsymbol{\theta_i})$, where $\boldsymbol{\theta_i} = (\alpha_i, \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{im+1})$.

Step 5. Compute $g_i = g(\boldsymbol{\theta_i})$; $i = 1, 2, \ldots, M$.

Step 6. Calculate the weights $w_j = \frac{g_j}{\sum_{i=1}^{M} g_i}$; $j = 1, 2, \ldots, M$

Step 7. Compute the BE of $h(\boldsymbol{\theta})$ under the squared error loss function as $\widehat{h}_{BE}(\boldsymbol{\theta}) = \frac{\sum_{j=1}^{M} g_j h_j}{\sum_{i=1}^{M} g_i}$.

Step 8. To construct $100(1-\gamma)\%$ $(0 < \gamma < 1)$ CRI for $h(\boldsymbol{\theta})$, arrange h_j's in ascending order to obtain $(h_{(1)}, h_{(2)}, \ldots, h_{(M)})$ and arrange corresponding w_j's accordingly to get $(w_{(1)}, w_{(2)}, \ldots, w_{(M)})$. Note that $(w_{(1)}, w_{(2)}, \ldots, w_{(M)})$ may not be ordered.

Step 9. A $100(1-\gamma)\%$ CRI for $h(\boldsymbol{\theta})$ can be obtained as $(h_{j_1},\, h_{j_2})$, where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, M\}, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w_{(i)} \leq 1-\gamma < \sum_{i=j_1}^{j_2+1} w_{(i)}. \tag{15}$$

The $100(1-\gamma)\%$ highest posterior density (HPD) CRI of $h(\boldsymbol{\theta})$ becomes $\big(h_{(j_1^*)},\, h_{(j_2^*)}\big)$, where $1 \leq j_1^* < j_2^* \leq M$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w_{(i)} \leq 1-\gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{(i)}, \quad \text{and} \quad h_{(j_2^*)} - h_{(j_1^*)} \leq h_{(j_2)} - h_{(j_1)},$$

for all j_1 and j_2 satisfying (15).

4 Simulation studies and data analysis :

4.1 Simulation studies

To demonstrate the effectiveness of the proposed Bayesian framework, an extensive simulation study has been carried out. A multi-step SSALT experiment consisting of three stress levels, is considered and the parameter values are taken as $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$. In addition, different sample sizes and time points for step acceleration $(n = 30,\ 40,\ 50)$; $(\tau_1, \tau_2) = (0.4, 0.6),\ (0.4, 0.7),\ (0.5, 0.6))$ are considered for illustration. We have considered both the complete sample $(r = n)$ case and 20% censoring $(r = 0.8n)$ case for our analysis purpose. When it comes to prior selection, both the non-informative prior(NIP) and informative prior(IP) assumptions are considered. While considering NIP, the hyperparameter values are taken as follows: $a = b = a_0 = b_0 = 0.001$, $a_1 = a_2 = a_3 = 1$. For IP assumptions, the hyperparameter values are taken as follows: $a = 5, b = 2, , b_0 = 1, a_0 = 6, a_1 = 2, a_2 = 4, a_3 = 6$. In each of the cases, we have computed the BEs and the associated symmetric and HPD CRIs using the importance sampling algorithm.

The average BEs and the corresponding mean squared errors(MSEs) of α, λ_1, λ_2, λ_3, are reported in Table 1 and Table 2, based on NIP and IP assumptions, respectively. Again for fixed n, τ_1 and τ_2, as we increase r, the MSEs of λ_3 decrease in every instance, as expected. It is observed that for fixed r, τ_1, τ_2, as sample size increases, as expected, the MSEs decrease. Performance of the BEs with respect to the IP assumptions are better than that with respect to NIP assumptions. In fact, the performance is significantly better in small sizes. For the same set of parameter values, order restricted MLEs and the associated MSEs are computed using the method of generalized isotonic regression as discussed in Pal et al. [27]. The results are reported in Table 3. It is observed that the performance of the BEs with respect to the NIP assumptions are better than the MLEs in terms of MSEs. For interval estimation, 95% symmetric and HPD CRIs of all the model parameters are computed using Algorithm 1. The average lengths and the associated coverage probabilities(CPs) are reported in Tables 4-7. Both the CRIs perform well and as sample size increases, for fixed r, τ_1, τ_2, the ALs decrease. However, HPD CRIs provide shorter ALs compared to the symmetric CRIs. All the results are based on 5000 replications.

Table 1: AEs and MSEs of Bayes Estimators with NIP assumptions.
Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	r	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	16	0.4	0.6	2.0326	0.3908	1.1501	0.2147	2.1710	0.4389	3.9958	2.2432
		0.4	0.7	2.0307	0.3824	1.0761	0.1014	1.9791	0.2330	3.9831	3.1486
		0.5	0.6	2.3028	0.3229	1.1650	0.2131	2.3813	0.7458	3.9685	2.0480
	20	0.4	0.6	2.1009	0.3154	1.1694	0.1556	2.1541	0.3555	3.8510	1.4886
		0.4	0.7	2.0917	0.3223	1.1496	0.1061	2.1279	0.3415	3.9400	2.2549
		0.5	0.6	2.3320	0.2938	1.1443	0.1459	2.3452	0.5221	3.8693	1.1164
30	24	0.4	0.6	2.1720	0.2736	1.0296	0.1173	1.9934	0.2715	3.5536	1.1784
		0.4	0.7	2.2046	0.2538	1.0393	0.1013	2.0259	0.2328	3.7121	1.9042
		0.5	0.6	2.3867	0.2543	1.0174	0.1154	2.0990	0.2599	3.5246	0.9704
	30	0.4	0.6	2.3682	0.1856	1.1104	0.1137	2.0033	0.2279	3.3739	0.6757
		0.4	0.7	2.4038	0.1952	1.0898	0.0876	2.0272	0.2231	3.5611	1.0948
		0.5	0.6	2.4415	0.2258	1.0387	0.1095	2.1165	0.2392	3.4610	0.7119
40	32	0.4	0.6	2.2810	0.2363	0.9766	0.1053	1.8973	0.1899	3.2458	0.5858
		0.4	0.7	2.2829	0.2123	0.9869	0.0886	1.9949	0.1786	3.5637	1.2890
		0.5	0.6	2.4026	0.2207	0.9814	0.1028	2.0156	0.2228	3.3794	0.6086
	40	0.4	0.6	2.5187	0.1514	1.1153	0.1050	1.9764	0.1722	3.1500	0.3500
		0.4	0.7	2.4738	0.1560	1.0986	0.0874	2.0153	0.1623	3.2541	0.5737
		0.5	0.6	2.4970	0.1995	1.0200	0.0953	2.0239	0.2012	3.3088	0.4637

Table 2: AEs and MSEs of Bayes Estimators with IP assumptions. Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	r	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	16	0.4	0.6	2.1862	0.1715	1.1602	0.0682	2.0718	0.1694	3.4325	0.6380
		0.4	0.7	2.1950	0.1749	1.1281	0.0656	1.9511	0.1477	3.4254	0.7036
		0.5	0.6	2.3703	0.1494	1.1025	0.0590	2.2579	0.1943	3.4996	0.7347
	20	0.4	0.6	2.2234	0.1569	1.1862	0.0675	2.0992	0.1684	3.4320	0.6125
		0.4	0.7	2.2359	0.1521	1.1775	0.0647	2.0558	0.1470	3.4877	0.6735
		0.5	0.6	2.3779	0.1283	1.1105	0.0593	2.2518	0.1826	3.4471	0.6262
30	24	0.4	0.6	2.3010	0.1290	1.0699	0.0583	1.9924	0.1418	3.3123	0.4892
		0.4	0.7	2.3237	0.1283	1.0819	0.0509	2.0031	0.1374	3.3898	0.5820
		0.5	0.6	2.4432	0.1371	1.0232	0.0575	2.0902	0.1467	3.3341	0.5198
	30	0.4	0.6	2.3753	0.1145	1.1072	0.0588	2.0134	0.1399	3.2682	0.3968
		0.4	0.7	2.3706	0.1081	1.1062	0.0506	2.0233	0.1379	3.3231	0.4716
		0.5	0.6	2.4779	0.1272	1.0392	0.0583	2.0995	0.1364	3.3029	0.4211
40	32	0.4	0.6	2.3632	0.1235	1.0216	0.0584	1.9475	0.1263	3.1549	0.3605
		0.4	0.7	2.3832	0.1177	1.0376	0.0542	2.0099	0.1246	3.3022	0.4822
		0.5	0.6	2.4597	0.1282	1.0012	0.0580	2.0213	0.1301	3.2268	0.3368
	40	0.4	0.6	2.4904	0.1000	1.0929	0.0580	1.9858	0.1210	3.1786	0.3445
		0.4	0.7	2.4710	0.1028	1.0836	0.0525	2.0039	0.1205	3.1786	0.3445
		0.5	0.6	2.5091	0.1251	1.0190	0.0563	2.0293	0.1258	3.1960	0.2843

Table 3: MLEs and MSEs of model parameters with $\alpha_1 = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α_1		λ_1		α_2		λ_2	
n	τ	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	16	0.4	0.6	2.3569	0.6214	1.4174	0.9416	2.1407	0.8667	3.6723	2.3568
		0.4	0.7	2.3537	0.5016	1.3746	0.7817	2.0673	0.5046	3.8763	3.4861
		0.5	0.6	2.6367	0.8255	1.4086	1.0327	2.4654	1.1765	3.6315	2.1948
	20	0.4	0.6	2.4424	0.4484	1.4705	0.8742	2.1512	0.7617	3.5319	1.6441
		0.4	0.7	2.4078	0.4368	1.4180	0.7573	2.1357	0.5820	3.6851	2.5088
		0.5	0.6	2.6719	0.7131	1.3738	0.8563	2.4234	0.9627	3.4476	1.4494
30	24	0.4	0.6	2.4853	0.5496	1.3300	0.8447	2.1485	0.6936	3.5049	1.4282
		0.4	0.7	2.4971	0.5229	1.3101	0.7479	2.1481	0.4623	3.6565	2.4004
		0.5	0.6	2.6893	0.7797	1.2961	0.7882	2.2644	0.8723	3.4266	1.2757
	30	0.4	0.6	2.5399	0.4618	1.3375	0.7330	2.1333	0.5823	3.3432	0.9168
		0.4	0.7	2.5207	0.4221	1.2906	0.6035	2.1241	0.4027	3.4112	1.3827
		0.5	0.6	2.6538	0.5864	1.2451	0.6180	2.2305	0.6998	3.3127	0.8606
40	32	0.4	0.6	2.5633	0.5421	1.2954	0.8058	2.1832	0.6159	3.3568	0.9495
		0.4	0.7	2.5425	0.5064	1.2575	0.6761	2.1404	0.3626	3.4758	1.6565
		0.5	0.6	2.6696	0.6235	1.2734	0.6824	2.2001	0.7078	3.3164	0.8325
	40	0.4	0.6	2.5906	0.4054	1.2760	0.6462	2.1534	0.4810	3.2383	0.6057
		0.4	0.7	2.5629	0.4146	1.2310	0.5718	2.1300	0.3275	3.3519	1.1095
		0.5	0.6	2.6564	0.4801	1.2240	0.5196	2.1520	0.5570	3.2220	0.5531

Table 4: CP and AL of 95% symmetric CRI with NIP assumptions. Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	r	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL
20	16	0.4	0.6	0.9362	2.2934	0.9978	2.4067	0.9908	3.0279	0.9584	5.1981
		0.4	0.7	0.9474	2.2857	0.9980	2.1302	0.9878	2.5285	0.9814	6.0972
		0.5	0.6	0.9760	2.5622	0.9965	2.3640	0.9860	3.4724	0.9340	4.8166
	20	0.4	0.6	0.9512	2.1958	0.9995	2.3033	0.9910	2.7713	0.9432	3.9775
		0.4	0.7	0.9488	2.1761	0.9900	2.1713	0.9804	2.5005	0.9424	5.0089
		0.5	0.6	0.9734	2.5069	0.9982	2.2506	0.9902	3.2300	0.9346	4.0795
30	24	0.4	0.6	0.9434	1.9995	0.9960	1.9220	0.9808	2.2357	0.9612	3.2302
		0.4	0.7	0.9612	2.0399	0.9954	1.9384	0.9746	2.0882	0.9672	4.4900
		0.5	0.6	0.9745	2.2092	0.9965	1.8199	0.9945	2.5293	0.9590	3.1858
	30	0.4	0.6	0.9724	1.8626	0.9964	1.9519	0.9840	2.1469	0.9458	2.4976
		0.4	0.7	0.9684	1.9342	0.9974	1.9029	0.9714	1.9604	0.9556	3.1079
		0.5	0.6	0.9598	1.9003	0.9854	1.5845	0.9876	2.1612	0.9580	2.5626
40	32	0.4	0.6	0.9166	1.6615	0.9782	1.5420	0.9582	1.6898	0.9268	2.1949
		0.4	0.7	0.9506	1.7886	0.9826	1.6911	0.9634	1.7230	0.9548	3.3814
		0.5	0.6	0.9598	1.9003	0.9854	1.5845	0.9876	2.1612	0.9580	2.5626
	40	0.4	0.6	0.9620	1.5086	0.9740	1.7428	0.9800	1.8061	0.9414	1.9202
		0.4	0.7	0.9704	1.6517	0.9830	1.7515	0.9714	1.6544	0.9412	2.2725
		0.5	0.6	0.9776	1.8977	0.9826	1.6516	0.9804	2.1650	0.9552	2.3064

Table 5: CP and AL of 95% symmetric CRI with IP assumptions. Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	r	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL
20	16	0.4	0.6	0.9796	2.0288	0.9995	1.9971	0.9996	2.3983	0.9970	3.6539
		0.4	0.7	0.9654	2.0462	0.9998	1.8673	0.9972	2.1131	0.9996	3.9882
		0.5	0.6	0.9928	2.2657	0.9968	1.9284	0.9992	2.6941	0.9940	3.5725
	20	0.4	0.6	0.9786	1.9527	0.9995	1.9649	0.9978	2.3196	0.9878	3.2140
		0.4	0.7	0.9782	1.9526	0.9994	1.8980	0.9948	2.1219	0.9920	3.5954
		0.5	0.6	0.9904	2.1462	0.9982	1.8713	0.9902	2.5848	0.9826	3.1780
30	24	0.4	0.6	0.9776	1.8262	0.9970	1.7789	0.9932	2.0182	0.9864	2.8130
		0.4	0.7	0.9852	1.8526	0.9996	1.7770	0.9810	1.8734	0.9944	3.4373
		0.5	0.6	0.9870	2.0270	0.9965	1.6659	0.9976	2.2748	0.9778	2.8179
	30	0.4	0.6	0.9868	1.7516	0.9990	1.7758	0.9944	1.9520	0.9794	2.3452
		0.4	0.7	0.9856	1.7956	0.9982	1.7596	0.9870	1.7955	0.9832	2.7312
		0.5	0.6	0.9914	1.9684	0.9992	1.6653	0.9982	2.2396	0.9748	2.5525
40	32	0.4	0.6	0.9588	1.5672	0.9846	1.5448	0.9897	1.6425	0.9684	2.1070
		0.4	0.7	0.9788	1.6487	0.9894	1.6372	0.9822	1.6334	0.9834	2.8617
		0.5	0.6	0.9832	1.7733	0.9868	1.5019	0.9874	2.0063	0.9824	2.3567
	40	0.4	0.6	0.9852	1.4900	0.9872	1.6350	0.9898	1.7035	0.9628	1.8660
		0.4	0.7	0.9866	1.5911	0.9890	1.6390	0.9814	1.5616	0.9732	2.1559
		0.5	0.6	0.9812	1.7676	0.9870	1.5238	0.9862	2.0085	0.9738	2.1506

Table 6: CP and AL of 95% HPD CRI with NIP assumptions.
Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	r	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL
20	16	0.4	0.6	0.9092	2.1724	0.9970	2.1814	0.9878	2.8437	0.9616	4.8840
		0.4	0.7	0.9044	2.1703	0.9970	1.9450	0.9752	2.3868	0.9828	5.6194
		0.5	0.6	0.9520	2.4174	0.9935	2.1480	0.9795	3.2847	0.9270	4.5571
	20	0.4	0.6	0.9118	2.0898	0.9992	2.1216	0.9906	2.6414	0.9402	3.7883
		0.4	0.7	0.9050	2.0706	0.9890	2.0138	0.9744	2.3985	0.9592	4.6955
		0.5	0.6	0.9500	2.3833	0.9960	2.0679	0.9902	3.0989	0.9408	3.8990
30	24	0.4	0.6	0.9208	1.8828	0.9908	1.7663	0.9620	2.1116	0.9464	3.0605
		0.4	0.7	0.9354	1.9241	0.9904	1.7849	0.9612	1.9874	0.9658	4.1999
		0.5	0.6	0.9570	2.0777	0.9835	1.6695	0.9870	2.4113	0.9355	3.0378
	30	0.4	0.6	0.9530	1.7545	0.9934	1.8194	0.9676	2.0484	0.9238	2.3832
		0.4	0.7	0.9484	1.8334	0.9936	1.7771	0.9560	1.8813	0.9420	2.9511
		0.5	0.6	0.9588	2.0642	0.9800	1.7126	0.9814	2.4150	0.9432	2.7512
40	32	0.4	0.6	0.9024	1.5378	0.9574	1.4310	0.9336	1.5767	0.9054	2.0488
		0.4	0.7	0.9234	1.6734	0.9808	1.5657	0.9462	1.6364	0.9430	3.1659
		0.5	0.6	0.9414	1.9003	0.9618	1.4570	0.9736	2.0548	0.9334	2.4397
	40	0.4	0.6	0.9330	1.4080	0.9778	1.7428	0.9588	1.7150	0.9286	1.8248
		0.4	0.7	0.9454	1.5463	0.9762	1.6440	0.9506	1.5829	0.9158	2.1565
		0.5	0.6	0.9542	1.7956	0.9720	1.5319	0.9790	2.0880	0.9408	2.2176

Table 7: CP and AL of 95% HPD CRI with IP assumptions.
Actual values: $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

				α		λ_1		λ_2		λ_3	
n	r	τ_1	τ_2	CP	AL	CP	AL	CP	AL	CP	AL
20	16	0.4	0.6	0.9674	1.9602	0.9974	1.8844	0.9990	2.3162	0.9954	3.5113
		0.4	0.7	0.9654	1.9844	0.9994	1.7774	0.9932	2.0496	0.9984	3.8280
		0.5	0.6	0.9870	2.1938	0.9998	1.8093	0.9994	2.6097	0.9926	3.4522
	20	0.4	0.6	0.9646	1.9002	0.9994	1.8715	0.9906	2.2537	0.9828	3.0941
		0.4	0.7	0.9620	1.9022	0.9990	1.8177	0.9882	2.0680	0.9926	3.4416
		0.5	0.6	0.9842	2.0916	0.9996	1.7718	0.9982	2.5222	0.9836	3.0874
30	24	0.4	0.6	0.9702	1.7273	0.9994	1.6736	0.9890	1.9312	0.9788	2.6872
		0.4	0.7	0.9750	1.7625	0.9988	1.6752	0.9844	1.8102	0.9918	3.2738
		0.5	0.6	0.9820	1.9315	0.9984	1.5588	0.9946	2.1983	0.9644	2.7169
	30	0.4	0.6	0.9770	1.6722	0.9978	1.6806	0.9870	1.8833	0.9616	2.2522
		0.4	0.7	0.9744	1.7255	0.9936	1.6739	0.9772	1.7425	0.9746	2.6159
		0.5	0.6	0.9822	1.8968	0.9984	1.5689	0.9956	2.1808	0.9640	2.4763
40	32	0.4	0.6	0.9462	1.4600	0.9807	1.4490	0.9731	1.5539	0.9422	1.9885
		0.4	0.7	0.9724	1.5458	0.9808	1.5373	0.9746	1.5688	0.9782	2.7159
		0.5	0.6	0.9764	1.6713	0.9812	1.4009	0.9816	1.9311	0.9662	2.2637
	40	0.4	0.6	0.9730	1.4038	0.9834	1.5454	0.9750	1.6298	0.9428	1.7790
		0.4	0.7	0.9732	1.5091	0.9868	1.5537	0.9660	1.5063	0.9544	2.0576
		0.5	0.6	0.9784	1.6890	0.9820	1.4310	0.9806	1.9510	0.9616	2.0828

4.2 Real life Data Analysis

In this section, we provide analyses of two real life multiple step-stress Fish data sets obtained from Greven et al. [13].

4.2.1 Fish data set 1: At least one failure is present at every stress level

With an initial swimming rate of 15 cm/sec, swimming performance of 14 fishes was investigated. The time at which a fish could not maintain its natural position was recorded as the failure time. To ensure the early failures, the flow rate was increased by 5 cm /sec every time after 110, 130, and 150 minutes. Here, the increased flow rate can be thought of as the stress factor. There are four stress levels and number of failures at each stress level is 6, 3, 3, and 2 respectively. The observed failure time data is presented in Table 8.

Table 8: Fish data set 1

Stress level	Failure times
s_1	83.50, 91.00, 91.00, 97.00, 107.00, 109.50
s_2	114.00, 115.41, 128.61
s_3	133.53, 138.58, 140.00
s_4	152.08, 155.10

For computational purpose, we have subtracted 80 from each data points, divide them by 100 and then analyze the data with $n = 14$, $r = 14$, $\tau_1 = 0.3$, $\tau_2 = 0.5$, $\tau_3 = 0.7$. We analyze the above failure data assuming Weibull **multi-step SSALT** model with common shape parameter but different scale parameters across the different stress levels. Bayesian analysis of this data set is considered on the basis of NIP assumptions. Under the above model assumptions, the order restricted BEs of the model parameters and the corresponding 95% HPD CRIs are provided below in Table 9.

Table 9: BEs and the corresponding associated 95% HPD CRIs of the model parameters in the Fish data set 1.

	BEs	95% HPD CRIs
$\widehat{\alpha}$	1.2087	(0.7540, 1.7788)
$\widehat{\lambda}_1$	1.8223	(0.6423, 3.4271)
$\widehat{\lambda}_2$	3.2224	(2.0147, 4.4942)
$\widehat{\lambda}_3$	5.4430	(2.9607, 9.8831)
$\widehat{\lambda}_4$	13.4219	(5.3086, 25.2114)

As a measure of goodness of fit, we compute the Kolmogorov-Smirnov (K-S) distance between the empirical distribution function (EDF) and the fitted distribution function (FDF) and also obtain the associated p-value. The K-S distance and the associated p-value based on the order restricted BEs are 0.1246 and 0.9728 respectively, which indicates a very good fit of the given data. The plot of the empirical v/s the fitted CDFs is shown in Figure 1.

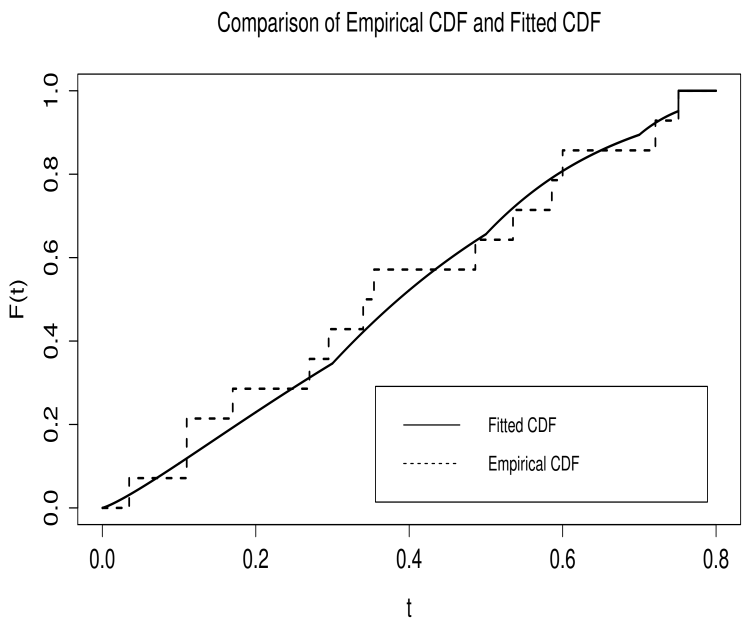


Figure 1: Plot of empirical and fitted CDFs of Fish data set 1.

4.2.2 Fish data set 2: No failure in at least one of the internal stress levels

A sample of 15 fishes were swum at initial flow rate 15 cm/sec. The time at which a fish could not maintain its natural position was recorded as the failure time. To ensure the early failures, the stress level was increased (flow rate by 5 cm/sec) at time 110, 130, 150 and 170 minutes. The observed failure time data is presented in Table 10 below.

Table 10: Fish data set 2

Stress level	Failure times
s_1	91.00, 93.00, 94.00, 98.20
s_2	115.81, 116.00, 116.50, 117.25, 126.75, 127.50
s_3	No failures
s_4	154.33, 159.50, 164.00
s_5	184.14, 188.33

There are five stress levels and number of failures at each stress level is 4, 6, 0, 3 and 2, respectively. Here, we consider the complete sample for analysis purpose and stop at the 15−th failure. For computational purpose, we have subtracted 80 from each data points, divide them by 150 and then analyze the data with $n = 15$, $r = 15$, $\tau_1 = 0.20$, $\tau_2 = 0.33$, $\tau_3 = 0.46$, $\tau_4 = 0.6$. We assume that the above failure data follow Weibull distribution at each stress level with common shape parameter but different scale parameters across the different stress levels. The Bayesian analysis of this data set is considered on the basis of NIP assumptions.

Under the above model assumptions, the order restricted BEs of the model parameters and the corresponding 95% HPD CRIs are provided below in Table 11.

Table 11: BEs and the corresponding associated 95% HPD CRIs of the model parameters in the Fish data set 2.

	BEs	95% HPD CRIs
$\widehat{\alpha}$	0.7733	(0.5602, 0.9762)
$\widehat{\lambda}_1$	1.0818	(0.3947, 1.9118)
$\widehat{\lambda}_2$	3.1624	(1.3018, 3.7457)
$\widehat{\lambda}_3$	3.5906	(1.7679, 4.6929)
$\widehat{\lambda}_4$	5.1163	(2.7916, 6.3943)
$\widehat{\lambda}_5$	9.0499	(3.6553, 12.2361)

As a measure of goodness of fit, we compute the K-S distance between the EDF and the FDF, and also obtain the associated p-value. The K-S distance and the associated p-value based on the order restricted BEs are 0.1630 and 0.7828 respectively, which indicates a good fit of the given data. The plot of the empirical v/s the fitted CDFs is shown in Figure 2.

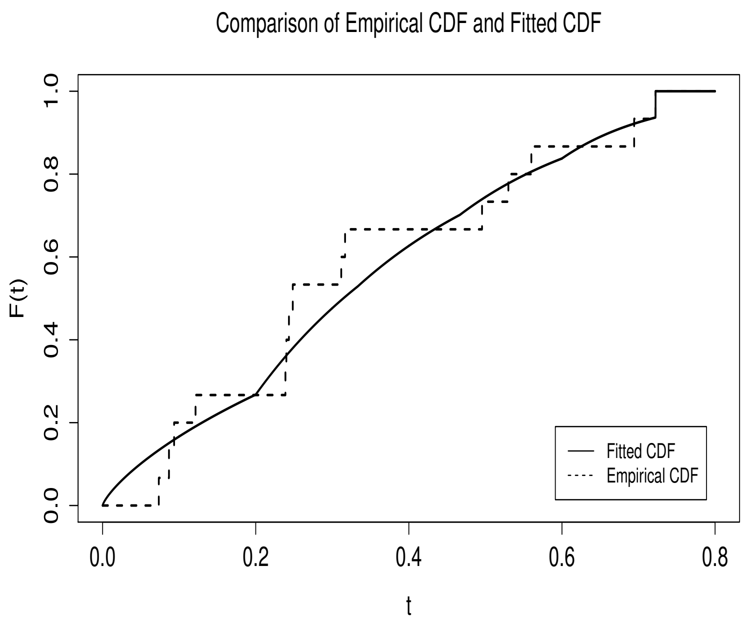


Figure 2: Plot of empirical and fitted CDFs of Fish data set 2.

5 Optimal Bayesian planning of a multi-step SSALT:

Optimal planning of SSALT has been the subject of many studies. For exponential distribution, optimal plans for simple SSALT are presented by Miller and Nelson [24], Bai et al. [6]. Yeo and Tang [36] and Tang [35] proposed a simple SSALT design for an optimum stress- changing time under low stress and an optimum low stress level considering the target acceleration factor. Design of SSALT is studied for other distributions also. In case of Weibull distribution, Ma and Meeker [22], Alhadeed and Yang [1], Li and Fard [20] considered optimal designs for both censored and complete data, while Alhadeed and Yang [2] considered similar issues for lognormal distribution. Optimum SSALT plans for log-location scale distributions based on the CEM are discussed by Ma and Meeker [23]. All these works are mostly based on simple step -stress model and when there is no order restriction on the mean failures.

A very commonly used optimization criterion is to minimize the asymptotic variance of the MLE of some reliability characteristic (mean time to failure (MTTF), reliability function, and so on) at the stressed condition. The optimal design heavily relies on the pre-estimates or the "planning values" of the model parameters which are obtained mainly through past experience, experts' comments, historical data, and so on. Thus, an uncertainty is always associated with these pre-estimates. Bayesian approach deals with the uncertainty in the planning values by assuming some joint prior distribution on the model parameters.

In this section, the optimal design considered is to determine the optimal stress-changing time points $\tau_1, \tau_2, \ldots, \tau_m$, for a multi-step SSALT experiment when the order restriction on the mean failure times is present. Recently, Samanta et al. [32] obtained an optimal value of the stress-changing time point for a simple SSALT set up when the underlying distribution is generalized exponential. However, design of the optimal plan clearly becomes more and more involved as the number of stress levels (m) increases. Alternate ways to tackle this is to make one of the following two assumptions.

(a) The failure of test units in any stress level is equiprobable, the probability being say p. In

other words, we have $F(\tau_{i+1}) - F(\tau_i) = p,\ i = 0, 1, \ldots, m$.. For $i = 0,\ \tau_i = 0$, and hence this condition boils down to $F(\tau_1) = p$, while for $i = m,\ \tau_{m+1} = \infty$, and hence $1 - F(\tau_m) = p$.
(b) The stress changing time points $\tau_1, \tau_2, \ldots, \tau_m$, are equidistant, i.e., $\tau_k = k\tau_1;\ \ k = 1, 2, \ldots, m$.
Both the assumptions drastically reduce the dimension of the optimality problem. However, one drawback in assuming equidistant stress-changing time points is that at every stress level, the experimenter needs to wait for τ_1 time to elevate the stress level. Performing ALT may not then be practical for large τ_1. On the other hand, by assumption (a), it is expected that as the stress level increases, the differences between the consecutive stress-changing time points $(\tau_{i+1} - \tau_i)$ will get narrower, thus reducing the total time of the experiment. Additionally, note that once we obtain the optimal value of either p or τ_1, obtaining the optimal values of other stress- changing time points is immediate. We propose to use a data independent criteria to obtain the optimal value of p (p_{opt}) based on assumption (a). Hence, p_{opt} is obtained by minimizing the sum

$$S = E_{data}[V_{posterior}(\widehat{\alpha})] + E_{data}[V_{posterior}(\widehat{\lambda}_1)] + \ldots + E_{data}[V_{posterior}(\widehat{\lambda}_{\overline{m+1}})]$$

subject to the condition $F(\tau_1) = p$, where $\widehat{\alpha},\ \widehat{\lambda}_1, \ldots, \widehat{\lambda}_{m+1}$ are the BEs of the model parameters. Since, closed- form expressions of the equation (10) is not available, we resort to Lindley's approximation (See Lindley [21] in this respect) of the ratios of integrals for calculating the posterior variances of the BEs of the unknown model parameters. For illustration purpose, we consider here a multi-step SSALT consisting of three stress levels. A detailed study of the Lindley's approximation for three stress levels, is provided in Appendix Section 7.2., and p_{opt} is obtained using the following algorithm.

Algorithm 2

Step 1. Given $n,\ \alpha,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ p$ generate step-stress data from $F(t)$.

Step 2. Applying Lindley's approximation (Appendix Section 7.2), calculate the posterior variances of the BEs of the model parameters i.e., $V_{posterior}(\widehat{\theta}_i)$ where $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{\theta}_4) = (\widehat{\alpha}, \widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)$.

Step 3. Repeat Step 1 and Step 2, say N times and hence obtain $V_{posterior}(\widehat{\theta}_i^{(j)})$;

$j = 1, 2, \ldots, \text{N},\ i = 1, 2, 3, 4.$

Step 4. A Monte Carlo approximation of $E_{data}[V_{posterior}(\widehat{\theta}_i)]$; $i = 1, 2, 3, 4$ is obtained by computing $\frac{1}{N}\sum_{j=1}^{N} V_{posterior}(\widehat{\theta}_i^{(j)})$.

Step 5. Calculate the sum S for $m = 2$.

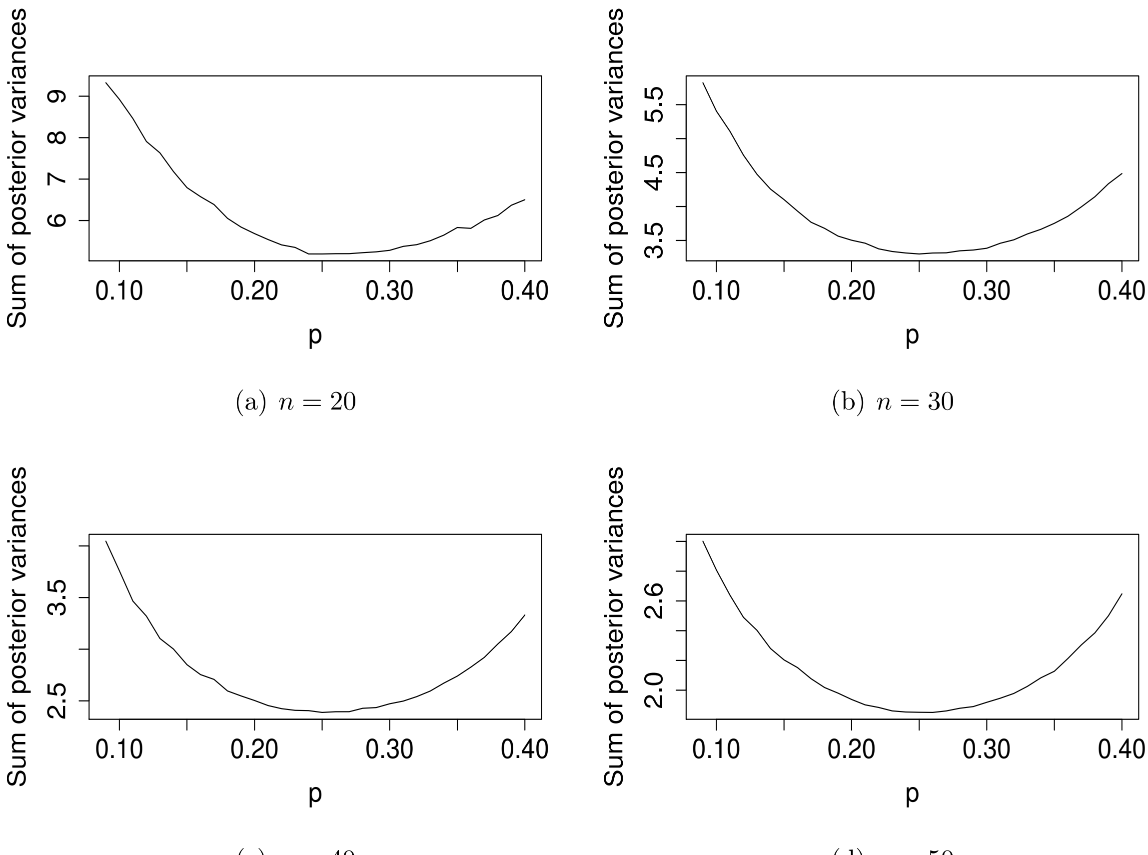
Step 6. Repeat Steps 1 - 5 for different values of $p \in (0, 1)$.

Step 7. Choose p_{opt} for which S is minimum.

In Table 12, the optimal values of p and hence τ_1, τ_2 are reported for both the NIP and IP assumptions and for different sample sizes and shape parameters. The plot of the associated sum of expected posterior variances is provided in Figures 3-6. It is observed that the sum of expected posterior variances initially decreases, reaches the minimum value and then increases in every instances. Additionally, the results seem to be robust.

Table 12: Optimal values of p and hence $\tau_1,\ \tau_2$ for different $n,\ \alpha$ and prior assumptions with $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

α	Prior assumptions	n	Optimal values		
			p	τ_1	τ_2
2.5	Non-informative	20	0.25	0.6075	0.7520
		30	0.25	0.6075	0.7520
		40	0.25	0.6075	0.7520
		50	0.26	0.6187	0.7683
	Informative	20	0.21	0.5609	0.6863
		30	0.21	0.5609	0.6863
		40	0.22	0.5729	0.7028
		50	0.22	0.5729	0.7028
3	Non-informative	20	0.26	0.6702	0.8028
		30	0.26	0.6702	0.8028
		40	0.27	0.6802	0.8171
		50	0.25	0.6601	0.7885
	Informative	20	0.19	0.5950	0.7009
		30	0.21	0.6177	0.7307
		40	0.22	0.6286	0.7453
		50	0.22	0.6286	0.7453



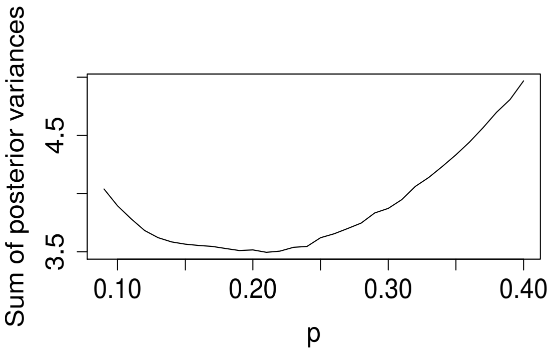
(a) $n = 20$

(b) $n = 30$

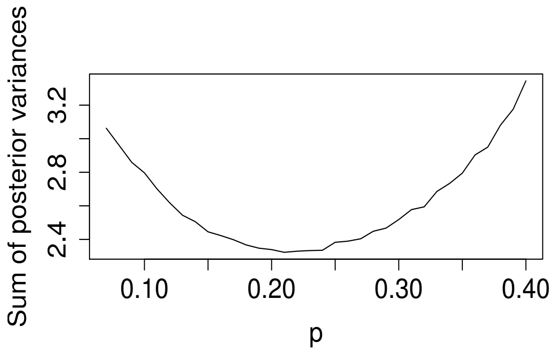
(c) $n = 40$

(d) $n = 50$

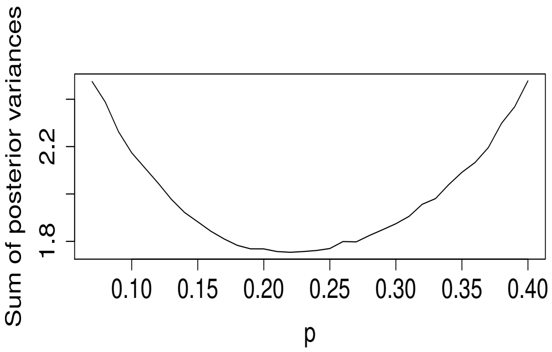
Figure 3: Plot of sum of expected posterior variances for different sample sizes with parameter values $\alpha = 2.5,\ \ \lambda_1 = 1,\ \ \lambda_2 = 2,\ \ \lambda_3 = 3$ and the NIP assumptions $a = .001,\ \ b = .001,\ \ a_0 = .001,\ \ b_0 = .001,\ \ a_1 = 1,\ \ a_2 = 1,\ \ a_3 = 1.$



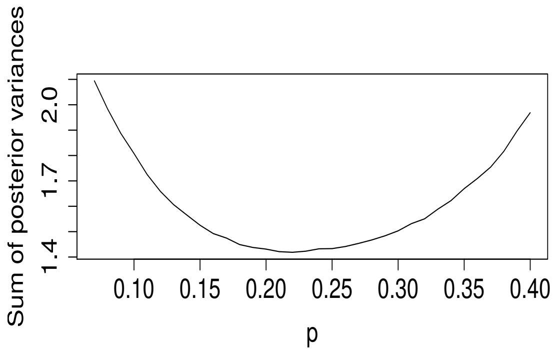
(a) $n = 20$



(b) $n = 30$

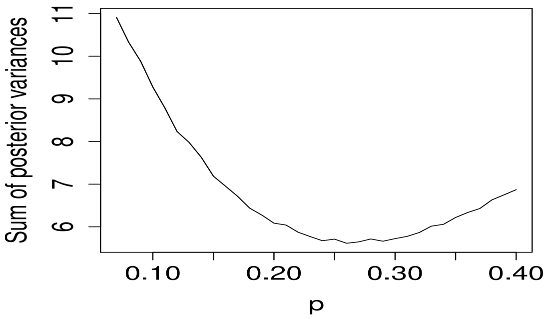


(c) $n = 40$

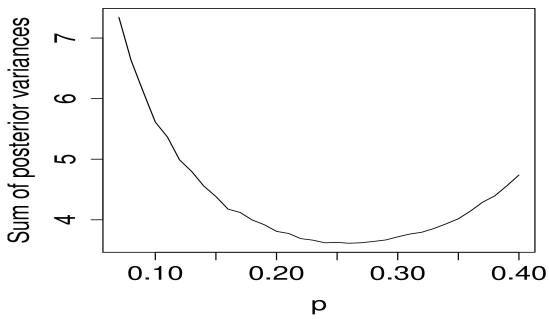


(d) $n = 50$

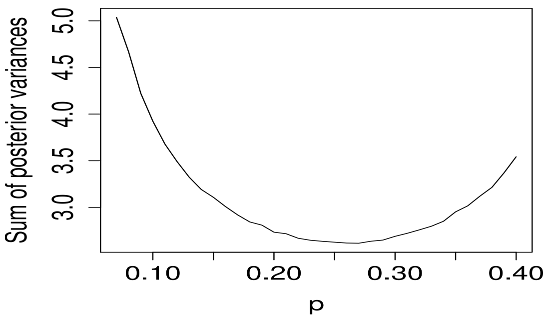
Figure 4: Plot of sum of expected posterior variances for different sample sizes with parameter values $\alpha = 2.5,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ and the IP assumptions $a = 5,\ b = 2,\ a_0 = 6,\ b_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_3 = 6.$



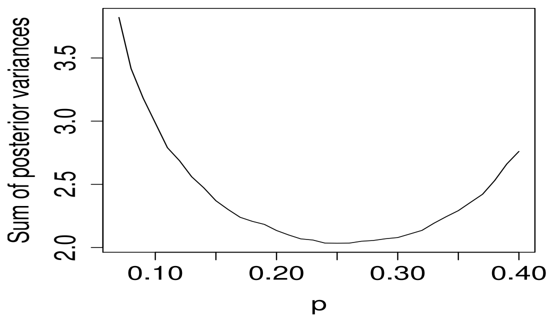
(a) $n = 20$



(b) $n = 30$

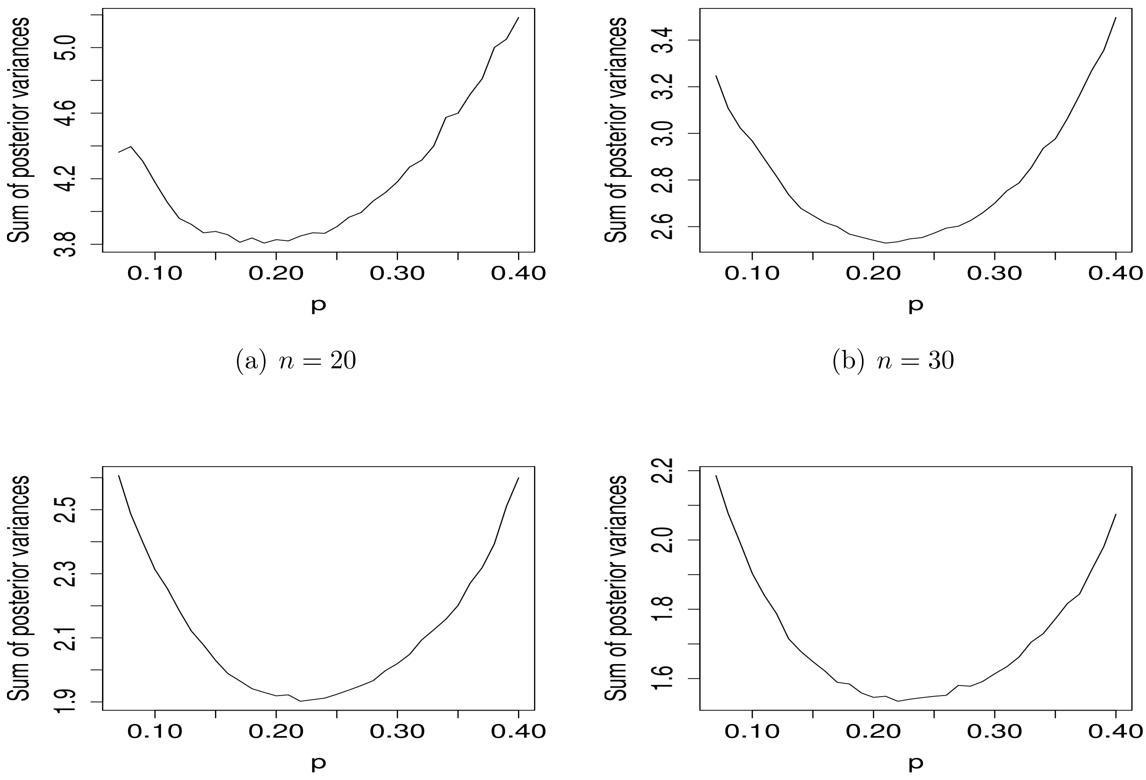


(c) $n = 40$



(d) $n = 50$

Figure 5: Plot of sum of expected posterior variances for different sample sizes with parameter values $\alpha = 3,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$ and the NIP assumptions $a = .001,\ b = .001,\ a_0 = .001,\ b_0 = .001,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1.$



(a) $n = 20$

(b) $n = 30$

(c) $n = 40$

(d) $n = 50$

Figure 6: Plot of sum of expected posterior variances for different sample sizes with parameter values $\alpha = 3, \ \lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$ and the IP assumptions $a = 6, \ b = 2, \ a_0 = 6, \ b_0 = 1, \ a_1 = 2, \ a_2 = 4, \ a_3 = 6.$

6 Conclusion:

Bayesian approach is a natural alternative to the maximum likelihood method when there exists uncertainty in the planning values. In this study, we develop a Bayesian approach for the multi-step SSALT assuming Weibull lifetime distribution, Type-II censoring and the failure rate based SSALT model. Assuming a fairly flexible prior on the parameters, the ordering in the mean failure time at the different stress levels is taken care of. The importance sampling technique is proposed to compute the BEs and the associated CRIs. Extensive simulation studies have been carried out for different sample sizes, censoring proportion and stress changing time points. It is observed that the proposed method works quite well. Finally, a novel optimal plan is proposed to determine the stress-changing time points. More importantly, the complexity in implementation of the proposed methodology does not grow with increase in the number of stress levels. Hence, it can be implemented quite easily in practice. The proposed methodology can well be extended to a general family of distributions-the proportional hazard class of distributions.

It may be mentioned that at different stress-levels the Weibull distribution has the same shape parameter seems to be a strong assumption. But analytically it is difficult to remove that assumption under the present set up, although it may not satisfy that assumption in practice all the time. The main problem is due to order restriction on the mean lifetime at different stress levels. If the shape parameters are same, then the restriction becomes equivalent to the restriction on the scale parameters. Although, the same is not true if the shape parameters are also different, and we donot have any simple necessary and sufficient conditions in terms of the shape and scale parameters. We need more work along that direction. At present, we have the following suggestions for practitioners. At the beginning we ignore the order restriction on the mean life time and assume that the shape parameters of the Weibull distributions can be different also. We assume the Dirichlet-Gamma prior without any order restriction on the scale parameters, and assume independent gamma priors on the shape parameters of the Weibull distribution. Perform the Bayesian analysis using the importance sampling method. Now based on the Bayes factor, we can perform the test that all the shape parameters are equal or not. Based on the observations, if the null hypothesis cannot be rejected, then we can perform the order restricted analysis as we have presented here.

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References

[1] Alhadeed, A. A. and Yang, S. S. Optimal simple step-stress plan for Khamis-Higgins model. *IEEE Transactions on Reliability*, 51:212–215, 2002.

[2] Alhadeed, A. A. and Yang, S. S. Optimal simple step-stress plan for cumulative exposure model using log normal distribution. *IEEE Transactions on Reliability*, 54(1):64–68, 2005.

[3] Bagdonavicius, V. Testing the hypothesis of the addative accumulation of damages. *Probability Theory and its Applications*, 23:403–408, 1978.

[4] Bagdonavicius, V. B. and Nikulin, M. Accelerated life models: modeling and satistical analysis. *Chapman and Hall CRC Press, Boca Raton, Florida*, 2002.

[5] Bai, D. S. and Kim, M. S. Optimum step-stress accelerated life test for Weibull distribution and Type-I censoring. *Naval Research Logistics*, 40:193–210, 2006.

[6] Bai, D. S., Kim, S. M., and Lee, S. H. Optimum simple step-stress accelerated life tests with censoring. *IEEE Transactions on Reliability*, 32(1):528–532, 1989.

[7] Balakrishnan, N. A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. *Metrika*, 69:351–396, 2009.

[8] Balakrishnan, N., Beutner, E., and Kateri, M. Order restricted inference for exponential step-stress models. *IEEE Transactions on Reliability*, 58:132–142, 2009.

[9] Bhattacharyya, G. K. and Soejoeti, Z. A tampered failure rate model for step-stress accelerated life test. *Communication in Statistics - Theory and Methods*, 18:1627–1643, 1989.

[10] Devroye, L. Non-uniform random variate generation. *Springer-Verlag, New York, 1985.*

[11] Dorp, J. R. V. and Mazzuchi, T. A. A general Bayes Weibull inference model for accelerated life testing. *Reliability Engineering & System Safety*, 90(2-3):140–147, 2005.

[12] Dorp, J. R. V., Mazzuchi, T. A., Fornell, G. E., and Pollock, L. R. A Bayes approach to step-stress accelerated life testing. *IEEE Transactions on Reliability*, 45(3):491–498, 1996.

[13] Greven, S., Bailer, A. J., Kupper, L. L., Muller, K. E., and Craft, J. L. A parametric model for studying organism fitness using step-stress experiments. *Biometrics*, 60(3): 793–799, 2004.

[14] Kateri, M. and Balakrishnan, N. Inference for a simple step-stress model with Type-II censoring, and weibull distributed lifetimes. *IEEE Transactions on Reliability*, 57: 616–626, 2008.

[15] Kateri, M. and Kamps, U. Inference in step-stress models based on failure rates. *Statistical Papers*, 56:639–660, 2015.

[16] Kateri, M. and Kamps, U. Hazard rate modeling of step-stress experiments. *Annual Review of Statistics and Its Application*, 4:147–168, 2017.

[17] Khamis, I. H. and Higgins, J. J. A new model for step-stress testing. *IEEE Transactions on Reliability*, 47:131–134, 1998.

[18] Komori, Y. Properties of the Weibull cumulative exposure model. *Journal of Applied Statistics*, 33:17–34, 2006.

[19] Kundu, D. and Pradhan, B. Bayesian analysis of progressively censored competing risks data. *Sankhya B*, 73(2):276–296, 2011.

[20] Li, C. and Fard, N. Optimum bivariate step-stress accelerated life test for censored data. 56:77–84, 2007.

[21] Lindley, D. V. Approximate Bayes method. *Trabajos de Estadistica*, 31:223–237, 1980.

[22] Ma, H. and Meeker, W. Q. Optimum simple step-stress accelerated life tests for Weibull distribution and Type i censoring. *Naval Research Logistics (NRL)*, 40(2):193–210, 1993.

[23] Ma, H. and Meeker, W. Q. Optimum step-stress accelerated life test plans for log location scale distributions. *Naval Research Logistics (NRL)*, 55(6):551–562, 2008.

[24] Miller, R. and Nelson, W. Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, 32(1):59–65, 1983.

[25] Mondal, S. and Kundu, D. Point and interval estimation of Weibull parameters based on joint progressively censored data. *Sankhya, Ser. B*, 117:1–25, 2017.

[26] Nelson, W. B. Accelerated life testing: step-stress models and data analysis. *IEEE Transactions on Reliability*, 29:103–108, 1980.

[27] Pal, A., Mitra, S., and Kundu, D. Order restricted classical inference of a Weibull multiple step-stress model. *Journal of Applied Statistics*, 2020. DOI: https://doi.org/10.1080/02664763.2020.1736526.

[28] Pena, E. A. and Gupta, A. K. Bayes estimation for the Marshall Olkin exponential distribution. *Journal of the Royal Statistical Society: Series B (Methodological)*, 52(2): 379–389, 1990.

[29] Samanta, D., Ganguly, A., Gupta, A., and Kundu, D. On classical and Bayesian order restricted inference for multiple exponential step stress model. *Statistics*, 53(1):177–195, 2019.

[30] Samanta, D., Gupta, A., and Kundu, D. Analysis of Weibull step-stress model in presence of competing risk. *IEEE Transactions on Reliability*, 68:420 – 438, 2019.

[31] Samanta, D. and Kundu, D. Order restricted inference of a multiple step-stress model. *Computational Statistics & Data Analysis*, 117:62 – 75, 2018.

[32] Samanta, D., Kundu, D., and Ganguly, A. Order restricted Bayesian analysis of a simple step stress model. *Sankhya B*, 80:195–221, 2018.

[33] Sedyakin, N. M. On one physical principle in reliability theory. *Technical Cybernatics*, 3:80–87, 1966.

[34] Sha, N. and Pan, R. Bayesian analysis for step-stress accelerated life testing using weibull proportional hazard model. *Statistical Papers*, 55(3):715–726, 2014.

[35] Tang, L. C. Handbook of reliability engineering. *Springer, London.*, 2003.

[36] Yeo, K. P. and Tang, L. C. Planning step-stress life-test with a target acceleration-factor. *IEEE Transactions on Reliability*, 48(1):61–67, 1999.

7 Appendix

7.1 Proof of result 1

$$E(\lambda_i) = \int_0^\infty \dots \int_0^\infty \frac{\Gamma(\sum\limits_{i=1}^{m+1} a_i)}{\Gamma(a_o)} (b_o\lambda)^{a_o - \sum\limits_{k=1}^{m+1} a_k} \lambda_i \prod_{k=1}^{m+1} \Big(\frac{b_o^{a_k}}{\Gamma(a_k)} \lambda_1^{a_k-1} e^{-b_o\lambda_k} \Big) d\lambda_1 \dots d\lambda_{m+1}$$

$$= K_1(\boldsymbol{a}, b_o) \int_0^\infty \dots \int_0^\infty \frac{\Gamma(\sum\limits_{i=1}^{m+1} a_i + 1)}{\Gamma(a_o+1)} (b_o\lambda)^{a_o+1-\sum\limits_{k=1}^{i} a_i - 1 - \sum\limits_{k=i+1}^{m+1} a_k} \Big(\frac{b_o^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1-1} e^{-b_o\lambda_1} \Big)$$
$$\times \dots \Big(\frac{b_o^{a_i+1}}{\Gamma(a_i+1)} \lambda_i^{a_i} e^{-b_o\lambda_i} \Big) \dots \Big(\frac{b_o^{a_{m+1}}}{\Gamma(a_{m+1})} \lambda_{m+1}^{a_{m+1}-1} e^{-b_o\lambda_{m+1}} \Big) d\lambda_1 \dots d\lambda_i \dots d\lambda_{m+1}.$$

The integration in the Step 2 equals to 1 as (7) is a proper density function and

$$E(\lambda_i) = K_1(\boldsymbol{a}, b_o) = \frac{a_o a_i}{b_o \sum_{i=0}^{m+1} a_i}, \text{ where}$$

$$K_1(\boldsymbol{a}, b_o) = \frac{\Gamma(a_o+1)}{\Gamma(\sum_{i=1}^{m+1} a_i + 1)} \frac{\Gamma(a_i+1)}{\Gamma a_i} \frac{\Gamma(\sum_{i=1}^{m+1} a_i)}{\Gamma a_o} \frac{1}{b_o}, \boldsymbol{a} = (a_0, a_1, \ldots, a_{m+1}).$$

Now we derive the expression for $V(\lambda_i)$. Note that $V(\lambda_i) = E(\lambda_i^2) - E(\lambda_i)^2$

$$\begin{aligned} E(\lambda_i^2) &= \int_0^\infty \cdots \int_0^\infty \frac{\Gamma(\sum_{i=1}^{m+1} a_i)}{\Gamma(a_o)} (b_o\lambda)^{a_o - \sum_{k=1}^{m+1} a_k} \lambda_i^2 \prod_{k=1}^{m+1} \left(\frac{b_o^{a_k}}{\Gamma(a_k)} \lambda_1^{a_k-1} e^{-b_o\lambda_k} \right) d\lambda_1 \ldots d\lambda_{m+1} \\ &= K_2(\boldsymbol{a}, b_o) \int_0^\infty \cdots \int_0^\infty \frac{\Gamma(\sum_{i=1}^{m+1} a_i + 2)}{\Gamma(a_o+2)} (b_o\lambda)^{a_o + 2 - \sum_{k=1}^{i} a_i - 2 - \sum_{k=i+1}^{m+1} a_k} \left(\frac{b_o^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1-1} e^{-b_o\lambda_1} \right) \\ &\quad \times \cdots \left(\frac{b_o^{a_i+2}}{\Gamma(a_i+2)} \lambda_i^{a_i+1} e^{-b_o\lambda_i} \right) \cdots \left(\frac{b_o^{a_{m+1}}}{\Gamma(a_{m+1})} \lambda_{m+1}^{a_{m+1}-1} e^{-b_o\lambda_{m+1}} \right) d\lambda_1 \ldots d\lambda_i \ldots d\lambda_{m+1}. \end{aligned}$$

Along the same lines of the derivation of $E(\lambda_i)$, we have

$$E(\lambda_i^2) = K_2(\boldsymbol{a}, b_o) = \frac{a_o(a_o+1)a_i(a_i+1)}{b_o^2 (\sum_{i=0}^{m+1} a_i)(\sum_{i=0}^{m+1} a_i + 1)},$$

where

$$K_2(\boldsymbol{a}, b_o) = \frac{\Gamma(a_o+2)}{\Gamma(\sum_{i=1}^{m+1} a_i + 2)} \frac{\Gamma(a_i+2)}{\Gamma a_i} \frac{\Gamma(\sum_{i=1}^{m+1} a_i)}{\Gamma a_o} \frac{1}{b_o^2}, \ \boldsymbol{a} = (a_0, a_1, \ldots, a_{m+1}).$$

Thus,

$$V(\lambda_i) = \frac{a_o a_i}{b_o^2 \sum_{i=0}^{m+1} a_i} \left\{ \frac{(a_o+1)(a_i+1)}{(\sum_{i=0}^{m+1} a_i + 1)} - \frac{a_o a_i}{\sum_{i=0}^{m+1} a_i} \right\}$$

7.2 Lindley's approximation

For determining the optimal choice of p, we consider here three levels of stress. Assuming $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha, \lambda_1, \lambda_2, \lambda_3)$, the Lindley's approximation (See Lindley [21]) for the

BE of any parametric function $q(\boldsymbol{\theta})$ is given by

$$\begin{aligned} E_{\boldsymbol{\theta}|\mathcal{D}}\{q(\theta_1,\theta_2,\theta_3,\theta_4)\} = q(\widehat{\theta}_1,\widehat{\theta}_2,\widehat{\theta}_3,\widehat{\theta}_4) + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} u_{ij}\sigma_{ij} \\ + \sum_{i=1}^{4}\sum_{j=1}^{4} u_i\rho_j\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} L_{ijk}U_k\sigma_{ij} \end{aligned} \tag{16}$$

where,

$$\begin{aligned} L_{ijk} &= \frac{\partial^3 l}{\partial\theta_i\partial\theta_j\partial\theta_k\theta_l};\;\; i,j,k,l = 1(1)4, \text{where } \; l = \log L(\boldsymbol{\theta}|\mathcal{D}), \\ u_i &= \frac{\partial q}{\partial\theta_i};\;\; i = 1(1)4, \\ u_{ij} &= \frac{\partial^2 q}{\partial\theta_i\partial\theta_j};\;\; i,j = 1(1)4, \\ \sigma_{ij} &= (i,j)\text{-th element of the inverse of observed Fisher information matrix.,} \\ \rho_i &= \frac{\partial \log\pi(\boldsymbol{\theta})}{\partial\theta_i};\;\; i = 1(1)4, \\ U_k &= \sum_{i=1}^{4} u_i\sigma_{ki};\;\; k = 1(1)4;\;\; \text{and} \end{aligned}$$

$\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{\theta}_4$ are the order restricted MLEs of the model parameters.

7.2.1 Different Elements of Lindley's Approximation

The elements of the observed Fisher information matrix

$$I(\alpha,\lambda_1,\lambda_2,\lambda_3) = \begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_3} \\ -\frac{\partial^2 l}{\partial\lambda_1\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_1^2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_3} \\ -\frac{\partial^2 l}{\partial\lambda_2\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_2^2} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_3} \\ -\frac{\partial^2 l}{\partial\lambda_3\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_3\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_3\partial\lambda_2} & -\frac{\partial^2 l}{\partial\lambda_3^2} \end{bmatrix}$$

can be expressed as follows.

$$\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} &= -\frac{n}{\alpha^2} - \lambda_1 D_1''(\alpha) - \lambda_2 D_2''(\alpha) - \lambda_3 D_3''(\alpha), \\
\frac{\partial^2 l}{\partial \lambda_1^2} &= -\frac{n_1}{\lambda_1^2}, \frac{\partial^2 l}{\partial \lambda_2^2} = -\frac{n_2}{\lambda_2^2}, \frac{\partial^2 l}{\partial \lambda_3^2} = -\frac{n_3}{\lambda_3^2}, \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda_1} &= \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} = -D_1'(\alpha), \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda_2} &= \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} = -D_2'(\alpha), \\
\frac{\partial^2 l}{\partial \alpha \partial \lambda_3} &= \frac{\partial^2 l}{\partial \lambda_3 \partial \alpha} = -D_3'(\alpha), \\
\frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} &= \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_1} = \frac{\partial^2 l}{\partial \lambda_2 \partial \lambda_3} = \frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_2} = \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_3} = \frac{\partial^2 l}{\partial \lambda_3 \partial \lambda_1} = 0.
\end{aligned}$$

$$\begin{aligned}
L_{111} &= \frac{2n}{\alpha^3} - \lambda_1 D_1'''(\alpha) - \lambda_2 D_2'''(\alpha) - \lambda_3 D_3'''(\alpha), \\
L_{222} &= \frac{2n_1}{\lambda_1^3}, \ L_{333} = \frac{2n_2}{\lambda_2^3}, \ L_{444} = \frac{2n_3}{\lambda_3^3}, \\
L_{112} &= -D_1''(\alpha), \ L_{113} = -D_2''(\alpha), \ L_{114} = -D_3''(\alpha), \\
L_{221} &= L_{223} = L_{224} = 0, \\
L_{331} &= L_{332} = L_{334} = 0, \\
L_{441} &= L_{442} = L_{443} = 0, \\
L_{231} &= L_{232} = L_{233} = L_{234} = 0, \\
L_{241} &= L_{242} = L_{243} = L_{244} = 0, \\
L_{341} &= L_{342} = L_{343} = L_{344} = 0.
\end{aligned}$$

where

$$
\begin{aligned}
D_1'(\alpha) &= \sum_{i=1}^{\overline{n}_1} t_{i:n}^{\alpha} \ln t_{i:n} + (n-\overline{n}_1)\tau_1^{\alpha} \ln \tau_1, \\
D_2'(\alpha) &= \sum_{i=\overline{n}_1+1}^{\overline{n}_2} t_{i:n}^{\alpha} \ln t_{i:n} + (n-\overline{n}_2)\tau_2^{\alpha} \ln \tau_2 - (n-\overline{n}_1)\tau_1^{\alpha} \ln \tau_1, \\
D_3'(\alpha) &= \sum_{i=\overline{n}_2+1}^{\overline{n}_3} t_{i:n}^{\alpha} \ln t_{i:n} - (n-\overline{n}_2)\tau_2^{\alpha} \ln \tau_2, \\
D_1''(\alpha) &= \sum_{i=1}^{\overline{n}_1} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n-\overline{n}_1)\tau_1^{\alpha} (\ln \tau_1)^2, \\
D_2''(\alpha) &= \sum_{i=\overline{n}_1+1}^{\overline{n}_2} t_{i:n}^{\alpha} (\ln t_{i:n})^2 + (n-\overline{n}_2)\tau_2^{\alpha} (\ln \tau_2)^2 - (n-\overline{n}_1)\tau_1^{\alpha} (\ln \tau_1)^2, \\
D_3''(\alpha) &= \sum_{i=\overline{n}_2+1}^{\overline{n}_3} t_{i:n}^{\alpha} (\ln t_{i:n})^2 - (n-\overline{n}_2)\tau_2^{\alpha} (\ln \tau_2)^2 \\
D_1'''(\alpha) &= \sum_{i=1}^{\overline{n}_1} t_{i:n}^{\alpha} (\ln t_{i:n})^3 + (n-\overline{n}_1)\tau_1^{\alpha} (\ln \tau_1)^3, \\
D_2'''(\alpha) &= \sum_{i=\overline{n}_1+1}^{\overline{n}_2} t_{i:n}^{\alpha} (\ln t_{i:n})^3 + (n-\overline{n}_2)\tau_2^{\alpha} (\ln \tau_2)^3 - (n-\overline{n}_1)\tau_1^{\alpha} (\ln \tau_1)^3, \\
D_3'''(\alpha) &= \sum_{i=\overline{n}_2+1}^{\overline{n}_3} t_{i:n}^{\alpha} (\ln t_{i:n})^3 - (n-\overline{n}_2)\tau_2^{\alpha} (\ln \tau_2)^3.
\end{aligned}
$$

It is to note that the L_{ijk} is independent of the order in which i, j and k appear in the suffixes. Now,

$$
\begin{aligned}
\log(\pi) = \; & (a_0 - a_1 - a_2 - a_3)\log(\lambda_1+\lambda_2+\lambda_3) - b_0(\lambda_1+\lambda_2+\lambda_3) \\
& + \log g(\lambda_1,\lambda_2,\lambda_3) + (a-1)\log\alpha - b\alpha; \quad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
g(\lambda_1,\lambda_2,\lambda_3) &= \lambda_1^{a_1-1}\lambda_2^{a_2-1}\lambda_3^{a_3-1} + \lambda_1^{a_1-1}\lambda_3^{a_2-1}\lambda_2^{a_3-1} + \lambda_2^{a_1-1}\lambda_3^{a_2-1}\lambda_1^{a_3-1} \\
& \lambda_2^{a_1-1}\lambda_1^{a_2-1}\lambda_3^{a_3-1} + \lambda_3^{a_1-1}\lambda_2^{a_2-1}\lambda_1^{a_3-1} + \lambda_3^{a_1-1}\lambda_1^{a_2-1}\lambda_2^{a_3-1}
\end{aligned}
$$

$$\begin{aligned}
\rho_1 &= \frac{\partial \log \pi}{\partial \alpha} = \frac{a-1}{\alpha} - b, \\
\rho_2 &= \frac{\partial \log \pi}{\partial \lambda_1} = \frac{(a_0 - a_1 - a_2 - a_3)}{\lambda_1 + \lambda_2 + \lambda_3} - b_0 + \frac{g_1'(\lambda_1 + \lambda_2 + \lambda_3)}{g(\lambda_1 + \lambda_2 + \lambda_3)}, \\
\rho_3 &= \frac{\partial \log \pi}{\partial \lambda_2} = \frac{(a_0 - a_1 - a_2 - a_3)}{\lambda_1 + \lambda_2 + \lambda_3} - b_0 + \frac{g_2'(\lambda_1 + \lambda_2 + \lambda_3)}{g(\lambda_1 + \lambda_2 + \lambda_3)}, \\
\rho_4 &= \frac{\partial \log \pi}{\partial \lambda_3} = \frac{(a_0 - a_1 - a_2 - a_3)}{\lambda_1 + \lambda_2 + \lambda_3} - b_0 + \frac{g_3'(\lambda_1 + \lambda_2 + \lambda_3)}{g(\lambda_1 + \lambda_2 + \lambda_3)}; \quad \text{where}
\end{aligned}$$

$$g_j'(\lambda_1 + \lambda_2 + \lambda_3) = \frac{\partial g(\lambda_1 + \lambda_2 + \lambda_3)}{\partial \lambda_j}; \;\; j = 1(1)3.$$

For calculating the posterior variance of the model parameters, say θ_j, we need to take $q(\boldsymbol{\theta}) = \theta_j$ and $q(\boldsymbol{\theta}) = \theta_j^2$

- **Calculation of the posterior variance of** α.

  When $q(\boldsymbol{\theta}) = \alpha$, $u_1 = 1,\ u_2 = 0,\ u_3 = 0,\ u_4 = 0;\ \ u_{ij} = 0;\ i, j = 1(1)4$.

  When $q(\boldsymbol{\theta}) = \alpha^2$, $u_1 = 2\alpha,\ u_2 = 0,\ u_3 = 0,\ u_4 = 0;\ \ u_{11} = 2,\ \ u_{ij} = 0;\ i, j = 1(1)4$ and $(i, j) \neq (1, 1)$

- **Calculation of the posterior variance of** λ_1.

  When $q(\boldsymbol{\theta}) = \lambda_1$, $u_1 = 0,\ u_2 = 1,\ u_3 = 0,\ u_4 = 0;\ \ u_{ij} = 0;\ i, j = 1(1)4$.

  When $q(\boldsymbol{\theta}) = \lambda_1^2$, $u_1 = 0,\ u_2 = 2\lambda_1,\ u_3 = 0,\ u_4 = 0;\ \ u_{22} = 2,\ \ u_{ij} = 0;\ i, j = 1(1)4$ and $(i, j) \neq (2, 2)$.

- **Calculation of the posterior variance of** λ_2.

  When $q(\boldsymbol{\theta}) = \lambda_2$, $u_1 = 0,\ u_2 = 0,\ u_3 = 1,\ u_4 = 0;\ \ u_{ij} = 0;\ i, j = 1(1)4$.

  When $q(\boldsymbol{\theta}) = \lambda_2^2$, $u_1 = 0,\ u_2 = 0,\ u_3 = 2\lambda_2,\ u_4 = 0;\ \ u_{33} = 2,\ \ u_{ij} = 0;\ i, j = 1(1)4$ and $(i, j) \neq (3, 3)$.

- **Calculation of the posterior variance of** λ_3.

  When $q(\boldsymbol{\theta}) = \lambda_3$, $u_1 = 0,\ u_2 = 0,\ u_3 = 0,\ u_4 = 1;\ \ u_{ij} = 0;\ i, j = 1(1)4$.

  When $q(\boldsymbol{\theta}) = \lambda_3^2$, $u_1 = 0,\ u_2 = 0,\ u_3 = 0,\ u_4 = 2\lambda_3;\ \ u_{44} = 2,\ \ u_{ij} = 0;\ i, j = 1(1)4$ and $(i, j) \neq (4, 4)$.