Bivariate Log Birnbaum-Saunders Distribution

Debasis Kundu[1]

Abstract

Univariate Birnbaum-Saunders distribution has received a considerable amount of attention in recent years. Rieck and Nedelman (1991, 'A log-linear model for the Birnbaum-Saunders distribution', Technometrics, 51-60) introduced a log Birnbaum-Saunders distribution. The main aim of this paper is to introduce bivariate log Birnbaum-Saunders distribution. The proposed model is symmetric and it has five parameters. It can be obtained using Gaussian copula. Different properties can be obtained using copula structure. It is observed that the maximum likelihood estimators cannot be obtained explicitly. Two dimensional profile likelihood approach may be adopted to compute the maximum likelihood estimators. We propose some alternative estimators also, which can be obtained quite conveniently. The analysis of one data set is performed for illustrative purposes. Finally, it is observed that this model can be used as a bivariate log-linear model, and its multivariate generalization is also quite straight forward.

Key Words and Phrases: Birnbaum-Saunders distribution; sinh distribution; copula; TP_2 properties; asymptotic distribution.

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[1] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India. E-mail: kundu@iitk.ac.in, Phone no. 91-512-2597141, Fax no. 91-512-2597500.

1 Introduction

Birnbaum-Saunders [2, 3] proposed a two-parameter failure time model for analyzing fatigue failure time data. From now on, we call this distribution as the Birnbaum-Saunders distribution. Recently Birnbaum- Saunders distribution has been used quite extensively in analyzing failure time data. The cumulative distribution function (CDF) of a two-parameter Birnbaum-Saunders random variable T can be written as

$$F_T(t;\alpha,\beta) = \Phi(a(t;\alpha,\beta)); \quad 0 < t < \infty, \quad \alpha,\beta > 0, \tag{1}$$

where

$$a(t;\alpha,\beta) = \left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{\beta}{t}\right)^{1/2}\right\}\right], \tag{2}$$

and $\Phi(\cdot)$ is the CDF of a standard normal random variable. Here α is the shape parameter and β is the scale parameter. Birnbaum-Saunders distribution can be obtained using the Central Limit Theorem (CLT) for independent identically distributed random variables with finite second moments. The probability density function (PDF) of the Birnbaum-Saunders distribution function is unimodal, and it is right skewed. An extensive review on Birnbaum-Saunders distribution till 1995 can be obtained in Johnson *et al.* [8], and for some other references, the readers are referred to Ng *et al.* [22, 23], Kundu *et al.* [11] Balakrishnan *et al.* [1], and the references cited therein.

Recently, Patriota [24] introduced scale mixture Birnbaum-Saunders distribution using generalized CLT. Several other extensions of the Birnbaum-Saunders distribution and new inference procedures of the unknown parameters, have been proposed by several authors, see for example the articles by Cordeiro and Lemonte [5], Leiva *et al.* [12, 13], Lemonte [17], Lemonte *et al.* [18, 19].

Rieck and Nedelman [27] introduced log Birnbaum-Saunders distribution and showed that it can be obtained as a special case of the sinh-normal distribution. The log Birnbaum-

Saunders distribution has several interesting properties, and it can be both unimodal and bimodal. Therefore, log Birnbaum-Saunders distribution is more flexible than many other unimodal symmetric models. It plays an important role in the log-linear model for life data with Birnbaum-Saunders distribution. Recently, Lemonte [14] proposed a log-Birnbaum-Saunders regression model with asymmetric error. A brief account of log Birnbaum-Saunders distribution is provided in the next section.

The main aim of this paper is to introduce a bivariate log Birnbaum-Saunders distribution along the same manner as the bivariate Birnbaum-Saunders distribution of Kundu *et al.* [10]. It can be obtained as a special case of the bivariate sinh-normal distribution. The bivariate log Birnbaum-Saunders is a symmetric distribution, and it can be obtained using the bivariate Gaussian copula. It is observed that the joint PDF of the log Birnbaum-Saunders random variable is symmetric and it can be both unimodal and bimodal depending on the parameter values. Using the copula structure different dependency measures and dependency properties can be established. It is further observed that the generation from a bivariate log Birnbaum-Saunders distribution is very simple, therefore simulation experiments can be performed quite effectively.

We discuss the maximum likelihood estimation procedure of the unknown parameters and it is observed that the maximum likelihood estimators (MLEs) cannot be obtained in explicit forms. We propose to use the profile log-likelihood method to compute the MLEs. We also provide some other estimation procedures which can be obtained easily compared to the MLEs. We provide the analysis of one data set for illustrative purposes. It is observed that this model can be used as a bivariate log-linear model. Finally we indicate the p dimensional generalization of the log Birnbaum-Saunders distribution.

Rest of the paper is organized as follows. In Section 2, we provide some preliminaries. Bivariate log Birnbaum-Saunders distribution is introduced and its different properties are discussed in Section 3. Different inferential issues are discussed in Section 4, and the analysis of one data set is provided in Section 5. Bivariate log-linear model using the proposed model is introduced in Section 6, finally we conclude the paper in Section 7.

2 Preliminaries

2.1 Univariate and Bivariate Birnbaum-Saunders Distributions

A random variable T is said to have a two-parameter Birnbaum-Saunders distribution if it has the CDF as given in (1). The corresponding PDF takes the following form;

$$f_T(t;\alpha,\beta) = \phi(a(t;\alpha,\beta))A(t;\alpha,\beta), \tag{3}$$

where $\phi(\cdot)$ is the PDF of a standard normal distribution, and

$$A(t;\alpha,\beta) = \frac{d}{dt}a(t;\alpha,\beta) = \frac{1}{2\alpha\beta}\left\{\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right\} = \frac{t+\beta}{2\alpha\sqrt{\beta}t^{3/2}}. \tag{4}$$

From now on, a two-parameter Birnbaum-Saunders random variable with CDF (1) or PDF (3) will be denoted by BS(α,β).

Recently, Kundu *et al.* [10] introduced a bivariate Birnbaum-Saunders distribution using a similar transformation as in (1) in two dimension. The random vector $(T_1,T_2)^T$ is said to have a bivariate Birnbaum-Saunders distribution, if its joint CDF is of the following form;

$$F_{T_1,T_2}(t_1,t_2) = P(T_1 \le t_1, T_2 \le t_2) = \Phi_2(a(t_1;\alpha_1,\beta_1), a(t_2;\alpha_2,\beta_2);\rho); \quad t_1>0, t_2>0, \tag{5}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2 > 0$, $-1<\rho<1$, and $\Phi_2(u,v;\rho)$ is the CDF of a bivariate standard normal random vector with correlation coefficient ρ. A bivariate Birnbaum-Saunders random vector with CDF (5) will be denoted by BVBS$(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho)$.

If $(T_1,T_2)^T \sim$ BVBS$(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho)$, then it has the PDF

$$f_{T_1,T_2}(t_1,t_2) = \phi_2(a(t_1;\alpha_1,\beta_1), a(t_2;\alpha_2,\beta_2);\rho) \times \prod_{i=1}^{2} A(t_i;\alpha_i,\beta_i), \tag{6}$$

where

$$\phi_2(u, v; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(u^2 + v^2 - 2\rho uv)\right\}, \tag{7}$$

and it is the standard bivariate normal probability density function with correlation coefficient ρ. The density surface of a bivariate Birnbaum-Saunders distribution is unimodal and it can take different shapes, see Kundu *et al.* [10] for details.

2.2 Sinh-Normal and log-Birnbaum-Saunders Distribution

Rieck and Nedelman [27] introduced the Sinh-normal and log Birnbaum-Saunders distribution as follows. Let Y be a real valued random variable with a cumulative distribution function $F(\cdot)$ given by

$$P(Y \leq y) = F_Y(y; \alpha, \gamma, \sigma) = \Phi\left\{\frac{2}{\alpha}\sinh\left(\frac{y-\gamma}{\sigma}\right)\right\}, \quad \text{for} \quad -\infty < y < \infty. \tag{8}$$

Here $\alpha > 0$, $\sigma > 0$, $-\infty < \gamma < \infty$, and $\sinh(x)$ is the hyperbolic sine function of x, and it is defined as $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$. In this case Y is said to have a sinh-normal distribution, and it is denoted by $\text{SN}(\alpha, \gamma, \sigma)$. The PDF of sinh-normal distribution is given by

$$f_Y(y; \alpha, \gamma, \sigma) = \frac{2}{\alpha\sigma\sqrt{2\pi}} \times \cosh\left(\frac{y-\gamma}{\sigma}\right) \times \exp\left[\left(-\frac{2}{\alpha^2}\sinh^2\left(\frac{y-\gamma}{\sigma}\right)\right)\right]. \tag{9}$$

Here $\cosh(x)$ is the hyperbolic cosine function of x, and it is defined as $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. In this case α is the shape parameter, σ is the scale parameter and γ is the location parameter.

It is clear from (8) that if $Y \sim \text{SN}(\alpha, \gamma, \sigma)$, then

$$Z = \frac{2}{\alpha}\sinh\left(\frac{Y-\gamma}{\sigma}\right) \sim N(0, 1). \tag{10}$$

From (10), it follows that if $Z \sim N(0, 1)$, then

$$Y = \sigma \text{ arcsinh}\left(\frac{\alpha Z}{2}\right) + \gamma \sim SN(\alpha, \gamma, \sigma), \tag{11}$$

where $\text{arcsinh}(x) = \ln(x + \sqrt{x^2+1})$. The above representation (11) of the sinh-normal distribution can be used for generation purposes.

The random variable Y is said to have a standard sinh-normal distribution when $\gamma = 0$ and $\sigma = 1$. Therefore, if $Y \sim \text{SN}(\alpha, 0, 1)$, then the corresponding PDF becomes;

$$f_Y(y; \alpha, 0, 1) = \frac{2}{\alpha\sqrt{2\pi}} \times \cosh(y) \times \exp\left[\left(-\frac{2}{\alpha^2}\sinh^2(y)\right)\right]. \tag{12}$$

It is immediate from (12) that the $\text{SN}(\alpha, 0, 1)$ is symmetric about 0. The PDF (12) of Y, for different values of α when $\gamma = 0$, and $\sigma = 1$ has been presented by Rieck [26]. It has been observed by Rieck [26] that for $\alpha \leq 2$, the PDF is strongly unimodal and for $\alpha > 2$, it is bimodal. Furthermore, if $Y \sim \text{SN}(\alpha, \gamma, \sigma)$, then $U = 2\alpha^{-1}(Y - \gamma)/\sigma$ converges to the standard normal distribution as $\alpha \to 0$.

Note that if $T \sim \text{BS}(\alpha, \beta)$, $\ln T \sim \text{SN}(\alpha, \ln\beta, 2)$. Because of this reason this is also known as log Birnbaum-Saunders distribution. From now on if $Y \sim \text{SN}(\alpha, \ln\beta, 2)$, then it will be denoted by $\text{LBS}(\alpha, \beta)$. Therefore, if $Y \sim \text{LBS}(\alpha, \beta)$, then Y has the CDF and PDF as

$$F_Y(y; \alpha, \beta) = P(Y \leq y) = \Phi\left\{\frac{2}{\alpha}\sinh\left(\frac{y - \ln\beta}{2}\right)\right\}, \quad \text{for} \quad -\infty < y < \infty, \tag{13}$$

and

$$f_Y(y; \alpha, \beta) = \frac{1}{\alpha} \times \cosh\left(\frac{y - \ln\beta}{2}\right) \times \phi\left(\frac{2}{\alpha}\sinh\left(\frac{y - \ln\beta}{2}\right)\right), \tag{14}$$

respectively.

2.3 Bivariate Gaussian Copula

To every bivariate distribution function $F_{X_1,X_2}(\cdot,\cdot)$, with continuous marginals, $F_{X_1}(\cdot)$ and $F_{X_2}(\cdot)$, corresponds a unique function $C : [0,1] \times [0,1] \to [0,1]$, called a copula such that

$$F_{X_1,X_2}(x_1, x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)); \quad (x_1, x_2) \in (-\infty, \infty) \times (-\infty, \infty). \tag{15}$$

Conversely, it is possible to construct a bivariate distribution function having the desired marginal distribution functions and a chosen dependence structure, *i.e.* copula. We have the following relation

$$C(u,v) = F_{X_1,X_2}(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)). \tag{16}$$

The bivariate Gaussian copula is defined as follows;

$$C_G(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \phi_2(x,y;\rho)dxdy = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v);\rho). \tag{17}$$

Here $\phi(\cdot)$, $\Phi(\cdot)$ and $\Phi_2(\cdot)$ are same as defined before.

The bivariate Gaussian copula density can be obtained as

$$c_G(u,v;\rho) = \frac{\partial^2}{\partial u \partial v} C_G(u,v;\rho) = \frac{\phi_2(\Phi^{-1}(u), \Phi^{-1}(v);\rho)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \tag{18}$$

$$= \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v) - \rho^2(\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2)}{2(1-\rho^2)} \right). \tag{19}$$

It may be noted that the bivariate Gaussian copula cannot be obtained in explicit form, it has to computed numerically. To compute bivariate Gaussian copula one needs to compute $\Phi^{-1}(\cdot)$, and several very well known approximation exist and they can be used quite effectively for computational purposes.

It may be recalled that a non-negative function g defined in $\mathbb{R}^2$ is total positivity of order two, abbreviated by TP_2 if for all $x_1 < x_2$, $y_1 < y_2$, with $x, y \in \mathbb{R}$

$$g(x_1,y_1)g(x_2,y_2) \geq g(x_2,y_1)g(x_1,y_2). \tag{20}$$

If the equality (20) is reversed, it is called reverse rule of order two (RR_2).

The following result will be useful for future development, and it may have some independent interest also. The proof is quite straight forward, and therefore it is avoided.

Result 1: The Gaussian copula density is (a) TP_2 for $0 < \rho < 1$, (b) RR_2 if $-1 < \rho < 0$.

3 Bivariate Log Birnbaum-Saunders distributions

Now along the same line as the univariate sinh-normal distribution, first we will define bivariate sinh-normal distribution, before introducing bivariate log Birnbaum-Saunders distribution, as follows. The random vector (Y_1, Y_2) is said to have a bivariate sinh-normal distribution with parameters $\alpha_1 > 0$, $-\infty < \gamma_1 < \infty$, $\sigma_1 > 0$, $\alpha_2 > 0$, $-\infty < \gamma_2 < \infty$, $\sigma_2 > 0$, $-1 < \rho < 1$ if the joint CDF of (Y_1, Y_2) is;

$$P(Y_1 \le y_1, Y_2 \le y_2) = F_{Y_1,Y_2}(y_1, y_2) = \Phi_2 \left\{ \frac{2}{\alpha_1} \sinh\left(\frac{y_1 - \gamma_1}{\sigma_1}\right), \frac{2}{\alpha_2} \sinh\left(\frac{y_2 - \gamma_2}{\sigma_2}\right); \rho \right\} \tag{21}$$

for $-\infty < y_1,\ y_2 < \infty$. Here $\Phi_2(u, v; \rho)$ is the cumulative distribution function of a standard bivariate normal vector (Z_1, Z_2) with the correlation coefficient ρ. The joint PDF of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{4}{\alpha_1\alpha_2\sigma_1\sigma_2} \phi_2 \left\{ \frac{2}{\alpha_1} \sinh\left(\frac{y_1 - \gamma_1}{\sigma_1}\right), \frac{2}{\alpha_2} \sinh\left(\frac{y_2 - \gamma_2}{\sigma_2}\right); \rho \right\} \times \\ & \quad \cosh\left(\frac{y_1 - \gamma_1}{\sigma_1}\right) \times \cosh\left(\frac{y_2 - \gamma_2}{\sigma_2}\right). \end{aligned} \tag{22}$$

Here $\phi_2(u, v, \rho)$ is same as (7). A bivariate random vector $(Y_1, Y_2)^T$ with the PDF (22) will be denoted by BSN$(\alpha_1, \gamma_1, \sigma_1, \alpha_2, \gamma_2, \sigma_2, \rho)$. If $\gamma_1 = \gamma_1 = 0$ and $\sigma_1 = \sigma_2 = 1$, it will be called standard bivariate sinh-normal distribution. Note that if $(Y_1, Y_2) \sim$ BSN$(\alpha_1, \gamma_1, \sigma_1, \alpha_2, \gamma_2, \sigma_2, \rho)$, then it has the following stochastic representation;

$$Y_i = \sigma_i \ \text{arcsinh}\left(\frac{\alpha_i Z_i}{2}\right) + \gamma_i; \quad i = 1, 2, \tag{23}$$

where $(Z_1, Z_2)^T$ follows a standard bivariate normal distribution with correlation coefficient ρ.

Now we are in a position to define a bivariate log Birnbaum-Saunders distribution. If $(X_1, X_2) \sim$ BVBS$(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then $(Y_1, Y_2) = (\ln X_1, \ln X_2)$ is said to have a bivariate log Birnbaum-Saunders distribution. Note that $(Y_1, Y_2) \sim$ BSN$(\alpha_1, \ln \beta_1, 2, \alpha_2, \ln \beta_2, 2, \rho)$, and it will be denoted as BLBS$(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$. Therefore, the joint CDF and the joint PDF of a BLBS$(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$ are

$$P(Y_1 \le y_1, Y_2 \le y_2) = F_{Y_1,Y_2}(y_1, y_2) = \Phi_2 \left\{ \frac{2}{\alpha_1} \sinh\left(\frac{y_1 - \ln \beta_1}{2}\right), \frac{2}{\alpha_2} \sinh\left(\frac{y_2 - \ln \beta_2}{2}\right); \rho \right\} \tag{24}$$

and

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{1}{\alpha_1 \alpha_2} \phi_2 \left\{ \frac{2}{\alpha_1} \sinh\left(\frac{y_1 - \ln \beta_1}{2}\right), \frac{2}{\alpha_2} \sinh\left(\frac{y_2 - \ln \beta_2}{2}\right); \rho \right\} \times \\ & \cosh\left(\frac{y_1 - \ln \beta_1}{2}\right) \times \cosh\left(\frac{y_2 - \ln \beta_2}{2}\right). \end{aligned} \tag{25}$$

The contour plots of (22) for different values of α_1 and α_2 are presented in Figure 1. It is clear that it can take different shapes depending on α_1, α_2 and ρ. It is always symmetric along the axis $x = y$. It can be both unimodal and bimodal, therefore it can be used quite effectively to analyze bivariate data. The unimodality and bimodality of the joint PDF depend on the values of α_1, α_2 and ρ. When $\alpha_1 = \alpha_2 = 2$, it is bimodal if $|\rho|$ is large, but it is unimodal if $|\rho|$ is small. Therefore, the unimodality or bimodality of the PDF not only depends on α_1 and α_2, but it depends on ρ also.

From (24) it easily follows that if $(Y_1, Y_2) \sim$ BSN$(\alpha_1, \ln \beta_1, 2, \alpha_2, \ln \beta_2, 2, \rho)$

$$\left(\frac{2}{\alpha_1} \sinh\left(\frac{Y_1 - \ln \beta_1}{2}\right), \frac{2}{\alpha_2} \sinh\left(\frac{Y_2 - \ln \beta_2}{2}\right) \right)^T \sim BN(\rho). \tag{26}$$

BN(ρ) denotes bivariate standard normal distribution with correlation coefficient ρ. It is quite simple to generate samples from a bivariate log Birnbaum-Saunders distribution. We present the following algorithm to generate samples from BSN$(\alpha_1, \ln \beta_1, 2, \alpha_2, \ln \beta_2, 2, \rho)$.

Algorithm:

- Step 1: Generate independent U_1 and U_2 from $N(0, 1)$

- Step 2: Compute

$$\begin{aligned} Z_1 &= \frac{\sqrt{1+\rho}+\sqrt{1-\rho}}{2}U_1 + \frac{\sqrt{1+\rho}-\sqrt{1-\rho}}{2}U_2 \\ Z_2 &= \frac{\sqrt{1+\rho}-\sqrt{1-\rho}}{2}U_1 + \frac{\sqrt{1+\rho}+\sqrt{1-\rho}}{2}U_2 \end{aligned}$$

- Step 3: Obtain

$$Y_i = 2\ \ln\left[\frac{\alpha_i Z_i}{2} + \sqrt{1+\left(\frac{\alpha_i Z_i}{2}\right)^2}\right]; \quad i = 1,2.$$

The following theorem provides the marginal and conditional distributions of BLBS distribution.

THEOREM 3.1: If $(Y_1, Y_2) \sim \text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then

(a) $Y_i \sim \text{LBS}(\alpha_i, \beta_i), \quad i = 1{,}2.$

(b) The conditional PDF of Y_1 given $Y_2 = y_2$ is given by

$$\begin{aligned} f_{Y_1|Y_2=y_2}(y_1) &= \frac{1}{\alpha_1\sqrt{2\pi}\sqrt{1-\rho^2}}\cosh\left(\frac{y_1-\ln\beta_1}{2}\right) \\ &\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{2}{\alpha_1}\sinh\left(\frac{y_1-\ln\beta_1}{2}\right) - \frac{2\rho}{\alpha_2}\sinh\left(\frac{y_2-\ln\beta_2}{2}\right)\right]^2\right\} \end{aligned}$$

(c) The conditional CDF of Y_1 given $Y_2 = y_2$ is given by

$$P(Y_1 \le y_1 | Y_2 = y_2) = \Phi\left\{\frac{\frac{2}{\alpha_1}\sinh\left(\frac{y_1-\ln\beta_1}{2}\right) - \frac{2\rho}{\alpha_2}\sinh\left(\frac{y_2-\ln\beta_2}{2}\right)}{\sqrt{1-\rho^2}}\right\}$$

PROOF: Note that the proofs of (a) and (b) can be easily obtained from the definition itself. To prove part (c), take the transformation

$$u = \frac{2}{\alpha_1}\sinh\left(\frac{y_1-\ln\beta_1}{2}\right) \quad \text{and} \quad v = \frac{u - \frac{2\rho}{\alpha_2}\sinh\left(\frac{y_2-\ln\beta_2}{2}\right)}{\sqrt{1-\rho^2}},$$

and the result can be easily obtained.

COMMENT: From (b) it is clear that Y_1 and Y_2 are independent if and only if $\rho = 0$.

THEOREM 3.2: If $(Y_1, Y_2) \sim \text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then

(a) $(Y_1, -Y_2) \sim \text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2^{-1}, -\rho)$.

(b) $(-Y_1, Y_2) \sim \text{BLBS}(\alpha_1, \beta_1^{-1}, \alpha_2, \beta_2, -\rho)$.

(a) $(-Y_1, -Y_2) \sim \text{BLBS}(\alpha_1, \beta_1^{-1}, \alpha_2, \beta_2^{-1}, \rho)$.

PROOF: All the proofs can be obtained directly from the joint PDF of BLBS and taking proper transformations.

THEOREM 3.3: If $(Y_1, Y_2) \sim \text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for $\rho > 0$ ($\rho < 0$) it has the TP_2 (RR_2) property.

PROOF: Since both TP_2 and RR_2 properties are copula properties, the results follow from Result 1.

Using Theorem 3.3, and using the result of Holland and Wong [7], it follows that the local dependence of a BLBS distribution is positive if $\rho > 0$, and negative if $\rho < 0$. It also follows using the result of Shaked [28] and Theorem 3.3, that the conditional failure rate of Y_1 given $Y_2 = y_2$ is a decreasing (increasing) function in y_2 if $\rho > (<) 0$. Using the copula property, it easily follows that for Y_1 and Y_2, for all values of α_1, β_1, α_2, β_2, (a) Blomqvist's beta, (b) Kendall's tau and (c) Spearman's rho become

$$\beta = \frac{2}{\pi}\arcsin(\rho), \qquad \tau = \frac{2}{\pi}\arcsin(\rho), \qquad \rho_S = \frac{6}{\pi}\arcsin\left(\frac{\rho}{2}\right), \tag{27}$$

respectively.

THEOREM 3.4: If $(Y_1, Y_2) \sim \text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then for $\rho > 0$ (a) Y_1 is stochastically increasing in Y_2 (b) Y_2 is stochastically increasing in Y_1, for all values of α_1, β_1, α_2 and β_2.

PROOF: We will prove (a), (b) follows along the same line. Note that the results can be established if we can show that $C_G(u,v;\rho)$ is a concave function in u for fixed v when $\rho > 0$, see Nelsen ([21], page 197). It is equivalent to prove that $\frac{\partial}{\partial u} C_G(u,v;\rho)$ is a decreasing function in u (v). Using Meyer [20], we have

$$\frac{\partial}{\partial u} C(u,v;\rho) = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right). \tag{28}$$

Clearly, for $\rho > 0$, the right hand side is a decreasing function in u for fixed v and the result follows. ∎

4 Inference

Now, we discuss the maximum likelihood estimators of the unknown parameters based on a random sample of size n, namely, $\{(y_{1i}, y_{2i}); i = 1, \ldots, n\}$ from the BLBS$(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$. We further discuss some testing of hypotheses problem also.

4.1 Maximum Likelihood Estimators:

In this section we present the MLEs of the unknown parameters. The MLEs of the unknown parameters can be obtained by maximizing the log-likelihood function with respect to the unknown parameters. Let us denote $\boldsymbol{\theta} = (\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$. Based on the random sample, the log-likelihood function (without the additive constant) becomes;

$$\begin{aligned} l(\boldsymbol{\theta}) \;=\; & -n \ln \alpha_1 - n \ln \alpha_2 - \frac{n}{2} \ln(1-\rho^2) + \sum_{i=1}^{n} \ln\left\{\cosh\left(\frac{y_{1i} - \ln\beta_1}{2}\right)\right\} \\ & + \sum_{i=1}^{n} \ln\left\{\cosh\left(\frac{y_{2i} - \ln\beta_2}{2}\right)\right\} - \frac{1}{2(1-\rho^2)}\left\{\frac{4}{\alpha_1^2}\sum_{i=1}^{n}\sinh^2\left(\frac{y_{1i} - \ln\beta_1}{2}\right)\right. \\ & \left. + \frac{4}{\alpha_2^2}\sum_{i=1}^{n}\sinh^2\left(\frac{y_{2i} - \ln\beta_2}{2}\right) - \frac{8\rho}{\alpha_1\alpha_2}\sum_{i=1}^{n}\sinh\left(\frac{y_{1i} - \ln\beta_1}{2}\right)\sinh\left(\frac{y_{2i} - \ln\beta_2}{2}\right)\right\}. \end{aligned}$$

$$(29)$$

Therefore, the MLEs can be obtained by maximizing (29) with respect to $\alpha_1, \alpha_2, \beta_1, \beta_2$ and ρ. Clearly, they cannot be obtained in explicit forms, and we need to solve five non-linear equations to obtain the solutions. To avoid that, we adopt the following procedure. Note that for known β_1 and β_2, the MLEs of the unknown parameters can be obtained by maximizing

$$\begin{aligned} l_1(\alpha_1, \alpha_2, \rho) &= -n \ln \alpha_1 - n \ln \alpha_2 - \frac{n}{2} \ln(1-\rho^2) - \frac{1}{2(1-\rho^2)} \left\{ \frac{4}{\alpha_1^2} \sum_{i=1}^{n} \sinh^2\left(\frac{y_{1i} - \ln \beta_1}{2}\right) \right. \\ &\left. + \frac{4}{\alpha_2^2} \sum_{i=1}^{n} \sinh^2\left(\frac{y_{2i} - \ln \beta_2}{2}\right) - \frac{8\rho}{\alpha_1 \alpha_2} \sum_{i=1}^{n} \sinh\left(\frac{y_{1i} - \ln \beta_1}{2}\right) \sinh\left(\frac{y_{2i} - \ln \beta_2}{2}\right) \right\}, \end{aligned} \tag{30}$$

with respect to α_1, α_2 and ρ. By comparing with the log-likelihood function of a bivariate normal distribution, it can be easily seen that for fixed β_1 and β_2, the MLEs of α_1, α_2 and ρ can be obtained as

$$\widehat{\alpha}_1(\beta_1) = \left\{ \frac{4}{n} \sum_{i=1}^{n} \sinh^2\left(\frac{y_{1i} - \ln \beta_1}{2}\right) \right\}^{\frac{1}{2}}, \quad \widehat{\alpha}_2(\beta_2) = \left\{ \frac{4}{n} \sum_{i=1}^{n} \sinh^2\left(\frac{y_{2i} - \ln \beta_2}{2}\right) \right\}^{\frac{1}{2}}, \tag{31}$$

and

$$\widehat{\rho}(\beta_1, \beta_2) = \frac{\sum_{i=1}^{n} \sinh\left(\frac{y_{1i} - \ln \beta_1}{2}\right) \sinh\left(\frac{y_{2i} - \ln \beta_2}{2}\right)}{\left\{\frac{4}{n} \sum_{i=1}^{n} \sinh^2\left(\frac{y_{1i} - \ln \beta_1}{2}\right)\right\}^{\frac{1}{2}} \left\{\frac{4}{n} \sum_{i=1}^{n} \sinh^2\left(\frac{y_{2i} - \ln \beta_2}{2}\right)\right\}^{\frac{1}{2}}}. \tag{32}$$

Therefore, when β_1 and β_2 are unknown, the MLEs of β_1 and β_2 can be obtained by maximizing the profile log-likelihood function namely

$$\begin{aligned} l_{profile}(\beta_1, \beta_2) &= l(\widehat{\alpha}_1(\beta_1), \beta_1, \widehat{\alpha}_2(\beta_2), \beta_2, \widehat{\rho}(\beta_1, \beta_2)) \\ &= -n \ln \widehat{\alpha}_1(\beta_1) - n \ln \widehat{\alpha}_2(\beta_2) - \frac{n}{2} \ln(1 - \widehat{\rho}^2(\beta_1, \beta_2)) \\ &\quad + \sum_{i=1}^{n} \ln \left\{ \cosh\left(\frac{y_{1i} - \ln \beta_1}{2}\right) \right\} + \sum_{i=1}^{n} \ln \left\{ \cosh\left(\frac{y_{2i} - \ln \beta_2}{2}\right) \right\}, \end{aligned} \tag{33}$$

with respect to β_1 and β_2. Clearly, they cannot be obtained in explicit forms. Some iterative process like Newton-Raphson algorithm or Gauss-Newton algorithm may be used to compute the MLEs of β_1 and β_2 by maximizing (33). Once the MLEs of β_1 and β_2, say $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are obtained the MLEs of α_1, α_2 and ρ can be obtained as

$$\widehat{\alpha}_1 = \widehat{\alpha}_1(\widehat{\beta}_1), \quad \widehat{\alpha}_2 = \widehat{\alpha}_2(\widehat{\beta}_2), \quad \text{and} \quad \widehat{\rho} = \widehat{\rho}(\widehat{\beta}_1, \widehat{\beta}_2) \tag{34}$$

respectively.

One important question is whether the function (33) is unimodal or not. Due to complicated nature of the profile log-likelihood function it is very difficult to establish theoretically the uniqueness of the maximum of the function (33). In our numerical experiments it is observed that the profile log-likelihood function is unimodal.

It has already been mentioned that to compute the MLEs of β_1 and β_2, we need to maximize the profile log-likelihood function as provided in(33). It has to be performed using some iterative method, and for that we need some initial guesses of β_1 and β_2. Since $\ln \beta_1$ and $\ln \beta_2$ are location parameters of Y_1 and Y_2 respectively, we use the corresponding median as alternative estimators of β_1 and β_2, respectively. So let us denote

$$\widetilde{y}_1 = \text{median}\{y_{11}, \ldots, y_{1n}\} \quad \text{and} \quad \widetilde{y}_2 = \text{median}\{y_{21}, \ldots, y_{2n}\}, \tag{35}$$

and

$$\widetilde{\beta}_1 = e^{\widetilde{y}_1} \quad \text{and} \quad \widetilde{\beta}_2 = e^{\widetilde{y}_2}. \tag{36}$$

Therefore, alternative estimators of α_1, α_2 and ρ can be obtained as

$$\widetilde{\alpha}_1 = \widehat{\alpha}_1(\widetilde{\beta}_1), \quad \widetilde{\alpha}_2 = \widehat{\alpha}_2(\widetilde{\beta}_2), \quad \text{and} \quad \widetilde{\rho} = \widehat{\rho}(\widetilde{\beta}_1, \widetilde{\beta}_2), \tag{37}$$

and they can be obtained explicitly. We call these estimators as median based estimators (MBEs). Alternatively, instead of medians the corresponding means also can be used for estimating β's. For small sample sizes both of them can be obtained very quickly. It is observed that they behave in a very similar manner.

4.2 Testing of Hypotheses

In this section we discuss different testing of hypotheses problem which can be useful in practice.

$$\text{Test I} : H_0 : (\beta_1, \beta_2) = (a, b) \quad \text{vs.} \quad H_1 : (\beta_1, \beta_2) \neq (a, b), \tag{38}$$

where a and b are known. The above testing problem is an important one, as it tests the symmetry of the distribution around $(\ln a, \ln b)$. In particular when $a = b = 1$, it is of special interest. Now to test the hypothesis (38), consider the following transformation

$$u_{1i} = 2\sinh\left(\frac{y_{1i} - \ln a}{2}\right) \quad \text{and} \quad u_{2i} = 2\sinh\left(\frac{y_{2i} - \ln b}{2}\right); \quad i = 1, \ldots, n.$$

Due to (26), it follows that (u_{1i}, u_{2i}) has a bivariate normal distribution, each with mean zero. Therefore, we can use Hotelling T^2 statistic, see page 180 of Johnson and Wichern [9], to test the above hypothesis as follows. Consider

$$T^2 = n(\bar{\boldsymbol{u}}^T \boldsymbol{S}^{-1} \bar{\boldsymbol{u}}),$$

where

$$\bar{\boldsymbol{u}}^T = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{u}_i^T, \qquad \boldsymbol{S} = \frac{1}{n-1}\sum_{i=1}^{n} \boldsymbol{u_i}\boldsymbol{u}_i^T \quad \text{and} \quad \boldsymbol{u}_i^T = (u_{1i} \quad u_{2i}).$$

Under H_0, $T^2 \sim \mathrm{F}_{2,n-2}$.

Consider the following testing problem:

$$\text{Test II} : H_0 : \beta_1 = \beta_2 = \beta \quad \text{vs.} \quad H_1 : \beta_1 \neq \beta_2. \tag{39}$$

The testing of hypothesis problem (39) is an important one, as it tests whether the location parameters of the two marginals are same or not. First we consider the case when β is known. Consider the following statistic

$$t = \frac{\bar{d}}{s_d/\sqrt{n}}, \tag{40}$$

where

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i, \quad s_d^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2 \quad \text{and} \quad d_i = \sinh\left(\frac{y_{1i} - \ln\beta}{2}\right) - \sinh\left(\frac{y_{2i} - \ln\beta}{2}\right).$$

Under null hypothesis H_0, due to (26), t as defined in (40) has a t-distribution with $n-1$ degrees of freedom, see for example page 220 of Johnson and Wichern [9]. Therefore reject the null hypothesis at $\alpha\%$ significance level, if $|t| > t_{\alpha/2,n-1}$, where $t_{\alpha/2,n-1}$ denotes the upper $\alpha/2$-th percentile point of a t-distribution with $n-1$ degrees of freedom.

Now we will consider the case when β is not known. In this case we do not have any exact test, hence we propose to use the likelihood ratio test at least for large sample sizes. To construct the likelihood ratio test, we need to compute the MLEs under H_0. Under H_0, the MLEs of α_1, α_2 and ρ for a fixed β, say $\widetilde{\alpha}_1(\beta)$, $\widetilde{\alpha}_2(\beta)$ and $\widetilde{\rho}(\beta)$, can be obtained as

$$\widetilde{\alpha}_1(\beta) = \left\{\frac{4}{n}\sum_{i=1}^{n}\sinh^2\left(\frac{y_{1i} - \ln\beta}{2}\right)\right\}^{\frac{1}{2}}, \quad \widetilde{\alpha}_2(\beta) = \left\{\frac{4}{n}\sum_{i=1}^{n}\sinh^2\left(\frac{y_{2i} - \ln\beta}{2}\right)\right\}^{\frac{1}{2}}, \tag{41}$$

and

$$\widetilde{\rho}(\beta) = \frac{\sum_{i=1}^{n}\sinh\left(\frac{y_{1i}-\ln\beta}{2}\right)\sinh\left(\frac{y_{2i}-\ln\beta}{2}\right)}{\left\{\frac{4}{n}\sum_{i=1}^{n}\sinh^2\left(\frac{y_{1i}-\ln\beta}{2}\right)\right\}^{\frac{1}{2}}\left\{\frac{4}{n}\sum_{i=1}^{n}\sinh^2\left(\frac{y_{2i}-\ln\beta}{2}\right)\right\}^{\frac{1}{2}}}, \tag{42}$$

respectively, and the MLEs of β under H_0 can be obtained by maximizing the profile log-likelihood function of β, *i.e.*

$$\begin{aligned}
\widetilde{l}_{profile}(\beta) &= l(\widetilde{\alpha}_1(\beta), \beta, \widetilde{\alpha}_2(\beta), \beta, \widetilde{\rho}(\beta)) \\
&= -n\ln\widetilde{\alpha}_1(\beta) - n\ln\widetilde{\alpha}_2(\beta) - \frac{n}{2}\ln(1 - \widetilde{\rho}^2(\beta)) \\
&\quad + \sum_{i=1}^{n}\ln\left\{\cosh\left(\frac{y_{1i} - \ln\beta}{2}\right)\right\} + \sum_{i=1}^{n}\ln\left\{\cosh\left(\frac{y_{2i} - \ln\beta}{2}\right)\right\}.
\end{aligned} \tag{43}$$

If $\widetilde{\beta}$ maximizes (43), then asymptotically ($\rightarrow$), under H_0

$$-2\{\widetilde{l}_{profile}(\widetilde{\beta}) - l_{profile}(\widehat{\beta}_1, \widehat{\beta}_2)\} \rightarrow \chi_1^2.$$

Now let us consider the following testing of hypothesis problem;

$$\text{Test III} : H_0 : \alpha_1 = \alpha_2 \;\text{ vs. }\; H_1 : \alpha_1 \neq \alpha_2. \tag{44}$$

First we consider the case when β_1 and β_2 are known. In this case we consider the following transformation

$$v_{1i} = 2\sinh\left(\frac{y_{1i} - \ln\beta_1}{2}\right) \quad \text{and} \quad v_{2i} = 2\sinh\left(\frac{y_{2i} - \ln\beta_2}{2}\right); \quad i = 1, \ldots, n.$$

Now using (26), testing 44) is equivalent to testing the equality of variances in a bivariate normal case. Therefore, following Pitman [25], consider the following statistic

$$R = \frac{s_1 s_2 - 1}{\sqrt{(s_1 s_2 + 1)^2 - 4r^2 s_1 s_2}}, \tag{45}$$

where

$$s_1^2 = \sum_{i=1}^{n} v_{1i}^2, \quad s_2^2 = \sum_{i=1}^{n} v_{2i}^2 \quad \text{and} \quad r = \frac{\sum_{i=1}^{n} v_{1i} v_{2i}}{\sqrt{\sum_{i=1} v_{1i}^2}\sqrt{\sum_{i=1}^{n} v_{2i}^2}}.$$

Under the null hypothesis

$$t = \frac{R\sqrt{n-2}}{\sqrt{1-R^2}}$$

follows a t-distribution with $n-2$ degrees of freedom, see Pitman [25]. Therefore reject the null hypothesis at $\alpha\%$ significance level, if $|t| > t_{\alpha/2,n-2}$, where $t_{\alpha/2,n-2}$ denotes the upper $\alpha/2$-th percentile point of a t-distribution with $n-2$ degrees of freedom.

Now, we will consider the case when β_1 and β_2 are unknown. In this case we do not have any exact test. We propose to use the standard likelihood ratio test similarly as before to test the hypothesis (44). Finally we consider the following testing of hypothesis problem

$$\text{Test IV} : H_0 : \rho = 0 \ \text{ vs. } \ H_1 : \rho \neq 0. \tag{46}$$

It is an important one, as it tests the independence between the two components. In this case also if β_1 and β_2 are known, using the similar transformation as before, we will be able to use Fisher's z-test. But if β_1 and β_2 are unknown, we propose to use likelihood ratio test, and the details are avoided.

5 Simulation Results and Data Analysis

5.1 Simulation Results: Performances of MLEs and MBEs

In this section we perform some simulation experiments mainly to observe the behavior of the MLEs and MBEs, as proposed in Section 4, based on their average values and mean squared errors (MSEs) for different sample sizes and for different ρ values.

In all these cases we have kept α's and β's fixed. Based on the algorithm provided in Section 3, we have generated a sample of size n, from a BLBS$(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$ for a given $\alpha_1, \beta_1, \alpha_2, \beta_2$ and ρ. We further compute the MLEs and MBEs of the unknown parameters based on that random sample. We replicate the process 1000 times and report the average estimates and associated mean squared errors (MSEs). The results are reported in Tables 1 and 2. We have performed simulation experiments even for negative values of ρ also, but the results are very similar, hence they are not reported here.

Some of the points are quite clear from this simulation experiments. It is observed that for both the methods the biases of the estimators are quite small even for small sample sizes and the MSEs in all these cases decrease as sample size increases. The biases and MSEs of the two different estimators of α's and β's do not change significantly for different values of ρ, but the MSEs of ρ decrease as ρ increases. It is interesting to observe that the performance of the MBEs are also quite satisfactory. The average biases and MSEs of the MLEs and MBEs are very similar for α's and ρ, but the MSEs of β's are significantly larger for MBEs than MLEs.

Table 1: Average estimates and the MSEs (reported within brackets below) of the MLEs and MBEs for different sample sizes when $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$ and $\rho = 0.25$

	n	α_1	β_1	α_2	β_2	ρ
MLE	25	0.9748	0.9767	1.0067	1.0021	0.2451
		(0.0219)	(0.0201)	(0.0132)	(0.0130)	(0.0353)
	50	0.9837	0.9817	1.0019	1.0011	0.2503
		(0.0110)	(0.0104)	(0.0098)	(0.0099)	(0.0184)
	75	0.9889	0.9867	1.0024	1.0011	0.2505
		(0.0070)	(0.0067)	(0.0079)	(0.0078)	(0.0122)
	100	0.9930	0.9897	1.0053	1.0015	0.2506
		(0.0051)	(0.0050)	(0.0093)	(0.0064)	(0.0096)
MBE	25	0.9901	0.9925	0.9875	0.9765	0.2483
		(0.0243)	(0.0220)	(0.0616)	(0.0546)	(0.0346)
	50	0.9923	0.9907	1.0175	1.0178	0.2462
		(0.0115)	(0.0108)	(0.0323)	(0.0318)	(0.0182)
	75	0.9960	0.9934	0.9949	0.9948	0.2480
		(0.0072)	(0.0068)	(0.0214)	(0.0196)	(0.0120)
	100	0.9983	0.9949	1.0140	1.0103	0.2486
		(0.0052)	(0.0052)	(0.0164)	(0.0162)	(0.0095)

5.2 Simulation Results: Size and Power of the Different Asymptotic Tests

In this section we perform some simulation experiments mainly to observe the behavior of the two likelihood ratio tests (Test II and Test III), as proposed in Section 4, based on their size and power for different sample sizes and for different ρ values. In all cases we have considered 5% level of significance.

In case of testing (39), in our simulation experiments we have taken $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = 1$ in all the cases. We have reported the power of the likelihood ratio test based on 10000 replications for different β_2, n and ρ. We have reported the results in Tables 3 and 4. Similarly, in case of testing (44), we have considered $\beta_1 = \beta_2 = 1$ and $\alpha_1 = 1$ in all the cases. In this case also, we have reported the power of the likelihood ratio test based on

Table 2: Average estimates and the MSEs (reported within brackets below) of the MLEs and MBEs for different sample sizes when $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$ and $\rho = 0.75$

	n	α_1	β_1	α_2	β_2	ρ
MLE	25	0.9784	0.9802	1.0056	1.0023	0.7498
		(0.0220)	(0.0208)	(0.0115)	(0.0113)	(0.0081)
	50	0.9846	0.9832	1.0012	1.0001	0.7485
		(0.0110)	(0.0107)	(0.0088)	(0.0088)	(0.0042)
	75	0.9893	0.9878	1.0018	1.0007	0.7490
		(0.0070)	(0.0068)	(0.0074)	(0.0072)	(0.0027)
	100	0.9931	0.9909	1.0042	1.0019	0.7492
		(0.0051)	(0.0050)	(0.0063)	(0.0059)	(0.0021)
MBE	25	0.9938	0.9948	0.9839	0.9769	0.7352
		(0.0245)	(0.0232)	(0.0574)	(0.0536)	(0.0090)
	50	0.9933	0.9921	1.0162	1.0152	0.7403
		(0.0114)	(0.0110)	(0.0326)	(0.0322)	(0.0046)
	75	0.9960	0.9945	0.9950	0.9961	0.7438
		(0.0073)	(0.0070)	(0.0214)	(0.0207)	(0.0028)
	100	0.9982	0.9962	1.0132	1.0131	0.7454
		(0.0053)	(0.0051)	(0.0161)	(0.0167)	(0.0022)

10000 replications for different α_2, n and ρ. We have reported the results in Tables 5 and 6. From the Tables 3 and 4 (5 and 6), it is clear that as the sample size increases and $\beta_2 - \beta_1$ ($\alpha_2 - \alpha_1$) increases the power of the test increases. In case of Test II, it is observed that the power of the test decreases significantly if ρ increases, where as in case of Test III, the power of the test does not depend on ρ that much. In all the cases, the size of the tests are quite close to the nominal value.

5.3 Data Analysis

In this section we provide the analysis of a data set for illustrative purposes. The data set represents two different measures of stiffness of each of 30 boards. The first measurement involves sending a shock wave down the board, and the second measurement is determined

Table 3: Size and power of the likelihood ratio test for testing $H_0 : \beta_1 = \beta_2$ vs. $H_1 : \beta_1 \neq \beta_2$ (Test II), for different n and β_2, when $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 1$ and $\rho = 0.25$

n	$\beta_2 = 1$	$\beta_2 = 2$	$\beta_2 = 4$	$\beta_2 = 6$	$\beta_2 = 8$	$\beta_2 = 10$
10	0.06	0.12	0.48	0.70	0.81	0.86
15	0.06	0.13	0.54	0.77	0.86	0.91
20	0.06	0.13	0.59	0.83	0.91	0.95
25	0.05	0.14	0.64	0.86	0.93	0.96
30	0.05	0.14	0.67	0.89	0.95	0.97

Table 4: Size and powers of the likelihood ratio test for testing $H_0 : \beta_1 = \beta_2$ vs. $H_1 : \beta_1 \neq \beta_2$ (Test II), for different n and β_2, when $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 1$ and $\rho = 0.75$

n	$\beta_2 = 1$	$\beta_2 = 2$	$\beta_2 = 4$	$\beta_2 = 6$	$\beta_2 = 8$	$\beta_2 = 10$
10	0.03	0.07	0.10	0.15	0.18	0.21
15	0.03	0.07	0.12	0.16	0.20	0.23
20	0.04	0.08	0.12	0.18	0.21	0.25
25	0.05	0.08	0.15	0.20	0.23	0.25
30	0.05	0.09	0.17	0.22	0.25	0.26

while vibrating the board. The data set is originally from William Galligan, and it has been reported in Johnson and Wichern [9]. The data set is presented in Table 7 for convenience.

Preliminary data analysis indicates that the marginals are symmetric, and Johnson and Wichern [9] used normal distribution to fit this data set. We divide all the data points by 1000 for computational purposes. It is not going to make any difference in the inferential issues. We fit the bivariate log Birnbaum-Saunders distribution to the scaled data set. We obtain the estimates of the different unknown using the method suggested in the previous section. Before progressing further, first we plot the profile log-likelihood function of β_1 and β_2 as defined in (33) is provided in Figure 2.

It is clear that the profile log-likelihood function has the unique maximum. We obtain

Table 5: Size and power of the likelihood ratio test for testing $H_0 : \alpha_1 = \alpha_2$ vs. $H_1 : \alpha_1 \neq \alpha_2$ (Test III), for different n, and α_2, when $\beta_1 = \beta_2 = 1$, $\alpha_1 = 1$ and $\rho = 0.25$

n	$\alpha_2 = 1$	$\alpha_2 = 2$	$\alpha_2 = 4$	$\alpha_2 = 6$	$\alpha_2 = 8$	$\alpha_2 = 10$
10	0.06	0.25	0.75	0.95	0.96	0.97
15	0.06	0.35	0.79	0.95	0.96	0.98
20	0.06	0.42	0.82	0.96	0.97	0.99
25	0.05	0.48	0.84	0.96	0.98	1.00
30	0.05	0.55	0.85	0.98	0.99	1.00

Table 6: Size and power of the likelihood ratio test for testing $H_0 : \alpha_1 = \alpha_2$ vs. $H_1 : \alpha_1 \neq \alpha_2$ (Test III), for different n, and α_2, when $\beta_1 = \beta_2 = 1$, $\alpha_1 = 1$ and $\rho = 0.75$

n	$\alpha_2 = 1$	$\alpha_2 = 2$	$\alpha_2 = 4$	$\alpha_2 = 6$	$\alpha_2 = 8$	$\alpha_2 = 10$
10	0.07	0.32	0.81	0.95	0.96	0.99
15	0.07	0.33	0.81	0.95	0.97	0.99
20	0.06	0.35	0.83	0.96	0.97	1.00
25	0.06	0.37	0.85	0.96	0.99	1.00
30	0.05	0.39	0.87	0.97	0.99	1.00

the initial estimates of α_1, α_2, β_1, β_2 and ρ as: 0.3308, 0.3294, 6.4430, 5.3656 and 0.9159, respectively. Using these initial estimates, we obtain the MLEs as

$$\widehat{\alpha}_1 = 0.3269, \quad \widehat{\alpha}_2 = 0.3201, \quad \widehat{\beta}_1 = 6.7819, \quad \widehat{\beta}_2 = 5.7989, \quad \widehat{\rho} = 0.9163, \tag{47}$$

and the corresponding log-likelihood value is 95.9133. The associated 95% bootstrap confidence intervals are (0.3053, 0.3365), (0.3019, 0.3263), (4.1123,9.4139), (3.9856,8.3866), (0.8434,0.9865), respectively.

We perform the following testing of hypothesis:

$$H_0 : \rho = 0 \quad \text{vs.} \quad H_1 : \rho \neq 0.$$

This is an important problem, since under the null hypothesis the model reduces to an independent model. Moreover, this test will reveal (or not) that the new model is better than

Table 7: Stiffness data set.

Obs. No.	Y_1	Y_2	Obs. No.	Y_1	Y_2	Obs. No.	Y_1	Y_2
1.	1889	1651	11.	2403	2048	21.	2119	1700
2.	1645	1627	12.	1976	1916	22.	1712	1712
3.	1943	1685	13.	2104	1820	23.	2983	2794
4.	1745	1600	14.	1710	1591	24.	2046	1907
5.	1840	1841	15.	1867	1685	25.	1859	1649
6.	1954	2149	16.	1325	1170	26.	1419	1371
7.	1828	1634	17.	1725	1594	27.	2276	2189
8.	1899	1614	18.	1633	1513	28.	2061	1867
9.	1856	1493	19.	1727	1412	29.	2168	1896
10.	1655	1675	20.	2326	2301	30.	1490	1382

the independent bivariate sinh-normal model introduced by Diaz-Garcia and Dominguez-Molina [6]. We perform the likelihood ratio test. The MLEs of α_1, α_2, β_1 and β_2 under H_0 become 0.3233, 0.3210, 6.7679, 5.7879, respectively and the corresponding log-likelihood value becomes 68.4476. Therefore, the value of the test statistic is 2(95.9133 - 68.4476) = 54.9314, and the associated $p < 0.0001$. Hence, we reject the null hypothesis.

Now the natural question is how good the model fits the data. We use the following result: If $(Y_1, Y_2) \sim \text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$ then $(Z_1, Z_2) \sim \text{BN}(\rho)$, where

$$Z_1 = \frac{1}{\alpha_1}\left(\sqrt{\frac{e^{Y_1}}{\beta_1}} - \sqrt{\frac{\beta_1}{e^{Y_1}}}\right) \quad \text{and} \quad Z_2 = \frac{1}{\alpha_2}\left(\sqrt{\frac{e^{Y_2}}{\beta_2}} - \sqrt{\frac{\beta_2}{e^{Y_2}}}\right).$$

Hence, if $\{(y_{1i}, y_{2i}); i = 1, \ldots, n\}$ is a random sample from $\text{BLBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$, then

$$T = n(\bar{z}_1, \bar{z}_2)\begin{bmatrix} 1 & \widehat{\rho} \\ \widehat{\rho} & 1 \end{bmatrix}^{-1}\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix},$$

follows asymptotically chi-square distribution with 2 degrees of freedom. Here

$$\bar{z}_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\widehat{\alpha}_1}\left(\sqrt{\frac{e^{y_{1i}}}{\widehat{\beta}_1}} - \sqrt{\frac{\widehat{\beta}_1}{e^{y_{1i}}}}\right) \quad \text{and} \quad \bar{z}_2 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\widehat{\alpha}_2}\left(\sqrt{\frac{e^{y_{2i}}}{\widehat{\beta}_2}} - \sqrt{\frac{\widehat{\beta}_2}{e^{y_{2i}}}}\right).$$

In this case we obtain $T = 0.0134$ and the associated $p = 0.99$. It indicates that the proposed model provides a very good fit to the data set.

6 Bivariate Log-Linear Model

In this section we suggest how the proposed model can be used as a bivariate log-linear model. Rieck and Nedelman [27] proposed a univariate log-linear model using the Birnbaum-Saunders distribution. In this section we will extend the univariate model proposed by Rieck and Nedelman [27] to the bivariate case.

Suppose $(T_{11}, T_{21}), (T_{12}, (T_{22}), \ldots, (T_{1n}, T_{2n})$ are independent random variables, where $(T_{1i}, T_{2i}) \sim \text{BVBS}(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$. The distribution of T_{1i} is assumed to depend on a set of p explanatory variables say $\mathbf{x}_i^1 = (x_{i1}^1, \ldots, x_{ip}^1)$ as follows

$$\beta_1^i = \exp(\mathbf{x}_i^{1T}\boldsymbol{\zeta}) = \exp(x_{i1}^1\zeta_1 + \ldots + x_{ip}^1\zeta_p); \quad \text{for} \;\; i = 1, \ldots, n,$$

where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_p)$ is a vector of p unknown parameters. The shape parameter α_1 is independent of the covariates $\boldsymbol{x}_i^1$. Similarly it is assumed that the distribution of T_{2i} depends on a set of q explanatory variables say $\mathbf{x}_i^2 = (x_{i1}^2, \ldots, x_{iq}^2)$ as follows

$$\beta_2^i = \exp(\mathbf{x}_i^{2T}\boldsymbol{\eta}) = \exp(x_{i1}^2\eta_1 + \ldots + x_{ip}^2\eta_q); \quad \text{for} \;\; i = 1, \ldots, n,$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_q)$ is a vector of q unknown parameters. The shape parameter α_2 is independent of the covariates $\boldsymbol{x}_i^2$.

Therefore, if we assume $(Y_{1i}, Y_{2i}) = (\ln T_{1i}, \ln T_{2i})$, for $i = 1, \ldots, n$, then we have

$$\begin{pmatrix} Y_{1i} \\ Y_{2i} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_i^{1T}\boldsymbol{\zeta} \\ \mathbf{x}_i^{2T}\boldsymbol{\eta} \end{pmatrix} + \begin{pmatrix} \epsilon_{1i} \\ \epsilon_{2i} \end{pmatrix}, \tag{48}$$

here $(\epsilon_{1i}, \epsilon_{2i}) \sim \text{BLBS}(\alpha_1, 1, \alpha_2, 1, \rho)$, and they are independently distributed. Note that the proposed model generalizes the independent bivariate log-Birnbaum-Saunders regression model proposed by Lemonte [15].

This model may be used quite effectively for paired data when covariates are present. It may be useful for analyzing any bivariate survival data also in presence of covariates. The problem of interest is to estimate the unknown parameters namely $\alpha_1, \alpha_2, \boldsymbol{\zeta}, \boldsymbol{\eta}$ and ρ based on a random sample of size n, *i.e.* $\{(y_{1i}, y_{2i}, \mathbf{x}_i^1, \mathbf{x}_i^2); i = 1, \ldots, n\}$. The log-likelihood function of the observed sample (without the additive constant), can be written as follows:

$$\begin{aligned} l(\alpha_1, \alpha_2, \boldsymbol{\zeta}, \boldsymbol{\eta}, \rho) &= -n(\ln \alpha_1 + \ln \alpha_2) + \sum_{i=1}^{n} \ln \phi_2 \left(\frac{2}{\alpha_1} z_{1i}(\boldsymbol{\zeta}), \frac{2}{\alpha_2} z_{2i}(\boldsymbol{\eta}); \rho \right) \\ &\quad + \sum_{i=1}^{n} (u_{1i}(\boldsymbol{\zeta}) + u_{2i}(\boldsymbol{\eta})), \end{aligned} \tag{49}$$

here for $i = 1, \ldots, n$,

$$z_{1i} = \sinh\left(\frac{y_{1i} - \boldsymbol{x}_i^{1T} \boldsymbol{\zeta}}{2} \right), \qquad z_{2i} = \sinh\left(\frac{y_{2i} - \boldsymbol{x}_i^{2T} \boldsymbol{\eta}}{2} \right),$$

$$u_{1i} = \ln\cosh\left(\frac{y_{1i} - \boldsymbol{x}_i^{1T} \boldsymbol{\zeta}}{2} \right), \qquad u_{2i} = \ln\cosh\left(\frac{y_{2i} - \boldsymbol{x}_i^{2T} \boldsymbol{\eta}}{2} \right).$$

In this case also, as without covariate case, it can be seen that for fixed $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$, the MLEs of α_1, α_2 and ρ, can be obtained as

$$\widehat{\alpha}_1(\boldsymbol{\zeta}) = \left\{ \frac{4}{n} \sum_{i=1}^{n} z_{1i}^2(\boldsymbol{\zeta}) \right\}^{1/2}, \quad \widehat{\alpha}_2(\boldsymbol{\eta}) = \left\{ \frac{4}{n} \sum_{i=1}^{n} z_{2i}^2(\boldsymbol{\eta}) \right\}^{1/2},$$

and

$$\widehat{\rho}(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \frac{\sum_{i=1}^{n} z_{1i}(\boldsymbol{\zeta}) z_{2i}(\eta)}{\left\{ \frac{4}{n} \sum_{i=1}^{n} z_{1i}^2(\boldsymbol{\zeta}) \right\}^{1/2} \left\{ \frac{4}{n} \sum_{i=1}^{n} z_{2i}^2(\boldsymbol{\eta}) \right\}^{1/2}},$$

respectively. Finally, the MLEs of $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ can be obtained by maximizing the profile log-likelihood function namely,

$$\begin{aligned} l_2(\boldsymbol{\zeta}, \boldsymbol{\eta}) &= -n(\ln \widehat{\alpha}_1(\boldsymbol{\zeta}) + \ln \widehat{\alpha}_2(\boldsymbol{\eta})) + \sum_{i=1}^{n} \ln \phi_2 \left(\frac{2}{\alpha_1} z_{1i}(\boldsymbol{\zeta}), \frac{2}{\alpha_2} z_{2i}(\boldsymbol{\eta}); \widehat{\rho}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \right) \\ &\quad + \sum_{i=1}^{n} (u_{1i}(\boldsymbol{\zeta}) + u_{2i}(\boldsymbol{\eta})), \end{aligned} \tag{50}$$

with respect to $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$. Efficient numerical optimization method is required to estimate the unknown parameters. It is not an easy problem, more work is needed along that direction.

7 Conclusions

In this paper we have proposed a bivariate log Birnbaum-Saunders distribution. It is a natural extension of the univariate log Birnbaum-Saunders distribution. The model is symmetric, and it has five parameters. It is more flexible than the bivariate normal distribution, because it can be bimodal and it can have heavy tails depending on the parameter values. We have investigated several properties of the distribution. We discuss some inferential issues and perform the analysis of a data set using the model for illustrative purposes. Finally we have proposed the bivariate log-linear model based on bivariate log Birnbaum-Saunders distribution. There are several open issues related to the bivariate log-linear model. It may be mentioned that along the same line p-variate log Birnbaum-Saunders distribution can be defined. Work is in progress, and it will be reported later.

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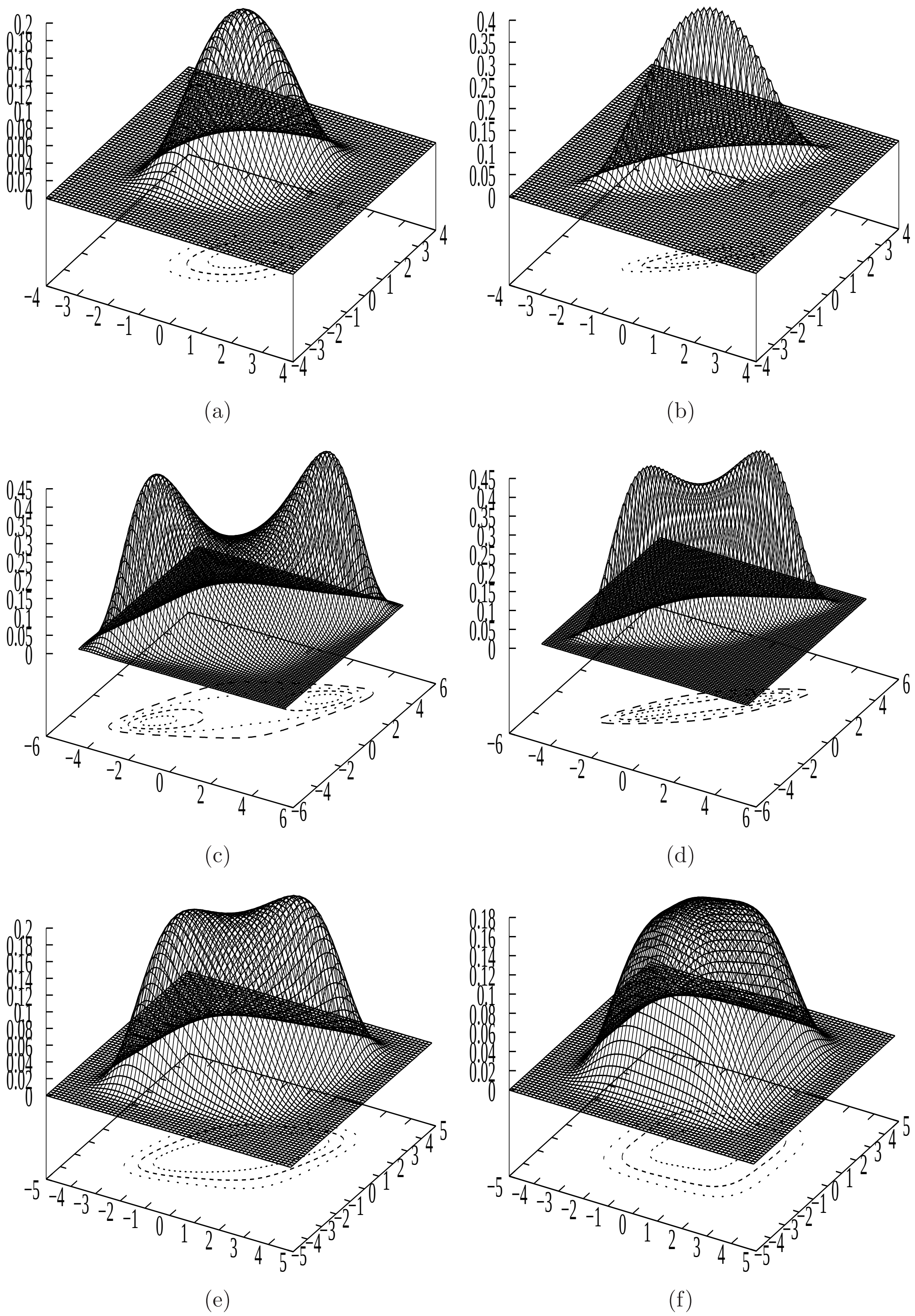

Figure 1: Contour plots of the of the joint PDF of the BLBS distribution for different parameter values: $(\alpha_1, \beta_1, \alpha_2, \beta_2, \rho)$: (a) (1,1,1,1,0.5) (b) (1,1,1,1,0.9) (c) (3,1,3,1,0.75) (d) (2,1,2,1,0.9) (e)(2,1,2,1,0.5) (f) (2,1,2,1,0.1)

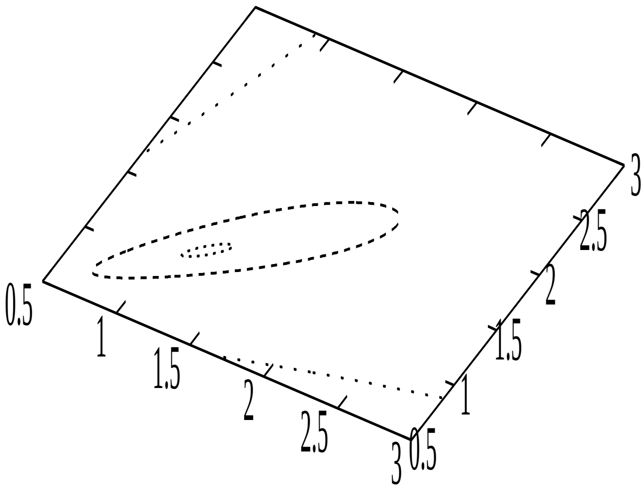

Figure 2: Profile log-likelihood function of β_1 and β_2 for the data set
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