Bayesian Analysis of Different Hybrid & Progressive Life Tests *

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Abstract

Type-I and Type-II censoring schemes are the widely used censoring schemes available for life testing experiments. A mixture of Type-I and Type-II censoring schemes is known as hybrid censoring scheme. Different hybrid censoring schemes have been introduced in recent years. In the last few years progressive censoring scheme also has received considerable attention. In this paper we mainly consider the Bayesian inference of the unknown parameters of two-parameter exponential distribution under different hybrid and progressive censoring schemes. It is observed that in general the Bayes estimate and the associated credible interval of any function of the unknown parameters cannot be obtained in explicit form. We propose to use the Monte Carlo sampling procedure to compute the Bayes estimate and also to construct the associated credible interval. Monte Carlo Simulation experiments have been performed to see the effectiveness of the proposed method in case of Type-I hybrid censored samples. The performances are quite satisfactory. One data analysis has been performed for illustrative purposes.

Key Words and Phrases: Type-I and Type-II censoring schemes; hybrid censoring scheme; progressive censoring scheme; prior distribution; posterior analysis.

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1 Introduction

In life testing experiments often the data are censored. Type-I and Type-II are the two most popular censoring schemes which are in use for any life testing experiment. Epstein [10] first introduced a hybrid censoring scheme (HCS), which is a mixture of Type-I and Type-II censoring schemes, and we will call this as the Type-I HCS. Like conventional Type-I censoring, the disadvantage of Type-I HCS is that the inference results are obtained under the condition that the number of observed failures is at least one, and in addition there may be very few failures before the experiment stops. In that case, the efficiency of the estimator will be very low. To avoid this problem, Childs *et al.* [8] proposed an alternative HCS named as Type-II HCS, which guarantees a minimum number of failures during the experiment.

Since then several HCSs have been introduced in the literature. For example, Childs *et al.* [5] introduced the generalized Type-I and Type-II HCSs, Balakrishnan *et al.* [4] introduced unified HCS, Kundu and Joarder [11] and Childs *et al.* [9] introduced the progressive HCS. Another censoring scheme which has received considerable attention in recent years is the progressive censoring scheme. It is also a more general censoring mechanism than the traditional Type-I or Type-II censoring schemes, see for example the monograph by Balakrishnan and Aggarwala [3] and also the recent review article by Balakrishnan [2] in this respect. Moreover, other than the standard Type-I or Type-II progressive censoring schemes, several other progressive schemes, like Type-I hybrid progressive censoring scheme, Type-II progressive hybrid censoring scheme, adaptive progressive censoring scheme have been introduced by several authors, and analysis have been performed under the assumption of specific lifetime distribution. But most of the analysis have been performed under the frequentist context. The main aim of this paper is to consider the Bayesian inference of the unknown parameter(s) of a two-parameter exponential distribution when the data are obtained from different censoring schemes.

Rest of the paper is organized as follows. In Section 2, we briefly mention about the different censoring schemes, and the prior information of the unknown parameters. In Section 3, we provide the posterior analysis and the Bayes estimators in details for Type-I HCSs. Monte Carlo simulation results are presented in Section 4. In Section 5 we provide the analysis of a Type-I hybrid censored data. In Section 6 we have indicated how the proposed method can be implemented for other censoring schemes also, and finally we conclude the paper in Section 7.

2 Different Censoring Schemes and Priors

2.1 Different Hybrid Censoring Schemes

Consider the following experiment. A total of n units is placed on a life testing experiment. The lifetimes of the sample units are independent and identically distributed (*i.i.d.*) random variables . Let the ordered lifetimes of these items be denoted by $X_{1:n}, \cdots, X_{n:n}$ respectively. In all the cases it is assumed that the failed items are not replaced.

Type-I HCS:

The test is terminated when a pre-chosen number, $r < n$, out of n items are failed, or when a pre-determined time, T, on test has been reached, *i.e.* the test is terminated at a random time $T_* = \min\{X_{r:n}, T\}$. For Type-I HCS, the available data will be of the form;

Case I: $\{x_{1:n} < \cdots < x_{r:n}\}$ if $x_{r:n} \leq T$

Case II: $\{x_{1:n} < \cdots < x_{d:n}\}$ if $x_{r:n} > T$,

here d denotes the number of observed failures that occur before time point T. For more details on Type-I HCS, the readers are referred to Epstein [10] or Chen and Bhattacharya [7].

Type-II HCS:

In Type-II HCS, the experiment is terminated at a random time $T^* = \max\{X_{r:n}, T\}$, here r and T are pre-determined as mentioned before. For more details on Type-II hybrid censoring schemes, see Childs *et al.* [8].

Generalized Type-I HCS:

Fix $r, k \in \{1, 2, \cdots, n\}$ and $T \in (0, \infty)$, such that $k < r < n$. If the k -th failure occurs before time T, terminate the experiment at $\min\{X_{r:n}, T\}$. If the k-th failure occurs after time T, terminate the experiment at $X_{k:n}$. Therefore, it is clear that this HCS modifies the Type-I HCS by allowing the experiment to continue beyond time T if very few failures had been observed up to time point T. Under this censoring scheme, the experimenter would like to observe r failures, but is willing to accept a bare minimum of k failures.

Generalized Type-II HCS:

Before starting the experiment, fix $r \in \{1, 2, \cdots, n\}$, and $T_1, T_2 \in (0, \infty)$, where $T_1 < T_2$. If the r-th failure occurs before the time point T_1, terminate the experiment at T_1. If the r-th failure occurs between T_1 and T_2, terminate the experiment at $X_{r:n}$. Otherwise, terminate the experiment at T_2. This hybrid censoring scheme modifies the Type-II HCS by guaranteeing that the experiment will be completed by time T_2. Therefore, T_2 represents the absolute longest that the experimenter allows the experiment to continue. For more details about generalized Type-I and Type-II HCSs, see Childs, Chandrasekhar and Balakrishnan [5].

2.2 Different Progressive Censoring Schemes

In this subsection we briefly describe different progressive censoring schemes and the data available in each scheme.

Type-I Progressive Censoring Scheme

Let $T_1, \cdots, T_k$ be pre-fixed k time points, and $R_1, \cdots, R_{k-1}$ be pre-fixed $(k-1)$ non-negative integers. Let n_1 be the number of failures before time point T_1. At the time point T_1, R_1 units are chosen randomly from the remaining $n - n_1$ units and removed from the experiment. The experiment continues, and suppose n_2 units fail between T_1 and T_2. At the time point T_2, from $n - n_1 - R_1 - n_2$ remaining units, R_2 units are chosen randomly and removed from the experiment and so on. Finally at the time of T_k, all the remaining units, say R_k, are removed and the experiment stops. It is assumed that n_j is the number of failures between T_{j-1} and T_j, for $j = 1, \cdots, k$, where $T_0 = 0$. Clearly, the maximum experimental time is T_k, and we have the following relation $\sum_{j=1}^{k} n_j + \sum_{j=1}^{k} R_j = n$. It should be noted that for Type-I progressive censoring scheme, it is always assumed, see Balakrishnan [2], the feasibility of the progressive censoring scheme, *i.e.*, the number of units still on test at each censoring time is larger than the corresponding number of units planned to be removed.

Type-II Progressive Censoring Scheme

Let $R_1, \cdots, R_m$ be m prefixed non-negative integers such that: $m + \sum_{j=1}^{m} R_j = n$. At the point of the first failure say $x_{1:n}$, R_1 units are chosen at random from the remaining $(n-1)$ units are removed from the experiment. Similarly, at the time of the second failure, say $x_{2:n}$, R_2 units are chosen at random from the remaining $n - R_1 - 2$ surviving units and removed, and so on. Finally at the time of the m-th failure say $x_{m:n}$, the rest of the R_m units are removed and the experiment stops.

Type-II Progressive HCS

The integer $m < n$ is pre-fixed at the beginning of the experiment, and $R_1, \cdots, R_m$ are pre-fixed integers, satisfying $R_1 + \cdots + R_m + m = n$. The time point T is also fixed before hand. At the time of first failure, say $x_{1:n}$, R_1 of the remaining units are randomly removed.

Similarly, at the time of the second failure, say $x_{2:n}$, R_2 of the remaining units are removed and so on. If the m-th failure occurs at $x_{m:n}$ before time point T, the experiment stops at the time point $x_{m:n}$. On the other hand if the m-th failure does not occur before time point T, and only J failures occur before the time point T, where $0 \leq J \leq m$, then at the time point T, all the remaining R^*_J units are removed and the experiment stops at the time point T. Note that $R^*_J = n - (R_1 + \cdots + R_J) - J$. Therefore, in presence of Type-II progressively HCS, we have one of the following types of observations;

Case I: $\{x_{1:n} < \cdots < x_{m:n}\}$ if $x_{m:n} < T$

Case II: $\{x_{1:n} < \cdots < x_{J:n}\}$ if $x_{J:n} < T < X_{J+1:n}$. Here $J < m$. For detailed description of the Type-II progressively HCS, see Kundu and Joarder [11].

2.3 Model Assumption and Prior Information

It is assumed that the failure times of the experimental units are independent and identically distributed two-parameter exponential random variables with the following probability density function (PDF)

$$f(x; \lambda, \mu) = \lambda e^{-\lambda(x-\mu)}; \quad x > \mu, \quad -\infty < \mu < \infty, \quad \lambda > 0.$$

We make the following prior assumption on λ and μ. Note that for known μ, λ has a conjugate gamma prior. It is assumed that λ has a gamma distribution with the shape and scale parameters $a > 0$ and $b > 0$ respectively, *i.e.* it has the following PDF

$$\pi_1(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}; \quad \lambda > 0.$$

It is further assumed that μ has a uniform prior over (M_1, M_2), where $M_1 < M_2$, *i.e.* it has the following PDF

$$\pi_2(\mu) = \frac{1}{M_2 - M_1}; \quad M_1 < \mu < M_2.$$

3 Bayes Estimates and Credible Intervals: Type-I HCS

Based on the observations from a Type-I HCS, the likelihood function can be written as

$$l(\text{Data}|\lambda, \mu) \propto \lambda^{d^*} e^{-\lambda\left(\sum_{i=1}^{d^*}(x_{i:n}-\mu)+(n-d^*)(U-\mu)\right)}.$$

Here for Case I, $d^* = r$, and $U = x_{r:n}$, and for Case II, $d^* = d$, $0 < d \le r-1$, and $U = T$. Therefore, based on the priors $\pi_1(\cdot)$ and $\pi_2(\cdot)$ as mentioned above the joint posterior density function of λ and μ becomes

$$l(\mu, \lambda|\text{Data}) \propto \lambda^{a+d^*-1} e^{-\lambda(b+\sum_{i=1}^{d^*}(x_{i:n}-\mu)+(n-d^*)(U-\mu))}; \quad \lambda > 0, \quad M_1 < \mu < M_3, \tag{1}$$

where $M_3 = \min\{M_2, x_{1:n}\}$. Note that $l(\mu, \lambda|\text{Data})$ is integrable over the region $D = \{(\mu, \lambda) : \lambda > 0, M_1 < \mu < M_3\}$ if $a + d^* > 0$.

If we want to compute the Bayes estimate of any function of μ and λ, say $g(\mu, \lambda)$, with respect to the squared error loss function, it will be posterior expectation of $g(\mu, \lambda)$, *i.e.*

$$\widehat{g}_B(\mu, \lambda) = \frac{\int_0^\infty \int_{M_1}^{M_3} g(\mu, \lambda) l(\mu, \lambda|\text{Data}) d\mu d\lambda}{\int_0^\infty \int_{M_1}^{M_3} l(\mu, \lambda|\text{Data}) d\mu d\lambda}, \tag{2}$$

provided it exists and is finite. Unfortunately (2) cannot be obtained in explicit form in most of the cases. Even when the integration (2) can be performed explicitly, it may not be possible to construct the corresponding credible interval. For example, let us consider the p-th percentile point, say η_p, of the two-parameter exponential distribution, *i.e.*

$$g(\mu, \lambda) = \eta_p = \mu - \frac{1}{\lambda} \ln(1-p).$$

The Bayes estimate of η_p with respect to the squared error loss function exists when

$a + d^* - 1 > 0$ and $d^* > 0$, and it is

$$\widehat{\eta}_p = \begin{cases} A_0 - \frac{A_0 - M_1}{a+d^*-2} \times \frac{\left(\frac{A_0-M_1}{A_0-M_3}\right)^{a+d^*-2} - 1}{\left(\frac{A_0-M_1}{A_0-M_3}\right)^{a+d^*-1} - 1} \times \{a + d^* - 1 + n\log(1-p)\} & \text{if } a + d^* \neq 2 \\ \\ A_0 - (A_0 - M_1)\left(\frac{A_0-M_1}{A_0-M_3} - 1\right)^{-1} (n\log(1-p) + 1)\log\left(\frac{A_0-M_1}{A_0-M_3}\right) & \text{if } a + d^* = 2, \end{cases}$$

where $A_0 = \frac{1}{n} \times \left(b + \sum_{i=1}^{d^*} x_{i:n} + (n - d^*)U\right)$. However, the posterior distribution of η_p cannot be obtained explicitly, and hence finding the credible interval analytically is not a trivial issue. We propose to use the Monte Carlo sampling to construct the associated symmetric credible interval (SCI) of η_p.

Note that (1) can be written as

$$l(\mu, \lambda|\text{Data}) = l(\lambda|\mu, \text{Data}) \times l(\mu|\text{Data}),$$

where

$$\begin{aligned} \lambda|\{\mu, \text{Data}\} &\sim \text{Gamma}\left(a + d^*,\, n(A_0 - \mu)\right), \\ l(\mu|\text{Data}) &= \frac{c(a + d^* - 1)}{(A_0 - \mu)^{a+d^*}}; \quad M_1 < \mu < M_3, \end{aligned} \tag{3}$$

here

$$A_0 = \frac{1}{n} \times \left(b + \sum_{i=1}^{d^*} x_{i:n} + (n - d^*)U\right), \quad c = \left\{\frac{1}{(A_0-M_3)^{a+d^*-1}} - \frac{1}{(A_0-M_1)^{a+d^*-1}}\right\}^{-1}.$$

Moreover, the posterior distribution of $\mu|\text{Data}$ has a compact and invertible cumulative distribution function as

$$F(x) = P(\mu \leq x|\text{Data}) = c\left\{\frac{1}{(A_0 - x)^{a+d^*-1}} - \frac{1}{(A_0 - M_1)^{a+d^*-1}}\right\}; \quad M_1 \leq x < M_3, \tag{4}$$

and hence random numbers can be generated quite easily from $l(\mu|\text{Data})$. Now we suggest to use the following procedure to compute the Bayes estimate of $g(\mu, \lambda)$, and also to construct the associated credible interval.

STEP 1: Generate μ_1 from (4).

STEP 2: Generate λ_1 from $l(\lambda|\mu, \text{Data})$ as given in (3).

STEP 3: Continue the process, and obtain $\{(\mu_1, \lambda_1), \cdots, (\mu_N, \lambda_N)\}$, and also obtain $g(\mu_1, \lambda_1)$, $\cdots$, $g(\mu_N, \lambda_N)\}$.

STEP 4: Compute the Bayes estimate of $g(\mu, \lambda)$ as

$$\widehat{g}_B(\mu, \lambda) = \frac{1}{N}\sum_{i=1}^{N} g(\mu_i, \lambda_i).$$

STEP 5: To construct a 100(1-α)% SCI of $g(\mu, \lambda)$, first order $g(\mu_i, \lambda_i)$ for $i = 1, \cdots, N$, say $g_1 < g_2 < \cdots < g_N$, and obtain the SCI as $(g_{[N\alpha/2]}, g_{[N(1-\alpha/2)]})$. Here $[x]$ denotes the largest integer less than or equal to x.

Similar methodology can be applied for other censoring schemes also, and we will briefly mention all the cases in Section 6 for completeness purposes.

4 Monte Carlo Simulations for Type-I Hybrid Censoring

In this section we present some simulation results to show how the proposed Bayes estimate and the associate credible interval behave for different sample sizes in case of Type-I hybrid censoring. We consider $g(\mu, \lambda) = \eta_{0.90}$, and through out we assume $a = 2$, $b = 0.1$, $M_1 = -100$, $M_2 = 100$, and $n = 10$. In all the cases we have considered $\mu = 0$ and $\lambda = 10$. We computed the Bayes estimate both theoretically and by Monte Carlo sampling. We computed the 90%, 95% and 99% SCIs using Monte Carlo sampling as suggested in the previous section. We report the average estimates and mean squared errors of the Bayes estimates, and the coverage percentages, the average lengths of SCIs based on 5000 replications and $N = 5000$ in each case. The results are reported in Table 1.

It is observed that in each case as T increases the biases and the MSEs decrease, it verifies the consistency properties of the estimates.In all the cases the coverage percentages are also quite close to the nominal levels even for very small sizes. For fixed sample size n and effective sample size, the coverage percentages decrease as T increases. For fixed n as r increases the biases, MSEs and the length of the credible intervals decrease. Moreover, in all the cases the theoretical and simulated Bayes estimates match very well.

Table 1: The average estimate (AE) and the corresponding MSE of $\eta_{0.90}$ and average lengths (AL) of the three different credible intervals and the associated coverage percentages (CP)

r	T	Sim.		Theo.		90%		95%		99%	
		AE	MSE	AE	MSE	CP	AL	CP	AL	CP	AL
	0.050	0.25	0.0226	0.25	0.0227	90.46	0.394	96.16	0.517	99.72	0.865
	0.075	0.24	0.0099	0.24	0.0100	89.76	0.334	96.22	0.428	99.70	0.671
5	0.100	0.23	0.0089	0.23	0.0089	89.88	0.324	95.60	0.414	99.62	0.646
	0.125	0.23	0.0083	0.23	0.0083	90.14	0.322	95.92	0.411	99.50	0.640
	0.150	0.23	0.0083	0.23	0.0083	90.14	0.322	95.92	0.411	99.50	0.640
	0.050	0.27	0.0186	0.27	0.0187	92.60	0.389	96.30	0.506	99.12	0.827
	0.075	0.25	0.0107	0.25	0.0107	90.16	0.308	95.88	0.389	99.16	0.588
10	0.100	0.24	0.0071	0.24	0.0071	89.74	0.262	94.74	0.326	99.24	0.478
	0.125	0.24	0.0064	0.24	0.0064	89.22	0.246	95.12	0.306	99.04	0.442
	0.150	0.23	0.0055	0.23	0.0055	90.20	0.238	94.86	0.295	98.92	0.424
	0.050	0.26	0.0161	0.26	0.0161	91.96	0.377	96.06	0.488	99.24	0.791
	0.075	0.25	0.0095	0.25	0.0095	91.08	0.303	95.98	0.382	99.24	0.575
15	0.100	0.24	0.0067	0.24	0.0068	90.96	0.265	95.66	0.330	99.28	0.482
	0.125	0.24	0.0064	0.24	0.0064	90.94	0.246	95.50	0.304	98.92	0.439
	0.150	0.24	0.0051	0.24	0.0051	90.08	0.228	94.86	0.281	98.88	0.401

5 Data Analysis

For illustrative purposes, we present the analysis of Type-I hybrid censoring data set. The data set has been obtained from Bain [1]. In this case 20 items are put on a life-test and they are observed for 150 hours. During that period 13 items fail with the following lifetime, measured in hours: 3, 19, 23, 26, 37, 38, 41, 45, 58, 84, 90, 109, and 138. In this case $n = 20$,

$r = 13$, and $T = 150$.

For this data set we obtain the Bayes estimates of $\eta_{0.90}$ and the associated SCIs with $a = 5$, $b = 0.1$, $M_1 = -100$, and $M_2 = 100$. The results are as follows: $\widehat{\eta}_{0.90}$(Theoretical) $= 92.80$, and $\widehat{\eta}_{0.90}$(Simulation) $= 93.29$. The associated 90%, 95%, and 99% SCIs are (62.26, 137.39), (58.50, 149.28), and (51.64, 179.40) respectively.

Now we will provide the empirical Bayes estimator of $\eta_{0.90}$. Note that in empirical Bayes analysis a popular choice of the hyper-parameters are argument maximum of the integrated posterior density function. In this case for $a + d^* > 1$, the integrated posterior density function, say $I(a,\, b,\, M_1,\, M_2)$, can be written as

$$\begin{aligned} I(a,\, b,\, M_1,\, M_2) &= \iint_D l(\mu,\, \lambda|\text{Data}) d\lambda d\mu \\ &= \frac{b^a\,\Gamma(a+d^*-1)}{n\,(M_1 - M_2)\,\Gamma(a)} \times \left[\frac{1}{(b+A_1)^{a+d^*-1}} - \frac{1}{(b+A_2)^{a+d^*-1}}\right], \end{aligned}$$

where $D = \{(\mu,\, \lambda) : M_1 < \mu < M_3,\, \lambda > 0\}$, $A_1 = \sum_{i=1}^{d^*} x_{i:n} + (n - d^*)U - nM_3$, and $A_2 = \sum_{i=1}^{d^*} x_{i:n} + (n - d^*)U - nM_1$. Here we assume M_1 and M_2 are known and want to maximize $I(a,\, b,\, M_1,\, M_2)$ with respect to a and b only. When M_1 and M_2 are known, we denote $I(a,\, b,\, M_1,\, M_2)$ by $I(a,\, b)$ for simplicity. For fixed a, the value of b, say $b^*(a)$, which maximizes the integrated posterior density function, is a positive solution of the equation

$$h(x) = 0,$$

where

$$\begin{aligned} h(b) &= a(b+A_1)(b+A_2)^{a+d^*} - a(b+A_1)^{a+d^*}(b+A_2) \\ &\quad + (a+d^*-1)b(b+A_1)^{a+d^*} - (a+d^*-1)b(b+A_2)^{a+d^*}. \end{aligned}$$

Analytically we could not prove that $I(a,\, b)$ does not have a maximum for finite (a, b).

However, the contour plot of $\log\{I(a,\, b)\}$ (see Figure 1), suggests that $I(a,\, b)$ does not possess a maximum.

Next empirical Bayes estimator of $\eta_{0.90}$ is considered when $M_1 \to -\infty$ and $M_2 \to \infty$. In this case the integrated posterior density function exists if $d^* > 1$, and it is given by

$$\begin{aligned} I_1(a,\, b) &= \int_{-\infty}^{x_{1:n}} \int_0^{\infty} l(\mu,\, \lambda|\text{Data}) d\lambda d\mu \\ &= \frac{b^a\, \Gamma(a + d^* - 1)}{n\, \Gamma(a)} \times \frac{1}{(b + A_1)^{a+d^*-1}}. \end{aligned} \tag{5}$$

In this case for fixed a, the value of b, say $b^*(a)$, which maximizes (5) is given by

$$b^*(a) = \frac{Aa}{d^* - 1}, \tag{6}$$

when $d^* > 1$. It can be shown, see in the Appendix, that $I_1(a, b^*(a))$ is an increasing function of a. Contour plot of $\log\{I_1(a,\, b)\}$ is reported in the Figure 1 along with the contour plot of $\log\{I(a,\, b)\}$ with $M_1 = -100$ and $M_2 = 100$. These two contour plots are not distinguishable as we take quite large range for the prior distribution of μ.

Since it seems, there does not exist any maximizers of the integrated posterior density function, we consider some large values of a and b for data analysis purpose, and the results are as follows:

Case 1: $a = 250$ and $b = 13354.17$: $\widehat{\eta}_{0.90}$(Theoretical) = 123.75, $\widehat{\eta}_{0.90}$(Simulation) = 123.62. The associated 90%, 95%, and 99% SCIs are (110.79, 137.39), (108.37, 140.49), and (103.56, 145.73).

Case 2: $a = 500$ and $b = 26708.33$: $\widehat{\eta}_{0.90}$(Theoretical) = 123.29, $\widehat{\eta}_{0.90}$(Simulation) = 123.18. The associated 90%, 95%, and 99% SCIs are (113.20, 132.86), (111.36, 135.11), and (107.41, 138.53).

Case 3: $a = 1000$ and $b = 53416.67$: $\widehat{\eta}_{0.90}$(Theoretical) = 123.35, $\widehat{\eta}_{0.90}$(Simulation) = 123.24.

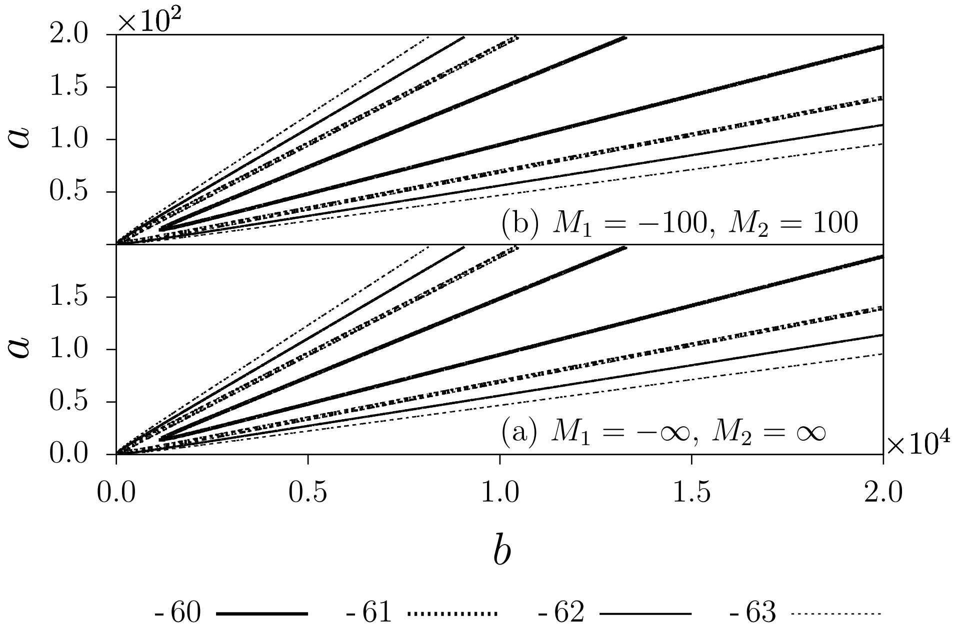

Figure 1: Contour plots of the logarithm of the integrated posterior density functions.

The associated 90%, 95%, and 99% SCIs are (115.43, 130.39), (113.38, 131.94), and (109.30, 134.60).

6 Other Censoring Schemes

The results corresponding to Type-II HCS, Generalized Type-I HCS and Generalized Type-II HCS can be obtained in a very similar way as the Type-I HCS. Now we will briefly discuss the Bayesian inference of the unknown parameters based on the observations obtained from different progressive censoring schemes.

Type-I Progressive Censoring Scheme

Based on the observations from a Type-I Progressive Censoring Scheme (PCS), the likelihood function can be written as

$$l(\text{Data}|\lambda,\mu) \propto \lambda^m e^{-\lambda W(\mu)},$$

here

$$W(\mu) = \sum_{i=1}^{m}(x_{i:n}-\mu) + \sum_{j=1}^{k} R_j(T_j-\mu) = \sum_{i=1}^{m} x_{i:n} + \sum_{j=1}^{k} R_j T_j - n\mu.$$

The posterior density function of λ and μ can be written as

$$l(\mu,\lambda|\text{Data}) = l(\lambda|\mu,\text{Data}) \times l(\mu|\text{Data}); \quad \lambda > 0,\ M_1 < \mu < M_3,$$

here

$$\lambda|\{\mu,\text{Data}\} \sim \text{Gamma}\left(a+m, b+W(\mu)\right),$$

$$l(\mu|\text{Data}) = \frac{c_1\,(a+m-1)}{(A_3-\mu)^{a+m}}; \quad M_1 < \mu < M_3,$$

where

$$A_3 = \frac{1}{n} \times \left(b + \sum_{i=1}^{m} x_{i:n} + \sum_{j=1}^{k} R_j T_j\right) \quad \text{and} \quad c_1 = \left\{\frac{1}{(A_3-M_3)^{a+m-1}} - \frac{1}{(A_3-M_1)^{a+m-1}}\right\}^{-1}.$$

Therefore, in this case also the Bayes estimate and the associated credible interval can be constructed in a very similar way.

Type-II Progressive Censoring Scheme

Based on the data obtained from a Type-II PCS, the likelihood function in this case can be written as

$$l(\text{Data}|\lambda,\mu) \propto \lambda^m e^{-\lambda W(\mu)},$$

here

$$W(\mu) = \sum_{i=1}^{m}(x_{i:n}-\mu) + \sum_{i=1}^{m} R_j(x_{i:n}-\mu) = \sum_{i=1}^{m}(R_i+1)x_{i:n} - n\mu.$$

The posterior density function of λ and μ can be written as

$$l(\mu,\lambda|\text{Data}) = l(\lambda|\mu,\text{Data}) \times l(\mu|\text{Data}); \quad \lambda > 0,\ M_1 < \mu < M_3,$$

where

$$\lambda|\{\mu, \text{Data}\} \sim \text{Gamma}\left(a+m, b+W(\mu)\right),$$

$$l(\mu|\text{Data}) = \frac{c_2\,(a+m-1)}{(A_4-\mu)^{a+m}}; \quad M_1 < \mu < M_3,$$

$$A_4 = \frac{1}{n} \times \left(b + \sum_{i=1}^{m}(R_i+1)x_{i:n}\right), \quad \text{and} \quad c_2 = \left\{\frac{1}{(A_4-M_3)^{a+m-1}} - \frac{1}{(A_4-M_1)^{a+m-1}}\right\}^{-1}.$$

Therefore, in this case also the Bayes estimate and the associated credible interval can be constructed in a very similar way.

Type-II Progressive HCS

Based on the observations from a Type-II Progressive HCS, the likelihood function can be written as

$$l(\text{Data}|\lambda, \mu) \propto \lambda^{d^*} e^{-\lambda W(\mu)},$$

where for Case I, $d^* = m$ and $W = \sum_{i=1}^{m}(1+R_i)(x_{i:m:n}-\mu)$, and for Case II, $d^* = J$, and $W = \sum_{i=1}^{J}(1+R_i)(x_{i:m:n}-\mu) + (T-\mu)R_J^*$. The posterior density function of λ and μ can be written as

$$l(\mu, \lambda|\text{Data}) = l(\lambda|\mu,\ \text{Data}) \times l(\mu|\text{Data}); \quad \lambda > 0,\ M_1 < \mu < M_3,$$

where

$$\lambda|\{\mu, \text{Data}\} \sim \text{Gamma}\left(a+d^*,\ b+W(\mu)\right)$$

$$l(\mu|\text{Data}) = \frac{c_3\,(a+d^*-1)}{(A_5-\mu)^{a^*+d}}; \quad M_1 < \mu < M_3,$$

$$A_5 = \begin{cases} \frac{1}{n} \times \left(b + \sum_{i=1}^{m}(1+R_i)x_{i:m:n}\right) & \text{for} \quad \text{Case I} \\ \frac{1}{n} \times \left(b + \sum_{i=1}^{J}(1+R_i)x_{i:m:n} + R_J^* T\right) & \text{for} \quad \text{Case II} \end{cases},$$

and

$$c_3 = \left\{\frac{1}{(A_5-M_3)^{a+d^*-1}} - \frac{1}{(A_5-M_1)^{a+d^*-1}}\right\}^{-1}.$$

Therefore, in this case also the Bayes estimate and the associated credible interval can be constructed in a very similar way.

7 Conclusions

In this paper we have considered the Bayesian inference of the two-parameter exponential model when the data are hybrid or progressively censored. We have assumed a uniform prior on the location parameter and gamma prior on the scale parameter. The Bayes estimates may not be obtained explicitly in many cases, even when they exist, and we have suggested to use the Monte Carlo sampling to compute simulation consistent Bayes estimators and also to construct the credible intervals. Monte Carlo simulation results suggest that the proposed Bayes estimators work quite well.

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Appendix

In this section we provide a formal proof that $J(a) = nI_1(a, b^*(a))$, where $b^*(a) = \dfrac{Aa}{d^*-1}$ is an increasing function of a. Note that

$$J(a) = \frac{\Gamma(a+d^*-1)\left(\frac{Aa}{d^*-1}\right)^a}{\Gamma(a)\left(\frac{Aa}{d^*-1}+A\right)^{a+d^*-1}}.$$

Now we will show that $\log J(a)$ is an increasing function of a. Let us consider

$$\frac{d\ln J(a)}{da} = \sum_{i=0}^{d^*-2} \frac{1}{a+i} - \log\left(1+\frac{d^*-1}{a}\right). \tag{7}$$

We will show that the right hand side of (7) is positive and we will show it by induction on d^*. Note that for $d^*=2$, the right hand side of (7) is clearly positive. Now consider $d^*=3$, and let

$$f(a) = \log(1 + \frac{2}{a}) - \log(1 + \frac{1}{a}) - \frac{1}{a+1}.$$

Using $x = \dfrac{1}{a}$, consider the function

$$g(x) = f\left(\frac{1}{a}\right) = \log(1+2x) - \log(1+x) - \frac{x}{x+1},$$

therefore for $x \geq 0$,

$$g'(x) = \frac{2}{1+2x} - \frac{1}{1+x} - \frac{1}{(1+x)^2} = -\frac{x}{(1+2x)(1+x)^2} \leq 0.$$

This implies that $g(x)$ is a decreasing function of x for $x \geq 0$. Since $g(0) = 0$, $g(x) \leq 0$ for $x \geq 0$. Moreover, since $\log(1+x) \leq x$, for $x \geq 0$, we have

$$\log(1+2x) \leq \log(1+x) + \frac{x}{1+x} \leq x + \frac{x}{1+x}. \tag{8}$$

From (8) it immediately follows

$$\frac{1}{a} + \frac{1}{a+1} - \log(1 + \frac{2}{a}) \geq 0.$$

Hence $\log J(a)$ is an increasing function of a for $d^* = 3$. Let it be true for $d^* = m$ and will prove that it is true for $d^* = m+1$ also. Let

$$\begin{aligned} f_m(a) &= \log(1 + \frac{m-1}{a}) - \log(1 + \frac{1}{a}) - \sum_{i=1}^{m-2} \frac{1}{a+i} \\ f_{m+1}(a) &= \log(1 + \frac{m}{a}) - \log(1 + \frac{1}{a}) - \sum_{i=1}^{m-1} \frac{1}{a+i} \\ &= f_m(a) + h_m(a), \end{aligned}$$

where

$$h_m(a) = \log\left(1 + \frac{m}{a}\right) - \log\left(1 + \frac{m-1}{a}\right) - \frac{1}{a+m-1}.$$

Using $x = \dfrac{1}{a}$ we consider the new function

$$g_m(x) = h_m\left(\frac{1}{a}\right) = \log(1+mx) - \log(1+(m-1)x) - \frac{x}{1+(m-1)x}.$$

Since for $x \geq 0$,

$$g'_m(x) = \frac{m}{1+mx} - \frac{m-1}{1+(m-1)x} - \frac{1}{(1+(m-1)x)^2} = \frac{-x}{(1+mx)(1+(m-1)x)} \leq 0,$$

$g_m(x)$ is an decreasing function of x. As $g_m(0) = 0$, $g_m(x) \leq 0$ for all $x \geq 0$, $\Rightarrow h_m(a) \leq 0$ for $a \geq 0$. Since $f_m(a) \leq 0$ due to induction hypothesis, $f_{m+1}(a) \leq 0$. Therefore,

$$\log\left(1+\frac{m}{a}\right) \leq \log\left(1+\frac{1}{a}\right) + \sum_{i=1}^{m-1} \frac{1}{a+i} \leq \sum_{i=0}^{m-1} \frac{1}{a+i},$$

hence

$$\sum_{i=0}^{m-1} \frac{1}{a+i} - \log\left(1+\frac{m}{a}\right) \geq 0.$$

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References

[1] Bain, L.J. (1978), *Statistical analysis of reliability and life-testing models*, Marcel and Dekker Inc., New York.

[2] Balakrishnan, N. (2007), "Progressive censoring methodology; an appraisal", (with discussions), *Test*, vol. 16, 211 - 289.

[3] Balakrishnan, N., and Aggarwala, R. (2000), *Progressive Censoring: Theory, Methods and Applications*, Boston, Birkhauser.

[4] Balakrishnan, N., Rasouli, A. and Sanjari F. N.(2008), "Exact likelihood inference based on an unified hybrid censored sample from the exponential distribution", *Journal of Statistical Computation and Simulation*, vol. 78, 475–488.

[5] Chandrasekhar, B., Childs, A. and Balakrishnan, N. (2004) "Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring", *Naval Research Logistics*, 994 - 1004.

[6] Chandrasekhar, B., Childs, A. and Balakrishnan, N. (2010) "Exact distribution of the MLEs of the parameters and of the quantiles of two-parameter exponential distribution under hybrid censoring", Technical Report, McMaster University.

[7] Chen, S. and Bhattacharyya, G.K. (1988), "Exact confidence bounds for an exponential parameter under hybrid censoring", *Communication in Statistics - Theory and Methods*, vol. 17, 1857 - 1870.

[8] Childs, A., Chandrasekhar, B., Balakrishnan, N. and Kundu, D. (2003), "Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution", *Annals of the Institute of Statistical Mathematics*, vol. 55, 319 - 330.

[9] Childs, A., Chandrasekhar, B. and Balakrishnan, N. (2008), "Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes", *Statistical models and methods for biomedical and technical systems*, 319–330, Birkhuser Boston, Boston, MA, 2008.

[10] Epstein, B. (1954), "Truncated life-test in exponential case", *Annals of Mathematical Statistics*, vol. 25, 555 - 564.

[11] Kundu, D. and Joarder, A. (2006), "Analysis of Type-II progressively hybrid censored data", *Computational Statistics and Data Analysis*, vol. 50, 2509 - 2528.