Inference on Weibull Parameters Under a Balanced Two-Sample Type-II Progressive Censoring Scheme

Shuvashree Mondal[1], Debasis Kundu[1,2]

Abstract

The progressive censoring scheme has received a considerable amount of attention in the last fifteen years. During the last few years joint progressive censoring scheme has gained some popularity. Recently, the authors Mondal and Kundu ("A New Two Sample Type-II Progressive Censoring Scheme", Communications in Statistics-Theory and Methods) introduced a balanced two sample Type-II progressive censoring scheme and provided the exact inference when the two populations are exponentially distributed. In this article we consider the case when the two populations follow Weibull distributions with the common shape parameter and different scale parameters. We obtain the maximum likelihood estimators of the unknown parameters. It is observed that the maximum likelihood estimators cannot be obtained in explicit forms, hence, we propose approximate maximum likelihood estimators, which can be obtained in explicit forms. We construct the asymptotic and bootstrap confidence intervals of the population parameters. Further we derive an exact joint confidence region of the unknown parameters. We propose an objective function based on the expected volume of this confidence region and using that we obtain the optimum progressive censoring scheme. Extensive simulations have been performed to see the performances of the proposed method, and one real data set has been analyzed for illustrative purposes.

Key Words and Phrases: Type-II censoring; progressive censoring; joint progressive censoring; maximum likelihood estimator; approximate maximum likelihood estimator; joint confidence region; optimum censoring scheme.

AMS Subject Classifications: 62N01, 62N02, 62F10.

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1 Introduction

In any life testing experiment censoring is inevitable. Different censoring schemes have been introduced in the literature to optimize time, cost and efficiency. Among different censoring schemes, Type-I and Type-II are the two most popular censoring schemes. But none of these censoring schemes allows removal of the experimental units during a life testing experiment. Progressive censoring scheme incorporates this flexibility in a life testing experiment. Progressive Type-II censoring scheme allows removal of experimental units during the experiment as well as ensures a certain number of failures to be observed during the experiment to make it efficient. Extensive work had been done on the different aspects of the progressive censoring since the introduction of the book by Balakrishnan and Aggarwala[2]. A comprehensive collection of different work related to progressive censoring scheme can be found in a recent book by Balakrishnan and Cramer[3].

But all these developments are mainly based on a single population. Recently two sample joint censoring schemes are becoming popular for a life testing experiment mainly to optimize time and cost. In a Type-II joint censoring scheme two samples are put on a life testing experiment simultaneously and the experiment is continued until a certain number of failures are observed. Balakrishnan and Rasouli[7] first considered the likelihood inference for two exponential populations under a Type-II joint censoring scheme. Ashour and Eraki[1] extended the results for multiple populations and when the lifetimes of different populations follow Weibull distributions.

Recently, Rasouli and Balakrishnan [21] introduced a joint progressive Type-II censoring (JPC) scheme and provided the exact likelihood inference for two exponential populations under this censoring scheme. Parsi and Ganjali [17] extended the results of Rasouli and Balakrishnan [21] for two Weibull populations. Mondal and Kundu[14] provided point and interval estimation for two Weibull populations under the JPC scheme. Doostparast and Ahmadi et al.[12] provided the Bayesian inference of the unknown parameters based on the data obtained from a JPC scheme under the LINEX loss function. Balakrishnan et al.[8] extended the JPC model to $K(\geq 2)$ populations and studied the exact likelihood inference of the unknown model parameters for the exponential distributions.

Mondal and Kundu[13] recently introduced a balanced joint progressive Type-II censoring (BJPC) scheme and it is observed that it has certain advantages over the JPC scheme originally introduced by Rasouli and Balakrishnan [21]. The scheme proposed by Mondal and Kundu[13] has a close connection with the self relocating design proposed by Srivastava[22]. Mondal and Kundu[13] provided the exact likelihood inference for the two exponential populations under a BJPC scheme. In practice, this scheme can be applied in an accelerated life testing problem, where the products are put under different stress levels. Again in acceptance sampling plan problem, this scheme can be used to make decision on the acceptance of the lots coming from different product lines. Therefore, in a single experiment, we can decide on multiple different products.

The main aim of this paper is to study likelihood inference of two Weibull populations under this new scheme. We provide the maximum likelihood estimators (MLEs) of the unknown parameters, and it is observed that the MLEs of the unknown parameters cannot be obtained in explicit forms. Due to this reason we propose to use approximate maximum likelihood estimators (AMLEs) of the unknown model parameters, which can be obtained in explicit forms. We propose to use the asymptotic distribution of the MLEs and bootstrap method to construct confidence intervals (CI) of the unknown model parameters. We also provide an exact joint confidence region of the parameter set. Further, we propose an objective function based on the expected volume of this confidence region and this has been used to find the optimum censoring scheme (OCS). Extensive simulations are performed to see the effectiveness of the different methods, and one real data set has been analyzed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2 we briefly describe the model and provide necessary assumptions. The MLEs and AMLEs are derived in Section 3. In Section 4 we provide the joint confidence regions of the unknown parameters. Next we propose the objective function in Section 5. In Section 6 we provide the simulation results and the analysis of a real data set. Finally we conclude the paper in Section 7.

2 Model Description and Model Assumption

The BJPC scheme proposed by Mondal and Kundu[13] can be briefly described as follows. Suppose there are two lines of similar products, say, line A and line B and it is important to study the relative merits of these two products. From each of the product lines we draw a sample of size m. Let k be the total number of failures to be observed from the life testing experiment and $R_1,\ldots,R_{k-1}$ are pre-specified non-negative integers satisfying $\sum_{i=1}^{k-1}(R_i+1) < m$. Under the BJPC scheme, two sets of samples from these two products are simultaneously put on a test. Suppose the first failure is coming from the product line A and the first failure time is denoted by W_1, then at W_1, R_1 units are removed randomly from the remaining $m-1$ surviving units of the product line A as well as R_1+1 units are chosen randomly from the remaining m surviving units of product line B and they are removed from the experiment. Next, if the second failure is coming from the product line B at time point W_2, R_2+1 units are withdrawn from the remaining $m-R_1-1$ units from the product line A and R_2 units withdrawn from the remaining $m-R_1-2$ units from the product line B randomly at W_2. The test is continued until k failures are observed with removal of all the remaining surviving units from both the product lines at the kth failure. In this life testing experiment a new set of random variable $Z_1,\ldots,Z_k$ is introduced where $Z_i = 1$ or 0 if the i-th failure comes from the product line A or B, respectively. Under the BJPC scheme, the data consist of $(\mathbf{W},\mathbf{Z})$ where $\mathbf{W} = (W_1,\ldots W_k)$ and $\mathbf{Z} = (Z_1,\ldots,Z_k)$. A schematic diagram of the BJPC scheme is provided in Figures 1 and 2.

A random variable X is said to follow Weibull distribution with the shape parameter $\alpha > 0$ and the scale parameter $\lambda > 0$ if it has the following probability density function (PDF)

$$f(x;\alpha,\lambda) = \begin{cases} \alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0, \end{cases} \tag{1}$$

and it will be denoted by $\text{WE}(\alpha,\lambda)$. We assume the lifetimes of m units of product line A, say $X_1,\ldots,X_m$, are independent identically distributed (i.i.d.) random variables from $\text{WE}(\alpha,\lambda_1)$ and the lifetimes of m units of product line B, say $Y_1,\ldots,Y_m$ are i.i.d. random variables from $\text{WE}(\alpha,\lambda_2)$.

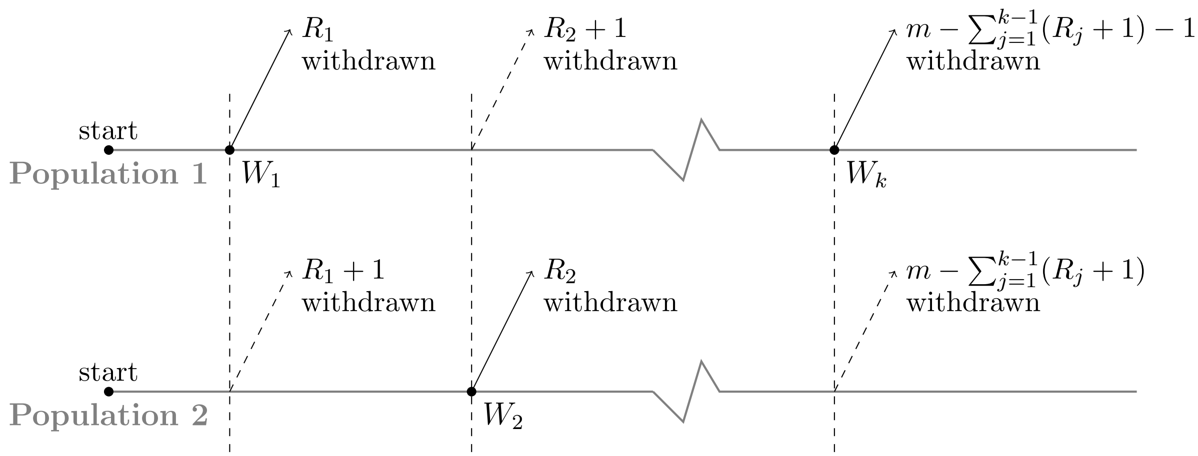

Figure 1: kth failure comes from Population 1

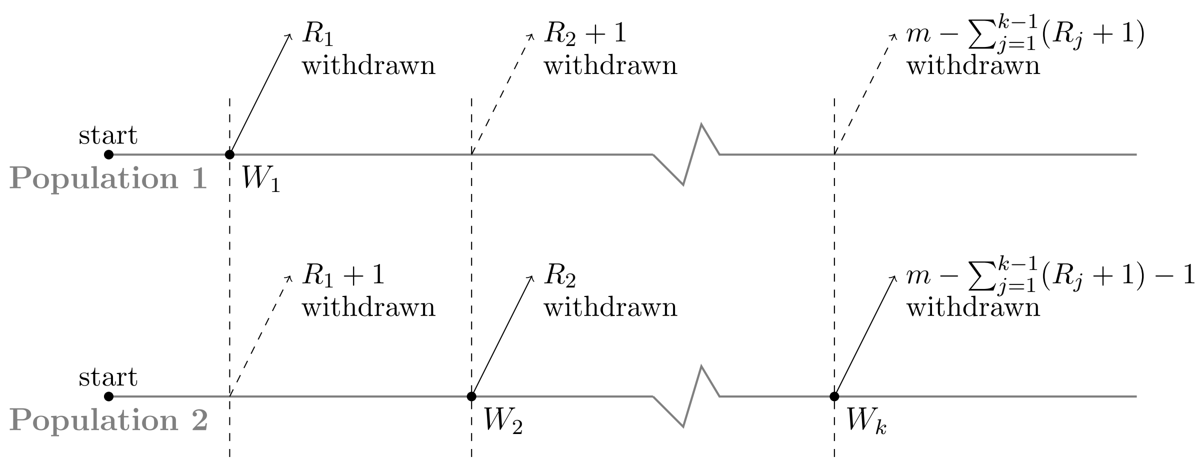

Figure 2: kth failure comes from Population 2

3 Point Estimations

3.1 Maximum Likelihood Estimators (MLEs)

The likelihood function of the unknown parameters $(\alpha, \lambda_1, \lambda_2)$ based on the observed data $(\mathbf{W}, \mathbf{Z})$, is given by

$$\begin{aligned} L(\alpha, \lambda_1, \lambda_2|\mathbf{w}, \mathbf{z}) &= \Big[\prod_{i=1}^{k}(m - \sum_{j=1}^{i-1}(R_j + 1))\Big] \times \alpha^k {\lambda_1}^{k_1} {\lambda_2}^{k_2} \times \Big[\prod_{i=1}^{k} {w_i}^{\alpha-1}\Big] \times \\ & \qquad e^{-(\lambda_1+\lambda_2)\left(\sum_{i=1}^{k-1}(R_i+1)_i w_i^{\alpha} + (m - \sum_{i=1}^{k-1}(R_i+1))w_k^{\alpha}\right)} \\ &= C\alpha^k {\lambda_1}^{k_1} {\lambda_2}^{k_2} \prod_{i=1}^{k} w_i^{\alpha-1} e^{-(\lambda_1+\lambda_2)A(\alpha)}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} C &= \prod_{i=1}^{k}(m - \sum_{j=1}^{i-1}(R_j + 1)), \\ k_1 &= \sum_{i=1}^{k} z_i, \quad k_2 = \sum_{i=1}^{k}(1 - z_i) = k - k_1, \\ A(\alpha) &= \sum_{i=1}^{k} c_i w_i^{\alpha}, \quad c_i = R_i + 1; i = 1, \ldots, k-1, c_k = m - \sum_{i=1}^{k-1}(R_i + 1). \end{aligned}$$

The log-likelihood function without the normalizing constant is given by

$$l(\alpha, \lambda_1, \lambda_2|\mathbf{w}, \mathbf{z}) = k\ln(\alpha) + k_1\ln(\lambda_1) + k_2\ln(\lambda_2) - (\lambda_1 + \lambda_2)A(\alpha) + (\alpha - 1)\sum_{i=1}^{k}\ln(w_i). \tag{3}$$

Hence, the normal equations can be obtained by taking partial derivatives of the log-likelihood function (3) and equating them to zero as given below

$$\frac{\partial l}{\partial \lambda_1} = \frac{k_1}{\lambda_1} - \sum_{i=1}^{k} c_i {w_i}^{\alpha} = 0, \tag{4}$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{k_2}{\lambda_2} - \sum_{i=1}^{k} c_i {w_i}^{\alpha} = 0, \tag{5}$$

$$\frac{\partial l}{\partial \alpha} = \frac{k}{\alpha} - (\lambda_1 + \lambda_2)\sum_{i=1}^{k} c_i \ln(w_i){w_i}^{\alpha} + \sum_{i=1}^{k}\ln w_i = 0. \tag{6}$$

For a given α, when $k_1 > 0$ and $k_2 > 0$ the MLEs of λ_1 and λ_2 can be obtained from (4) and (5) as follows:

$$\widehat{\lambda}_1(\alpha) = \frac{k_1}{A(\alpha)} \quad \text{and} \quad \widehat{\lambda}_2(\alpha) = \frac{k_2}{A(\alpha)}.$$

When α is also unknown, it is possible to obtain the MLE of α from (6) by substituting λ_1 and λ_2 with $\widehat{\lambda}_1(\alpha)$ and $\widehat{\lambda}_2(\alpha)$, respectively. Alternatively, the MLE of α can be obtained by maximizing the profile log-likelihood function of α, $l(\alpha, \widehat{\lambda}_1(\alpha), \widehat{\lambda}_2(\alpha)) = P(\alpha)$ (say), where,

$$P(\alpha) = k \ln(\alpha) - k \ln A(\alpha) + (\alpha - 1) \sum_{i=1}^{k} \ln(w_i). \tag{7}$$

The function $P(\alpha)$ as defined in (7) attains a unique maximum at some $\alpha^* \in (0, \infty)$ where, α^* is the unique solution of

$$\frac{1}{\alpha} - H(\alpha) + \frac{1}{k} \sum_{i=1}^{k} \ln(w_i) = 0, \tag{8}$$

where $H(\alpha) = \dfrac{A'(\alpha)}{A(\alpha)} = \dfrac{\sum_{i=1}^{k} c_i \ln(w_i) w_i^{\alpha}}{\sum_{i=1}^{k} c_i w_i^{\alpha}}$. The uni-modality of $P(\alpha)$ is shown in Appendix A.

Once the unique MLE of α, say $\widehat{\alpha}_{MLE}$, is obtained as a solution of (8), then the MLEs of λ_1 and λ_2 also can be obtained uniquely as $\widehat{\lambda}_{1_{MLE}} = \widehat{\lambda}_1(\widehat{\alpha}_{MLE})$ and $\widehat{\lambda}_{2_{MLE}} = \widehat{\lambda}_2(\widehat{\alpha}_{MLE})$, respectively, provided $k_1 > 0$ and $k_2 > 0$.

3.2 Approximate Maximum Likelihood Estimators

Since the MLEs cannot be obtained in explicit forms, we propose to use approximate MLEs (AMLEs) of the unknown model parameters which can be obtained in explicit forms. They are obtained by expanding the normal equations using the Taylor series expansion of the first order. It can be easily seen that for $i = 1, 2, \ldots, k$, the distribution of $(\lambda_1 + \lambda_2){W_i}^{\alpha}$ is independent of the parameters α, λ_1, λ_2 (discussed in Section 4). Let us define the following random variables:

$$U_i = \ln\left((\lambda_1 + \lambda_2){W_i}^{\alpha}\right) \quad \text{and} \quad V_i = \ln W_i; \quad i = 1, 2, \ldots, k.$$

Therefore, if $\theta = \ln\,(\lambda_1 + \lambda_2)$, then $U_i = \alpha V_i + \theta$, and the distribution of U_i is free from α, λ_1, λ_2, for $i = 1, 2, \ldots k$.

Now from (4) and (5) using u_i and v_i as defined above, we obtain

$$\sum_{i=1}^{k} c_i e^{u_i} = k, \tag{9}$$

and from (6) we have

$$\sum_{i=1}^{k} c_i v_i e^{u_i} = \frac{k}{\alpha} + \sum_{i=1}^{k} v_i. \tag{10}$$

Using the Taylor series expansion of order 1 of e^{u_i}, we obtain the AMLEs of the unknown model parameters as follows. The AMLE of α, say $\widehat{\alpha}_{AMLE}$, is the positive root of

$$\alpha^2 D_1 + \alpha D_2 = k, \tag{11}$$

where,

$$\begin{aligned}
D_1 &= (\sum_{i=1}^{k} c_i A_i v_i^2 - \frac{\left(\sum_{i=1}^{k} c_i A_i v_i\right)^2}{\sum_{i=1}^{k} c_i A_i}),\\
D_2 &= (\sum_{i=1}^{k} (c_i B_i - 1) v_i + \frac{(k - \sum_{i=1}^{k} c_i B_i) \sum_{i=1}^{k} c_i A_i v_i}{\sum_{i=1}^{k} c_i A_i}),\\
\xi_i &= E(U_i),\ A_i = e^{\xi_i}\ B_i = e^{\xi_i}(1 - \xi_i);\ i = 1, 2, \ldots, k.
\end{aligned}$$

As $c_i, A_i > 0$, using the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{k} c_i A_i v_i^2 \sum_{i=1}^{k} c_i A_i - \left(\sum_{i=1}^{k} c_i A_i v_i\right)^2 > 0.$$

Hence, we get $D_1 > 0$. As $D_1 > 0$, (11) must have one positive root and the AMLE of α is

$$\widehat{\alpha}_{AMLE} = \frac{-D_2 + \sqrt{D_2^2 + 4 D_1 k}}{2 D_1}. \tag{12}$$

Once the AMLE of α can be obtained, the AMLEs of λ_1, λ_2 are given by

$$\widehat{\lambda}_{1,AMLE} = \frac{k_1}{A(\widehat{\alpha}_{AMLE})}, \quad \widehat{\lambda}_{2,AMLE} = \frac{k_2}{A(\hat{\alpha}_{AMLE})}.$$

4 Exact Confidence Set

In this section we provide a methodology to construct an exact $100(1-\gamma)\%$ confidence set of α, λ_1 and λ_2. The following development is needed in that direction.

Suppose $G_1, G_2, \ldots, G_k$ are independent exponential random variables and

$$E(G_j) = \frac{1}{(\lambda_1 + \lambda_2)(m - \sum_{l=1}^{j-1}(R_l + 1))}; \quad j = 1, 2, \ldots, k.$$

Then $W_i^\alpha \stackrel{d}{=} \sum_{j=1}^{i} G_j$ for $i = 1, 2, \ldots, k$, where W_i's are same as defined before, and '$\stackrel{d}{=}$' means equal in distribution. The result follows from Lemma 2 of Mondal and Kundu[13] using $m = n$.

Let us use the following transformation:

$$\begin{aligned} S_1 &= m(\lambda_1 + \lambda_2)W_1^\alpha \\ S_2 &= (m - (R_1 + 1))(\lambda_1 + \lambda_2)(W_2^\alpha - W_1^\alpha) \\ &\vdots \\ S_k &= (m - \sum_{i=1}^{k-1}(R_i + 1))(\lambda_1 + \lambda_2)(W_k^\alpha - W_{k-1}^\alpha). \end{aligned}$$

It is evident that $S_1, S_2, \ldots, S_k$ are i.i.d. exponential random variables with mean one. Let us define

$$U = 2\sum_{i=2}^{k} S_i, \quad V = 2S_1, \quad T_1 = \frac{U}{(k-1)V}, \quad T_2 = U + V.$$

Observe that U and V are independent, $U \sim \chi^2_{2k-2}$, $V \sim \chi^2_2$, $T_1 \sim F_{2k-2,2}$ and $T_2 \sim \chi^2_{(2k)}$. Using Basu's theorem it follows that T_1 and T_2 are independently distributed. Note that

$$T_1 = \frac{U}{(k-1)V} = \frac{\sum_{i=1}^{k} S_i}{S_1(k-1)} - \frac{1}{k-1} = \frac{\sum_{i=1}^{k} c_i W_i^\alpha}{(k-1)mW_1^\alpha} - \frac{1}{k-1}. \tag{13}$$

$$T_2 = = 2(\lambda_1 + \lambda_2)\sum_{i=1}^{k} c_i W_i^\alpha. \tag{14}$$

From (13) it is clear that T_1 is a function of α, and from now on we denote it by $T_1(\alpha)$.

Also for, $0 < w_1 < w_2 < \ldots < w_k$,

$$t_1(\alpha) = \frac{\sum_{i=1}^{k} c_i w_i^\alpha}{(k-1)mw_1^\alpha} - \frac{1}{k-1},$$

is a strictly increasing function of α and $\lim_{\alpha \to 0} t_1(\alpha) = 0$, $\lim_{\alpha \to \infty} t_1(\alpha) = \infty$. Hence, the equation $t_1(\alpha) = t$ has a unique solution for $\alpha > 0$ and for all $t > 0$. Similar result is discussed in Lemma 1 in Wu and Shuo-Jye[24].

We introduce the following notations. Let $\varphi(\cdot) = t_1^{-1}(\cdot)$ and note that $\varphi(\cdot)$ is an increasing function. $F_{\gamma,\delta_1,\delta_2}$ denotes the upper γ-th quantile of F distribution with degrees of freedom δ_1, δ_2 and $\chi^2_{\gamma,\delta}$ denotes the upper γ-th quantile of χ^2 distribution with degrees of freedom δ.

Based on the above development a $100(1-\gamma)\%$ confidence interval of α can be given by

$$\Big[\varphi(F_{1-\gamma/2,2k-2,2}), \quad \varphi(F_{\gamma/2,2k-2,2})\Big] = B(\gamma) \quad \text{(say)}.$$

For a given α, a $100(1-\gamma)\%$ confidence set of (λ_1, λ_2) is given by

$$\left\{(\lambda_1, \lambda_2); \lambda_1 \geq 0, \lambda_2 \geq 0, \frac{\chi^2_{1-\gamma/2,2k}}{2\sum_{i=1}^{k} c_i w_i^{\alpha}} < \lambda_1 + \lambda_2 < \frac{\chi^2_{\gamma/2,2k}}{2\sum_{i=1}^{k} c_i w_i^{\alpha}}\right\} = C(\gamma; \alpha) \quad \text{(say)}.$$

The constructions of $B(\gamma)$ and $C(\gamma; \alpha)$ are given in Appendix B. Note that $C(\gamma; \alpha)$ is a trapezoid enclosed by four straight lines

$$i)\ \lambda_1 = 0, \quad ii)\ \lambda_2 = 0, \quad iii)\ \lambda_1 + \lambda_2 = \frac{\chi^2_{1-\gamma/2,2k}}{2\sum_{i=1}^{k} c_i w_i^{\alpha}} \quad iv)\ \lambda_1 + \lambda_2 = \frac{\chi^2_{\gamma/2,2k}}{2\sum_{i=1}^{k} c_i w_i^{\alpha}}.$$

Therefore, a $100(1-\gamma)\%$ joint confidence region of α, λ_1, λ_2 is given by

$$D(\gamma) = \Big\{(\alpha, \lambda_1, \lambda_2); \alpha \in B(\gamma_1), \ (\lambda_1, \lambda_2) \in C(\gamma_2; \alpha)\Big\},$$

here γ_1 and γ_2 are such that $1-\gamma = (1-\gamma_1)(1-\gamma_2)$.

5 Optimum Censoring Scheme

Finding an optimum censoring scheme is an important problem in any life testing experiment. In this section we propose a new objective function and based on which we provide an algorithm to find out the optimum censoring scheme.

In a progressive censoring scheme, for fixed sample size (m) and for fixed effective sample size (k), the *efficiency* of the estimators depends on the censoring scheme $\{R_1, \ldots, R_{k-1}\}$. In practical situation out of all possible censoring schemes, it is important to find out the optimal censoring scheme (OCS) i.e. the censoring scheme which provides maximum *information* about the unknown parameters. In this case, for fixed m and k, the possible set of censoring schemes consists of R_i's, $i = 1, \ldots, k-1$ such that $\sum_{i=1}^{k-1}(R_i + 1) < m$.

In case of Weibull and other lifetime distributions most of the available criteria to find the optimum censoring scheme, are based on the expected Fisher information matrix, i.e. the asymptotic variance covariance matrix of the MLEs, see for example Ng et al.[15], Pradhan and Kundu[18], Pradhan and Kundu[19], Balakrishnan and Cramer[3] and the references cited therein.

In this paper we propose an alternative objective function based on the volume of the exact joint confidence region of the parameters. First we will show that it is possible to determine the volume of the exact joint confidence region of α, λ_1 and λ_2. From the development in Section 4, the area $Area(C(\gamma_2;\alpha))$ of the trapezoid $C(\gamma_2;\alpha)$ is

$$Area(C(\gamma_2;\alpha)) = \frac{(\chi^2_{\gamma_2/2,2k})^2 - (\chi^2_{1-\gamma_2/2,2k})^2}{8A^2(\alpha)}.$$

The volume $V(D(\gamma))$ of the confidence region $D(\gamma)$ as constructed in previous section, becomes

$$V(D(\gamma)) = \frac{1}{8}((\chi^2_{\gamma_2/2,2k})^2 - (\chi^2_{1-\gamma_2/2,2k})^2)\int_{B(\gamma_1)} \frac{1}{(\sum\limits_{i=1}^{k} c_i w_i^{\alpha})^2}\, d\alpha. \tag{15}$$

Based on (15), we propose the objective function as $E(V(D(\gamma)))$ where the expectation is with respect to $(\mathbf{W},\mathbf{Z})$.

Therefore, for fixed m and k if $\mathcal{R}_1 = (R_{1,1}, R_{2,1}, \ldots, R_{k-1,1})$ and $\mathcal{R}_2 = (R_{1,2}, R_{2,2}, \ldots, R_{k-1,2})$ are two censoring plans then $\mathcal{R}_1$ is better than $\mathcal{R}_2$ if $\mathcal{R}_1$ provides smaller $E_{data}(V(D(\gamma)))$ than $\mathcal{R}_2$. The following algorithm can be used to compute $E(V(D(\gamma)))$, for fixed m, k and $R_1, \ldots, R_{k-1}$.

ALGORITHM:

- Step 1: Given m, k and $R_1, \ldots, R_{k-1}$ generate the data $(\mathbf{W},\mathbf{Z})$ under BJPC from two Weibull populations, $WE(\alpha,\lambda_1)$ and $WE(\alpha,\lambda_2)$.
- Step 2: Compute $V(D(\gamma))$ based on the data. This can be done by using trapezoidal rule.
- Step 3: Repeat Steps 1 and 2 say B times, and obtain B different $V(D(\gamma))$s. Take the average of these $V(D(\gamma))$s, which approximates $E_{data}(V(D(\gamma)))$.

6 SIMULATION STUDY AND DATA ANALYSIS

6.1 SIMULATION STUDY

In this section we compare the performances of the MLEs and AMLEs based on an extensive simulation experiment. We have taken the sample size $m = 25$ and different effective sample sizes namely, $k = 15, 20$. For different censoring schemes and for different parameter values we compute the average estimate (AE), variance estimates and mean square error (MSE) of the MLEs and AMLEs based on $10,000$ replications. The standard errors can be obtained from the square root of the variance estimates. We have also computed the 90% asymptotic and percentile bootstrap confidence intervals, and we have reported the average length (AL) and the coverage percentages (CP) in each case. Bootstrap confidence intervals are obtained based on 1000 bootstrap samples. The results are reported in Tables 1 to 6. We have used the following notations denoting different censoring schemes. For example, the progressive censoring scheme $R_1 = 2, R_2 = 0, R_3 = 0, R_4 = 0$, is denoted by $R = (2, 0_{(3)})$.

Some of the points are quite clear from this simulation experiment. It is observed that for fixed m (sample size) as k (effective sample size) increases, the biases and the standard errors decrease, consequently MSEs decrease for both MLEs and AMLEs as expected. The performances of the MLEs and AMLEs are very close to each other in all cases considered both in terms of biases and MSEs. Theoretically, it is quite complicated to compare the different properties of the MLEs and AMLEs. To conduct a comparative study between MLEs and AMLEs of the parameters, we plot histograms for both the estimators based on $50,000$ replications in Figures 3, 4, 5 for the scheme $k = 15, R = (7, 0_{(13)})$. Form these histograms, it is quite evident that the distributions of the MLEs and the AMLEs have very similar tail behavior, peakedness and skewness. Hence we recommend to use AMLEs in this case as they have explicit forms.

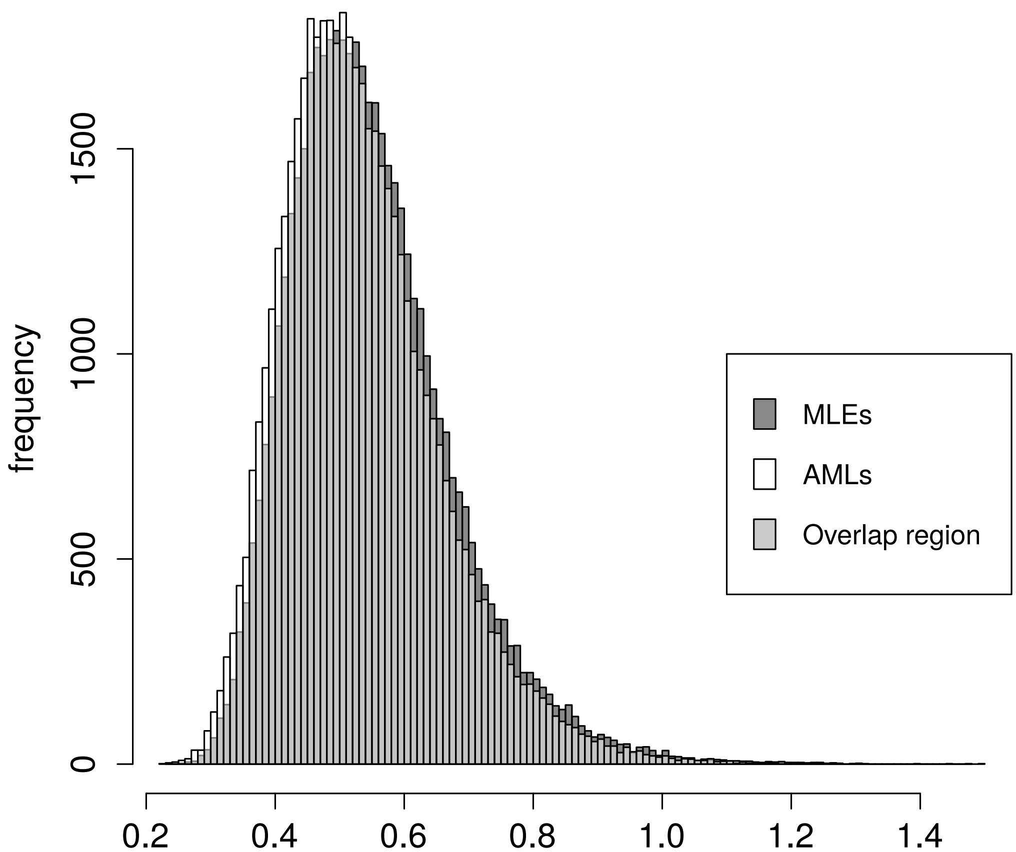

Figure 3: Histograms of the MLEs and the AMLEs of α with $m = 25, n = 25, k = 15, R = (7, 0_{(13)})$ and $\alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

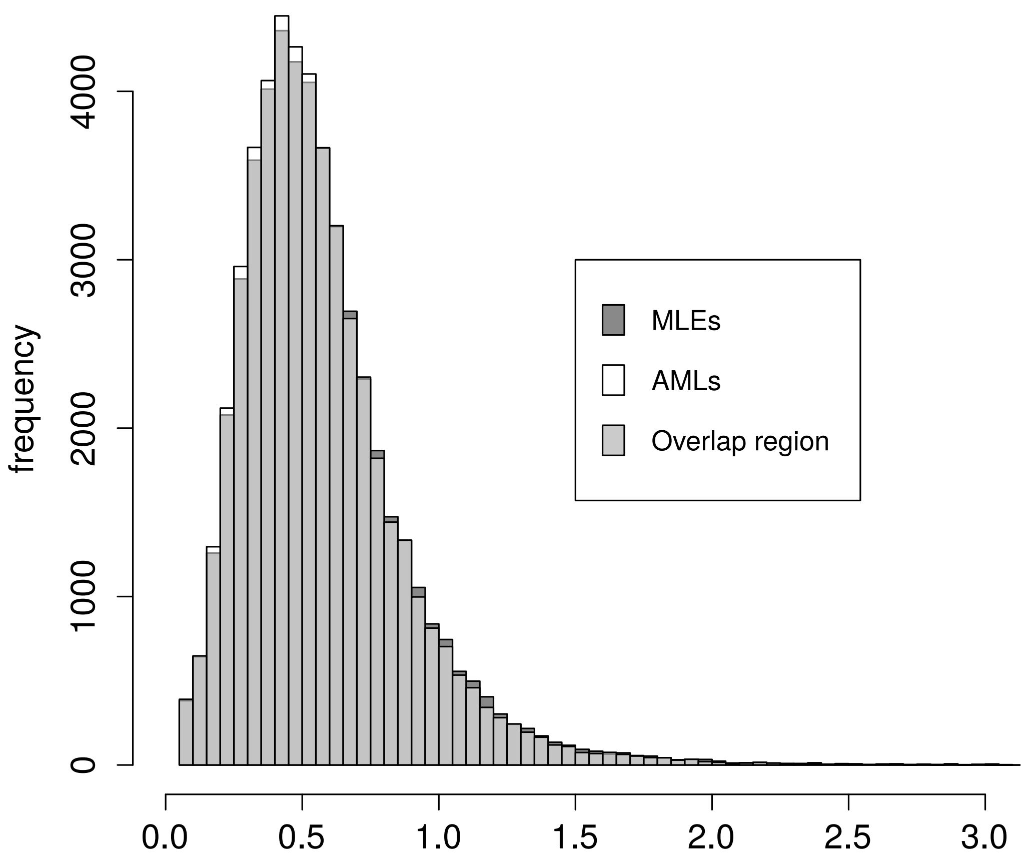

Figure 4: Histograms of the MLEs and the AMLEs of λ_1 with $m = 25, n = 25, k = 15, R = (7, 0_{(13)})$ and $\alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

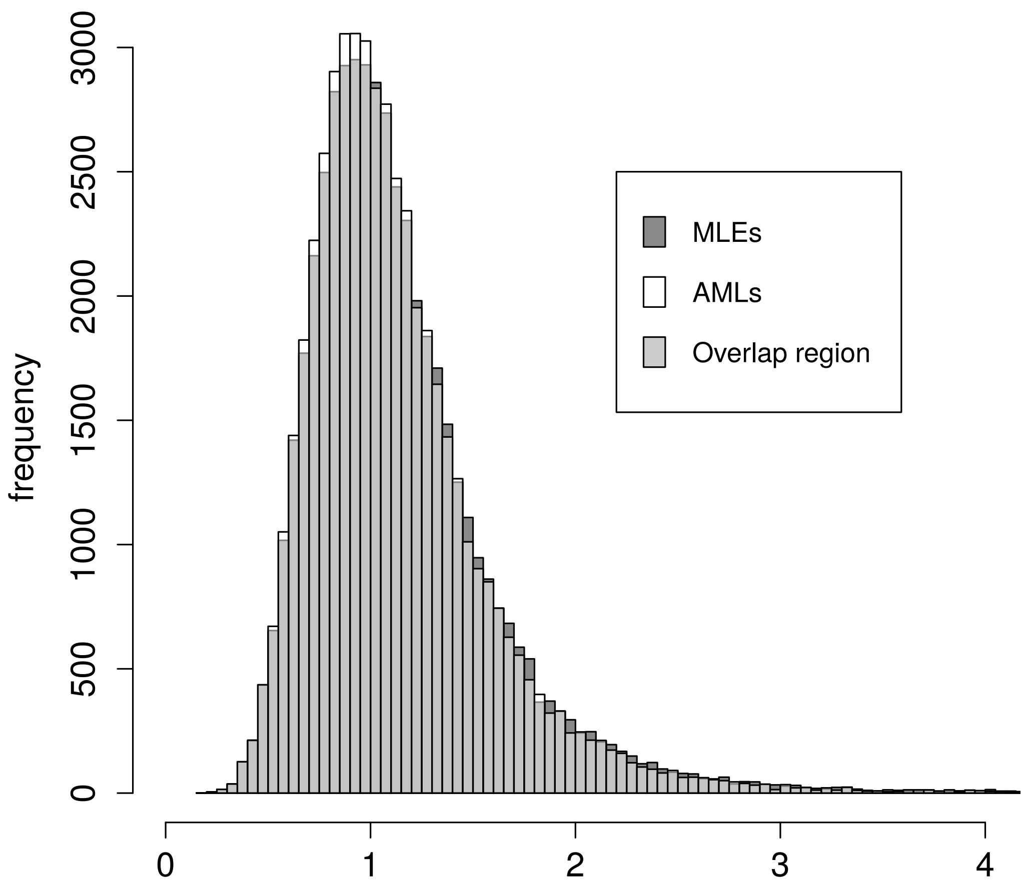

Figure 5: Histograms of the MLEs and the AMLEs of λ_2 with $m = 25, n = 25, k = 15, R = (7, 0_{(13)})$ and $\alpha = 0.5, \lambda_1 = 0.5, \lambda_2 - 1$

Now comparing the bootstrap and asymptotic confidence intervals in terms of the average lengths and coverage percentages, it is observed the performance of the bootstrap confidence intervals are not very satisfactory. Most of the times it cannot maintain the nominal level of the coverage percentages. Where as even for small sample sizes the performances of the asymptotic confidence intervals are quite satisfactory. In most of the cases considered the coverage percentages are very close to the nominal level. Hence, we recommend to use asymptotic confidence interval in this case.

We have further studied the relation between $E(V(D(\gamma)))$ and the expected time of test

(ETOT) for different censoring schemes and for different parameter values. $E(V(D(\gamma))$ is computed as described in Section 5 based on $B = 50,000$ samples with $\gamma = 0.1$ and $\gamma_1 = \gamma_2$. The ETOT i.e. $E(W_k)$ is computed by Monte-Carlo simulation based on 10,000 samples. The results are reported in Tables 7 to 9 for different parameter values and for different censoring schemes. We have also provided a scatter plot of ETOT vs. $E(V(D(\gamma)))$ for different censoring schemes in Figure 6. It is observed that the censoring schemes where more of the units are censored at the early stage of the experiment, provide smaller value of $E(V(D(\gamma)))$, than the schemes where units are censored at the later stage of the experiment. This result is quite justified as we censor more units at the early stage of the experiment, expected time of the experiment is increasing. With longer expected duration of the test, the data are expected to provide more information about the unknown model parameters. Hence the expected volume of the joint confidence set decreases. For fixed m, when $k = m$, the expected volume is minimum as expected.

Table 1: $m = 25, n = 25, \alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE			AMLE		
		AE	MSE	Variance estimate	AE	MSE	Variance estimate
k=15,R=$(7,0_{(13)})$	α	0.550	0.017	0.014	0.534	0.015	0.013
	λ_1	0.576	0.109	0.103	0.568	0.101	0.096
	λ_2	1.149	0.279	0.256	1.132	0.253	0.235
k=15,R=$(0_{(6)},7,0_{(7)})$	α	0.552	0.019	0.016	0.547	0.018	0.015
	λ_1	0.601	0.143	0.136	0.595	0.137	0.127
	λ_2	1.198	0.371	0.331	1.187	0.352	0.317
k=15,R=$(0_{(13)},7)$	α	0.564	0.024	0.019	0.559	0.023	0.019
	λ_1	0.628	0.184	0.167	0.622	0.176	0.161
	λ_2	1.248	0.514	0.452	1.236	0.491	0.435
k=20,R=$(3,0_{(18)})$	α	0.537	0.012	0.010	0.529	0.011	0.010
	λ_1	0.547	0.056	0.053	0.544	0.054	0.052
	λ_2	1.079	0.123	0.116	1.074	0.118	0.112
k=20,R=$(0_{(9)},3,0_{(9)})$	α	0.539	0.013	0.011	0.534	0.012	0.010
	λ_1	0.548	0.062	0.059	0.546	0.061	0.058
	λ_2	1.097	0.147	0.137	1.093	0.143	0.134
k=20,R=$(0_{(18)},3)$	α	0.538	0.012	0.010	0.529	0.011	0.010
	λ_1	0.542	0.055	0.053	0.539	0.054	0.052
	λ_2	1.083	0.130	0.123	1.078	0.125	0.118

Table 2: $m = 25, n = 25, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE			AMLE		
		AE	MSE	Variance estimate	AE	MSE	Variance estimate
k=15,R=(7,$0_{(13)}$)	α	1.096	0.071	0.061	1.064	0.063	0.058
	λ_1	0.575	0.103	0.097	0.566	0.095	0.090
	λ_2	1.154	0.292	0.268	1.136	0.264	0.245
k=15,R=($0_{(6)}$,7,$0_{(7)}$)	α	1.107	0.078	0.066	1.096	0.074	0.064
	λ_1	0.602	0.155	0.144	0.597	0.148	0.138
	λ_2	1.204	0.446	0.404	1.193	0.426	0.424
k=15,R=($0_{(13)}$,7)	α	1.126	0.101	0.085	1.116	0.097	0.083
	λ_1	0.620	0.210	0.195	0.614	0.201	0.188
	λ_2	1.244	0.660	0.600	1.232	0.625	0.571
k=20,R=(3,$0_{(18)}$)	α	1.082	0.057	0.050	1.073	0.055	0.049
	λ_1	0.557	0.066	0.062	0.554	0.064	0.061
	λ_2	1.109	0.162	0.150	1.105	0.158	0.146
k=20,R=($0_{(9)}$,3,$0_{(9)}$)	α	1.080	0.052	0.045	1.071	0.050	0.044
	λ_1	0.550	0.060	0.057	0.548	0.059	0.056
	λ_2	1.093	0.139	0.130	1.089	0.135	0.127
k=20,R=($0_{(18)}$,3)	α	1.085	0.058	0.050	1.076	0.056	0.050
	λ_1	0.555	0.066	0.062	0.553	0.065	0.062
	λ_2	1.113	0.172	0.159	1.109	0.167	0.155

Table 3: $m = 25, n = 25, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	MLE			AMLE		
		AE	MSE	Variance estimate	AE	MSE	Variance estimate
k=15,R=(7,$0_{(13)}$)	α	2.209	0.294	0.250	2.147	0.259	237
	λ_1	0.578	0.110	0.103	0.569	0.101	0.096
	λ_2	1.150	0.289	0.266	1.132	0.258	0.240
k=15,R=($0_{(6)}$,7,$0_{(7)}$)	α	2.220	0.319	0.270	2.197	0.304	0.265
	λ_1	0.597	0.132	0.122	0.592	0.126	0.117
	λ_2	1.192	0.388	0.351	1.181	0.365	0.332
k=15,R=($0_{(13)}$,7)	α	2.261	0.414	0.345	2.240	0.397	0.339
	λ_1	0.630	0.193	0.176	0.624	0.184	0.168
	λ_2	1.253	0.531	0.466	1.241	0.504	0.445
k=20,R=(3,$0_{(18)}$)	α	2.148	0.191	0.169	2.113	0.178	0.165
	λ_1	0.545	0.055	0.052	0.542	0.054	0.052
	λ_2	1.087	0.131	0.123	1.081	0.125	0.118
k=20,R=($0_{(9)}$,3,$0_{(9)}$)	α	2.158	0.207	0.182	2.140	0.199	0.179
	λ_1	0.548	0.060	0.057	0.546	0.059	0.056
	λ_2	1.098	0.141	0.131	1.094	0.137	0.128
k=20,R=($0_{(18)}$,3)	α	2.164	0.227	0.200	2.145	0.218	0.196
	λ_1	0.552	0.063	0.060	0.549	0.062	0.059
	λ_2	1.108	0.155	0.143	1.103	0.151	0.140

Table 4: AL and CP of CI's, $m = 25, n = 25, \alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	Bootstrap 90% CI		Asymptotic 90%CI	
		AL	CP	AL	CP
k=15,R=($7,0_{(13)}$)	α	0.435	83.1%	0.378	90.1%
	λ_1	1.141	87.8%	0.882	90.1%
	λ_2	1.812	84.5%	1.296	92.1%
k=15,R=($0_{(6)}$,$7,0_{(7)}$)	α	0.457	78.8%	0.378	89.8%
	λ_1	1.374	86.8%	0.937	90.6%
	λ_2	2.245	83.8%	1.430	92.9%
k=15,R=($0_{(13)},7$)	α	0.519	79.2%	0.431	89.7%
	λ_1	1.809	84.1%	1.049	91.1%
	λ_2	3.084	82.2%	1.667	93.8%
k=20,R=($3,0_{(18)}$)	α	0.365	82.9%	0.323	89.6%
	λ_1	0.811	88.6%	0.700	88.3%
	λ_2	1.241	86.4%	1.018	90.5%
k=20,R=($0_{(9)},3,0_{(9)}$)	α	0.366	83.2%	0.323	89.3%
	λ_1	0.836	89.6%	0.711	88.8%
	λ_2	1.285	87.5%	1.044	90.5%
k=20,R=($0_{(18)},3$)	α	0.392	84.7%	0.343	90.3%
	λ_1	0.852	88.7%	0.724	89.3%
	λ_2	1.355	85.7%	1.065	90.8%

Table 5: AL and CP of CI's, $m = 25, n = 25, \alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	Bootstrap 90% CI		Asymptotic 90%CI	
		AL	CP	AL	CP
k=15,R=($7,0_{(13)}$)	α	0.866	83.6%	0.759	89.9%
	λ_1	1.089	89.7%	0.869	89.4%
	λ_2	1.745	86.2%	1.293	92.0%
k=15,R=($0_{(6)}$,$7,0_{(7)}$)	α	0.914	78.4%	0.758	90.0%
	λ_1	1.525	86.8%	0.947	90.5%
	λ_2	2.683	82.8%	1.451	93.5%
k=15,R=($0_{(13)},7$)	α	1.027	81.7%	0.862	90.1%
	λ_1	1.639	88.4%	1.053	91.4%
	λ_2	2.810	84.4%	1.667	93.7%
k=20,R=($3,0_{(18)}$)	α	0.726	82.3%	0.643	90.3%
	λ_1	0.808	87.2%	0.697	88.5%
	λ_2	1.222	86.1%	1.012	90.3%
k=20,R=($0_{(9)},3,0_{(9)}$)	α	0.7355	82.5%	0.648	89.9%
	λ_1	0.835	88.9%	0.712	89.2%
	λ_2	1.292	86.0%	1.039	90.7%
k=20,R=($0_{(18)},3$)	α	0.789	81.9%	0.684	90.2%
	λ_1	0.920	86.7%	0.723	89.7%
	λ_2	1.419	84.8%	1.063	90.7%

Table 6: AL and CP of CI's, $m = 25, n = 25, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	Parameter	Bootstrap 90% CI		Asymptotic 90%CI	
		AL	CP	AL	CP
k=15,R=$(7,0_{(13)})$	α	1.774	81.4%	1.515	90.1%
	λ_1	1.169	87.6%	0.873	89.8%
	λ_2	1.875	85.7%	1.300	91.9%
k=15,R=$(0_{(6)}\ ,7,0_{(7)})$	α	1.813	79.7%	1.521	89.1%
	λ_1	1.332	88.1%	0.948	90.0%
	λ_2	2.227	83.1%	1.440	93.0%
k=15,R=$(0_{(13)},7)$	α	2.101	78.6%	1.722	90.0%
	λ_1	1.765	86.4%	1.074	91.2%
	λ_2	3.010	83.5%	1.708	93.7%
k=20,R=$(3,0_{(18)})$	α	1.461	80.9%	1.294	89.8%
	λ_1	0.821	88.0%	0.699	88.9%
	λ_2	1.237	86.6%	1.014	90.0%
k=20,R=$(0_{(9)},3,0_{(9)})$	α	1.478	81.0%	1.296	89.8%
	λ_1	0.844	88.7%	0.713	89.0%
	λ_2	1.294	86.2%	1.044	90.5%
k=20,R=$(0_{(18)},3)$	α	1.555	83.3%	1.374	89.3%
	λ_1	0.883	89.6%	0.724	89.3%
	λ_2	1.386	86.0%	1.066	91.5%

Table 7: $\alpha = 0.5, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	$E(Vol_{0.1})$	$ETOT$
m=25,k=20,R=$(5,0_{(18)})$	12.463	6.420
m=25,k=20,R=$(0,5,0_{(17)})$	12.583	6.383
m=25,k=20,R=$(0_{(2)},5,0_{(16)})$	12.845	6.369
m=25,k=20,R=$(0_{(3)},5,0_{(15)})$	13.032	6.245
m=25,k=20,R=$(0_{(4)},5,0_{(14)})$	13.243	6.181
m=25,k=20,R=$(0_{(8)},5,0_{(10)})$	14.614	6.043
m=25,k=20,R=$(0_{(14)},5,0_{(4)})$	17.319	5.092
m=25,k=20,R=$(0_{(16)},5,0_{(2)})$	20.768	4.458
m=25,k=20,R=$(0_{(17)},5,0)$	20.918	3.884
m=25,k=20,R=$(_{(18)},5)$	22.883	3.023
m=30,k=25,R=$(5,0_{(23)})$	9.616	7.181
m=30,k=25,R=$(0,5,0_{(22)})$	9.718	7.145
m=30,k=25,R=$(0_{(2)},5,0_{(21)})$	9.743	7.081
m=30,k=25,R=$(0_{(3)},5,0_{(20)})$	9.834	7.074
m=30,k=25,R=$(0_{(5)},5,0_{(18)})$	9.908	6.986
m=30,k=25,R=$(0_{(8)},5,0_{(15)})$	10.304	6.954
m=30,k=25,R=$(0_{(12)},5,0_{(11)})$	10.680	6.675
m=30,k=25,R=$(0_{(15)},5,0_{(8)})$	11.217	6.454
m=30,k=25,R=$(0_{(18)},5,0_{(5)})$	12.045	5.934
m=30,k=25,R=$(0_{(23)},5)$	14.197	3.451

Table 8: $\alpha = 1, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	$E(Vol_{0.1})$	$ETOT$
m=25,k=20,R=$(5,0_{(18)})$	24.360	2.393
m=25,k=20,R=$(0,5,0_{(17)})$	25.214	2.384
m=25,k=20,R=$(0_{(2)},5,0_{(16)})$	25.524	2.374
m=25,k=20,R=$(0_{(3)},5,0_{(15)})$	26.034	2.366
m=25,k=20,R=$(0_{(4)},5,0_{(14)})$	26.513	2.359
m=25,k=20,R=$(0_{(8)},5,0_{(10)})$	28.065	2.300
m=25,k=20,R=$(0_{(14)},5,0_{(4)})$	36.513	2.120
m=25,k=20,R=$(0_{(16)},5,0_{(2)})$	38.831	1.963
m=25,k=20,R=$(0_{(17)},5,0)$	40.552	1.816
m=25,k=20,R=$(0_{(18)},5)$	46.317	1.582
m=30,k=25,R=$(5,0_{(23)})$	19.331	2.549
m=30,k=25,R=$(0,5,0_{(22)})$	19.414	2.536
m=30,k=25,R=$(0_{(2)},5,0_{(21)})$	19.538	2.524
m=30,k=25,R=$(0_{(3)},5,0_{(20)})$	19.895	2.523
m=30,k=25,R=$(0_{(5)},5,0_{(18)})$	20.094	2.522
m=30,k=25,R=$(0_{(8)},5,0_{(15)})$	20.665	2.488
m=30,k=25,R=$(0_{(12)},5,0_{(11)})$	21.478	2.445
m=30,k=25,R=$(0_{(15)},5,0_{(8)})$	22.538	2.374
m=30,k=25,R=$(0_{(18)},5,0_{(5)})$	23.846	2.274
m=30,k=25,R=$(0_{(23)},5)$	28.662	1.692

Table 9: $\alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$

Censoring scheme	$E(Vol_{0.1})$	$ETOT$
m=25,k=20,R=$(5,0_{(18)})$	49.927	1.523
m=25,k=20,R=$(0,5,0_{(17)})$	50.653	1.522
m=25,k=20,R=$(0_{(2)},5,0_{(16)})$	51.265	1.515
m=25,k=20,R=$(0_{(3)},5,0_{(15)})$	51.578	1.512
m=25,k=20,R=$(0_{(4)},5,0_{(14)})$	52.719	1.508
m=25,k=20,R=$(0_{(8)},5,0_{(10)})$	57.245	1.492
m=25,k=20,R=$(0_{(14)},5,0_{(4)})$	67.359	1.424
m=25,k=20,R=$(0_{(16)},5,0_{(2)})$	77.601	1.365
m=25,k=20,R=$(0_{(17)},5,0)$	88.436	1.322
m=25,k=20,R=$(0_{(18)},5)$	89.061	1.229
m=30,k=25,R=$(5,0_{(23)})$	38.486	1.572
m=30,k=25,R=$(0,5,0_{(22)})$	38.571	1.572
m=30,k=25,R=$(0_{(2)},5,0_{(21)})$	38.781	1.569
m=30,k=25,R=$(0_{(3)},5,0_{(20)})$	39.411	1.568
m=30,k=25,R=$(0_{(5)},5,0_{(18)})$	40.220	1.561
m=30,k=25,R=$(0_{(8)},5,0_{(15)})$	41.074	1.560
m=30,k=25,R=$(0_{(12)},5,0_{(11)})$	43.549	1.538
m=30,k=25,R=$(0_{(15)},5,0_{(8)})$	45.367	1.517
m=30,k=25,R=$(0_{(18)},5,0_{(5)})$	48.084	1.486
m=30,k=25,R=$(0_{(23)},5)$	55.966	1.277

Figure 6: The ETOT and $E(D(\gamma))$ for $m = 22, k = 20, \alpha = 2, \lambda_1 = 0.5, \lambda_2 = 1$ where $s1$: R $= (5, 0_{(18)})$, $s2$: R $= (0_{(4)}, 5, 0_{(14)})$, $s3$: R $= (0_{(8)}, 5, 0_{(10)})$, $s4$: R $= (0_{(14)}, 5, 0_{(4)})$, $s5$: R $= (0_{(16)}, 5, 0_{(2)})$, $s6$: R $= (0_{(18)}, 5)$.

6.2 Data Analysis

In this section we perform the analysis of a real data set to illustrate how the propose methods work in practice. We have used the following data set originally obtained from Proschan[20] and here the data indicate the failure times (in hour) of air-conditioning system of two airplanes. The data are provided below.

Plane 7914: 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

Plane 7913: 1, 4, 11, 16, 18,18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 141, 163, 191, 206, 216.

From the above data sets we have generated two BJPC samples with the censoring schemes Scheme 1: $k = 10$ and $R = (14, 0_{(8)})$ and Scheme 2: $k = 10$, $R = (2_{(7)}, 0_{(2)})$. The generated data sets are provided below.

SCHEME 1:

$$\mathbf{w} = (1, 4, 5, 13, 15, 16, 22, 36, 80, 97) \quad \mathbf{z} = (0, 0, 1, 1, 1, 0, 1, 1, 0, 0);$$

Based on the above BJPC sample, the MLEs, AMLEs, and the two different 90% confidence intervals are provided in Table 10. We also compute the estimated standard errors of the MLEs and the AMLEs in Table 10. These standard errors are obtained based on the 10000 bootstrap samples. In Figure 7 we have provided the profile log-likelihood function $P(\alpha)$ of the shape parameter α and it is clear that $P(\alpha)$ attains a unique maximum. To get an idea about the joint confidence region of α, λ_1, λ_2, we have provided the confidence set of (λ_1, λ_2) for different values of α in Figure 8 where $\gamma = 0.1$ and $\gamma_1 = \gamma_2$.

Table 10: real data analysis(scheme-1)

Parameter	MLE		AMLE		90% Asymptotic CI		90% Bootstrap CI	
	Estimate	Standard error	Estimate	Standard error	LL	UL	LL	UL
α	0.9835	0.2661	0.9822	0.2592	0.6508	1.3160	0.7253	1.5900
λ_1	0.0175	0.0186	0.0176	0.0212	0	0.0426	0.0016	0.0559
λ_2	0.0175	0.0182	0.0176	0.0212	0	0.0426	0.0016	0.0562

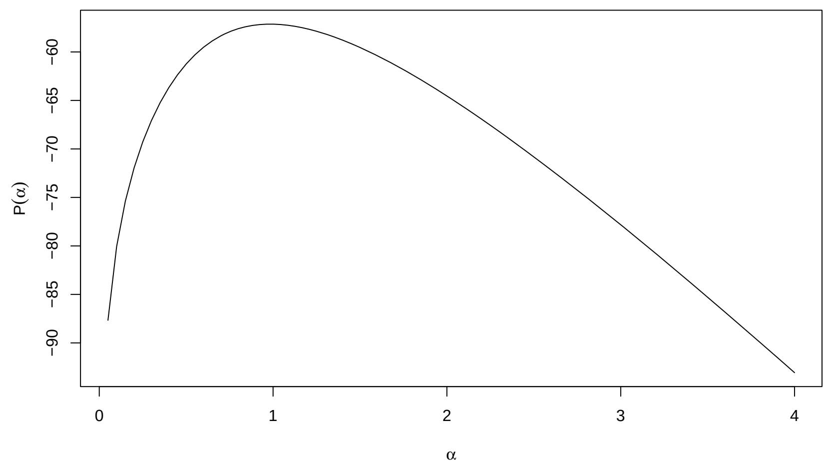

Figure 7: profile-loglikelihood function of shape parameter α for scheme-1

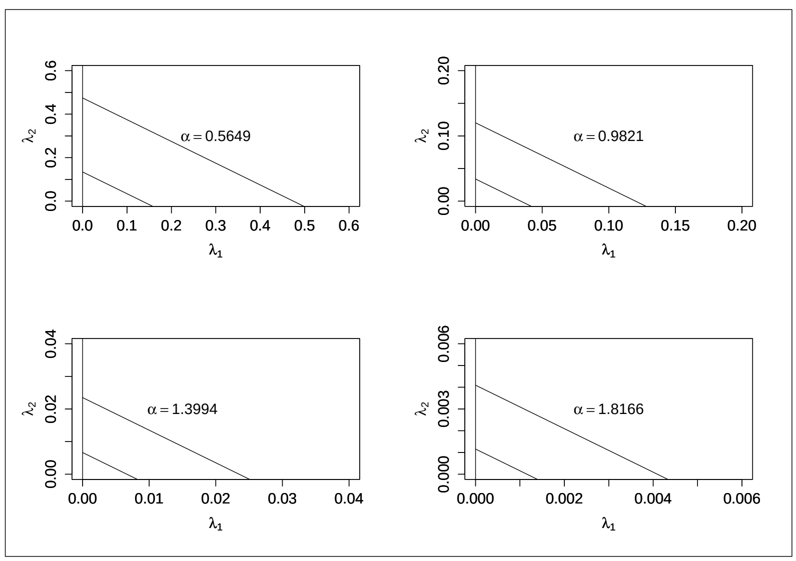

Figure 8: Confidence region of λ_1 and λ_2 for different values α for scheme-1

Scheme 2:

$$\mathbf{w} = (1, 3, 4, 5, 5, 13, 14, 31, 44, 51), \mathbf{z} = (0, 1, 0, 1, 1, 1, 1, 0, 1, 0);$$

In this case the estimates along with the estimated standard errors and the associated confidence intervals are reported in Table 11. The standard errors are computed based on 10000 bootstrap samples. The profile log-likelihood function $P(\alpha)$ has been provided in Figure 9 and it indicates that it attains a unique maximum. The confidence set of (λ_1, λ_2) for different values of α is provided in Figure 10 with $\gamma = 0.1$ and $\gamma_1 = \gamma_2$.

Table 11: real data analysis(scheme-2)

Parameter	MLE		AMLE		90% Asymptotic CI		90% Bootstrap CI	
	Estimate	Standard error	Estimate	Standard error	LL	UL	LL	UL
α	1.1740	0.3550	1.1612	0.3464	0.7533	1.5947	0.9105	2.0364
λ_1	0.0137	0.0128	0.0142	0.0139	0	0.0335	0.0012	0.0369
λ_2	0.0091	0.0092	0.0095	0.0103	0	0.0230	0.0007	0.0253

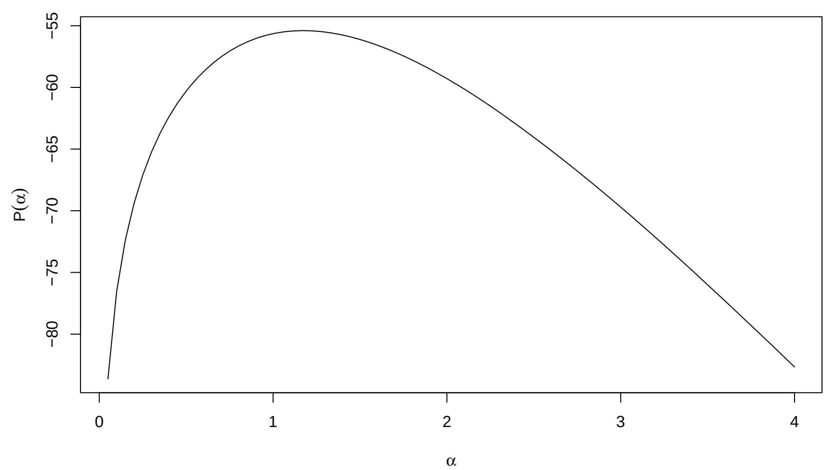

Figure 9: profile-loglikelihood function of shape parameter α for scheme-2

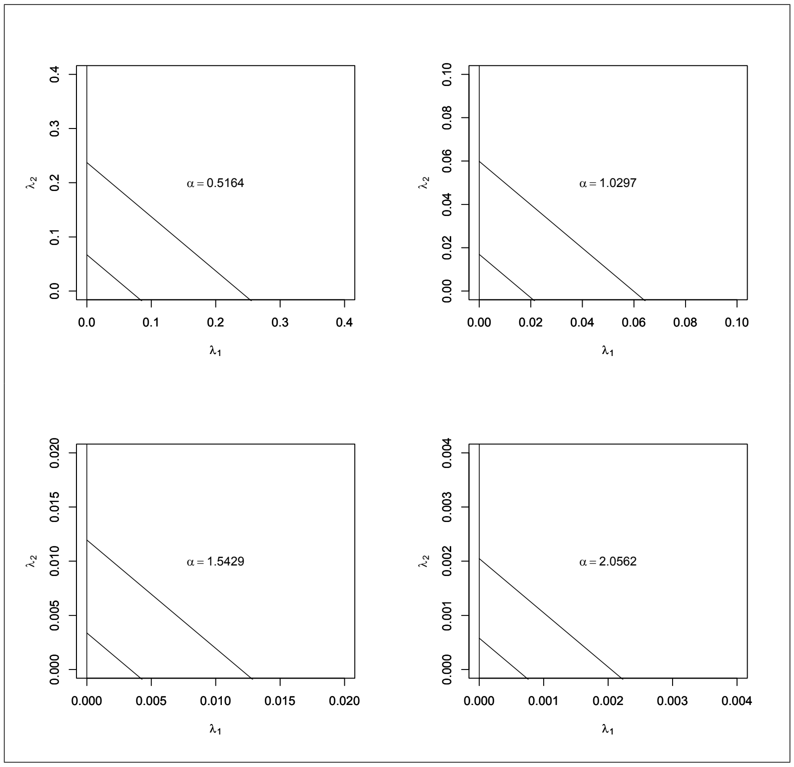

Figure 10: Confidence region of λ_1 and λ_2 for different values of α for Scheme-2

7 Conclusion

In this paper we analyze the new joint progressive censoring (BJPC) for two populations. It is assumed that the lifetimes of the two populations follow Weibull distributions with the same shape parameter but different scale parameters. We have obtained the MLEs of the unknown parameters and since they cannot be obtained in explicit forms we have proposed to use AMLEs which can be obtained explicitly. Based on extensive simulation experiments it is observed that the performances of the MLEs and the AMLEs are very similar in nature. We have obtained the asymptotic and the bootstrap confidence intervals and it is observed that the asymptotic confidence intervals perform quite well even for small sample sizes. Further we have constructed an exact joint confidence region of the unknown model parameters and based on the expected volume of the joint confidence region we have proposed an objective function and it has been used to obtain optimum censoring scheme. Note that all the developments in this paper are mainly based on the classical approach. It will be important to develop the necessary Bayesian inference. It may be mentioned that in this paper we have considered the sample sizes to be equal from both the populations, although most of the results can be extended even when they are not equal.

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Appendix A

To show $P(\alpha)$ is uni-modal:

$P(\alpha) \to -\infty \quad \text{as} \quad \alpha \to 0$

$$\begin{aligned}
\lim_{\alpha\to\infty} P(\alpha) &= -k \lim_{\alpha\to\infty} \frac{\left(\sum_{i=1}^{k-1}(R_i+1)\ln(w_i){w_i}^{\alpha} + (m-\sum_{i=1}^{k-1}(R_i+1))\ln(w_i){w_k}^{\alpha}\right)}{\left(\sum_{i=1}^{k-1}(R_i+1){w_i}^{\alpha} + (m-\sum_{i=1}^{k-1}(R_i+1)){w_k}^{\alpha}\right)} + \sum_{i=1}^{k}\ln(w_i) \\
&= -k\ln(w_k) + \sum_{i=1}^{k}\ln(w_i) \\
&= \sum_{i=1}^{k}\ln(\frac{w_i}{w_k}) < 0
\end{aligned}$$

$\Rightarrow P(\alpha) \to -\infty \quad \text{as} \quad \alpha \to \infty$. This concludes the MLE of α, α^* is attained in $(0,\infty)$.

According to Balakrishnan and Kateri [6] $H(\alpha)$ is increasing function of α and $\frac{1}{\alpha}$ is decreasing in α resulting unique solution of (8).

Appendix B

Construction of $B(\gamma)$:

$T_1(\alpha) \sim F_{2k-2,2}$, $P(F_{1-\gamma/2,2k-2,2} < T_1(\alpha) < F_{\gamma/2,2k-2,2}) = 1-\gamma$. As $t_1(\alpha)$ is an increasing function of α and $\varphi(t)$ is the unique solution of $t_1(\alpha) = t$,

$$\begin{aligned}
& P(\varphi(F_{1-\gamma/2,2k-2,2}) < \alpha < \varphi(F_{\gamma/2,2k-2,2})) = 1-\gamma \\
\Longrightarrow \quad & P(\varphi(F_{1-\gamma/2,2k-2,2}), < \alpha < \varphi(F_{\gamma/2,2k-2,2}) = 1-\gamma.
\end{aligned}$$

Construction of $C(\gamma;\alpha)$:

$T_2 \sim \chi^2_{2k}$, using (14) we obtain, $P(\chi^2_{1-\gamma/2,2k} < 2(\lambda_1+\lambda_2)\sum_{i=1}^{k} c_i W_i^{\alpha} < \chi^2_{\gamma/2,2k}) = 1-\gamma$.

Hence,

$$P\left((\lambda_1,\lambda_2);\lambda_1\geq 0,\lambda_2\geq 0,\frac{\chi^2_{1-\gamma/2,2k}}{2\sum_{i=1}^{k}c_iw_i^{\alpha}}<\lambda_1+\lambda_2<\frac{\chi^2_{\gamma/2,2k}}{2\sum_{i=1}^{k}c_iw_i^{\alpha}}\right)=1-\gamma.$$

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