On Estimation of $R = P(Y < X)$ for Exponential Distribution Under Progressive Type-II Censoring

Buğra Saraçoğlu[1] & Ismail Kinaci[1] & Debasis Kundu[2]

Abstract

This paper deals with the estimation of the stress-strength parameter $R = P(Y < X)$, when X and Y are independent exponential random variables, and the data obtained from both distributions are progressively type-II censored. The uniformly minimum variance unbiased estimator and the maximum likelihood estimator are obtained for the stress strength parameter. Based on the exact distribution of the maximum likelihood estimator of R, an exact confidence interval of R has been obtained. Bayes estimate of R and the associated credible interval are also obtained under the assumption of independent inverse gamma priors. An extensive computer simulation is used to compare the performances of the proposed estimators. One data analysis has been performed for illustrative purpose.

Key Words and Phrases: Maximum likelihood estimator; Uniformly minimum variance unbiased estimator; Confidence intervals; Bayes estimator; Prior distribution; Posterior analysis; Credible intervals.

[1] Department of Statistics, Faculty of Science, Selcuk University, Konya, Turkey.

[2] Department of Mathematics and Statistics, IIT Kanpur, India.

1 Introduction

In this paper we consider the statistical inference of the stress-strength parameter $R = P(X < Y)$ when X and Y are independent exponential random variables. It is further assumed that we observe progressively type-II censored samples from both exponential distributions. Estimation of the stress-strength parameter has received considerable attention in the statistical literature, starting with the pioneering work of Birnbaum [4]. Birnbaum [4] provided an interesting connection between the classical Mann-Whitney statistic and the stress-strength model. Since then, work has been accomplished on the estimation and inference of the stress-strength parameter for different distributions from the frequentist and Bayesian points of view. The monograph by Kotz *et al.* [9] provided an excellent review of the development of this model till that time. For some of the recent references, the readers may refer to Kundu and Gupta [11, 12], Raqab and Kundu [14], Kundu and Raqab [13], Raqab *et al.* [15], Saraçoğlu and Kaya [16], Saraçoğlu *et al.* [17], Krishnamoorthy *et al.* [10], Kim and Chung [8] and the references cited therein.

Although extensive works have been done on the developments for the stress-strength models under complete samples, not much attention has been paid to when the data are censored, for an example, Jiang and Wong [7]. But in many practical situations it may not be unrealistic. Consider the following example of comparison of two treatments discussed in page 202-203 of Kotz *et al.* [9]. This is an old technique motivated by the close relation between Wilcoxon type test and stress-strength model. Suppose we want to compare the effectiveness of two drugs, namely Drug-I and Drug-II and suppose X and Y represent the remission times of these two drugs. We are interested about $P(Y < X)$. Here the terminology stress-strength may not be appropriate, but analytically they are equivalent. In a situation like this, we observe censored samples from both treatment groups, rather than complete samples.

Another example which was discussed on page 209 by Kotz *et al.* [9] and by Surles and Padgett [18] is the comparison of carbon fiber strengths at different gauge lengths. One is interested in $P(Y < X)$, where X and Y represent the strengths of the fiber at two different gauge lengths. In this example, it may also be possible that one observes censored samples rather than complete samples from both populations.

Among different censoring schemes, type-I and type-II are the two most popular censoring schemes. In type-I censoring scheme, the experiment is stopped at a pre-fixed time point, and in type-II censoring scheme, the experiment is stopped whenever a fixed number of failures (pre-fixed) has been observed. Unfortunately none of these censoring schemes allows the removal of active units during the experiment. Progressive censoring scheme allows the experimenter to remove active units during the experiment. Combining the type-II censoring and progressive censoring schemes, the progressive type-II censoring can be described as follows: Given $m < n$, and $R_1, \cdots, R_m$ non-negative integers such that

$$R_1 + \cdots + R_m = n - m. \tag{1}$$

Let n items be on the life test at the same time. At the time of the first failure, R_1 items from the rest of the active $n - 1$ are chosen at random and removed. Similarly, at the time of the second failure R_2 of the remaining $n - R_1 - 2$ items are chosen at random and removed, and so on. Finally, at the time of the m-th failure all the remaining active items are removed. Note that the conventional type-II censoring scheme is a special case of the progressive type-II censoring scheme, and it can be obtained by using $R_1 = \cdots = R_{m-1} = 0$ and $R_m = n - m$. There are several advantages of the progressive censoring scheme than the usual type-I and type-II censoring schemes. Due to this reason, extensive work has been done on the development of the progressive censoring methodology during the last few years. See for examples the monograph by Balakrishnan and Aggarwala [2] and the recent review article by Balakrishnan [1] on different aspects of progressive censoring schemes.

Now, we can formulate the problem as follows. It is assumed that X and Y are independent exponential random variables with means θ_1 and θ_2 respectively, *i.e.* the probability density functions (pdfs) of X and Y for $x > 0$ and $y > 0$ are;

$$f_X(x) = \frac{1}{\theta_1}\, e^{-\frac{x}{\theta_1}} \qquad \text{and} \qquad f_Y(y) = \frac{1}{\theta_2}\, e^{-\frac{y}{\theta_2}}, \tag{2}$$

respectively. From now on, an exponential random variable with mean θ will be denoted by $Exp(\theta)$. There are two progressive censoring schemes, namely $\{n_1, m_1, R_1, R_2, \ldots, R_{m_1}\}$ and $\{n_2, m_2, S_1, S_2, \ldots, S_{m_2}\}$ for X and Y, respectively. We observe progressively censored sample $\{X_{1:m_1:n_1}, \ldots, X_{m_1:m_1:n_1}\}$ from X and $\{Y_{1:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2}\}$ from Y. Based on the above progressively censored samples, our problem is to estimate $R = P(Y < X) = \dfrac{\theta_1}{\theta_1 + \theta_2}$.

In this case, it is possible to obtain the uniformly minimum variance unbiased estimator (UMVUE) of R. Although the UMVUE can be obtained in explicit form, due to its complicated nature, deriving the exact distribution of the UMVUE is not very simple. The maximum likelihood estimator (MLE) of R can be obtained very conveniently, and the exact distribution of the MLE of R can be easily obtained. Based on the exact distribution of the MLE of R, it is possible to construct exact confidence interval of R. The Bayesian inference of R is also investigated under the assumption of independent inverse gamma priors.

The rest of the paper is organized as follows. In Section 2, the UMVUE of R is derived. The maximum likelihood estimator of R and its exact distribution are provided in Section 3. In this section, exact confidence interval based on the MLE of R is also discussed. Bayes estimate and the associated credible interval are provided in Section 4. Simulation results and data analysis are presented in Sections 5 and 6, respectively. Finally, concluding remarks are given in Section 7.

2 UMVUE of R

In this section, the UMVUE of R is derived. Let $\{X_{1:m_1:n_1},\ldots,X_{m_1:m_1:n_1}\}$ be a progressively censored sample from $Exp(\theta_1)$ under the progressive censoring scheme $\{n_1,m_1,R_1,R_2,\ldots,R_{m_1}\}$. Similarly, let $\{Y_{1:m_2:n_2},\ldots,Y_{m_2:m_2:n_2}\}$ be a progressively censored sample from $Exp(\theta_2)$ under the progressive censoring scheme is $\{n_2,m_2,S_1,S_2,\ldots,S_{m_2}\}$. The joint pdf of $X_{1:m_1:n_1},\ldots,X_{m_1:m_1:n_1}$ is

$$\begin{aligned} f_{X_{1:m_1:n_1},\ldots,X_{m_1:m_1:n_1}}(x_1,\ldots,x_{m_1}) &= \\ \frac{c}{\theta_1^{m_1}}\exp\left[-\frac{1}{\theta_1}\left(\sum_{i=1}^{m}(R_i+1)x_{i:m_1:n_1}\right)\right];\ 0 &< x_1<\ldots<x_{m_1}<\infty, \end{aligned} \tag{3}$$

where $c=n_1(n_1-R_1-1)\cdots(n_1-R_1-\cdots-R_{m_1-1}-m_1+1)$ is the normalizing constant, see for an example Balakrishnan and Aggarwala [2]. It is immediate from Eq. (3) that $U=\sum_{i=1}^{m_1}(R_i+1)X_{i:m_1:n_1}$ is a complete sufficient statistics for θ_1. Consider the following transformation;

$$\begin{aligned} Z_1 &= n_1X_{1:m_1:n_1} \\ Z_2 &= (n_1-R_1-1)\left(X_{2:m_1:n_1}-X_{1:m_1:n_1}\right) \\ &\vdots \\ Z_{m_1} &= \Big(n_1-R_1-\ldots-R_{m_{1-1}}-(m_1-1)\Big)\left(X_{m_1:m_1:n_1}-X_{m_1-1:m_1:n_1}\right). \end{aligned} \tag{4}$$

It has been shown by Balakrishnan and Aggarwala [2] that Z_i's are independent and identically distributed (*i.i.d.*) exponential random variables with mean θ_1. Moreover,

$$\sum_{i=1}^{m_1}Z_i=\sum_{i=1}^{m_1}(R_i+1)X_{i:m_1:n_1}=U. \tag{5}$$

Therefore, it is immediate that U has a gamma distribution with the shape parameter m_1 and the scale parameter θ_1, *i.e.* it has the pdf

$$f_U(u)=\frac{1}{\theta_1^{m_1}\Gamma(m_1)}u^{m_1-1}exp\left(-\frac{u}{\theta_1}\right);\qquad 0<u<\infty. \tag{6}$$

We have the following lemma.

LEMMA 1: The conditional pdf of $X_{1:m_1:n_1}$ given $U = u$, is

$$f_{X_{1:m_1:n_1}|U=u}(x) = n_1(m_1 - 1)\ \frac{(u - n_1 x)^{m_1-2}}{u^{m_1-1}}; \quad 0 < x < \frac{u}{n_1}, \tag{7}$$

and the conditional pdf of $Y_{1:m_2:n_2}$ given $V = v$, is

$$f_{Y_{1:m_2:n_2}|V=v}(y) = n_2(m_2 - 1)\ \frac{(v - n_2 y)^{m_2-2}}{v^{m_2-1}}; \quad 0 < y < \frac{v}{n_2}. \tag{8}$$

PROOF: We will prove the first part, second part follows along the same line. Note that

$$f_{X_{1:m_1:n_1}|U=u}(x) = \frac{f_{X_{1:m_1:n_1},U}(x,u)}{f_U(u)}, \tag{9}$$

where $f_{X_{1:m_1:n_1},U}(x,u)$ is the the joint pdf of $X_{1:m_1:n_1}$ and U and $f_U(u)$ is the pdf of U. It is obvious that U is a complete sufficient statistics for θ_1. Suppose we denote $W = \sum_{i=2}^{m_1} Z_i$, then clearly W and Z_1 are independent. The joint pdf of X_1 and U can be easily obtained from the joint pdf of W and Z_1, by using the the transformation $Z_1 = n_1 X_{1:m_1:n_1}$ and $U = W + Z_1$. Finally the result is found using Eq.(6). ∎

The following theorem provides the UMVUE of R.

THEOREM 1: Based on the complete sufficient statistics U and V, as defined before for θ_1 and θ_2 respectively, the UMVUE of R, say $\widetilde{R}$, for $m_1 \geq 2$, and $m_2 \geq 2$, can be expressed as follows;

$$\widetilde{R} = \begin{cases} Q_1(n_1, m_1, m_2, u, v) & if \quad u < v \\ Q_2(n_2, m_1, m_2, u, v) & if \quad u > v \end{cases}, \tag{10}$$

where

$$\begin{aligned} Q_1(n_1, m_1, m_2, u, v) &= 1 - \frac{n_1(m_1 - 1)}{u^{m_1-1} v^{m_2-1}} \int_0^{u/n_1} (u - n_1 x)^{m_1-2} (v - n_1 x)^{m_2-1} dx \\ &= 1 - \sum_{k=0}^{m_2-1} (-1)^k \left(\frac{u}{v}\right)^k \frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}} \end{aligned} \tag{11}$$

and

$$Q_2(n_2,m_1,m_2,u,v) = \frac{n_2(m_2-1)}{u^{m_1-1}v^{m_2-1}}\int_0^{v/n_2}(u-n_2y)^{m_1-1}(v-n_2y)^{m_2-2}dy.$$
$$= \sum_{k=0}^{m_1-1}(-1)^k\left(\frac{v}{u}\right)^k\frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}}. \tag{12}$$

Proof: The proof of Theorem 1 can be obtained in a routine matter as it was obtained by Tong [19] for complete sample case. Observe that $X_{1:m_1:n_1} \sim$ Exp (θ_1/n_1) and $Y_{1:m_2:n_2} \sim$ Exp (θ_2/n_2), therefore,

$$\phi(X_1,Y_1) = \begin{cases} 1 & if \quad n_2Y_{1:m_2:n_2} < n_1X_{1:m_1:n_1} \\ 0 & if \quad n_2Y_{1:m_2:n_2} > n_1X_{1:m_1:n_1} \end{cases} \tag{13}$$

is an unbiased estimator of R. Therefore,

$$\widetilde{R} = E(\phi(X_1,Y_1)|U=u,V=v) = \int\int_{\mathcal{A}} f_{X_1|U=u}(x)f_{Y_1|V=v}(y)dxdy, \tag{14}$$

where $\mathcal{A} = \{(x,y), 0<x<u/n_1, 0<y<v/n_2, n_1x>n_2y\}$, $f_{X_1|U=u}(x)$ and $f_{Y_1|V=v}(y)$ are same as defined in Lemma 1. For $u<v$,

$$\widetilde{R} = \int_0^{u/n_1}\int_0^{n_1x/n_2}(u-n_1x)^{m_1-2}(v-n_2y)^{m_2-2}dydx$$
$$= 1-\frac{n_1(m_1-1)}{u^{m_1-1}v^{m_2-1}}\int_0^{u/n_1}(u-n_1x)^{m_1-2}(v-n_1x)^{m_2-1}dx. \tag{15}$$

By using transformation $\dfrac{n_1x}{u}=t$, $\widetilde{R}$ is written as follows;

$$\widetilde{R} = 1-(m_1-1)\int_0^1(1-t)^{m_1-2}(1-\frac{ut}{v})^{m_2-1}dt. \tag{16}$$

By using the expansion

$$(1-\frac{ut}{v})^{m_2-1} = \sum_{k=0}^{m_2-1}(-1)^k\binom{m_2-1}{k}\left(\frac{ut}{v}\right)^k,$$

$\widetilde{R}$ is obtained as follows;

$$\widetilde{R} = 1-\sum_{k=0}^{m_2-1}(-1)^k\left(\frac{u}{v}\right)^k\frac{\binom{m_2-1}{k}}{\binom{m_1+k-1}{k}}. \tag{17}$$

Similarly for $u > v$,

$$\widetilde{R} = \sum_{k=0}^{m_1-1} (-1)^k \left(\frac{v}{u}\right)^k \frac{\binom{m_1-1}{k}}{\binom{m_2+k-1}{k}}. \tag{18}$$

∎

3 Maximum Likelihood Estimation of R

In this section, the maximum likelihood estimator (MLE) of R and its distribution are derived. It is clear that based on the progressively censored samples $\{X_{1:m_1:n_1}, \ldots, X_{m_1:m_1:n_1}\}$ and $\{Y_{1:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2}\}$, the MLEs of θ_1 and θ_2 are

$$\widehat{\theta}_1 = \frac{U}{m_1} \qquad \widehat{\theta}_2 = \frac{V}{m_2}, \tag{19}$$

respectively. Based on the joint distribution of U and V provided in the previous section, it is clear that $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are UMVUE for θ_1 and θ_2, respectively. Since $R = \dfrac{\theta_1}{\theta_1 + \theta_2}$, the MLE of R, say $\widehat{R}$, becomes

$$\widehat{R} = \frac{\widehat{\theta}_1}{\widehat{\theta}_1 + \widehat{\theta}_2} = \frac{U}{U + \left(\frac{m_1}{m_2}\right) V} = \frac{1}{1 + \left(\frac{\theta_2}{\theta_1}\right) W}, \tag{20}$$

where $W = \dfrac{m_1\theta_1 V}{m_2\theta_2 U}$. It should be noticed that U and V are independent gamma random variables with shape and scale parameters as m_1, θ_1 and m_2, θ_2, respectively. Hence, W has a F distribution with degrees of freedom $2m_2$ and $2m_1$. The pdf of $\widehat{R}$ can be obtained as

$$f_{\widehat{R}}(r) = k \times \frac{\left(\frac{1-r}{r}\right)^{m_2-1}}{\left(1 + \frac{m_2\theta_2(1-r)}{m_1\theta_1 r}\right)^{m_1+m_2}}; \qquad 0 < r < 1, \tag{21}$$

where

$$k = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)} \times \left(\frac{\theta_2}{\theta_1}\right)^{m_2-1} \times \left(\frac{m_2}{m_1}\right)^{m_2}.$$

The first and second moments of $\widehat{R}$ can be found, respectively, as follows;

$$E(\widehat{R}) \;=\; \frac{\Gamma(m_1+3)\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_1+m_2+3)} \left(\frac{m_1}{m_2}\right)^{m_1} \left(\frac{\theta_1}{\theta_2}\right)^{m_1+1} \times$$

$$\times F_{2,1}\left(\left(m_1+m_2, m_1+3\right), m_1+m_2+3, 1-\frac{m_2\theta_2-m_1\theta_1}{m_2\theta_2}\right) \tag{22}$$

$$E(\widehat{R}^2) = \frac{\Gamma(m_1+4)\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_1+m_2+4)}\left(\frac{m_1}{m_2}\right)^{m_1}\left(\frac{\theta_1}{\theta_2}\right)^{m_1+1}\times$$
$$\times F_{2,1}\left(\left(m_1+m_2, m_1+4\right), m_1+m_2+4, 1-\frac{m_2\theta_2-m_1\theta_1}{m_2\theta_2}\right), \tag{23}$$

where $F_{2,1}$ is the hypergeometric function. This function is given as follows;

$$\begin{aligned} F_{2,1}(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt \\ &= \sum_{k=0}^{\infty}\frac{\Gamma(c)\Gamma(a+k)\Gamma(b+k)z^k}{\Gamma(a)\Gamma(b)\Gamma(c+k)k!}, \end{aligned} \tag{24}$$

see for an example, Bailey [3]. Variance of $\widehat{R}$ is found with $Var(\widehat{R}) = E(\widehat{R}^2) - \left(E(\widehat{R})\right)^2$.

Using the fact

$$\frac{R}{1-R}\times\frac{1-\widehat{R}}{\widehat{R}} = W \sim F_{2m_2,2m_1}, \tag{25}$$

100(1 - α)% confidence interval of R can be obtained as

$$\left[\frac{F_{2m_2,2m_1,\alpha/2}\times\left(\frac{\widehat{R}}{1-\widehat{R}}\right)}{1+F_{2m_2,2m_1,\alpha/2}\times\left(\frac{\widehat{R}}{1-\widehat{R}}\right)}, \frac{F_{2m_2,2m_1,1-\alpha/2}\times\left(\frac{\widehat{R}}{1-\widehat{R}}\right)}{1+F_{2m_2,2m_1,1-\alpha/2}\times\left(\frac{\widehat{R}}{1-\widehat{R}}\right)}\right], \tag{26}$$

where $F_{2m_2,2m_1,\alpha/2}$ and $F_{2m_2,2m_1,1-\alpha/2}$ are the lower and upper $\alpha/2$-th percentile points of a F distribution with $2m_2$ and $2m_1$ degrees of freedom.

4 Bayes Estimation

In this section, the Bayes estimator of R is derived. The exponential distributions are re-parameterized in terms of $\lambda_1 = \frac{1}{\theta_1}$ and $\lambda_2 = \frac{1}{\theta_2}$, respectively. Therefore, $R = \frac{\lambda_2}{\lambda_1+\lambda_2}$. For constructing Bayes estimate of R, we assume independent gamma priors on λ_1 and λ_2, with the pdfs

$$\pi(\lambda_1) = \frac{b_1^{a_1}}{\Gamma(a_1)}\,\lambda_1^{a_1-1}e^{-b_1\lambda_1}; \quad \lambda_1 > 0 \tag{27}$$

and

$$\pi(\lambda_2) = \frac{b_2^{a_2}}{\Gamma(a_2)}\ \lambda_2^{a_2-1} e^{-b_2\lambda_2}; \qquad \lambda_2 > 0, \tag{28}$$

respectively. Here the hyper-parameters $a_1 > 0, b_1 > 0, a_2 > 0, b_2 > 0$. The posterior pdfs of λ_1 and λ_2 are as follows;

$$\lambda_1|Data \sim \text{Gamma}\left(a_1 + m_1,\ (b_1 + \sum_{i=1}^{m_1}(R_i + 1)x_i)\right) \tag{29}$$

$$\lambda_2|Data \sim \text{Gamma}\left(a_2 + m_2,\ (b_2 + \sum_{i=1}^{m_2}(S_i + 1)y_i)\right). \tag{30}$$

Since λ_1 and λ_2 are independent *a posterior*, the posterior pdf of R is;

$$f_{R|Data}(r) = C \times \frac{r^{a_2+m_2-1}(1-r)^{a_1+m_1-1}}{((1-r)(b_1 + \sum_{i=1}^{m_1}(R_i+1)x_i) + r(b_2 + \sum_{i=1}^{m_2}(S_i+1)y_i))^{m_1+m_2+a_1+a_2}}, \tag{31}$$

for $0 < r < 1$; 0 otherwise, where

$$C = \frac{\Gamma(m_1 + m_2 + a_1 + a_2)}{\Gamma(a_1 + m_1)\Gamma(a_2 + m_2)} \times (b_1 + \sum_{i=1}^{m_1}(R_i + 1)x_i)^{a_1+m_1} \times (b_2 + \sum_{i=1}^{m_2}(S_i + 1)y_i))^{a_2+m_2}. \tag{32}$$

Therefore, if the squared error loss function is used, then the Bayes estimate of R is the posterior mean R, say $\widehat{R}_{BAYES}$, which can be obtained as follows.

$$\begin{aligned}
\widehat{R}_{BAYES} &= \int_0^1 r f_{R|Data}(r)dr \\
&= \left(\frac{b_2+v}{b_1+u}\right)^{a_2+m_2} \frac{a_2+m_2}{s} \times \\
&\quad \times F_{2,1}\left((1 + m_2 + a_2, s), 1 + s, 1 - \frac{(b_1+u) - (b_2+v)}{b_1+u}\right),
\end{aligned} \tag{33}$$

where $s = a_1 + a_2 + m_1 + m_2$ and $F_{2,1}$ is defined in Equation (24). On the other hand, if the absolute error loss function is used, then the Bayes estimate of R is the posterior median. Unfortunately, the posterior mean and median cannot be obtained explicitly. Hence, a numerical technique will be used.

Although the posterior mean and median cannot be obtained explicitly, the posterior mode could be derived in explicit form. It is easy to show that the derivative of $f_{R|Data}(r)$ could be expressed as

$$\frac{d}{dr} f_{R|Data}(r) = \frac{c r^{A_2-1}(1-r)^{A_1-1}}{((1-r)B_1 + rB_2)^{A_1+A_2+3}} \, g(r),$$

where

$$g(r) = \left[(A_2(1-r) - A_1 r)((1-r)B_1 + rB_2) - (A_1 + A_2 + 2)(B_2 - B_1) r(1-r) \right], \qquad (34)$$

$B_1 = (b_1 + \sum_{i=1}^{m_1}(R_i+1)x_i)$, $B_2 = (b_2 + \sum_{i=1}^{m_2}(S_i+1)y_i)$, $A_1 = a_1 + m_1 - 1$ and $A_2 = a_2 + m_2 - 1$. Since $\lim_{r \to 0^+} g(r) > 0$ and $\lim_{r \to 1^-} g(r) < 0$, it easily follows that the $f_{R|Data}(r)$ has a unique mode over $0 < r < 1$, and the posterior mode can be obtained as the unique root of the quadratic equation $g(r) = 0$ over $0 < r < 1$.

Now, consider the following loss function

$$L(a,b) = \begin{cases} 0 & if \quad |a-b| \le c \\ 1 & f \quad |a-b| > c. \end{cases} \qquad (35)$$

It is known that the Bayes estimate with respect to the above loss function Eq. (35) is the midpoint of the modal interval of length 2c of the posterior distribution, see Ferguson ([6], page 51, problem 5). Therefore, the posterior mode is an approximate Bayes estimate of R with respect to the loss function Eq. (35) when the constant c is small.

4.1 Credible Interval

It has been mentioned that the posterior mean and the posterior median cannot be obtained in explicit form. In this section, a simulation procedure to generate a sample from the posterior density function of R and to compute the Bayes estimate of R along with the associated credible interval will be described.

Since the support of the posterior density function of R is bounded, the acceptance rejection method could be used to generate samples from the posterior distribution of R. Therefore, the algorithm used to compute the Bayes estimate and the credible interval is established as follows;

ALGORITHM:

- Step 1: Find the mode of the posterior density function $f_R(r|data)$ by solving (34), say r^*.
- Step 2: Since $f_R(r|data) \le f_R(r^*|data)$, for $0 < r < 1$, generate B samples, say $r_1, \cdots, r_B$, from (31) using acceptance rejection principle, see Devroye [5].
- Step 3: Calculate the sample mean as the Bayes estimate of R under the squared error loss function.
- Step 4: Order $r_1, \cdots, r_B$ and compute the lower and upper $\alpha/2$-th percentile points of $r_1, \cdots, r_B$ as the lower and upper bounds of 100(1-α)% confidence interval, respectively.

5 Numerical Experiments

In this section, the Monte Carlo simulation is conducted to compare the performances of MLE, UMVUE, and Bayes estimator under different progressive censoring schemes. Two sets of population parameter values, (i) $\theta_1 = 20$, $\theta_2 = 10$, (ii) $\theta_1 = 10$, $\theta_2 = 20$ are considered. For a given n and m, three different progressive censoring schemes are used to generate the progressively censored samples. Two are extreme ones: (i) the usual type-II censoring scheme (*i.e.* $n - m$ remaining items are removed at the m-th failure) and (ii) the censoring scheme removing $n - m$ items at the first failure at random, that may be called type-III censoring scheme. The other censoring scheme lies in between these two extremes. The last censoring scheme has all the R_i's taken the same number and may be called as type-IV censoring scheme. A typical example of type-IV progressive censoring is given as $n = 10$, $m = 5$ and $R_i = 1$, $i = 1, 2, 3, 4, 5$. For fixed n and m, it is well known that the expected experimental time for the case of type-II censoring scheme is the lowest and the expected experimental time for the type-III censoring scheme is the largest. The expected experimental time for each of the other progressive censoring scheme is between the two experimental times from the two extremes.

For given (n_1, m_1) and progressive censoring scheme $\{n_1, m_1, R_1, \cdots, R_m\}$ for the first population, and (n_2, m_2) and progressive censoring scheme $\{n_2, m_2, S_1, \cdots, S_{m_2}\}$ for the second population, the simulation is replicated 1000 times. In each simulation, the MLE and UMVUE and the boundaries for exact 95% confidence intervals are obtained and a sample of size $B = 1000$ is generated from the posterior distribution to obtain the Bayes estimate under the squared error loss function and the 95% credible interval based on the improper priors, with $a_1 = a_2 = b_1 = b_2 = 0$ (here assume that $\Gamma(0) = 1$). Then the mean squared errors (MSEs) for all the estimators, and average biases for MLEs and Bayes estimators are calculated based on the estimates from all 1000 simulations and reported in Tables 3 and 4. The average of boundaries for all the confidence intervals along with the corresponding coverage percentages are calculated based on all 1000 simulations. Since the coverage percentages for all the cases are very close to the nominal level, only the averages of boundaries for all confidence intervals are reported in Tables 5 and 6. It may be mentioned that when $a_1 = a_2 = b_1 = b_2 = 0$, although the priors are not proper, the corresponding posterior density functions are proper ones. In viewing the tables, it is clear that all three estimators work quite well. In all the cases, it is observed that as the effective sample size increases the performances become better. The performances of the MLEs and the Bayes estimators based on improper priors are very similar. However, the UMVUEs have larger errors than the other two estimators. The length of the exact confidence intervals and the corresponding credible intervals very close in all cases. From the computational point of view, the MLEs are easiest to be obtained. Therefore, it is suggested to use the MLE for all practical purposes.

6 Data Analysis

In this section, the analysis of a pair of real data sets is presented for illustrative purposes. Tables 1 and 2 show the breaking strengths of jute fiber at two different gauge lengths. These two data sets were used by Xia *et al.* [20].

Table 1: Breaking strength of jute fiber of gauge length 10 mm.

693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48
108.94	50.16	671.49	183.16	257.44	727.23	291.27	101.15
376.42	163.40	141.38	700.74	262.90	353.24	422.11	43.93
590.48	212.13	303.90	506.60	530.55	177.25		

Table 2: Breaking strength of jute fiber of gauge length 20 mm.

71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16
662.66	45.58	578.62	756.70	594.29	166.49	99.72	707.36
765.14	187.13	145.96	350.70	547.44	116.99	375.81	581.60
119.86	48.01	200.16	36.75	244.53	83.55		

First, it was checked whether exponential distribution can be used or not to analyze these data sets. The maximum likelihood estimators of θ_1 and θ_2 are 356.7297 and 340.7400, respectively. The Kolmogorov-Smirnov (KS) distances between the empirical distribution functions and the fitted distribution functions have been used to check the goodness-of-fit. The Kolmogorov-Smirnov Z values are 0.958 and 0.727 and the associated p values are 0.317 and 0.666, respectively. Based on the p values, one cannot reject the hypothesis that the data are coming from exponential distributions.

The following notations have been used; X: the breaking strength of jute fiber with 10 mm, and Y: the breaking strength of jute fiber with 20 mm. Based on the complete data set the MLE and UMVUE of $R = P(Y < X)$ are $\widehat{R} = 0.5177$ and $\widetilde{R} = 0.5180$, respectively and the associated 95% exact confidence interval as provided by (26), is (0.3917, 0.6615). The Bayes estimate of R with respect to improper priors is $\widehat{R}_B = 0.5051$. The associated 95% credible interval is (0.3716, 0.6502).

For illustrative purposes, three different progressively censored samples have been generated from the above data sets with $m_1 = m_2 = 15$ in all the cases, and (i) Scheme-1: (type-II, type-II), (ii) Scheme-2: (type-III, type-III) and (iii) Scheme 3 (type-IV, type-IV). Based on Scheme 1, the MLE, UMVUE and Bayes estimates are 0.5952, 0.5986, and 0.5817 respectively. The associated 95% exact confidence interval and the credible interval are (0.4149, 0.7531) and (0.3918, 0.7399) respectively. Similarly, based on Scheme 2, the MLE, UMVUE and Bayes estimates are 0.5180, 0.5186, 0.5075 respectively. The associated 95% confidence interval and the credible interval are (0.3413, 0.6903) and (0.3291, 0.6798) respectively. Finally based on Scheme 3, the MLE, UMVUE and Bayes estimates are 0.5624, 0.5646, 0.5615 respectively. The associated 95% confidence interval and the credible interval are (0.3826, 0.7271) and (0.3623, 0.6997) respectively. Clearly, the estimates obtained using type-III censored data, are closer to the estimates obtained by complete sample as expected.

7 Conclusions

In this paper, the estimation of the stress strength parameter for two exponential distributions under progressive censoring has been considered. The UMVUE and MLE of the stress strength parameter have been derived. Bayes estimator based on inverse gamma priors has been proposed. Extensive simulations are performed to check the performances of the different estimators and it is observed that all the three estimators behave in a similar manner.

Since MLE is easiest to obtain computationally, it has been proposed to use the MLE in practice.

It may be mentioned that although it has been assumed that the samples are from exponential distributions, but it may be extended to some other distributions also, for an example, Weibull or gamma distribution. Work is in progress, and it will be reported later.

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Table 3: Biases and MSEs for MLEs and Bayes estimators, and MSEs for UMVUEs when $\theta_1 = 20$, $\theta_2 = 10$

(n_1, m_1)	(n_2, m_2)	R	S	MLE		UMVUE	Bayes	
				Bias	MSE	MSE	Bias	MSE
(10, 5)	(10, 5)	II	II	-0.01379	0.01978	0.02268	-0.01287	0.01897
(10, 5)	(10, 5)	II	III	-0.01391	0.01931	0.02217	-0.01399	0.01957
(10, 5)	(10, 5)	III	II	-0.01542	0.01922	0.02207	-0.01514	0.01899
(10, 5)	(10, 5)	III	III	-0.01512	0.01955	0.02241	-0.01516	0.01947
(10, 5)	(10, 5)	IV	IV	-0.01421	0.01967	0.02258	-0.01429	0.01923
(20, 5)	(10, 5)	II	II	-0.01581	0.01942	0.02226	-0.01479	0.01971
(20, 5)	(10, 5)	II	III	-0.01544	0.01931	0.02213	-0.01531	0.01929
(20, 5)	(10, 5)	III	II	-0.01191	0.01937	0.02228	-0.01181	0.01941
(20, 5)	(10, 5)	III	III	-0.01330	0.01893	0.02172	-0.01319	0.01901
(20, 5)	(10, 5)	IV	IV	-0.01463	0.01916	0.02206	-0.01396	0.01954
(10, 5)	(20, 5)	II	II	-0.01228	0.01908	0.02193	-0.01219	0.01919
(10, 5)	(20, 5)	II	III	-0.01359	0.01975	0.02265	-0.01369	0.01966
(10, 5)	(20, 5)	III	II	-0.01401	0.01937	0.02211	-0.01409	0.01923
(10, 5)	(20, 5)	III	III	-0.01284	0.01911	0.02197	-0.01291	0.01900
(10, 5)	(20, 5)	IV	IV	-0.01263	0.01909	0.02196	-0.01248	0.01914
(20, 5)	(20, 5)	II	II	-0.01231	0.01937	0.02224	-0.01228	0.01925
(20, 5)	(20, 5)	II	III	-0.01325	0.01893	0.02175	-0.01335	0.01889
(20, 5)	(20, 5)	III	II	-0.01384	0.01981	0.02274	-0.01390	0.01971
(20, 5)	(20, 5)	III	III	-0.01355	0.01977	0.02273	-0.01345	0.01969
(20, 5)	(20, 5)	IV	IV	-0.01298	0.01958	0.02268	-0.01297	0.01949
(20, 10)	(10, 5)	II	II	0.00015	0.01386	0.01577	-0.00010	0.01371
(20, 10)	(10, 5)	II	III	-0.00049	0.01434	0.01631	-0.00037	0.01410
(20, 10)	(10, 5)	III	II	-0.00161	0.01430	0.01629	-0.00156	0.01417
(20, 10)	(10, 5)	III	III	-0.00171	0.01410	0.01606	-0.00167	0.01399
(20, 10)	(10, 5)	IV	IV	-0.00098	0.01392	0.01582	-0.00112	0.01381
(10, 5)	(20, 10)	II	II	-0.02360	0.01582	0.01681	-0.02351	0.01511
(10, 5)	(20, 10)	II	III	-0.01926	0.01534	0.01645	-0.01939	0.01528
(10, 5)	(20, 10)	III	II	-0.02152	0.01619	0.01727	-0.02148	0.01587
(10, 5)	(20, 10)	III	III	-0.02060	0.01573	0.01680	-0.02071	0.01581
(10, 5)	(20, 10)	IV	IV	-0.02226	0.01578	0.01680	-0.02199	0.01542
(20, 10)	(20, 10)	II	II	-0.00754	0.00967	0.01032	-0.00749	0.00973
(20, 10)	(20, 10)	II	III	-0.00658	0.00972	0.01038	-0.00671	0.00961
(20, 10)	(20, 10)	III	II	-0.00636	0.00971	0.01037	-0.00629	0.00968
(20, 10)	(20, 10)	III	III	-0.00722	0.00998	0.01065	-0.00712	0.00971
(20, 10)	(20, 10)	IV	IV	-0.00741	0.00978	0.01040	-0.00733	0.00973
(30, 10)	(20, 10)	II	II	-0.00715	0.00978	0.01045	-0.00699	0.00968
(30, 10)	(20, 10)	II	III	-0.00880	0.01011	0.01079	-0.00848	0.00899
(30, 10)	(20, 10)	III	II	-0.00867	0.00992	0.01057	-0.00887	0.00999
(30, 10)	(20, 10)	III	III	-0.00706	0.00982	0.01049	-0.00717	0.00971
(30, 10)	(20, 10)	IV	IV	-0.00711	0.00980	0.01047	-0.00709	0.00969
(20, 10)	(30, 10)	II	II	-0.00615	0.00964	0.01030	-0.00629	0.00951
(20, 10)	(30, 10)	II	III	-0.00683	0.00981	0.01048	-0.00675	0.00978
(20, 10)	(30, 10)	III	II	-0.00859	0.01016	0.01083	-0.00845	0.00901
(20, 10)	(30, 10)	III	III	-0.00520	0.00968	0.01035	-0.00515	0.00979
(20, 10)	(30, 10)	IV	IV	-0.00567	0.00966	0.01033	-0.00598	0.00965
(30, 10)	(30, 10)	II	II	-0.00930	0.01000	0.01066	-0.00927	0.00989
(30, 10)	(30, 10)	II	III	-0.00601	0.00977	0.01044	-0.00591	0.00961
(30, 10)	(30, 10)	III	II	-0.00732	0.01001	0.01069	-0.00698	0.00989
(30, 10)	(30, 10)	III	III	-0.00819	0.00963	0.01028	-0.00829	0.00101
(30, 10)	(30, 10)	IV	IV	-0.00882	0.00987	0.01037	-0.00901	0.00889
(30, 15)	(30, 10)	II	II	-0.00178	0.00802	0.00857	-0.00161	0.00781
(30, 15)	(30, 10)	II	III	-0.00137	0.00793	0.00847	-0.00125	0.00764
(30, 15)	(30, 10)	III	II	-0.00420	0.00787	0.00840	-0.00401	0.00769
(30, 15)	(30, 10)	III	III	-0.00139	0.00813	0.00869	-0.00129	0.00799
(30, 15)	(30, 10)	IV	IV	-0.00158	0.00809	0.00860	-0.00143	0.00787
(30, 10)	(30, 15)	II	II	-0.01002	0.00845	0.00878	-0.00918	0.00876
(30, 10)	(30, 15)	II	III	-0.01009	0.00848	0.00882	-0.00991	0.00831
(30, 10)	(30, 15)	III	II	-0.00930	0.00827	0.00862	-0.00927	0.00821
(30, 10)	(30, 15)	III	III	-0.00971	0.00830	0.00864	-0.00959	0.00801
(30, 10)	(30, 15)	IV	IV	-0.00981	0.00839	0.00869	-0.00934	0.00843
(30, 15)	(30, 15)	II	II	-0.00545	0.00655	0.00684	-0.00559	0.00671
(30, 15)	(30, 15)	II	III	-0.00555	0.00668	0.00697	-0.00535	0.00676
(30, 15)	(30, 15)	III	II	-0.00384	0.00649	0.00678	-0.00369	0.00659
(30, 15)	(30, 15)	III	III	-0.00464	0.00638	0.00667	-0.00451	0.00647
(30, 15)	(30, 15)	IV	IV	-0.00502	0.00643	0.00671	-0.00556	0.00665

Table 4: Biases and MSEs for MLEs and Bayes estimators, and MSEs for UMVUEs when $\theta_1 = 10$, $\theta_2 = 20$

(n_1, m_1)	(n_2, m_2)	R	S	MLE		UMVUE	Bayes	
				Bias	MSE	MSE	Bias	MSE
(10, 5)	(10, 5)	II	II	0.01552	0.01952	0.02236	0.01547	0.01941
(10, 5)	(10, 5)	II	III	0.01240	0.01960	0.02252	0.01231	0.01955
(10, 5)	(10, 5)	III	II	0.01426	0.01968	0.02262	0.01433	0.01975
(10, 5)	(10, 5)	III	III	0.01135	0.01943	0.02231	0.01128	0.01937
(10, 5)	(10, 5)	IV	IV	0.01401	0.01948	0.02233	0.01386	0.01939
(20, 5)	(10, 5)	II	II	0.01303	0.01917	0.02204	0.01295	0.01901
(20, 5)	(10, 5)	II	III	0.01268	0.01885	0.02164	0.01259	0.01898
(20, 5)	(10, 5)	III	II	0.01211	0.01928	0.02214	0.01201	0.01917
(20, 5)	(10, 5)	III	III	0.01548	0.01935	0.02220	0.01529	0.01915
(20, 5)	(10, 5)	IV	IV	0.01401	0.01926	0.02212	0.01356	0.01911
(10, 5)	(20, 5)	II	II	0.01211	0.01928	0.02214	0.01227	0.01939
(10, 5)	(20, 5)	II	III	0.01546	0.01940	0.02226	0.01525	0.01931
(10, 5)	(20, 5)	III	II	0.01352	0.01945	0.02235	0.01360	0.01949
(10, 5)	(20, 5)	III	III	0.01296	0.01875	0.02154	0.01271	0.01869
(10, 5)	(20, 5)	IV	IV	0.01251	0.01892	0.02189	0.01245	0.01901
(20, 5)	(20, 5)	II	II	0.01374	0.01985	0.02275	0.01361	0.01999
(20, 5)	(20, 5)	II	III	0.01348	0.01944	0.02232	0.01351	0.01937
(20, 5)	(20, 5)	III	II	0.01282	0.01969	0.02263	0.01274	0.01955
(20, 5)	(20, 5)	III	III	0.01558	0.01971	0.02259	0.01565	0.01984
(20, 5)	(20, 5)	IV	IV	0.01442	0.01979	0.02261	0.01425	0.01989
(20, 10)	(10, 5)	II	II	0.02105	0.01572	0.01678	0.02091	0.01555
(20, 10)	(10, 5)	II	III	0.02178	0.01580	0.01684	0.02167	0.01593
(20, 10)	(10, 5)	III	II	0.02157	0.01607	0.01715	0.02149	0.01591
(20, 10)	(10, 5)	III	III	0.02036	0.01597	0.01707	0.02023	0.01556
(20, 10)	(10, 5)	IV	IV	0.02079	0.01581	0.01699	0.02067	0.01555
(10, 5)	(20, 10)	II	II	0.00149	0.01382	0.01574	0.00138	0.01367
(10, 5)	(20, 10)	II	III	-0.00169	0.01353	0.01539	-0.00168	0.01348
(10, 5)	(20, 10)	III	II	-0.00185	0.01357	0.01543	-0.00171	0.01343
(10, 5)	(20, 10)	III	III	0.00022	0.01373	0.01564	0.00019	0.01341
(10, 5)	(20, 10)	IV	IV	0.00119	0.01379	0.01568	0.00121	0.01359
(20, 10)	(20, 10)	II	II	0.00778	0.00987	0.01053	0.00766	0.00943
(20, 10)	(20, 10)	II	III	0.00764	0.00988	0.01055	0.00751	0.00976
(20, 10)	(20, 10)	III	II	0.00785	0.00982	0.01074	0.00772	0.00973
(20, 10)	(20, 10)	III	III	0.00482	0.00965	0.01038	0.00469	0.00973
(20, 10)	(20, 10)	IV	IV	0.00601	0.00979	0.01042	0.00587	0.00962
(30, 10)	(20, 10)	II	II	0.00685	0.00979	0.01046	0.00664	0.00955
(30, 10)	(20, 10)	II	III	0.00896	0.00994	0.01059	0.00891	0.01014
(30, 10)	(20, 10)	III	II	0.00704	0.01006	0.01074	0.00698	0.00991
(30, 10)	(20, 10)	III	III	0.00628	0.00971	0.01038	0.00641	0.01010
(30, 10)	(20, 10)	IV	IV	0.00649	0.00973	0.01041	0.00657	0.00998
(20, 10)	(30, 10)	II	II	0.00613	0.00967	0.01034	0.00642	0.00951
(20, 10)	(30, 10)	II	III	0.00665	0.00982	0.01049	0.00659	0.00977
(20, 10)	(30, 10)	III	II	0.00709	0.01001	0.01069	0.00697	0.00901
(20, 10)	(30, 10)	III	III	0.00658	0.00984	0.01052	0.00669	0.01001
(20, 10)	(30, 10)	IV	IV	0.00638	0.00976	0.01042	0.00651	0.00987
(30, 10)	(30, 10)	II	II	0.00758	0.00991	0.01058	0.00768	0.00983
(30, 10)	(30, 10)	II	III	0.00794	0.01008	0.01075	0.00787	0.00997
(30, 10)	(30, 10)	III	II	0.00699	0.00994	0.01062	0.00717	0.01104
(30, 10)	(30, 10)	III	III	0.00575	0.00973	0.01041	0.00597	0.00945
(30, 10)	(30, 10)	II	II	0.00612	0.00981	0.01051	0.00661	0.00978
(30, 15)	(30, 10)	II	II	0.00989	0.00830	0.00863	0.01001	0.00891
(30, 15)	(30, 10)	II	III	0.01048	0.00845	0.00878	0.00998	0.00878
(30, 15)	(30, 10)	III	II	0.00949	0.00836	0.00870	0.01000	0.00873
(30, 15)	(30, 10)	III	III	0.00984	0.00849	0.00884	0.00979	0.00868
(30, 15)	(30, 10)	IV	IV	0.00986	0.00840	0.00877	0.00988	0.00879
(30, 10)	(30, 15)	II	II	0.00484	0.00819	0.00874	0.00428	0.00831
(30, 10)	(30, 15)	II	III	0.00230	0.00812	0.00868	0.00228	0.00801
(30, 10)	(30, 15)	III	II	0.00071	0.00785	0.00839	0.00068	0.00765
(30, 10)	(30, 15)	III	III	0.00299	0.00791	0.00845	0.00312	0.00805
(30, 10)	(30, 15)	IV	IV	0.00386	0.00801	0.00859	0.00414	0.00826
(30, 15)	(30, 15)	II	II	0.00550	0.00660	0.00688	0.00534	0.00651
(30, 15)	(30, 15)	II	III	0.00380	0.00662	0.00692	0.00372	0.00678
(30, 15)	(30, 15)	III	II	0.00510	0.00661	0.00690	0.00491	0.00673
(30, 15)	(30, 15)	III	III	0.00461	0.00657	0.00686	0.00445	0.00643
(30, 15)	(30, 15)	IV	IV	0.00514	0.00659	0.00686	0.00499	0.00647

Table 5: Exact 95% confidence intervals and credible intervals based on non-informative priors when $\theta_1 = 10$, $\theta_2 = 20$

(n_1, m_1)	(n_2, m_2)	R	S	Confidence Intervals		Credible Intervals	
				Lower	Upper	Lower	Upper
(10, 5)	(10, 5)	II	II	0.13595	0.63745	0.11589	0.61748
(10, 5)	(10, 5)	II	III	0.13498	0.63677	0.11491	0.61672
(10, 5)	(10, 5)	III	II	0.13487	0.63718	0.11483	0.63722
(10, 5)	(10, 5)	III	III	0.13527	0.63618	0.11523	0.61622
(10, 5)	(10, 5)	IV	IV	0.13329	0.63432	0.11325	0.61429
(20, 5)	(10, 5)	II	II	0.13512	0.63656	0.11499	0.61652
(20, 5)	(10, 5)	II	III	0.13553	0.63766	0.11560	0.61771
(20, 5)	(10, 5)	III	II	0.13456	0.63566	0.11452	0.61569
(20, 5)	(10, 5)	III	III	0.13519	0.63756	0.11521	0.61780
(20, 5)	(10, 5)	IV	IV	0.13608	0.63870	0.11599	0.61868
(10, 5)	(20, 5)	II	II	0.13589	0.63846	0.11591	0.61849
(10, 5)	(20, 5)	II	III	0.13495	0.63684	0.11491	0.61679
(10, 5)	(20, 5)	III	II	0.13594	0.63735	0.11596	0.61741
(10, 5)	(20, 5)	III	III	0.13522	0.63731	0.11525	0.61735
(10, 5)	(20, 5)	IV	IV	0.13623	0.63853	0.11619	0.61851
(20, 5)	(20, 5)	II	II	0.13399	0.63414	0.11402	0.61419
(20, 5)	(20, 5)	II	III	0.13591	0.63738	0.11593	0.61741
(20, 5)	(20, 5)	III	II	0.13584	0.63780	0.11581	0.61776
(20, 5)	(20, 5)	III	III	0.13606	0.63827	0.11600	0.61821
(20, 5)	(20, 5)	IV	IV	0.13539	0.63717	0.11542	0.61724
(20, 10)	(10, 5)	II	II	0.14712	0.58650	0.12709	0.56653
(20, 10)	(10, 5)	II	III	0.14772	0.58823	0.12776	0.56828
(20, 10)	(10, 5)	III	II	0.14765	0.58842	0.12762	0.56839
(20, 10)	(10, 5)	III	III	0.14705	0.58759	0.12699	0.56754
(20, 10)	(10, 5)	IV	IV	0.14601	0.58550	0.12608	0.56555
(10, 5)	(20, 10)	II	II	0.16043	0.61193	0.14039	0.59188
(10, 5)	(20, 10)	II	III	0.15854	0.60930	0.13861	0.59105
(10, 5)	(20, 10)	III	II	0.15849	0.60842	0.13852	0.58998
(10, 5)	(20, 10)	III	III	0.16059	0.61281	0.14053	0.59276
(10, 5)	(20, 10)	IV	IV	0.15978	0.61166	0.14002	0.59159
(20, 10)	(20, 10)	II	II	0.17704	0.54813	0.15698	0.52816
(20, 10)	(20, 10)	II	III	0.17897	0.55124	0.15903	0.53116
(20, 10)	(20, 10)	III	II	0.17754	0.54880	0.15756	0.52891
(20, 10)	(20, 10)	III	III	0.17858	0.55067	0.15854	0.53059
(20, 10)	(20, 10)	IV	IV	0.17814	0.54956	0.15817	0.52961
(30, 10)	(20, 10)	II	II	0.17790	0.54951	0.15797	0.52948
(30, 10)	(20, 10)	II	III	0.17827	0.54965	0.15831	0.52958
(30, 10)	(20, 10)	III	II	0.17782	0.54993	0.15778	0.52996
(30, 10)	(20, 10)	III	III	0.17745	0.54853	0.15751	0.52856
(30, 10)	(20, 10)	IV	IV	0.17843	0.54990	0.15839	0.52997
(20, 10)	(30, 10)	II	II	0.17728	0.54860	0.15731	0.52858
(20, 10)	(30, 10)	II	III	0.17772	0.54865	0.15775	0.52861
(20, 10)	(30, 10)	III	II	0.17805	0.54949	0.15801	0.52953
(20, 10)	(30, 10)	III	III	0.17843	0.55033	0.15841	0.53029
(20, 10)	(30, 10)	IV	IV	0.17863	0.55038	0.15859	0.53033
(30, 10)	(30, 10)	II	II	0.17818	0.54991	0.15822	0.52996
(30, 10)	(30, 10)	II	III	0.17845	0.54975	0.15838	0.52978
(30, 10)	(30, 10)	III	II	0.17788	0.54954	0.15791	0.52951
(30, 10)	(30, 10)	III	III	0.17799	0.54935	0.15804	0.52940
(30, 10)	(30, 10)	IV	IV	0.17816	0.54992	0.15818	0.55001
(30, 15)	(30, 10)	II	II	0.18557	0.52304	0.16560	0.50308
(30, 15)	(30, 10)	II	III	0.18554	0.52266	0.16555	0.50271
(30, 15)	(30, 10)	III	II	0.18453	0.52107	0.16449	0.50103
(30, 15)	(30, 10)	III	III	0.18455	0.52148	0.16449	0.50143
(30, 15)	(30, 10)	IV	IV	0.18615	0.52362	0.16621	0.50365
(30, 10)	(30, 15)	II	II	0.19635	0.53996	0.17629	0.51994
(30, 10)	(30, 15)	II	III	0.19512	0.53863	0.17509	0.51859
(30, 10)	(30, 15)	III	II	0.19505	0.53822	0.17501	0.51826
(30, 10)	(30, 15)	III	III	0.19592	0.53935	0.17595	0.51940
(30, 10)	(30, 15)	IV	IV	0.19532	0.53840	0.17538	0.51843
(30, 15)	(30, 15)	II	II	0.20378	0.50839	0.18381	0.48842
(30, 15)	(30, 15)	II	III	0.20495	0.51036	0.18491	0.49029
(30, 15)	(30, 15)	III	II	0.20533	0.51084	0.18527	0.49079
(30, 15)	(30, 15)	III	III	0.20436	0.50960	0.18432	0.48973
(30, 15)	(30, 15)	IV	IV	0.20351	0.50785	0.18349	0.48788

Table 6: Exact 95% confidence intervals and credible intervals based on non-informative priors when $\theta_1 = 20$, $\theta_2 = 10$

(n_1, m_1)	(n_2, m_2)	R	S	Confidence Intervals		Credible Intervals	
				Lower	Upper	Lower	Upper
(10, 5)	(10, 5)	II	II	0.36255	0.86401	0.38258	0.88398
(10, 5)	(10, 5)	II	III	0.36329	0.86502	0.38327	0.88508
(10, 5)	(10, 5)	III	II	0.36286	0.86515	0.38291	0.88519
(10, 5)	(10, 5)	III	III	0.36382	0.86476	0.38386	0.88482
(10, 5)	(10, 5)	IV	IV	0.36568	0.86674	0.38572	0.88671
(20, 5)	(10, 5)	II	II	0.36344	0.86488	0.38352	0.88492
(20, 5)	(10, 5)	II	III	0.36234	0.86447	0.38229	0.88451
(20, 5)	(10, 5)	III	II	0.36434	0.86546	0.38438	0.88540
(20, 5)	(10, 5)	III	III	0.36243	0.86483	0.38248	0.88485
(20, 5)	(10, 5)	IV	IV	0.36135	0.86398	0.38131	0.88402
(10, 5)	(20, 5)	II	II	0.36156	0.86407	0.38161	0.88401
(10, 5)	(20, 5)	II	III	0.36318	0.86507	0.38321	0.88502
(10, 5)	(20, 5)	III	II	0.36264	0.86403	0.38268	0.88399
(10, 5)	(20, 5)	III	III	0.36269	0.86481	0.38271	0.88478
(10, 5)	(20, 5)	IV	IV	0.36148	0.86372	0.38143	0.88369
(20, 5)	(20, 5)	II	II	0.36582	0.86596	0.38588	0.88603
(20, 5)	(20, 5)	II	III	0.36264	0.86410	0.38262	0.88405
(20, 5)	(20, 5)	III	II	0.36227	0.86419	0.38231	0.88422
(20, 5)	(20, 5)	III	III	0.36178	0.86398	0.38182	0.88393
(20, 5)	(20, 5)	IV	IV	0.36383	0.86465	0.38378	0.88462
(20, 10)	(10, 5)	II	II	0.41352	0.85292	0.43349	0.87303
(20, 10)	(10, 5)	II	III	0.41173	0.85231	0.43167	0.87236
(20, 10)	(10, 5)	III	II	0.41162	0.85235	0.43165	0.87231
(20, 10)	(10, 5)	III	III	0.41245	0.85302	0.43241	0.87298
(20, 10)	(10, 5)	IV	IV	0.41452	0.85404	0.41450	0.87402
(10, 5)	(20, 10)	II	II	0.38812	0.83957	0.40816	0.85963
(10, 5)	(20, 10)	II	III	0.39068	0.84152	0.41054	0.86113
(10, 5)	(20, 10)	III	II	0.39158	0.84151	0.41162	0.86148
(10, 5)	(20, 10)	III	III	0.38723	0.83948	0.40717	0.85941
(10, 5)	(20, 10)	IV	IV	0.38834	0.84028	0.40828	0.86031
(20, 10)	(20, 10)	II	II	0.45187	0.82302	0.47191	0.84310
(20, 10)	(20, 10)	II	III	0.44923	0.82119	0.46928	0.84112
(20, 10)	(20, 10)	III	II	0.45128	0.82242	0.47131	0.84238
(20, 10)	(20, 10)	III	III	0.44929	0.82148	0.47012	0.84151
(20, 10)	(20, 10)	IV	IV	0.45041	0.82188	0.47052	0.84191
(30, 10)	(20, 10)	II	II	0.45051	0.82217	0.47048	0.84211
(30, 10)	(20, 10)	II	III	0.45038	0.82179	0.47041	0.84181
(30, 10)	(20, 10)	III	II	0.45111	0.82218	0.47117	0.84223
(30, 10)	(20, 10)	III	III	0.45147	0.82258	0.47151	0.82263
(30, 10)	(20, 10)	IV	IV	0.45019	0.82157	0.47021	0.84159
(20, 10)	(30, 10)	II	II	0.45138	0.82278	0.47141	0.84273
(20, 10)	(30, 10)	II	III	0.45137	0.82226	0.47141	0.84231
(20, 10)	(30, 10)	III	II	0.45041	0.82195	0.47048	0.84193
(20, 10)	(30, 10)	III	III	0.44967	0.82161	0.47001	0.84158
(20, 10)	(30, 10)	IV	IV	0.44959	0.82141	0.46989	0.84138
(30, 10)	(30, 10)	II	II	0.45009	0.82178	0.47016	0.84183
(30, 10)	(30, 10)	II	III	0.45023	0.82155	0.47019	0.84161
(30, 10)	(30, 10)	III	II	0.45046	0.82212	0.47051	0.84207
(30, 10)	(30, 10)	III	III	0.45061	0.82201	0.47058	0.84196
(30, 10)	(30, 10)	IV	IV	0.45004	0.82185	0.47011	0.84191
(30, 15)	(30, 10)	II	II	0.47968	0.81423	0.48006	0.83419
(30, 15)	(30, 10)	II	III	0.47734	0.81447	0.49728	0.83451
(30, 15)	(30, 10)	III	II	0.47891	0.81549	0.49901	0.83551
(30, 15)	(30, 10)	III	III	0.47852	0.81545	0.49863	0.83540
(30, 15)	(30, 10)	IV	IV	0.47638	0.81385	0.47632	0.81379
(30, 10)	(30, 15)	II	II	0.46012	0.80365	0.48009	0.82361
(30, 10)	(30, 15)	II	III	0.46137	0.80483	0.48143	0.82486
(30, 10)	(30, 15)	III	II	0.46178	0.80497	0.48173	0.82505
(30, 10)	(30, 15)	III	III	0.46065	0.80411	0.48072	0.82414
(30, 10)	(30, 15)	IV	IV	0.46168	0.80471	0.48161	0.82465
(30, 15)	(30, 15)	II	II	0.49164	0.79628	0.51167	0.81621
(30, 15)	(30, 15)	II	III	0.48964	0.79507	0.51005	0.81511
(30, 15)	(30, 15)	III	II	0.48916	0.79467	0.50921	0.81463
(30, 15)	(30, 15)	III	III	0.49047	0.79564	0.51051	0.81569
(30, 15)	(30, 15)	IV	IV	0.49219	0.79651	0.51221	0.81648