An Extension of the Freund's Bivariate Distribution to Model Load Sharing Systems

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Abstract

Several authors have considered the analysis of load sharing parallel systems. The main characteristics of such a two component system is that after the failure of one component the surviving component has to shoulder extra load, and hence it is more likely lo fail at an earlier time than what is expected under the original model. In other cases, the failure of one component may releases extra resources to the other component, thus delays the system failure. Freund (1961) introduced a bivariate extension of the exponential distribution which is applicable to a two-component load sharing systems. It is based on the assumptions that the lifetime distributions of the components are exponential random variables before and after the change. In this paper, we introduce a new class of bivariate distribution using the proportional hazard model. It is observed that the bivariate model proposed by Freund (1961) is a particular case of our model. We study different properties of the proposed model. Different statistical inferences have also been developed. We have considered four different special cases namely when the base distributions are exponential, Weibull, linear failure rate and Pareto III distributions. One data analysis has been performed for illustrative purposes. Finally we propose some generalizations.

Key Words and Phrases Proportional hazard model; Bivariate exponential distribution; Bivariate hazard function; Maximum likelihood estimator; Fisher information matrix.

1 Introduction

Most reliability models are intended for components that operate independently within a system. In many systems, the performance of the functioning of a component depends on whether the other components within the system are working or not. The main characteristic of a load sharing system is that after failure of one component, the surviving component has to shoulder extra load, and hence it is expected to fail earlier that what is expected under the original situations. In some other situations, the failure of one component may release extra resources to the surviving component, and it increases the system lifetime.

It has been observed by Gross et al. (1971) that two organ sub systems in a human body typically show this behavior. If one organ fails, the surviving organ is subjected to higher failure rate. For example, if a patient get his/ her kidney removed due to illness, then the second kidney shows a higher failure rate. Similar phenomenon can be observed in the behavior of human eyes also. Another typical situation of this type can be observed in nuclear power industry. For example, in a nuclear power industry, components are redundantly added to system mainly to safeguard against core meltdown. If the failure of one backup system adversely affects the operation of another, then the probability of core meltdown can increase significantly.

Freund (1961) proposed a bivariate distribution which is designed for the life testing of a two-component load sharing system. In a two-component load sharing system, it is assumed that the system can function even after one of the components has failed, although the lifetime distribution of the other component (functioning) changes due to the over loading. Therefore, the bivariate model proposed by Freund (1961) can be used to model survival times of a system with two identical or nearly identical components, for example two-engine plane, organisms with paired organs, such as kidneys, eyes, lungs, or a cooling system which relies on two adjacent pumps to circulate coolant through the same ducts etc.

Freund (1961) in the development of his model assumed that the lifetime distributions of the individual components are exponential random variables. It is assumed that if one of the components fails, the life time distribution of the other component follows exponential random variable with a different scale parameter. Since exponential distribution has only constant failure rate it is known to have its own limitations. Due to this reason Freund's model has been extended to a model where it has been assumed that the lifetime distributions of the components are no longer exponential random variables, see for example Lu (1989) and Spurrier and Weier (1981) in this respect. In both these cases although the models are more flexible than the Freund's model, they are not very easy to handle analytically due to their complicated structure. The readers who are interested about the different load sharing systems, may refer to Deshpande, Dewan and Naik-Nimbalkar (2010), Kim and Kvam (2004), Kvam and Pena (2005), Lynch (1999), Shaked (1984) and see the references cited therein.

The main idea of this paper is to introduce a class of bivariate distributions to model two-component load sharing system, which is more flexible than the model proposed by Freund (1961), and which are easy to use in practice. In this case, we do not make any specific assumptions on the lifetime distributions of the two components. Instead, we have assumed that the lifetime distributions of the two components are from a proportional hazard class of distributions. It may by noted that a class of distributions is said to be from a proportional hazard class of distributions, if the survival function of any member of this class is of the form $(S_0(x))^\alpha$, $\alpha > 0$. Where $S_0(x)$ is the base line survival function. It is called a proportional hazard class because the hazard function of any member of this class of distribution functions, is proportional to the hazard function of the base line distribution function, see for example Kalbfleish and Prentice (2002). In this present formulation, we can take any base line distribution function. The bivariate distribution proposed by Freund (1961) can be obtained as a special case of the proposed model. We call this bivariate distribution as the extended Freund's bivariate (EFB) distribution. It is observed that in certain cases the proposed EFB distribution provides a better fit than the bivariate distribution proposed by Freund (1961). Hence, it provides the practitioner a wide range of choices for the bivariate load sharing distributions.

It is observed that the joint probability density function (PDF) and the joint survival function (SF) can be obtained in explicit forms. Hence the implementation of the proposed model is quite simple. The joint PDF can take different shapes depending on the base line distribution and other associated parameters values. Several properties of the proposed EFB model have been established. The marginal and the conditional distributions also can be obtained in convenient forms. If the base line distribution is completely known, the maximum likelihood estimators (MLEs) can be obtained explicitly. If it is assumed that the base line distribution has a specific form with some unknown parameters, the MLEs of the unknown parameters can be obtained by maximizing the profile likelihood function. The expected Fisher information matrix for different specific base line distributions have been provided. If the base line distribution is known, the generation from a EFB model model is quite straight forward. We discuss four specific examples in details, namely when the base line distributions are (i) exponential, (ii) Weibull, (iii) linear failure rate and (iv) Pareto III. We perform the analysis of one data set for illustrative purposes by using four different EFB models, and proposed to choose the best fitted model using Kolmogorov-Smirnov distance measure. It is observed that the proposed best fitted EFB model works quite well in analyzing the data set. Finally we propose some generalizations also.

Rest of the paper is organized as follows. In Section 2, we introduce the model, and provide the physical interpretation of the model. In Section 3, we present different properties of the model. The MLEs and their properties are presented in Section 4. The analysis of a data set is presented in Section 5. We provide the conclusions and propose some generalizations in Section 6.

2 EFB Model and Some Special Cases

The bivariate random variable (Y_1, Y_2) is said to have EFB distribution, if the the joint PDF of Y_1 and Y_2 has the following form;

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \theta_1'\theta_2 f_0(y_1)f_0(y_2)(S_0(y_1))^{\theta_1'-1}(S_0(y_2))^{\theta_1+\theta_2-\theta_1'-1} & \text{if} \quad y_1 > y_2 > 0 \\ \theta_1\theta_2' f_0(y_1)f_0(y_2)(S_0(y_1))^{\theta_1+\theta_2-\theta_2'-1}(S_0(y_2))^{\theta_2'-1} & \text{if} \quad y_2 > y_1 > 0, \end{cases} \tag{1}$$

and 0 otherwise. Here $\theta_1 > 0$, $\theta_2 > 0$, $\theta_1' > 0$, $\theta_2' > 0$, $f_0(\cdot)$ and $S_0(\cdot)$ are the density function and survival function of the distribution function $F_0(\cdot)$. It is further assumed that $F_0(0) = 0$. From now on it will be denoted by EFB$(F_0, \theta_1, \theta_2, \theta_1', \theta_2')$. Note that when

$\theta_1 = \theta_2 = \theta_1' = \theta_2' = 1$, then Y_1 and Y_2 become independent.

It may be easily verified that (1) is indeed a proper bivariate density function, as $f_{Y_1,Y_2}(y_1, y_2) \geq 0$, for all $0 < y_1, y_2 < \infty$, and

$$\begin{aligned}
\int_0^\infty \int_{y_2}^\infty f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 &= \int_0^\infty \int_{y_2}^\infty \theta_1'\theta_2 f_0(y_1)f_0(y_2)(S_0(y_1))^{\theta_1'-1}(S_0(y_2))^{\theta_1+\theta_2-\theta_1'-1}dy_1dy_2 \\
&= \frac{\theta_2}{\theta_1+\theta_2} \\
\int_0^\infty \int_{y_1}^\infty f_{Y_1,Y_2}(y_1,y_2)dy_2dy_1 &= \int_0^\infty \int_{y_1}^\infty \theta_1\theta_2' f_0(y_1)f_0(y_2)(S_0(y_1))^{\theta_1+\theta_2-\theta_2'-1}(S_0(y_2))^{\theta_2'-1}dy_2dy_1 \\
&= \frac{\theta_1}{\theta_1+\theta_2}.
\end{aligned}$$

The following interpretation can be provided of the joint PDF (1). Suppose X_1 and X_2 are random variables representing the lifetimes of two components A and B respectively in a two-component system, when they first put on a test. If component B fails before A, *i.e.* if $X_2 < X_1$, the lifetime distribution of A changes, and suppose we denote it by X_1^*. Finally the system fails when component A fails, and in this case one observes the bivariate random variable (X_1^*, X_2), where $X_1^* > X_2$. Similarly, if A fails before B, *i.e.* $X_1 < X_2$, the lifetime distribution B changes and it will be denoted by X_2^*. In this case also, similarly as before, finally the system fails when component B fails eventually, and at the end one observes the bivariate random variable (X_1, X_2^*). If we denote the lifetime distributions of the components A and B as (Y_1, Y_2), then one observes $Y_1 = X_1^*$, $Y_2 = X_2$, if $Y_1 > Y_2$, and $Y_1 = X_1$, $Y_2 = X_2^*$, if $Y_1 < Y_2$.

It is assumed that X_1 and X_2 are independently distributed having survival functions $(S_0(\cdot))^{\theta_1}$, $\theta_1 > 0$ and $(S_0(\cdot))^{\theta_2}$, $\theta_2 > 0$ respectively. It is further assumed that X_1^* and X_2^* have the survival functions $(S_0(\cdot))^{\theta_1'}$ and $(S_0(\cdot))^{\theta_2'}$ respectively. Now let us look at the joint PDF of Y_1 and Y_2 for $y_1 > y_2$.

$$f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 = P(y_1 \leq Y_1 \leq y_1 + dy_1, y_2 \leq Y_2 \leq y_2 + dy_2)$$

$$
\begin{aligned}
&= P(y_1 \le X_1^* \le y_1 + dy_1, y_2 \le X_2 \le y_2 + dy_2) \\
&= P(y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2, X_1 > X_2, y_1 \le X_1^* \le y_1 + dy_1) \\
&= P(y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2) \\
&\quad \times P(X_1 > X_2 |\ y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2) \\
&\quad \times P(y_1 \le X_1^* \le y_1 + dy_1 |\ y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2, X_1 > X_2)
\end{aligned}
$$

Now note that

$$
P(y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2) = (\theta_1 + \theta_2)(S_0(y_2))^{\theta_1+\theta_2-1} f_0(y_2) dy_2, \tag{2}
$$

$$
\begin{aligned}
P(X_1 > X_2 |\ y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2) &= \frac{P(X_1 > X_2,\ y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2)}{P(y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2)} \\
&= \frac{P(X_1 > X_2,\ y_2 \le X_2 \le y_2 + dy_2)}{P(y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2)} \\
&= \frac{\theta_2 f_0(y_2)(S_0(y_2)^{\theta+\theta_2-1}}{(\theta+\theta_2) f_0(y_2)(S_0(y_2))^{\theta_1+\theta_2-1}} = \frac{\theta_2}{\theta_1+\theta_2}, \quad (3)
\end{aligned}
$$

and

$$
\begin{aligned}
&P(y_1 \le X_1^* \le y_1 + dy_1 |\ y_2 \le \min\{X_1, X_2\} \le y_2 + dy_2, X_1 > X_2) = \\
&P(y_1 \le X_1^* \le y_1 + dy_1 |\ y_2 \le X_2 \le y_2 + dy_2, X_1^* > y_2) = \frac{\theta_1'(S_0(y_1))^{\theta_1'-1} f_0(y_1)}{(S_0(y_2))^{\theta_1'}}. \quad (4)
\end{aligned}
$$

Therefore, combining (2), (3) and (4) we immediately obtain for $y_1 > y_2$

$$
f_{Y_1,Y_2}(y_1, y_2) dy_1 dy_2 = \theta_1' \theta_2 f_0(y_1) f_0(y_2)(S_0(y_1))^{\theta_1'-1}(S_0(y_2))^{\theta_1+\theta_2-\theta_1'-1} dy_1 dy_2.
$$

Similarly, we obtain $f_{Y_1,Y_2}(y_1, y_2)$ for $y_1 < y_2$ also.

Now we take different special cases of the base line distribution function.

EXAMPLE 1: If we take the base line distribution as the exponential distribution, *i.e.* $S_0(y) = e^{-y}$, then (1) becomes the Freund's (1961) model. In this case the joint PDF of (1) becomes;

$$
f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \theta_1' \theta_2 e^{-\theta_1' y_1} e^{-(\theta_1+\theta_2-\theta_1')y_2} & \text{if} \quad y_1 > y_2 > 0 \\ \theta_1 \theta_2' e^{-(\theta_1+\theta_2-\theta_2')y_1} e^{-\theta_2' y_2} & \text{if} \quad y_2 > y_1 > 0. \end{cases} \tag{5}
$$

It will be denoted by EFB$(EXP, \theta_1, \theta_2, \theta_1', \theta_2')$.

Example 2: If we take Weibull as the base line distribution, *i.e.* $S_0(y) = e^{-y^\alpha}$, $\alpha > 0$, then (1) becomes;

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \theta_1'\theta_2\alpha^2 y_1^{\alpha-1} y_2^{\alpha-1} e^{-\theta_1' y_1^\alpha} e^{-(\theta_1+\theta_2-\theta_1')y_2^\alpha} & \text{if} \quad y_1 > y_2 > 0 \\ \theta_1\theta_2'\alpha^2 y_1^{\alpha-1} y_2^{\alpha-1} e^{-(\theta_1+\theta_2-\theta_2')y_1^\alpha} e^{-\theta_2' y_2^\alpha} & \text{if} \quad y_2 > y_1 > 0. \end{cases} \tag{6}$$

It will be denoted by EFB$(WE(\alpha), \theta_1, \theta_2, \theta_1', \theta_2')$.

Example 3: If we take linear failure rate distribution as the base line distribution, *i.e.* $S_0(y) = e^{-(y+\alpha y^2)}$, for $\alpha > 0$, then (1) becomes;

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \theta_1'\theta_2(1+2\alpha y_1)(1+2\alpha y_2) e^{-\theta_1'(y_1+\alpha y_1^2)} e^{-(\theta_1+\theta_2-\theta_1')(y_2+\alpha y_2^2)} & \text{if} \quad y_1 > y_2 > 0 \\ \theta_1\theta_2'(1+2\alpha y_1)(1+2\alpha y_2) e^{-(\theta_1+\theta_2-\theta_2')(y_1+\alpha y_1^2)} e^{-\theta_2'(y_2+\alpha y_2^2)} & \text{if} \quad y_2 > y_1 > 0. \end{cases} \tag{7}$$

It will be denoted by EFB$(LFR(\alpha), \theta_1, \theta_2, \theta_1', \theta_2')$.

Example 4: If we take the base line distribution as a Pareto III distribution with $S_0(y) = (1 + y^\alpha)^{-1}$ for $\alpha > 0$, then (1) becomes;

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \theta_1'\theta_2\alpha^2 y_1^{\alpha-1} y_2^{\alpha-1} (1+y_1^\alpha)^{-(\theta_1'+1)} (1+y_2^\alpha)^{-(\theta_1+\theta_2-\theta_1'+1)} & \text{if} \quad y_1 > y_2 > 0 \\ \theta_1\theta_2'\alpha^2 y_1^{\alpha-1} y_2^{\alpha-1} (1+y_1^\alpha)^{-(\theta_1+\theta_2-\theta_2'+1)} (1+y_2^\alpha)^{-(\theta_2'+1)} & \text{if} \quad y_2 > y_1 > 0. \end{cases} \tag{8}$$

It will be denoted by EFB$(PAR(\alpha), \theta_1, \theta_2, \theta_1', \theta_2')$. In Figures 1 and 2, the surface plot of the different PDFs for different parameter values and for different base line distributions are provided. It clearly indicates that they can take variety of shapes depending on the base line distribution and the parameter values.

It may be mentioned that some of the very general bivariate models which have been obtained by minimization or maximization process for example the class of bivariate models with proportional reversed hazard marginals by Kundu and Gupta (2010) or the bivariate Kumaraswamy distribution by Barreto-Souza and Lemonte (2013), do not generate the

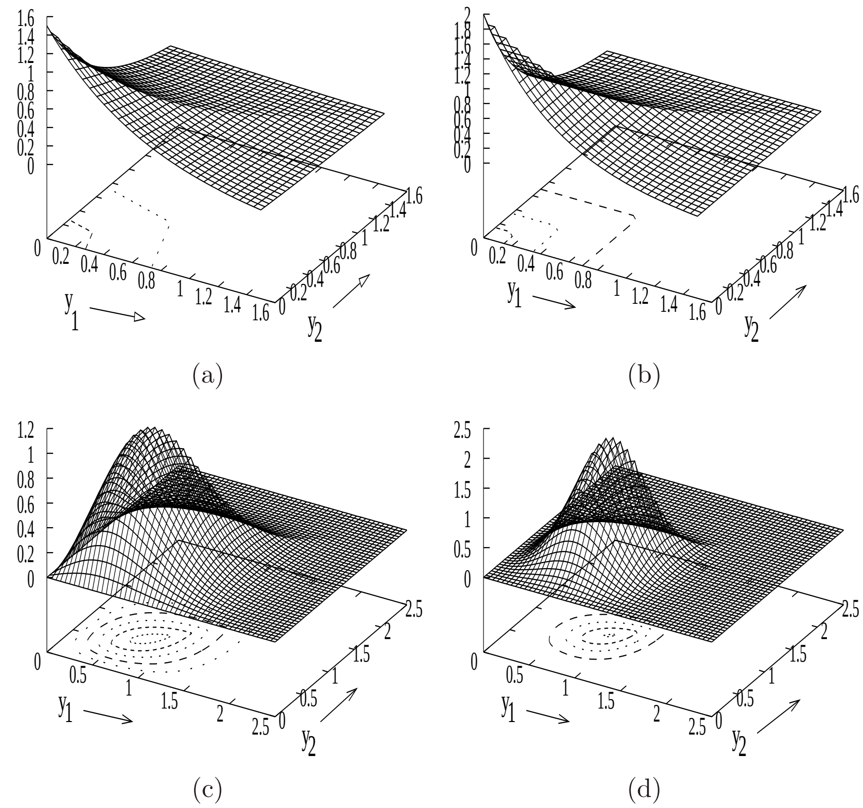

Figure 1: Contour plots of the joint PDF of the different EFB distributions are presented for different parameter values and for different base distributions: (a) EFB(EXP,1.0,1.0,1.5,1.5), (b) EFB(EXP,1.0,1.0,2.0,2.0), (c) EFB(WE(2.0),1.0,1.0,1.5,1.5), (d) EFB(WE(3.0),1.0,1.0,1.5,1.5).

proposed EFB model as their process of generations are completely different.

3 Properties

We have the following result for the joint survival function and the marginal distributions.

Theorem 3.1: Let $(Y_1, Y_2) \sim \text{EFB}(F, \theta_1, \theta_2, \theta_1', \theta_2')$. The joint survival function of (Y_1, Y_2) is

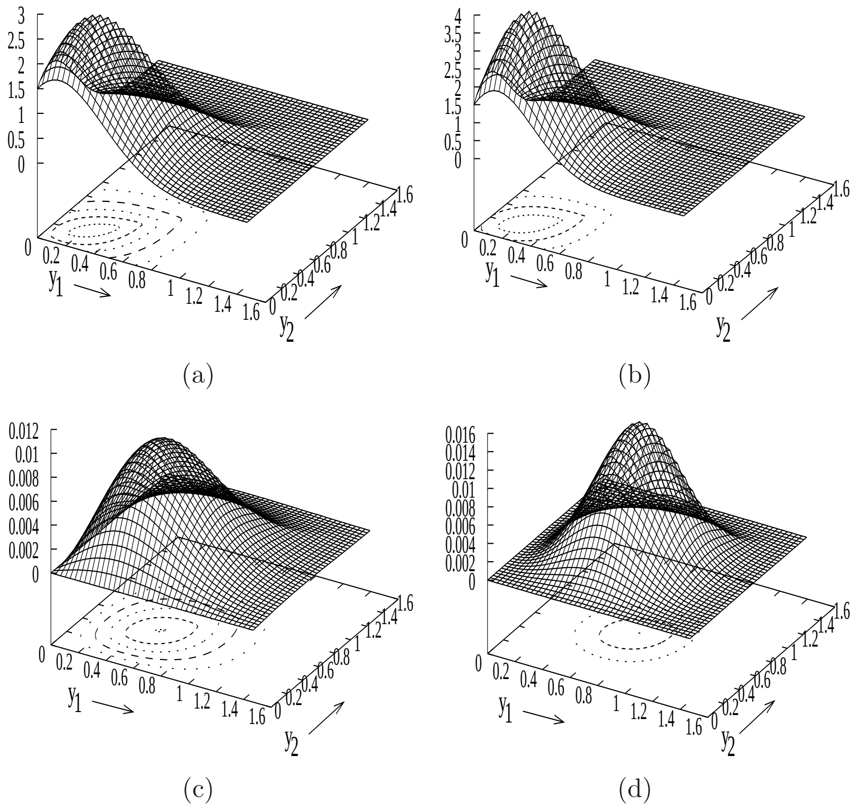

Figure 2: Contour plots of the joint PDF of the different EFB distributions are presented for different parameter values and for different base distributions: (a) EFB(LFR(2.0),1.0,1.0,1.5,1.5), (b) EFB(LFR(3.0),1.0,1.0,1.5,1.5), (c) EFB(PAR(2.0),1.0,1.0,1.5,1.5), (d) EFB(PAR(3.0),1.0,1.0,1.5,1.5).

Case 1: $\theta_1 + \theta_2 \neq \theta_1'$, $\theta_1 + \theta_2 \neq \theta_2'$

$$S(y_1, y_2) = \begin{cases} (S_0(y_1))^{\theta_1+\theta_2} + \frac{\theta_2 (S_0(y_1))^{\theta_1'}}{\theta_1+\theta_2-\theta_1'} \left((S_0(y_2))^{\theta_1+\theta_2-\theta_1'} - (S_0(y_1))^{\theta_1+\theta_2-\theta_1'}\right) & \text{if} \quad y_1 > y_2 \\ (S_0(y_2))^{\theta_1+\theta_2} + \frac{\theta_1 (S_0(y_2))^{\theta_2'}}{\theta_1+\theta_2-\theta_1'} \left((S_0(y_1))^{\theta_1+\theta_2-\theta_2'} - (S_0(y_2))^{\theta_1+\theta_2-\theta_2'}\right) & \text{if} \quad y_2 > y_1 \end{cases} \tag{9}$$

Case 2: $\theta_1 + \theta_2 = \theta_1'$, $\theta_1 + \theta_2 \neq \theta_2'$.

For $y_1 < y_2$, $S(y_1, y_2)$ is same as in Case 1. For $y_1 > y_2$

$$S(y_1, y_2) = (S_0(y_1))^{\theta_1+\theta_2}(1 + \theta_2(\ln S_0(y_2) - \ln S_0(y_1))) \tag{10}$$

Case 3: $\theta_1 + \theta_2 \neq \theta'_1$, $\theta_1 + \theta_2 = \theta'_2$.

For $y_1 > y_2$, $S(y_1, y_2)$ is same as in Case 1. For $y_1 < y_2$

$$S(y_1, y_2) = (S_0(y_2))^{\theta_1+\theta_2}(1 + \theta_1(\ln S_0(y_1) - \ln S_0(y_2))) \tag{11}$$

Case 4: $\theta_1 + \theta_2 = \theta'_1 = \theta'_2$.

For $y_1 > y_2$ ($y_1 < y_2$), $S(y_1, y_2)$ is same as in Case 2 (Case 3).

Proof of Theorem 3.1: It can be obtained in a routine manner, and it is not provided here. ∎

Theorem 3.2: Let $(Y_1, Y_2) \sim \text{EFB}(F, \theta_1, \theta_2, \theta'_1, \theta'_2)$. Then

(i) $Y = \min\{Y_1, Y_2\}$ has the survival function $P(Y > y) = (S_0(y))^{\theta_1+\theta_2}$.

(ii) $P(Y_1 > Y_2) = \dfrac{\theta_2}{\theta_1 + \theta_2}$

Proof of Theorem 3.2: (i) easily follows from Theorem 3.1, and (ii) follows from (1). ∎

Theorem 3.3: Let $(Y_1, Y_2) \sim \text{EFB}(F, \theta_1, \theta_2, \theta'_1, \theta'_2)$. The marginal survival functions of Y_1 and Y_2 are

Case 1: $\theta_1 + \theta_2 \neq \theta'_1$, $\theta_1 + \theta_2 \neq \theta'_2$

$$S_{Y_1}(y_1) = \frac{\theta_2}{\theta_1 + \theta_2 - \theta'_1}(S_0(y_1))^{\theta'_1} + \frac{\theta_1 - \theta'_1}{\theta_1 + \theta_2 - \theta'_1}(S_0(y_1))^{\theta_1+\theta_2} \quad \text{if} \quad y_1 > 0 \tag{12}$$

$$S_{Y_2}(y_2) = \frac{\theta_1}{\theta_1 + \theta_2 - \theta'_2}(S_0(y_2))^{\theta'_2} + \frac{\theta_2 - \theta'_2}{\theta_1 + \theta_2 - \theta'_2}(S_0(y_2))^{\theta_1+\theta_2} \quad \text{if} \quad y_2 > 0. \tag{13}$$

Case 2: $\theta_1 + \theta_2 = \theta'_1$, $\theta_1 + \theta_2 \neq \theta'_2$. In this case $S_{Y_2}(\cdot)$ is same as in Case 1. But

$$S_{Y_1}(y_1) = (S_0(y_1))^{\theta_1+\theta_2}(1 - \theta_2 \ln S_0(y_1)) \tag{14}$$

Case 3: $\theta_1 + \theta_2 \neq \theta'_1$, $\theta_1 + \theta_2 = \theta'_2$. In this case $S_{Y_1}(\cdot)$ is same as in Case 1. But

$$S_{Y_2}(y_2) = (S_0(y_2))^{\theta_1+\theta_2}(1 - \theta_1 \ln S_0(y_1)) \tag{15}$$

Case 4: $\theta_1 + \theta_2 = \theta'_1 = \theta'_2$.

For $y_1 > y_2$ $(y_1 < y_2)$, $S_{Y_1}(y_1)(S_{Y_2}(y_2)$ is same as in Case 2 (Case 3).

Proof of Theorem 3.3: It can be obtained easily from Theorem 3.1. ∎

Comment: Note that from the marginal distribution functions, the marginal PDFs can be obtained. Hence the conditional PDFs also can be obtained from the joint PDF and from the marginal PDFs. It is not pursued here.

Comment: One natural question is whether the joint survival function $S(y_1, y_2)$ has a convenient copula representation or not. Since $S^{-1}_{Y_1}(\cdot)$ and $S^{-1}_{Y_2}(\cdot)$ do not have closed form, the joint survival function do not have explicit copula representation in general.

Theorem 3.4: Let $(Y_1, Y_2) \sim \text{EFB}(F, \theta_1, \theta_2, \theta'_1, \theta'_2)$. If $\theta'_1 + \theta'_2 > \theta_1 + \theta_2$, then (Y_1, Y_2) has total positivity of order two (TP_2) property.

Proof of Theorem 3.4: Note that (Y_1, Y_2) has TP_2 property, if and only if for any $y_{11}, y_{12}, y_{21}, y_{22}$, whenever, $0 < y_{11} < y_{12}$ and $0 < y_{21} < y_{22}$, we have

$$f(y_{11}, y_{21})f(y_{12}, y_{22}) \geq f(y_{12}, y_{21})f(y_{11}, y_{22}). \tag{16}$$

To prove (16) let us consider the case $0 < y_{11} < y_{21} < y_{12} < y_{22}$. Therefore, proving (16) is equivalent to prove

$$(S_0(y_{21}))^{\theta'_1+\theta'_2-\theta_1-\theta_2} \geq (S_0(y_{12}))^{\theta'_1+\theta'_2-\theta_1-\theta_2}, \tag{17}$$

which is true. Along the same line for other cases also (16) can be proved. ∎

Now we provide a simple characterization of the EFB class of distributions. From Theorem 3.1, it is clear that $S(y_1, y_2)$ can be written in the form $R(S_0(y_1), S_0(y_2))$, where the exact form of $R(u, v)$, can be easily obtained from Theorem 3.1. Note that $R(u, v)$ is a proper survival function in $[0, 1] \times [0, 1]$. The following notations are consistent and it follows that

$$\begin{aligned} S(y_1, 0) &= R(S_0(y_1), S_0(0)) = R(S_0(y_1), 1) = S_{Y_1}(y_1) \\ S(0, y_2) &= R(S_0(0), S_0(y_2)) = R(1, S_0(y_2)) = S_{Y_2}(y_2) \\ S(0, 0) &= R(S_0(0), S_0(0)) = R(1, 1) = 1. \end{aligned}$$

We have the following general result

Theorem 3.5: The survival function $S(y_1, y_2) = R(S_0(y_1), S_0(y_2))$ satisfies the functional equation

$$R(S_0(t)S_0(y_1), S_0(t)S_0(y_2)) = R(S_0(y_1), S_0(y_2))R(S_0(t), S_0(t)) \tag{18}$$

for all $0 < S_0(t), S_0(y_1), S_0(y_2) < 1$, if and only if it is of the form

$$R(S_0(y_1), S_0(y_2)) = S(y_1, y_2) = \begin{cases} [S_0(y_1)]^c R\left(1, \frac{S_0(y_2)}{S_0(y_1)}\right) & \text{if} \quad y_2 \geq y_1 \\ [S_0(y_2)]^c R\left(\frac{S_0(y_1)}{S_0(y_2)}, 1\right) & \text{if} \quad y_1 \geq y_2, \end{cases} \tag{19}$$

for some $c > 0$.

Proof of Theorem 3.5: See in the Appendix.

Now we present the characterization result.

Theorem 3.6: The EFB$(F, \theta_1, \theta_2, \theta'_1, \theta'_2)$ distribution is characterized by the functional equation

$$R(S_0(t)S_0(y_1), S_0(t)S_0(y_2)) = R(S_0(y_1), S_0(y_2))R(S_0(t), S_0(t)),$$

for all $0 < S_0(t), S_0(y_1), S_0(y_2) < 1$.

Proof of Theorem 3.6: It mainly follows from Theorem 3.5, and observing the following fact that for EFB$(F,\theta_1,\theta_2,\theta_1',\theta_2')$, the survival function (Theorem 3.1) for all four cases, can be written as follows:

$$R(S_0(y_1),S_0(y_2)) = S(y_1,y_2) = \begin{cases} [S_0(y_1)]^c R\left(1, \frac{S_0(y_2)}{S_0(y_1)}\right) & \text{if} \quad y_2 \geq y_1 \\ [S_0(y_2)]^c R\left(\frac{S_0(y_1)}{S_0(y_2)}, 1\right) & \text{if} \quad y_1 \geq y_2. \end{cases}$$

∎

Corollary 3.1: The PDF of example 1, as given in (5) satisfies the bivariate lack of memory property of Galambos and Kotz (1978).

Corollary 3.2: The survival function corresponding to the PDF of example 2, as given in (6) satisfies the relation

$$S((y_1+t)^{1/\alpha},(y_2+t)^{1/\alpha}) = S_0(t,t)S(y_1,y_2).$$

It is the bivariate extension of the univariate characterization of the Weibull distribution.

Now we would like to provide the bivariate hazard function of the FEB distributions. It may be mentioned that there are several ways of defining the bivariate hazard rates. In this paper we consider the joint bivariate hazard function in the sense of Johnson and Kotz (1975) and it is defined as follows:

$$h(y_1,y_2) = \left(-\frac{\partial}{\partial y_1}, -\frac{\partial}{\partial y_2}\right) \ln S_{Y_1,Y_2}(y_1,y_2) = (h_1(y_1,y_2), h_2(y_1,y_2)) \quad \text{(say)}, \qquad (20)$$

and it is known that the bivariate hazard function $h(y_1,y_2)$ uniquely defines the joint PDF. It is difficult to establish the results regarding bivariate hazard function for a general EFB models. The following results can be obtained for specific cases.

Theorem 3.7: Suppose $(Y_1,Y_2) \sim$ EFB$(EXP,\theta_1,\theta_2,\theta_1',\theta_2')$.

(i) If (a)$\theta_1+\theta_2 > \theta_1'$, $\theta_1 < \theta_1'$, (b) $\theta_1+\theta_2 > \theta_2'$, $\theta_2 < \theta_2'$, then for fixed y_2 (y_1), $h_1(y_1,y_2)$ $(h_2(y_1,y_2))$ is a decreasing function of y_1 (y_2).

(ii) If $\theta_1' > \theta_1 + \theta_2$ and $\theta_2' > \theta_1 + \theta_2$, then for fixed y_2 (y_1), $h_1(y_1, y_2)$ $(h_2(y_1, y_2))$ is an increasing function of y_1 (y_2).

PROOF: To prove (i), first consider the case $y_1 > y_2$. In this case after some calculation we can obtain

$$h_1(y_1, y_2) = \frac{(\theta_1 + \theta_2)(\theta_1 - \theta_1')e^{-(\theta_1+\theta_2-\theta_1')y_1} + \theta_1'\theta_2 e^{-(\theta_1+\theta_2-\theta_1')y_2}}{(\theta_1 - \theta_1')e^{-(\theta_1+\theta_2-\theta_1')y_1} + \theta_2 e^{-(\theta_1+\theta_2-\theta_1')y_2}}.$$

Using condition (a), it can be easily shown that $\dfrac{\partial h_1(y_1, y_2)}{\partial y_1} < 0$, for fixed $y_2 < y_1$. Similarly, for $y_1 < y_2$, using condition (b), it can be shown that $\dfrac{\partial h_2(y_1, y_2)}{\partial y_2} < 0$, for fixed $y_1 < y_2$. Hence, the result follows. The proof of (ii) can be obtained along the same line. ∎

THEOREM 3.8: Suppose $(Y_1, Y_2) \sim \text{EFB}(WE(\alpha), \theta_1, \theta_2, \theta_1', \theta_2')$.

(i) If (a) $\theta_1 + \theta_2 > \theta_1'$, $\theta_1 < \theta_1'$, (b) $\theta_1 + \theta_2 > \theta_2'$, $\theta_2 < \theta_2'$, then for fixed y_2 (y_1), $h_1(y_1, y_2)$ $(h_2(y_1, y_2))$ is a decreasing function of y_1 (y_2) for all $\alpha > 0$.

(ii) If $\theta_1' > \theta_1 + \theta_2$ and $\theta_2' > \theta_1 + \theta_2$, then for fixed y_2 (y_1), $h_1(y_1, y_2)$ $(h_2(y_1, y_2))$ is an increasing function of y_1 (y_2) for all $\alpha > 0$.

PROOF: The proof of Theorem 3.8 can be obtained from Theorem 3.7, by substituting $u_1 = y_1^\alpha$, $u_2 = y_2^\alpha$, and observing the fact that $\dfrac{\partial u_1}{\partial y_1} > 0$ and $\dfrac{\partial u_2}{\partial y_2} > 0$. ∎

THEOREM 3.9: Suppose $(Y_1, Y_2) \sim \text{EFB}(LFR(\alpha), \theta_1, \theta_2, \theta_1', \theta_2')$.

(i) If (a) $\theta_1 + \theta_2 > \theta_1'$, $\theta_1 < \theta_1'$, (b) $\theta_1 + \theta_2 > \theta_2'$, $\theta_2 < \theta_2'$, then for fixed y_2 (y_1), $h_1(y_1, y_2)$ $(h_2(y_1, y_2))$ is a decreasing function of y_1 (y_2) for all $\alpha > 0$.

(ii) If $\theta_1' > \theta_1 + \theta_2$ and $\theta_2' > \theta_1 + \theta_2$, then for fixed y_2 (y_1), $h_1(y_1, y_2)$ $(h_2(y_1, y_2))$ is an increasing function of y_1 (y_2) for all $\alpha > 0$.

PROOF: The proof of Theorem 3.9 can be obtained from Theorem 3.7, by substituting $u_1 = y_1 + \alpha y_1^2$, $u_2 = y_2 + y_2^2$, and observing the fact that $\frac{\partial u_1}{\partial y_1} > 0$ and $\frac{\partial u_2}{\partial y_2} > 0$. ∎

THEOREM 3.10: If $(Y_1, Y_2) \sim \text{EFB}(PAR(\alpha), \theta_1, \theta_2, \theta_1', \theta_2')$, and (a) $\theta_1 > \theta_1'$, (b) $\theta_2 > \theta_2'$, then for fixed y_2 (y_1), $h_1(y_1, y_2)$ $(h_2(y_1, y_2))$ is an increasing function of y_1 (y_2) for all $0 < \alpha < 1$.

PROOF: In this case also, the proof can be obtained in a routine manner. Consider the case $y_1 > y_2$, in this case after some calculation we can obtain

$$h_1(y_1, y_2) = \frac{\alpha y_1^{\alpha-1}(\theta_1+\theta_2)(\theta_1-\theta_1')(1+y^{\alpha})^{-(\theta_1+\theta_2+1)} + \alpha y_1^{\alpha-1}\theta_1'\theta_2(1+y_2^{\alpha})^{-(\theta_1+\theta_2-\theta_1')}}{(\theta_1-\theta_1')(1+y_1^{\alpha})^{-(\theta_1+\theta_2)} + \theta_2(1+y_2^{\alpha})^{-(\theta_1+\theta_2-\theta_1')}}.$$

Using condition (a), it can be easily shown that $\frac{\partial h_1(y_1, y_2)}{\partial y_1} < 0$, for fixed $y_2 < y_1$. Similarly, for $y_1 < y_2$, using condition (b), it can be shown that $\frac{\partial h_2(y_1, y_2)}{\partial y_2} < 0$, for fixed $y_1 < y_2$. Hence, the result follows. ∎

Now we mention how to generate samples from (1) based on the assumption that it is known how to generate samples from the proportional hazard class of distributions for any $\alpha > 0$. It may be noted that if the base line distribution/ survival function is easily invertible, then it is very simple to generate samples from the corresponding proportional hazard class of distribution for any $\alpha > 0$. The following algorithm can be used to generate samples from $\text{EFB}(F, \theta_1, \theta_2, \theta_1', \theta_2')$

ALGORITHM:

- Generate u, v and w independently from a uniform $(0, 1)$ distribution.
- If $u \le \frac{\theta_1}{\theta_1 + \theta_2}$, consider $y_1 = S_0^{-1}((1-v)^{1/\theta_1})$, and $y_2 = S_0^{-1}(S_0(y_1)(1-w)^{1/\theta_2'})$
- If $u > \frac{\theta_1}{\theta_1 + \theta_2}$, consider $y_2 = S_0^{-1}((1-v)^{1/\theta_2})$, and $y_1 = S_0^{-1}(S_0(y_2)(1-w)^{1/\theta_1'})$

Therefore, it is immediate that the generation from the EFB model is quite straight forward when $S_0^{-1}(\cdot)$ has explicit form, otherwise we need to solve it numerically. We will now briefly discuss how it can be done in three different cases mentioned above. In all these cases, u, v and w are generated independently from a uniform $(0,1)$ distribution.

Example 1:

- If $u \le \frac{\theta_1}{\theta_1+\theta_2}$, $\quad y_1 = -\frac{1}{\theta_1}\ln(1-v)$, and $y_2 = y_1 - \frac{1}{\theta_2'}\ln(1-w)$
- If $u > \frac{\theta_1}{\theta_1+\theta_2}$, $\quad y_2 = -\frac{1}{\theta_2}\ln(1-v)$, and $y_1 = y_2 - \frac{1}{\theta_1'}\ln(1-w)$

Example 2:

- If $u \le \frac{\theta_1}{\theta_1+\theta_2}$, $\quad y_1 = \left[-\frac{1}{\theta_1}\ln(1-v)\right]^{1/\alpha}$, and $y_2 = y_1 + \left[-\frac{1}{\theta_2'}\ln(1-w)\right]^{1/\alpha}$
- If $u > \frac{\theta_1}{\theta_1+\theta_2}$, $\quad y_2 = \left[-\frac{1}{\theta_2}\ln(1-v)\right]^{1/\alpha}$, and $y_1 = y_2 + \left[-\frac{1}{\theta_1'}\ln(1-w)\right]^{1/\alpha}$

Example 3:

- If $u \le \frac{\theta_1}{\theta_1+\theta_2}$,

$$y_1 = \frac{-1+\sqrt{1-4\alpha\ln(1-v)/\theta_1}}{2\alpha} \quad \text{and} \quad y_2 = y_1 + \frac{-1+\sqrt{1-4\widetilde{\alpha}_1\ln(1-w)/\widetilde{\theta}_2}}{2\widetilde{\alpha}_1}$$

  where

$$\widetilde{\alpha}_1 = \frac{\alpha}{2\alpha y_1+1} \quad \text{and} \quad \widetilde{\theta}_2 = (2\alpha y_1+1)\theta_2'$$

- If $u > \frac{\theta_1}{\theta_1+\theta_2}$,

$$y_2 = \frac{-1+\sqrt{1-4\alpha\ln(1-v)/\theta_2}}{2\alpha} \quad \text{and} \quad y_1 = y_2 + \frac{-1+\sqrt{1-4\widetilde{\alpha}_2\ln(1-w)/\widetilde{\theta}_1}}{2\widetilde{\alpha}_2}$$

  where

$$\widetilde{\alpha}_2 = \frac{\alpha}{2\alpha y_2+1} \quad \text{and} \quad \widetilde{\theta}_1 = (2\alpha y_2+1)\theta_1'$$

Example 4:

- If $u \leq \dfrac{\theta_1}{\theta_1+\theta_2}$, $\quad y_1 = \left[(1-v)^{-1/\theta_1} - 1\right]^{1/\alpha}$, and $y_2 = \left[(1+y_1^{\alpha})(1-w)^{-1/\theta_2'} - 1\right]^{1/\alpha}$
- If $u > \dfrac{\theta_1}{\theta_1+\theta_2}$, $\quad y_2 = \left[(1-v)^{-1/\theta_2} - 1\right]^{1/\alpha}$, and $y_1 = \left[(1+y_2^{\alpha})(1-w)^{-1/\theta_1'} - 1\right]^{1/\alpha}$

From these examples, it is clear that if we know how to generate samples, from the base line distribution function, it is very easy to generate samples from the corresponding EFB models.

4 Statistical Inferences

4.1 Maximum Likelihood Estimators

In this section it is assumed we have a sample of size n, $\{(y_{1i}, y_{2i}), i = 1, \cdots, n\}$ from (1) and based on the sample we want to estimate the unknown parameters, and then discuss their properties.

First it is assumed that $F_0(\cdot)$, the base line distribution, is completely known, and therefore, we need to find the maximum likelihood estimators (MLEs) of the unknown parameters $\theta_1, \theta_2, \theta_1', \theta_2'$. We use the following notations: $I_1 = \{i; y_{1i} > y_{2i}\}$, $I_2 = \{i; y_{1i} < y_{2i}\}$, n_1 and n_2 denote the number of elements in I_1 and I_2 respectively. Based on the observations, the log-likelihood function becomes;

$$\begin{aligned} l(\theta_1, \theta_2, \theta_1', \theta_2') &= n_1 \ln \theta_1' + n_1 \ln \theta_2 + \theta_1' \sum_{i \in I_1} \ln(S_0(y_{1i})) + (\theta_1 + \theta_2 - \theta_1') \sum_{i \in I_1} \ln(S_0(y_{2i})) + \\ &\quad n_2 \ln \theta_1 + n_2 \ln \theta_2' + \theta_2' \sum_{i \in I_2} \ln(S_0(y_{2i})) + (\theta_1 + \theta_2 - \theta_2') \sum_{i \in I_2} \ln(S_0(y_{1i})) + \\ &\quad \sum_{i \in I} \ln f_0(y_{1i}) + \sum_{i \in I} \ln f_0(y_{2i}) - \sum_{i \in I} \ln(S_0(y_{1i})) - \sum_{i \in I} \ln(S_0(y_{2i})) \\ &= n_2 \ln \theta_1 + \theta_1 (\sum_{i \in I_1} \ln(S_0(y_{2i})) + \sum_{i \in I_2} \ln(S_0(y_{1i}))) + \end{aligned}$$

$$
\begin{aligned}
& n_1 \ln \theta_2 + \theta_2 (\sum_{i \in I_1} \ln(S_0(y_{2i})) + \sum_{i \in I_2} \ln(S_0(y_{1i}))) + \\
& n_1 \ln \theta_1' + \theta_1' (\sum_{i \in I_1} \ln(S_0(y_{1i})) - \sum_{i \in I_1} \ln(S_0(y_{2i}))) + \\
& n_2 \ln \theta_2' + \theta_2' (\sum_{i \in I_2} \ln(S_0(y_{2i})) - \sum_{i \in I_2} \ln(S_0(y_{1i}))) + \\
& \sum_{i \in I} \ln f_0(y_{1i}) + \sum_{i \in I} \ln f_0(y_{2i}) - \sum_{i \in I} \ln(S_0(y_{1i})) - \sum_{i \in I} \ln(S_0(y_{2i})). \qquad (21)
\end{aligned}
$$

It is clear that when $n_1 = 0$ the MLE of θ_1' does not exist, and similarly when $n_2 = 0$, the MLE of θ_2' does not exist. Therefore, it is assumed that $n_1 > 0$ and $n_2 > 0$. The MLEs of θ_1, θ_2, θ_1' and θ_2' can be obtained as

$$
\begin{aligned}
\widehat{\theta}_1 &= -\frac{n_2}{\sum_{i \in I_1} \ln S_0(y_{2i}) + \sum_{i \in I_2} \ln S_0(y_{1i})} \\
\widehat{\theta}_2 &= -\frac{n_1}{\sum_{i \in I_1} \ln S_0(y_{2i}) + \sum_{i \in I_2} \ln S_0(y_{1i})} \\
\widehat{\theta}_1' &= -\frac{n_1}{\sum_{i \in I_1} \ln S_0(y_{1i}) - \sum_{i \in I_1} \ln S_0(y_{2i})} \\
\widehat{\theta}_2' &= -\frac{n_2}{\sum_{i \in I_2} \ln S_0(y_{2i}) - \sum_{i \in I_2} \ln S_0(y_{1i})}.
\end{aligned}
$$

Now we discuss how to compute the MLEs of the unknown parameters for different examples provided in Section 2.

Example 1: In this case the base line distribution is completely know, therefore, the MLEs are as follows:

$$
\widehat{\theta}_1 = \frac{n_2}{\sum_{i \in I_1} y_{2i} + \sum_{i \in I_2} y_{1i}}, \quad \widehat{\theta}_2 = \frac{n_1}{\sum_{i \in I_1} y_{2i} + \sum_{i \in I_2} y_{1i}} \qquad (22)
$$

$$
\widehat{\theta}_1' = \frac{n_1}{\sum_{i \in I_1} (y_{1i} - y_{2i})}, \quad \widehat{\theta}_2' = \frac{n_2}{\sum_{i \in I_2} (y_{2i} - y_{1i})}. \qquad (23)
$$

Example 2: In this case if α is known, the base line distribution is completely known. In that case the MLEs of the unknown parameters can be obtained as

$$
\widehat{\theta}_1(\alpha) = \frac{n_2}{\sum_{i \in I_1} y_{2i}^\alpha + \sum_{i \in I_2} y_{1i}^\alpha}, \quad \widehat{\theta}_2(\alpha) = \frac{n_1}{\sum_{i \in I_1} y_{2i}^\alpha + \sum_{i \in I_2} y_{1i}^\alpha} \qquad (24)
$$

$$
\widehat{\theta}_1'(\alpha) = \frac{n_1}{\sum_{i \in I_1} (y_{1i}^\alpha - y_{2i}^\alpha)}, \quad \widehat{\theta}_2'(\alpha) = \frac{n_2}{\sum_{i \in I_2} (y_{2i}^\alpha - y_{1i}^\alpha)}. \qquad (25)
$$

Clearly, if α is not known, first we obtain the MLE of α by maximizing the profile log-likelihood function of α, which without the additive constant is

$$g(\alpha) = n_2 \ln \widehat{\theta}_1(\alpha) + n_1 \ln \widehat{\theta}_2(\alpha) + n_1 \ln \widehat{\theta}'_1(\alpha) + n_2 \ln \widehat{\theta}'_2(\alpha) + 2n \ln \alpha + \alpha(\sum_{i \in I} (\ln y_{1i} + \ln y_{2i}). \quad (26)$$

The right hand side of (26) is a mixture of both convex and concave functions, hence, it is difficult to prove analytically that $g(\alpha)$ is unimodal. But in all our numerical experiments it is observed empirically that $g(\alpha)$ is unimodal. Hence, it has the unique maximum. Once the MLE of α is obtained, the MLEs of θ_1, θ_2, θ'_1 and θ'_2 can be obtained as $\widehat{\theta}_1(\widehat{\alpha})$, $\widehat{\theta}_2(\widehat{\alpha})$, $\widehat{\theta}'_1(\widehat{\alpha})$, $\widehat{\theta}'_2(\widehat{\alpha})$, respectively.

Example 3: In this case also, if α is known, the MLEs of the unknown parameters can be obtained as

$$\begin{aligned}
\widehat{\theta}_1(\alpha) &= \frac{n_2}{\sum_{i \in I_1} (y_{2i} + \alpha y_{2i}^2) + \sum_{i \in I_2} (y_{1i} + \alpha y_{1i}^2)}, && (27)\\
\widehat{\theta}_2(\alpha) &= \frac{n_1}{\sum_{i \in I_1} (y_{2i} + \alpha y_{2i}^2) + \sum_{i \in I_2} (y_{1i} + \alpha y_{1i}^2)}, && (28)\\
\widehat{\theta}'_1(\alpha) &= \frac{n_1}{\sum_{i \in I_1} (y_{1i} - y_{2i} + \alpha(y_{1i}^2 - y_{2i}^2))}, && (29)\\
\widehat{\theta}'_2(\alpha) &= \frac{n_2}{\sum_{i \in I_2} (y_{2i} - y_{1i} + \alpha(y_{2i}^2 - y_{1i}^2))}. && (30)
\end{aligned}$$

For unknown α the profile log-likelihood function of α without additive constant can be written as

$$\begin{aligned}
g(\alpha) &= n_2 \ln \widehat{\theta}_1(\alpha) + n_1 \ln \widehat{\theta}_2(\alpha) + n_1 \ln \widehat{\theta}'_1(\alpha) + n_2 \ln \widehat{\theta}'_2(\alpha) \\
&\quad + \sum_{i \in I} (\ln(1 + 2\alpha y_{1i}) + \ln(1 + 2\alpha y_{2i})) - \sum_{i=1}^{n} (y_{1i}^{\alpha} + y_{2i}^{\alpha} + \alpha(y_{1i}^2 + y_{2i}^2)). \qquad (31)
\end{aligned}$$

In this also, it is observed empirically, that $g(\alpha)$ is an unimodal function, and hence it has the unique maximum. Once the MLE of α is obtained, the MLEs of θ_1, θ_2, θ'_1 and θ'_2 can be easily obtained as before; $\widehat{\theta}_1(\widehat{\alpha})$, $\widehat{\theta}_2(\widehat{\alpha})$, $\widehat{\theta}'_1(\widehat{\alpha})$, $\widehat{\theta}'_2(\widehat{\alpha})$, respectively.

Example 4: If α is known, the MLEs of the unknown parameters can be obtained as

$$\widehat{\theta}_1(\alpha) = \frac{n_2}{\sum_{i\in I_1}\ln(1+y_{2i}^{\alpha}) + \sum_{i\in I_2}\ln(1+y_{1i}^{\alpha})}, \tag{32}$$

$$\widehat{\theta}_2(\alpha) = \frac{n_1}{\sum_{i\in I_1}\ln(1+y_{2i}^{\alpha}) + \sum_{i\in I_2}(1+y_{1i}^{\alpha})}, \tag{33}$$

$$\widehat{\theta}'_1(\alpha) = \frac{n_1}{\sum_{i\in I_1}(\ln(1+y_{1i}^{\alpha}) - \ln(1+y_{2i}^{\alpha}))}, \tag{34}$$

$$\widehat{\theta}'_2(\alpha) = \frac{n_2}{\sum_{i\in I_2}(\ln(1+y_{2i}^{\alpha}) - \ln(1+y_{1i}^{\alpha}))}. \tag{35}$$

For unknown α the profile log-likelihood function of α without additive constant can be written as

$$\begin{aligned} g(\alpha) &= n_2\ln\widehat{\theta}_1(\alpha) + n_1\ln\widehat{\theta}_2(\alpha) + n_1\ln\widehat{\theta}'_1(\alpha) + n_2\ln\widehat{\theta}'_2(\alpha) + 2n\ln\alpha \\ &\quad +(\alpha-1)\sum_{i\in I}(\ln y_{1i} + \ln y_{2i}) - \sum_{i=1}^{n}(\ln(1+y_{1i}^{\alpha}) + \ln(1+y_{2i}^{\alpha})). \end{aligned} \tag{36}$$

In this also, it is observed empirically, that $g(\alpha)$ is an unimodal function, and hence it has the unique maximum. Once the MLE of α is obtained, the MLEs of θ_1, θ_2, θ'_1 and θ'_2 can be easily obtained as before; $\widehat{\theta}_1(\widehat{\alpha})$, $\widehat{\theta}_2(\widehat{\alpha})$, $\widehat{\theta}'_1(\widehat{\alpha})$, $\widehat{\theta}'_2(\widehat{\alpha})$, respectively.

Now we will discuss the properties of the MLEs. This is a regular family of distributions. Therefore, the MLEs will be asymptotically normally distributed, and the asymptotic variance will be the negative inverse of the expected Fisher information matrix, which can be expressed as follows:

Example 1: When the base line distribution is completely known, the model has four unknown parameters. In this case the negative of the expected Fisher information matrix is

$$I_1 = \begin{bmatrix} \frac{\theta_2}{\theta_1^2(\theta_1+\theta_2)} & 0 & 0 & 0 \\ 0 & \frac{\theta_1}{\theta_2^2(\theta_1+\theta_2)} & 0 & 0 \\ 0 & 0 & \frac{\theta_1}{\theta_1'^2(\theta_1+\theta_2)} & 0 \\ 0 & 0 & 0 & \frac{\theta_2}{\theta_2'^2(\theta_1+\theta_2)} \end{bmatrix} \tag{37}$$

Examples 2-4: In all these cases the base line distribution has also one parameter. Therefore, all the models have total five unknown parameters. In this case the negative of the expected

Fisher information matrix in each case can be expressed as follows:

$$I_2 = \begin{bmatrix} \frac{\theta_2}{\theta_1^2(\theta_1+\theta_2)} & 0 & 0 & 0 & h_1 \\ 0 & \frac{\theta_1}{\theta_2^2(\theta_1+\theta_2)} & 0 & 0 & h_2 \\ 0 & 0 & \frac{\theta_1}{\theta_1'^2(\theta_1+\theta_2)} & 0 & h_3 \\ 0 & 0 & 0 & \frac{\theta_2}{\theta_2'^2(\theta_1+\theta_2)} & h_4 \\ h_1 & h_2 & h_3 & h_4 & h_5 \end{bmatrix}, \tag{38}$$

where h_j's cannot be obtained in explicit forms, they can be obtained in terms of integration. The expressions of $h_1, \ldots, h_5$, are provided in the appendix.

4.2 Testing of Hypothesis

In a load sharing system, one of the natural questions is whether the failure of one component results in a change in the lifetime distribution of the other component or not. In this respect we consider the following testing of hypothesis problem:

$$H_0 : \theta_1 = \theta_1', \theta_2 = \theta_2', \quad \text{vs.} \quad H_1 : \text{At least one is different.} \tag{39}$$

We propose to use the likelihood ratio test to test H_0 vs. H_1 as given in (39). First, we provide the MLEs of the unknown parameters under H_0, in all the three cases considered.

Example 1:

$$\widehat{\theta}_{10} = \frac{n}{\sum_{i\in I} y_{1i}}, \quad \text{and} \quad \widehat{\theta}_{20} = \frac{n}{\sum_{i\in I} y_{2i}}.$$

Example 2: For fixed α, the MLEs of θ_1 and θ_2 can be obtained as follows:

$$\widehat{\theta}_{10}(\alpha) = \frac{n}{\sum_{i\in I} y_{1i}^{\alpha}}, \quad \text{and} \quad \widehat{\theta}_{20}(\alpha) = \frac{n}{\sum_{i\in I} y_{2i}^{\alpha}}.$$

The MLE of α can be obtained by maximizing the profile log-likelihood function of α as given below:

$$g_0(\alpha) = n \ln \widehat{\theta}_{10}(\alpha) + n \ln \widehat{\theta}_{20}(\alpha) + 2n \ln \alpha + \alpha(\sum_{i\in I} (\ln y_{1i} + \ln y_{2i}). \tag{40}$$

Once the MLE of α, $\widehat{\alpha}_0$, is obtained by maximizing (40), the MLEs of θ_1 and θ_2 can be obtained as $\widehat{\theta}_{10}(\widehat{\alpha}_0)$ and $\widehat{\theta}_{20}(\widehat{\alpha}_0)$

Example 3:

For fixed α, the MLEs of θ_1 and θ_2 can be obtained as follows:

$$\widehat{\theta}_{10}(\alpha) = \frac{n}{\sum_{i \in I}(y_{1i} + \alpha y_{1i}^2)}, \quad \text{and} \quad \widehat{\theta}_{20}(\alpha) = \frac{n}{\sum_{i \in I}(y_{2i} + \alpha y_{2i}^2)}.$$

The MLE of α can be obtained by maximizing the profile log-likelihood function of α as given below:

$$g_0(\alpha) = n \ln \widehat{\theta}_{10}(\alpha) + n \ln \widehat{\theta}_{20}(\alpha) + \sum_{i \in I} (\ln(1 + 2\alpha y_{1i}) + \ln(1 + 2\alpha y_{2i})). \tag{41}$$

Once $\widehat{\alpha}_0$, the MLE of α, is obtained by maximizing (41), the MLEs of θ_1 and θ_2 can be obtained as $\widehat{\theta}_{10}(\widehat{\alpha}_0)$ and $\widehat{\theta}_{20}(\widehat{\alpha}_0)$

Example 4:

For fixed α, the MLEs of θ_1 and θ_2 can be obtained as follows:

$$\widehat{\theta}_{10}(\alpha) = \frac{n}{\sum_{i \in I} \ln(1 + y_{1i}^{\alpha})}, \quad \text{and} \quad \widehat{\theta}_{20}(\alpha) = \frac{n}{\sum_{i \in I} \ln(1 + y_{2i}^{\alpha})}.$$

The MLE of α can be obtained by maximizing the profile log-likelihood function of α as given below:

$$\begin{aligned} g_0(\alpha) &= n \ln \widehat{\theta}_{10}(\alpha) + n \ln \widehat{\theta}_{20}(\alpha) + 2n \ln \alpha + (\alpha - 1) \sum_{i \in I} (\ln y_{1i} + \ln y_{2i}) \\ &\quad - \sum_{i \in I} \left(\ln(1 + y_{1i}^{\alpha}) + \ln(1 + y_{2i}^{\alpha})\right). \end{aligned} \tag{42}$$

Once $\widehat{\alpha}_0$, the MLE of α, is obtained by maximizing (42), the MLEs of θ_1 and θ_2 can be obtained as $\widehat{\theta}_{10}(\widehat{\alpha}_0)$ and $\widehat{\theta}_{20}(\widehat{\alpha}_0)$

Once the MLEs under H_0 are obtained, the likelihood ratio test can be easily obtained to test H_0 vs. H_1 as given in (39).

5 Data Analysis

In this section we analyze one data set consists of three star players in a basketball team. The data set is obtained from the Basketball Association franchise Boston Celtics obtained during the second half of the 2001-2002 session. The data set is presented in the following Table 1. The data set represents the game times for each player's second personal foul for

Table 1: Time until second foul for the three star players

Game	Player I	Player II	Player III	Game	Player I	Player II	Player III
1	21.02	30.22	43.43	2	24.25	45.54	17.19
3	6.56	19.47	23.28	4	15.35	16.37	25.40
5	39.08	30.32	43.53	6	16.20	4.16	39.52
7	34.59	46.44	16.33	8	19.10	38.40	20.17
9	28.22	37.43	25.41	10	32.00	45.52	39.11
11	11.25	19.09	11.59	12	17.39	25.43	22.51
13	28.47	31.15	2.41	14	23.42	31.28	40.03
15	42.06	23.21	45.36	16	28.51	33.59	16.20
17	34.56	32.53	40.44	18	40.33	15.35	28.33
19	27.56	46.21	28.05	20	9.54	36.21	28.12
21	27.09	11.11	23.33	22	40.36	33.21	17.04
23	41.44	36.28	19.13	24	32.23	8.17	41.27
25	7.53	37.31	13.43	26	28.34	35.58	41.48
27	26.32	28.02	29.33	28	30.47	40.40	42.13

that particular game, in which all the three players have participated and committed at least two fouls at the end.

Kvam and Pena (2003) first analyzed this data set, see also Kvam and Pena (2005). Later Deshpande et al. (2007) also analyzed this same data set. It is expected that once a player committed at least two fouls the player may be out of the game for some time, and due to this reason the foul rate of the other star players might change. Kvam and Pena (2003) conjectured that once a star player committed two fouls, the foul rate of the other star players changes. Kvam and Pena (2003) assumed that the three star players compose a system, where as Deshpande et al. (2007) assumed that any two star players compose a system. We follow the approach of Deshpande et al. (2007), *i.e.* any two star players compose a system. We analyze the data set assuming three different systems namely, System I: (Player I, Player II), System II: (Player I, Player III) and System III: (Player II, Player III) respectively.

System I (Player I & Player II):

Exponential: The MLEs of θ_1, θ_2, θ_1' and θ_2' as $\widehat{\theta}_1 = 3.0933$, $\widehat{\theta}_2 = 1.4652$, $\widehat{\theta}_1' = 7.5624$ and $\widehat{\theta}_2' = 8.4909$. The associated 95% confidence intervals are (1.072, 5.103), (0.815, 2.132), (4.158, 10.963) and (3.304, 13.682) respectively.

Weibull: The MLEs of the different parameters in this case are $\widehat{\alpha} = 3.0071$, $\widehat{\theta}_1 = 42.2983$, $\widehat{\theta}_2 = 20.0360$, $\widehat{\theta}_1' = 33.4448$, $\widehat{\theta}_2' = 31.7244$. and the associated confidence intervals are (1.8639, 4.1503), (23.3107, 61.2859), (10.7908, 29.2812), (17.2735, 49.6161) respectively.

Linear Failure Rate: The MLEs of the different parameters in this case are $\widehat{\alpha} = 0.5515$, $\widehat{\theta}_1 = 2.7047$, $\widehat{\theta}_2 = 1.2812$, $\widehat{\theta}_1' = 5.8915$ and $\widehat{\theta}_2' = 6.4637$. The associated confidence intervals are as follows: (0.3202, 0.7828), (1.3634, 4.0460), (.6958, 1.8666), (3.1350, 8.6430), (3.3919, 9.5355) respectively.

Pareto III: The MLEs of the different parameters in this case are $\widehat{\alpha} = 2.9396$, $\widehat{\theta}_1 = 35.6781$, $\widehat{\theta}_2 = 19.8212$, $\widehat{\theta}_1' = 21.7844$ and $\widehat{\theta}_2' = 31.0153$. The associated confidence intervals are as follows: (1.9282, 3.9510), (24.4770, 46.8792), (13.8756, 25.7668), (15.5298, 28.0390), (20.7605, 41.2701) respectively.

Now the natural question is how well these distributions fit the bivariate data set. Unfortunately, it is well known that although we have several satisfactory goodness of fit tests for the univariate case, but for the bivariate case except for the bivariate normal distribution there does not exist any satisfactory goodness of fit test. Due to this reason, we test the marginals (Theorem 3.3) and also the maximum (Theorem 3.2) of the two observations. We report the results for the Kolmogorov-Smirnov distances and also the associated p values in Table 2. From Table 2 it is clear that the EFB model with Weibull base line survival function works the best among the three EFB models, and since all the p values are very high, it indicates that we cannot reject the null hypothesis that the data of System-I are coming from a EFB model with Weibull base line survival function.

Table 2: Kolmogorov-Smirnov distances and the associated p values for System I.

Variable	Exponential		Weibull		LFR		Pareto	
	K-S	p	K-S	p	K-S	p	K-S	p
Y_1	0.2709	0.0323	0.1287	0.7430	0.2480	0.0638	0.1876	0.3778
Y_2	0.2533	0.0550	0.1322	0.7120	0.2416	0.0762	0.2011	0.2054
$\max\{Y_1, Y_2\}$	0.2654	0.0455	0.1389	0.7345	0.2456	0.0712	0.2166	0.1765

System II (Player I & Player III)

Exponential: The MLEs are $\widehat{\theta}_1 = 3.1387$, $\widehat{\theta}_2 = 1.4867$, $\widehat{\theta}'_1 = 7.0373$ and $\widehat{\theta}'_2 = 10.6621$. The associated 95% confidence intervals are: (1.6848, 4.5926), (0.8633, 2.1101), (3.5806, 10.4940), (5.3209, 16.0033) respectively.

Weibull: The MLEs of the different parameters are $\widehat{\alpha} = 2.7218$, $\widehat{\theta}_1 = 29.8836$, $\widehat{\theta}_2 = 14.1554$, $\widehat{\theta}'_1 = 26.0310$ and $\widehat{\theta}'_2 = 32.4743$. The associated 95% confidence intervals are: (1.4066, 4.0370), (19.6691, 40.0981), (7.9006, 20.4102), (14.7725, 37.2895) respectively.

Linear Failure Rate: The MLEs of the different parameters are $\widehat{\alpha} = 0.5108$, $\widehat{\theta}_1 = 2.7718$, $\widehat{\theta}_2 = 1.3129$, $\widehat{\theta}'_1 = 5.5991$ and $\widehat{\theta}'_2 = 8.2852$. The associated 95% confidence intervals are (0.2976, 0.7240), (1.4573, 4.0863), (0.7340, 1.8918), (3.2512, 7.9470) respectively.

Pareto III: The MLEs of the different parameters in this case are $\widehat{\alpha} = 2.4320$, $\widehat{\theta}_1 = 13.7189$,

$\widehat{\theta}_2 = 11.8897$, $\widehat{\theta}'_1 = 26.0183$ and $\widehat{\theta}'_2 = 31.8613$. The associated confidence intervals are as follows: (1.3073, 3.5567), (10.4593, 16.9785), (8.8787, 14.9007), (18.0059, 34.0307), (20.9195, 42.8031) respectively.

We report the Kolmogorov-Smirnov distances of Y_1, Y_2 and $\max\{Y_1, Y_2\}$ with the corresponding fitted distributions and the associated p values in Table 3. It is clear that in this case also the EFB model with Weibull survival functions provides a best fit to System-II data set.

Table 3: Kolmogorov-Smirnov distances and the associated p values for System II.

Variable	Exponential		Weibull		LFR		Pareto	
	K-S	p	K-S	p	K-S	p	K-S	p
Y_1	0.2712	0.0320	0.1453	0.5954	0.2508	0.0590	0.2178	0.1668
Y_3	0.2668	0.0371	0.2015	0.2055	0.2364	0.0876	0.2241	0.1511
$\max\{Y_1, Y_2\}$	0.2685	0.0356	0.1815	0.3876	0.2456	0.0678	0.2315	0.1478

System III (Player II & Player III)

Exponential: The MLEs are $\widehat{\theta}_1 = 2.3222$, $\widehat{\theta}_2 = 2.3222$, $\widehat{\theta}'_1 = 5.9544$ and $\widehat{\theta}'_2 = 7.7489$. The associated 95% confidence intervals are ; (0.8644, 3.7800), (0.8644, 3.7800), (3.4966, 8.4122), (4.4610, 11.0368) respectively.

Weibull: The MLEs are $\widehat{\alpha} = 3.0235$, $\widehat{\theta}_1 = 31.7787$, $\widehat{\theta}_2 = 31.7787$, $\widehat{\theta}'_1 = 23.6955$ and $\widehat{\theta}'_2 = 30.1831$. The corresponding 95% confidence intervals are (1.7890, 4.2580), (19.2098, 44.3476), (19.2098, 44.3476), (13.7081, 33.6829) respectively.

Linear Failure Rate: The MLEs are $\widehat{\alpha} = 0.5746$, $\widehat{\theta}_1 = 2.0208$, $\widehat{\theta}_2 = 2.0208$, $\widehat{\theta}'_1 = 4.5123$ and $\widehat{\theta}'_2 = 5.8721$. The corresponding 95% confidence intervals are: (0.3601, 0.7891), (1.0330, 3.0086), ((1.0330, 2.0208), (2.3867, 6.6379), (3.2932, 8.4510) respectively.

Pareto III: The MLEs of the different parameters in this case are $\widehat{\alpha} = 2.8418$, $\widehat{\theta}_1 =$

26.0509, $\widehat{\theta}_2 = 26.0509$, $\widehat{\theta}'_1 = 21.4136$ and $\widehat{\theta}'_2 = 27.4781$. The associated confidence intervals are as follows: (1.8431, 3.8405), (16.9389, 35.1629), (16.9389, 35.1629), (13.9658, 28.8614), (17.9983, 36.9579), respectively.

The Kolmogorov-Smirnov distances and the associated p values of the different variables are provided in Table 4. The Kolmogorov-Smirnov distances and the associated p values suggest that EFB model with Weibull base line survival function provides the best fit among the three different EFB models.

Table 4: Kolmogorov-Smirnov distances and the associated p values for System III.

Variable	Exponential		Weibull		LFR		LFR	
	K-S	p	K-S	p	K-S	p	K-S	p
Y_2	0.2582	0.0478	0.1839	0.3000	0.2469	0.0657	0.2236	0.1578
Y_3	0.2823	0.0231	0.1931	0.2469	0.2545	0.0531	0.2155	0.1732
$\max\{Y_2, Y_3\}$	0.2678	0.0345	0.1897	0.2675	0.2489	0.0578	0.2178	0.1698

Finally we perform the likelihood ratio test for testing H_0 vs. H_1 as defined in (39) for all the three cases. It is observed that for System I and System II, the corresponding p values are very high, and therefore we cannot reject the null hypothesis. On the other hand for System III, the p value is 0.07, and therefore, we reject the null hypothesis only at the 10% level of significance. Interestingly, the conclusions match with the corresponding non-parametric test procedure proposed by Deshpande et al. (2007).

6 Conclusions

In this paper we have introduced a class of bivariate models which can be used to analysis two-component load sharing systems. The proposed model is an extension of the well known Freund's bivariate exponential model. Our model is a very flexible model, and it can be used quite effectively to analyze a two-component load-sharing system. Moreover the proposed model is very easy to use as it has explicit expressions for PDF and CDF. We have discussed four specific examples of our proposed model in details and developed different inferential procedures.

Although we have introduced the bivariate model (1) when the base line distribution has the support only on the positive real line, but the same definition can be used to define a new class of bivariate distributions when the support of $F_0(\cdot)$ is on $(-\infty, \infty)$. It may be noted that the physical interpretations what we have provided, may not be applicable in this case, but still it will be a proper bivariate distribution. Consider the following example when the base line distribution has the support on the whole real line. Suppose the base line distribution has the standard normal distribution function, then the joint PDF (1) becomes;

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2\pi}\theta_1'\theta_2 e^{-\frac{1}{2}(y_1^2+y_2^2)}(\Phi(-y_1))^{\theta_1'-1}(\Phi(-y_2))^{\theta_1+\theta_2-\theta_1'-1} & \text{if} \quad y_1 > y_2 \\ \frac{1}{2\pi}\theta_1\theta_2' e^{-\frac{1}{2}(y_1^2+y_2^2)}(\Phi(-y_1))^{\theta_1+\theta_2-\theta_2'-1}(\Phi(-y_2))^{\theta_2'-1} & \text{if} \quad y_2 > y_1. \end{cases} \tag{43}$$

It might be interesting to develop different properties of these types of general models. More work is needed in this direction.

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Appendix

Proof of Theorem 3.5:

If part:

Let the functional equation (18) be satisfied, then for $y_1 = y_2 = y$, (18) becomes

$$R(S_0(t)S_0(y), S_0(t)S_0(y)) = R(S_0(y), S_0(y))R(S_0(t), S_0(t)). \tag{44}$$

From Aczel (1966, page 41) and using the fact that $R(S_0(y_1), S_0(y_2))$ is a survival function, it follows that

$$R(S_0(t), S_0(t)) = [S_0(t)]^c; \quad c > 0. \tag{45}$$

Therefore, (18) can be written as

$$R(S_0(t)S_0(y_1), S_0(t)S_0(y_2)) = R(S_0(y_1), S_0(y_2))[S_0(t)]^c. \tag{46}$$

Now consider the case $y_1 \geq y_2$, then

$$\begin{aligned} R(S_0(y_1), S_0(y_2)) &= R\left(S_0(y_2)\frac{S_0(y_1)}{S_0(y_2)}, S_0(y_2)S_0(0)\right) \\ &= R(S_0(y_2), S_0(y_2))R\left(\frac{S_0(y_1)}{S_0(y_2)}, S_0(0)\right) \\ &= [S_0(y_2)]^c R\left(\frac{S_0(y_1)}{S_0(y_2)}, 1\right). \end{aligned}$$

Similarly, it can be proved for $y_1 \leq y_2$ also.

Only if part:

Let $R(S_0(y_1), S_0(y_2))$ be of the form (19). Therefore,

$$R(S_0(t)S_0(y_1), S_0(t)S_0(y_2)) = S(y_1, y_2) = \begin{cases} [S_0(t)S_0(y_1)]^c \quad R\left(1, \frac{S_0(y_2)}{S_0(y_1)}\right) & \text{if} \quad y_2 \geq y_1 \\ [S_0(t)S_0(y_2)]^c \quad R\left(\frac{S_0(y_1)}{S_0(y_2)}, 1\right) & \text{if} \quad y_1 \geq y_2 \end{cases} \tag{47}$$

Moreover, from (19), if $y_1 = y_2 = t$

$$R(S_0(t), S_0(t)) = S(t, t) = [S_0(t)]^c.$$

Therefore, (47) can be written as

$$R(S_0(t)S_0(y_1), S_0(t)S_0(y_2)) = S(y_1, y_2) = \begin{cases} R(S_0(t), S_0(t))[S_0(y_1)]^c \quad R\left(1, \frac{S_0(y_2)}{S_0(y_1)}\right) & \text{if} \quad y_2 \geq y_1 \\ R(S_0(t), S_0(t))[S_0(y_2)]^c \quad R\left(\frac{S_0(y_1)}{S_0(y_2)}, 1\right) & \text{if} \quad y_1 \geq y_2 \end{cases}$$

Now, the result follows by observing the fact

$$R(S_0(y_1), S_0(y_2)) = \begin{cases} [S_0(y_1)]^c \quad R\left(1, \frac{S_0(y_2)}{S_0(y_1)}\right) & \text{if} \quad y_2 \geq y_1 \\ [S_0(y_2)]^c \quad R\left(\frac{S_0(y_1)}{S_0(y_2)}, 1\right) & \text{if} \quad y_1 \geq y_2 \end{cases}$$

∎

The Expressions of $h_1, \ldots, h_5$.

We use the following notations:

$$g_1(y;\alpha) = -\frac{d}{d\alpha} \ln S_0(y), \quad g_2(y;\alpha) = -\frac{d^2}{d\alpha^2} \ln S_0(y), \quad g_3(y;\alpha) = -\frac{d^2}{d\alpha^2} \ln f_0(y).$$

Then

$$\begin{aligned} h_1 = h_2 &= \frac{\theta_2}{\theta_1+\theta_2} \int_0^\infty \int_0^{y_1} g_1(y_2;\alpha) f_{Y_1,Y_2}(y_1,y_2) dy_2 dy_1 \\ &\quad + \frac{\theta_1}{\theta_1+\theta_2} \int_0^\infty \int_0^{y_2} g_1(y_1;\alpha) f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \\ h_3 &= \frac{\theta_2}{\theta_1+\theta_2} \int_0^\infty \int_0^{y_1} \left(g_1(y_1;\alpha) - g_1(y_2;\alpha)\right) f_{Y_1,Y_2}(y_1,y_2) dy_2 dy_1 \\ h_4 &= \frac{\theta_1}{\theta_1+\theta_2} \int_0^\infty \int_0^{y_2} \left(g_1(y_2;\alpha) - g_1(y_1;\alpha)\right) f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \end{aligned}$$

$$\begin{aligned} h_5 &= \theta_2 \int_0^\infty \int_0^{y_1} g_2(y_2;\alpha) f_{Y_1,Y_2}(y_1,y_2) dy_2 dy_1 + \theta_1 \int_0^\infty \int_0^{y_2} g_2(y_1;\alpha) f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \\ &\quad + \frac{\theta_1'\theta_2}{\theta_1+\theta_2} \int_0^\infty \int_0^{y_1} \left(g_2(y_1;\alpha) - g_2(y_2;\alpha)\right) f_{Y_1,Y_2}(y_1,y_2) dy_2 dy_1 \\ &\quad + \frac{\theta_2'\theta_1}{\theta_1+\theta_2} \int_0^\infty \int_0^{y_2} \left(g_2(y_2;\alpha) - g_2(y_1;\alpha)\right) f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \\ &\quad + \int_0^\infty \int_0^\infty \left(g_3(y_1;\alpha) + g_3(y_2;\alpha) - g_2(y_1;\alpha) - g_2(y_2;\alpha)\right) f_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \end{aligned}$$

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