On A Bivariate Pareto Model

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Abstract

Lindley-Singpurwalla (1986)'s bivariate Pareto distribution is one of the most popular bivariate Pareto distribution. Sankaran and Nair (1993) proposed a new bivariate Pareto distribution which also has Pareto marginals and it contains Lindley-Singpurwalla's bivariate Pareto model as a special case. It has several other interesting properties also. In this paper we re-visit Sankaran and Nair's bivariate Pareto model. We discuss several other new properties. The maximum likelihood estimators and two stage estimators are also investigated. We analyze two data sets for illustrative purposes. It is observed that this model can be used quite effectively to analyze competing risks data. Finally we propose some generalizations.

Kew Words and Phrases: Pareto distribution; bivariate hazard rate; copula; maximum likelihood estimators; competing risks.

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1 Introduction

Lindley and Singpurwalla (1986) introduced the bivariate Pareto distribution, which has the following survival function for $\alpha_1 > 0, \alpha_2 > 0, \theta > 0$;

$$P(X_1 > x_1, X_2 > x_2) = (1 + \alpha_1 x_1 + \alpha_2 x_2)^{-\theta}; \quad x_1 > 0, x_2 > 0. \tag{1}$$

From now on, we call this distribution as the Lindley-Singpurwalla bivariate Pareto (LSBP) distribution, and it will be denoted by LSBP$(\alpha_1, \alpha_2, \theta)$. It has Pareto II marginal distributions namely

$$P(X_1 \le x_1) = 1 - (1 + \alpha_1 x_1)^{-\theta} \qquad \text{and} \qquad P(X_2 \le x_2) = 1 - (1 + \alpha_2 x_2)^{-\theta} \tag{2}$$

respectively. Lindley and Singpurwalla (1986) discussed several properties of the above distribution and established different bounds related to reliability. They have also provided some nice theoretical justification of the above model. Since then extensive work has been done related to this distribution by several authors. Interested readers may have a look at Nayak (1987), Barlow and Mendel (1992), Fang and Kotz (1990), Langseth (2007) and see the references cited therein.

Sankaran and Nair (1993) proposed a bivariate Pareto distribution which has the following survival function for $x_1 > 0, x_2 > 0$

$$S_{X_1,X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = (1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0 x_1 x_2)^{-\theta}. \tag{3}$$

Here, $\alpha_1 > 0, \alpha_2 > 0, \theta > 0$ and $0 \le \alpha_0 \le (\theta + 1)\alpha_1\alpha_2$. From now on we call this distribution as the Sankaran-Nair bivariate Pareto (SNBP) distribution, and it will be denoted by SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$.

It is immediate that the LSBP distribution can be obtained as a special case of the SNBP distribution. Sankaran and Nair (1993) discussed different properties of their proposed distribution. The marginals of SNBP distribution also has the Pareto II marginals similar to LSBP distribution. Because of the extra parameter α_0, it can be used more effectively in modeling bivariate reliability or survival data than LSBP model. They discussed different applications of SNBP model in the reliability set up, and finally propose a multivariate generalization. Although, they have discussed different properties and provided different applications of SNBP model, they did not consider any inferential issues related to this problem.

In this paper, we investigate several other properties of the SNBP model. The conditional distribution of X_1 given X_2 can be written as a mixture of a Pareto and a length biased Pareto distribution. Using the conditional distribution, the bivariate samples from SNBP model can be easily generated. It is observed that SNBP model can be obtained as a survival copula. We study several dependency properties and dependency measures of SNBP model. It is observed that SNBP model can be used quite effectively in analyzing competing risks data, and it will be investigated in details. Maximum likelihood method plays an important role to compute efficient estimators of any stochastic model. Unfortunately computing the maximum likelihood estimators (MLEs) involves solving a four dimensional optimization problem. We propose to use two-stage estimation procedure following the approach of Xu (1996), which involves solving a two-dimensional optimization problem only. Theoretical properties of this two-stage estimators can easily be obtained using the approach of Joe (2005). We discuss several inferential issues related to distribution. Two data sets have been analyzed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2, we discuss different properties of SNBP model. Different inferential issues are discussed in Section 3. Analysis of two data sets are presented in Section 4. We analyze a competing risks data using this model in Section 5. We propose some generalizations in Section 6, and finally conclude the paper in Section 7.

2 Properties

2.1 Joint, Marginal and Conditional Density Functions

If (X_1, X_2) follows ($\sim$) SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$, with the survival function (3), then it has the joint probability density function (pdf) for $x_1 > 0, x_2 > 0$;

$$f_{X_1,X_2}(x_1, x_2) = \frac{\theta\left[\theta(\alpha_1 + \alpha_0 x_2)(\alpha_2 + \alpha_0 x_1) + \alpha_1\alpha_2 - \alpha_0\right]}{[1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0 x_1 x_2]^{\theta+2}}, \tag{4}$$

with the same restrictions on the parameters as in (3). It is clear that when $\alpha_0 = 0$, SNBP becomes LSBP model. The joint pdf of (X_1, X_2) as given in (4) can take different shapes. Due to presence of α_0, it is more flexible than LSBP model. It can be easily shown by simple calculus that if $0 \leq \alpha_0 \leq \frac{1}{2}(\theta + 2)\alpha_1\alpha_2$, then the mode of $f_{X_1,X_2}(x_1, x_2)$ will be at (0,0), otherwise it will be different from (0,0). Therefore, it is clear that the joint pdf of LSBP model is always unimodal, but it may not be the case for SNBP model.

It is immediate that if $(X_1, X_2) \sim$ SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$, then for $i = 1, 2$

$$P(X_i \geq x) = (1 + \alpha_i x)^{-\theta}; \quad x_i > 0, \tag{5}$$

the corresponding pdf is

$$f_{X_i}(x) = \frac{\alpha_i\theta}{(1 + \alpha_i x)^{\theta+1}}; \quad x > 0. \tag{6}$$

Therefore, it has the same Pareto II marginals as the LSBP model, with the scale parameter α_i and the shape parameter θ. From now on a Pareto distribution with the scale parameter α and the shape parameter θ will be denoted by P(α, θ). If $(X_1, X_2) \sim$ SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$, the joint distribution function of (X_1, X_2) can be written as

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= P(X_1 \leq x_1, X_2 \leq x_2) \\ &= 1 - (1 + \alpha_1 x_1)^{-\theta} - (1 + \alpha_2 x_2)^{-\theta} + (1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0 x_1 x_2)^{-\theta}. \end{aligned}$$

From (4) it easily follows that for $\alpha_0 = \alpha_1\alpha_2$, then

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2).$$

Therefore, it indicates that X_1 and X_2 are independent. This is one special feature of the SNBP model that independent marginals can be obtained as a special case, which cannot be obtained in case of LSBP model.

The conditional pdf of $X_1|X_2 = x_2$ can be written as

$$\begin{aligned} f_{X_1|X_2=x_2}(x) &= \frac{(1+\alpha_2 x_2)^{\theta+1}}{\alpha_2} \times \frac{(\theta(\alpha_1+\alpha_0 x_2)(\alpha_2+\alpha_0 x) + \alpha_1\alpha_2 - \alpha_0)}{(1+\alpha_1 x + \alpha_2 x_2 + \alpha_0 x x_2)^{\theta+2}} \\ &= p f_U(x;\beta,\theta+1) + (1-p) f_V(x;\beta,\theta+1); \quad 0 < p < 1. \end{aligned}$$

Here U and V are Pareto and length biased Pareto distributions with the following pdfs for $x > 0$

$$f_U(x;\beta,\theta+1) = \frac{\beta(\theta+1)}{(1+\beta x)^{\theta+2}}, \qquad f_V(x;\beta,\theta+1) = \frac{\beta^2\theta(\theta+1)x}{(1+\beta x)^{\theta+2}}, \tag{7}$$

respectively. Here,

$$p = \frac{\theta\alpha_0\alpha_2 x_2 + \alpha_1\alpha_2(1+\theta) - \alpha_0}{\alpha_2(1+\theta)(\alpha_1+\alpha_0 x_2)}, \quad \text{and} \quad \beta = \frac{\alpha_1+\alpha_0 x_2}{1+\alpha_2 x_2}.$$

Therefore, it is immediate that the conditional pdf of $X_1|X_2 = x_2$ is a convex combination of a Pareto and a length biased Pareto distribution. Therefore, all the moments can be easily calculated. Moreover, it may be observed that the conditional survival function of X_1 given $X_2 = x_2$ can be written in a convenient form as

$$P(X_1 \geq x|X_2 = x) = pS_U(x;\beta,\theta+1) + (1-p)S_V(x;\beta,\theta+1), \tag{8}$$

where S_U and S_V are the survival functions of U and V respectively, *i.e.*

$$S_U(x;\beta,\theta+1) = \frac{1}{(1+x\beta)^{\theta+1}} \quad \text{and} \quad S_V(x;\beta,\theta+1) = \frac{x\beta\theta}{(1+x\beta)^{\theta+1}} + \frac{1}{(1+x\beta)^{\theta}}.$$

Now we will describe how to generate samples from a SNBP distribution. Since Pareto distribution has an explicit survival function and the inversion is straight forward, the generation from a Pareto distribution is immediate. Now to generate samples from a length biased Pareto distribution note that for $x > 0$

$$f_V(x;\beta,\theta+1) = \frac{\beta^2\theta(\theta+1)x}{(1+\beta x)^{\theta+2}} = (\theta+1) \times \frac{\beta x}{1+\beta x} \times \frac{\theta}{(1+\beta x)^{\theta+1}} \leq (\theta+1) f_U(x;\beta,\theta). \quad (9)$$

Therefore, using the acceptance-rejection method from Pareto distribution, length biased Pareto random variate can be easily generated. Since X_2 also has the Pareto distribution, the generation from SNBP can be easily carried out.

Now we discuss the correlation coefficient between X_1 and X_2. It is clear that when $\alpha_0 = \alpha_1\alpha_2$, the correlation coefficient ρ between X_1 and X_2 is 0, otherwise it cannot be obtained in explicit form. In Figure 1 we provide the correlation coefficient of X_1 and X_2, for different values of θ and α_0, when $\alpha_1 = \alpha_2 = 1$. Some of the interesting points are as

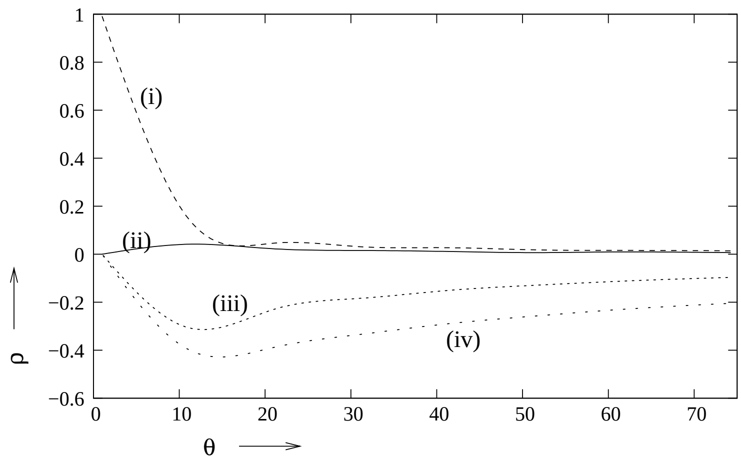

Figure 1: The correlation coefficient ρ of X_1 and X_2 as a function θ, when (i) $\alpha_0 = 0$, (ii) $\alpha_0 = 0.5$, (iii) $\alpha_0 = 10.0$, (iv) $\alpha_0 = 25$.

follows. When $\alpha_0 < \alpha_1\alpha_2$, $\rho > 0$ and for $\alpha_0 > \alpha_1\alpha_2$, $\rho < 0$. When $\alpha_0 = 0$, $\rho \uparrow 1$, as $\theta \downarrow 0$, and $\rho \downarrow 0$ and $\theta \uparrow \infty$. For $0 < \alpha_0 < \alpha_1\alpha_2$, $\rho \downarrow 0$, as $\theta \downarrow 0$, and $\rho \downarrow 0$ and $\theta \uparrow \infty$. Moreover, for $\alpha_0 > \alpha_1\alpha_2$, $\rho \uparrow 0$, as $\theta \downarrow 0$, and $\rho \uparrow 0$ and $\theta \uparrow \infty$. When $\alpha_0 > \alpha_1\alpha_2$, ρ reaches a minimum as θ varies from 0 to ∞. This minimum decrease as α_0 increases. It is our conjecture that as $\theta \uparrow \infty$, the minimum should decreases to -1. We could not show it in the Figure 1, as for very large θ the computation of the correlation becomes a computationally challenging problem. Another important point should be pointed out that although for LSBP model only positive correlation between X_1 and X_2 can be achieved, for SNBP model a wide range of correlation (both positive and negative) between X_1 and X_2 can be obtained.

It may be mentioned that the Shannon entropy of SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$ cannot be obtained in explicit form, although it can be obtained explicitly in case of LSBP$(\alpha_1, \alpha_2, \theta)$, and it is as given below:

$$H(\alpha_1, \alpha_2, \theta) = -\ln\theta - \ln(\theta+1) - \ln\alpha_1 - \ln\alpha_2 - \frac{\theta+2}{\theta} - \frac{\theta+2}{\theta+1}. \tag{10}$$

2.2 Ageing Properties and Bivariate Hazard Gradient

If $(X_1, X_2) \sim$ SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$, then for $\alpha_0 \leq \alpha_1\alpha_2$, it can be easily shown that

$$\frac{P(X_1 > x_1 + t, X_2 > x_2 + t)}{P(X_1 > x_1, X_2 > x_2)} \tag{11}$$

increases in x_1 and x_2 for $t > 0$. Therefore, in this case (X_1, X_2) has the multivariate decreasing failure rate (MDFR) property. Moreover, in this case for all $t, x_1, x_2 > 0$

$$P(X_1 > x_1 + t, X_2 > x_2 + t) \geq P(X_1 \geq x_1, X_2 \geq x_2) \times P(X_1 \geq t, X_2 \geq t). \tag{12}$$

Therefore, in this case (X_1, X_2) satisfies the new worse than used (MNWU) property.

Johnson and Kotz (1975) defined the bivariate hazard gradient as follows:

$$h_{X_1,X_2}(x_1, x_2) = \left(-\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}\right) \ln P(X_1 > x_1, X_2 > x_2). \tag{13}$$

If $(X_1, X_2) \sim$ SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$, then for all values of $x_1 > 0, x_2 > 0$, both the components of $h_{X_1,X_2}(x_1, x_2)$ are decreasing functions of x_1 and x_2.

2.3 Ordering and Total Positivity

If $(X_1, X_2) \sim \text{SNBP}(\alpha_0, \alpha_1, \alpha_2, \theta)$ and $(Y_1, Y_2), \sim \text{SNBP}(\beta_0, \beta_1, \beta_2, \xi)$, and $\alpha_0 \geq \beta_0$, $\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2$, $\theta \geq \xi$, then by using simple algebra it follows that for $u_1 > 0, u_2 > 0$

$$P(X_1 \geq u_1, X_2 \geq u_2) \leq P(Y_1 \geq u_1, Y_2 \geq u_2).$$

Therefore, $(X_1, X_2) \leq_{st} (Y_1, Y_2)$, *i.e.*, (X_1, X_2) is less than (Y_1, Y_2) in the usual stochastic ordering.

It is well known, see Marshall and Olkin (1979), that a random vector (X_1, X_2) has a total positivity of order two (TP_2) property, if and only if

$$S_{X_1,X_2}(x_1, x_2) S_{X_1,x_2}(y_1, y_2) \leq S_{X_1,X_2}(u_1, u_2) S_{X_1,X_2}(v_1, v_2) \tag{14}$$

where

$$u_1 = \max\{x_1, y_1\}, \quad u_2 = \max\{x_2, y_2\}, \quad v_1 = \min\{x_1, y_1\}, \quad v_2 = \min\{x_2, y_2\}.$$

If $(X_1, X_2) \sim \text{SNBP}(\alpha_0, \alpha_1, \alpha_2, \theta)$ and $\alpha_0 \leq \alpha_1\alpha_2$, then considering all possible cases namely (i) $x_1 < x_2 < y_1 < y_2$, (ii) $x_1 < y_1 < x_2 < y_2$, etc., it can be shown that if $(X_1, X_2) \sim \text{SNBP}(\alpha_0, \alpha_1, \alpha_2, \theta)$ and $\alpha_0 \leq \alpha_1\alpha_2$, then (X_1, X_2) satisfies (14). Therefore, (X_1, X_2) has TP_2 property.

2.4 Dependence

Several notions of positive and negative dependence for multivariate distributions of varying degree of strengths are available in the literature, see for example Colangelo, Hu and Shaked (2008), Joe (1997), Balakrishnan and Lai (2009) and the references cited therein.

A random vector (X_1, X_2) is said to be positive upper orthant dependent if for all $x_1 > 0$ and $x_2 > 0$

$$P(X_1 \geq x_1, X_2 \geq x_2) \geq P(X_1 \geq x_1) P(X_2 \geq x_2). \tag{15}$$

If $(X_1, X_2) \sim \text{SNBP}(\alpha_0, \alpha_1, \alpha_2, \theta)$, then for $\alpha_0 \leq \alpha_1\alpha_2$, (X_1, X_2) satisfies (15). Therefore, (X_1, X_2) is positive upper orthant dependent.

A random vector (X_1, X_2) is said to have right tail increasing property if for $i \neq j$

$$P(X_i > x_i | X_j > x_j) \tag{16}$$

is a non-decreasing in x_j for all $x_i > 0$. If $(X_1, X_2) \sim \text{SNBP}(\alpha_0, \alpha_1, \alpha_2, \theta)$, then for $\alpha_0 \leq \alpha_1\alpha_2$, (X_1, X_2) satisfies (16). Therefore, (X_1, X_2) has right tail increasing property.

Another bivariate dependence notion is the bivariate right corner set increasing (RCSI). A random vector (X_1, X_2) is said to have RCSI property if

$$P(X_1 > x_1, X_2 > x_2 | X_1 > \widetilde{x}_1, X_2 \geq \widetilde{x}_2) \tag{17}$$

increases in $\widetilde{x}_1, \widetilde{x}_2$ for every choice of (x_1, x_2). If $(X_1, X_2) \sim \text{SNBP}(\alpha_0, \alpha_1, \alpha_2, \theta)$, then for $\alpha_0 \leq \alpha_1\alpha_2$, (X_1, X_2) satisfies (17). Therefore, (X_1, X_2) has RCSI property.

2.5 Defining Through Copula

Note that the SNBP distribution can be obtained using the copula function also. To every bivariate distribution function F_{X_1,X_2} with absolute marginal distribution functions F_{X_1} and F_{X_2}, corresponds a unique function $C : [0,1] \times [0,1] \rightarrow [0,1]$, called a copula such that

$$F_{X_1,X_2}(x_1, x_2) = C\{F_{X_1}(x_1), F_{X_2}(x_2)\}, \quad \text{for} \quad (x_1, x_2) \in (-\infty, \infty) \times (-\infty, \infty). \tag{18}$$

Conversely, it is possible to construct a bivariate distribution function having the desired marginal distributions and a chosen dependence structure, *i.e.* copula. For a given copula C, there exists a unique survival copula $\overline{C}$, such that

$$\overline{C}(u, v) = u + v - 1 + C(1 - u, 1 - v) \tag{19}$$

and

$$S_{X_1,X_2}(x_1,x_2) = \overline{C}(S_{X_1}(x_1), S_{X_2}(x_2)). \tag{20}$$

Here S_{X_1,X_2}, S_{X_1}, S_{X_2} are the survival functions of F_{X_1,X_2}, F_{X_1} and F_{X_2} respectively, see Nelsen (2006) for details.

Let us consider the following survival copula

$$\overline{C}(u,v) = \left[u^{-1/\theta} + v^{-1/\theta} - 1 + \delta(u^{-1/\theta} - 1)(v^{-1/\theta} - 1)\right]^{-\theta}, \tag{21}$$

for $0 \le \delta \le (1+\theta)$ and $\theta > 0$. Then for the survival functions

$$S_{X_1}(x_1) = (1+\alpha_1 x_1)^{-\theta};\ x_1 > 0, \quad \text{and} \quad S_{X_2}(x_2) = (1+\alpha_2 x_2)^{-\theta};\ x_2 > 0,$$

and for the survival copula (21), the joint survival function of X_1 and X_2 becomes

$$S_{X_1,X_2}(x_1,x_2) = (1 + \alpha_1 x_1 + \alpha_2 x_2 + \delta(\alpha_1 x_1)(\alpha_2 x_2))^{-\theta}. \tag{22}$$

It is immediate that for $\alpha_0 = \delta\alpha_1\alpha_2$, (22) matches with (3).

2.6 Dependency Measures

In this subsection we explicitly calculate the medial correlation and the bivariate tail dependence using the copula property. The population version of the medial correlation coefficient for a pair (X_1, X_2) of continuous random variables was defined by Blomqvist (1950). If M_{X_1} and M_{X_2} denote the medians of X_1 and X_2 respectively, then M_{X_1,X_2}, the medial correlation coefficient of X_1 and X_2 is

$$M_{X_1,X_2} = P[(X_1 - M_{X_1})(X_2 - M_{X_2}) > 0] - M_{X_1,X_2} = P[(X_1 - M_{X_1})(X_2 - M_{X_2}) < 0].$$

It has been shown by Nelsen (2006) that the median correlation coefficient is a copula property, and $M_{X_1,X_2} = 4C\left(\frac{1}{2},\frac{1}{2}\right)$. Therefore, if (X_1, X_2) follows SNBP distribution, the medial correlation coefficient between X_1 and X_2 is $4\left[2^{1/\theta+1} - 1 + \delta(2^{1/\theta} - 1)^2\right]$

The concept of bivariate tail dependence relates to the amount of dependence in the upper quadrant (or lower quadrant) tail of bivariate distribution, see Joe (1997, page 33). In terms of original random variables X_1 and X_2, the upper tail dependence is defined as

$$\chi = \lim_{z \to 1} P(X_2 \geq F_{X_2}^{-1}(z) | X_1 \geq F_{X_1}^{-1}(z)).$$

Intuitively, the upper tail dependence exists, when there is a positive probability that some positive outliers may occur jointly. if $\chi \in (0,1]$, then X_1 and X_2 is said to be asymptotically dependent, and if $\chi = 0$, they are asymptotically independent. Coles, Hefferman and Tawn (1999) showed using copula function that

$$\chi = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u}.$$

In case of SNBP model it can be shown that $\chi = 0$, *i.e.* X_1 and X_2 are asymptotically independent.

3 Statistical Inference

3.1 Maximum Likelihood Estimation

In this section we describe the maximum likelihood estimators (MLEs) of the unknown parameters of SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$.

Suppose $\{(x_{i1}, x_{i2}); i = 1, \cdots, n\}$ is a random sample of size n from SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$. Based on the random sample, the log-likelihood function becomes;

$$\begin{aligned} l(\alpha_0, \alpha_1, \alpha_2, \theta) &= n \ln \theta + \sum_{i=1}^{n} \ln(\theta(\alpha_1 + \alpha_0 x_{2i})(\alpha_2 + \alpha_0 x_{1i}) + \alpha_1 \alpha_2 - \alpha_0) \\ & \quad -(\theta + 2) \sum_{i=1}^{n} \ln(1 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \alpha_0 x_{1i} x_{2i}). \end{aligned} \quad (23)$$

The MLEs of the unknown parameters can be obtained by maximizing (23) with respect to the unknown parameters α_0, α_1, α_2 and θ, such that $\alpha_1 > 0$, $\alpha_2 > 0$, $\theta > 0$, and $0 \leq$

$\alpha_0 < (\theta+1)\alpha_1\alpha_2$. To compute the MLEs, we need to solve a four dimensional optimization problem.

Alternatively, we propose to use a computationally efficient two-step estimation procedure as suggested by Xu (1996), see also Joe (1997, 2005) in this respect. In this two-step estimation procedure, the first stage involves the maximum likelihood from univariate marginals, and the second stage involves the maximum likelihood estimate of the dependence parameter, keeping the univariate parameters held fixed obtained from the first stage.

Using (18), the joint pdf of X_1 and X_2 can be written as

$$f_{X_1,X_2}(x_1,x_2;\alpha_1,\alpha_2,\theta,\alpha_0) = c(F_{X_1}(x_1;\alpha_1,\theta),F_{X_2}(x_2;\alpha_2,\theta);\alpha)f_{X_1}(x_1;\alpha_1,\theta)f_{X_2}(x_2;\alpha_2,\theta) \tag{24}$$

here $c(\cdot,\cdot)$ is the copula density function obtained from survival copula as follows;

$$c(u,v) = [g_1(u,v;\theta,\alpha_0)\times g_2(u,v;\theta,\alpha_0) - g_3(u,v;\theta,\alpha_0)] \times \frac{1}{\theta^2}(1-u)^{-\frac{1}{\theta}-1}(1-v)^{-\frac{1}{\theta}-1}, \tag{25}$$

where

$$\begin{aligned}
g_1(u,v;\theta,\alpha_0) &= \theta(\theta+1)\left((1-u)^{-\frac{1}{\theta}} + (1-v)^{-\frac{1}{\theta}} - 1 + \delta((1-u)^{-\frac{1}{\theta}}-1)((1-v)^{-\frac{1}{\theta}}-1)\right)^{-(\theta+2)} \\
g_2(u,v;\theta,\alpha_0) &= \left[1+\delta((1-u)^{-\frac{1}{\theta}}-1)\right]\left[1+\delta((1-v)^{-\frac{1}{\theta}}-1)\right] \\
g_2(u,v;\theta,\alpha_0) &= \delta\theta\left((1-u)^{-\frac{1}{\theta}} + (1-v)^{-\frac{1}{\theta}} - 1 + \delta((1-u)^{-\frac{1}{\theta}}-1)((1-v)^{-\frac{1}{\theta}}-1)\right)^{-(\theta+1)},
\end{aligned}$$

and $\delta = \dfrac{\alpha_0}{\alpha_1\alpha_2}$ same as defined before.

Therefore, the log-likelihood function can be written as

$$\begin{aligned}
l(\alpha_1,\alpha_2,\theta,\alpha_0) &= \sum_{i=1}^{n}\ln c(F_{X_1}(x_{1i};\alpha_1,\theta),F_{X_2}(x_{2i};\alpha_2,\theta);\alpha_0) + \sum_{i=1}^{n}\ln f_{X_1}(x_{1i};\alpha_1,\theta) \\
&\quad + \sum_{i=1}^{n}\ln f_{X_1}(x_{1i};\alpha_1,\theta).
\end{aligned} \tag{26}$$

The two-stage procedure works as follows: First obtain the estimates of α_1,α_2 and θ, by maximizing

$$\begin{aligned} g(\alpha_1, \alpha_2, \theta) &= \sum_{i=1}^{n} \ln f_{X_1}(x_{1i}; \alpha_1, \theta) + \sum_{i=1}^{n} \ln f_{X_1}(x_{1i}; \alpha_1, \theta) \\ &= n(\ln \alpha_1 + \ln \alpha_2) + 2n \ln \theta - (\theta + 1) \sum_{i=1}^{n} \left\{ \ln(1 + \alpha_1 x_{1i}) + \ln(1 + \alpha_2 x_{2i}) \right\}. \end{aligned} \tag{27}$$

If $\widehat{\alpha}_1$, $\widehat{\alpha}_2$ and $\widehat{\theta}$ maximize (27), then at the second stage obtain the estimate of α_0 by maximizing

$$h(\alpha_0) = \sum_{i=1}^{n} \ln c(F_{X_1}(x_{1i}; \widehat{\alpha}_1, \widehat{\theta}), F_{X_2}(x_{2i}; \widehat{\alpha}_2, \widehat{\theta}); \alpha_0) \tag{28}$$

with respect to α_0.

Note that for fixed α_1,α_2 the function $g(\alpha_1, \alpha_2, \theta)$ is maximized for

$$\widehat{\theta}(\alpha_1, \alpha_2) = \frac{2n}{\sum_{i=1}^{n} \left\{ \ln(1 + \alpha_1 x_{1i}) + \ln(1 + \alpha_2 x_{2i}) \right\}}. \tag{29}$$

Hence, $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ can be obtained by maximizing $g(\alpha_1, \alpha_2, \widehat{\theta}(\alpha_1, \alpha_2))$, with respect to α_1 and α_2. It involves solving a two-dimensional optimization problem. Therefore, two-stage estimation process involves solving a two-dimensional optimization and one-dimensional optimization problem. It saves significant amount of computational time. Two-stage estimators are consistent estimators of the unknown parameters. The asymptotic distribution of the two-stage estimators can be obtained in a routine manner as it has been obtained in Joe (2005), and it is not pursued here.

3.2 Testing of Hypotheses

We would like to perform two testing of hypotheses problems.

Problem 1: Testing whether X_1 and X_2 are independent or not, can be performed as follows:

$$\mathrm{H}_0 : \alpha_0 = \alpha_1 \alpha_2 \quad vs. \quad \mathrm{H}_1 : \alpha_0 \neq \alpha_1 \alpha_2. \tag{30}$$

Note that the MLEs of α_1, α_2 and θ under H_0 are same as the corresponding two-step estimators. The above testing of hypothesis problem can be performed using the standard likelihood ratio test, which has the asymptotic distribution as follows:

$$2(l_{SNBP}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\theta}, \widehat{\alpha}_0) - l_{SNBP}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\theta})) \longrightarrow \chi^2_1. \tag{31}$$

Problem 2: If we would like to test whether the submodel LNBP can be used to analyze the data, the following test can be performed:

$$\mathrm{H}_0 : \alpha_0 = 0 \quad vs. \quad \mathrm{H}_1 : \alpha_0 > 0. \tag{32}$$

In this case since α_0 is in the boundary under the null hypothesis, the standard results do not work. But using Theorem 3 of Self and Liang (1987), it follows that

$$2(l_{SNBP}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\theta}, \widehat{\alpha}_0) - l_{LSBP}(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\theta})) \longrightarrow \frac{1}{2} + \frac{1}{2}\chi^2_1. \tag{33}$$

4 Data Analysis

In this section we perform the analysis of two data sets (i) simulated data & (ii) real data for illustrative purposes.

Simulated Data:

A sample of size 30 has been generated from a SNBP(1,1,1,1) model, and it is presented in Table 1. We fit the SNBP model to the data set. We use the two step procedure as suggested in Section 3, and we obtain the estimates of α_1, α_2 and θ by maximizing $g(\alpha_1, \alpha_2, \theta)$, and they are

$$\widehat{\alpha}_1 = 0.6081, \quad \widehat{\alpha}_2 = 1.0164, \quad \widehat{\theta} = 1.8219.$$

Now we obtain the estimate of α_0 by argument maximum of $h(x)$. Because of the complicated nature of the function $h(\cdot)$, it is not possible to prove that the function is an unimodal

Table 1: The scores of Probability-I(X_1) and Inference-I(X_2)

No.	X_1	X_2	No.	X_1	X_2	No.	X_1	X_2
1.	0.252	8.400	2.	1.105	0.458	3.	0.427	1.602
4.	12.491	2.383	5.	0.260	0.106	6.	0.240	1.769
7.	4.888	0.758	8.	0.870	0.572	9.	0.036	0.254
10.	1.537	0.023	11.	1.508	0.535	12.	0.239	1.412
13.	0.173	0.011	14.	1.090	1.278	15.	6.002	0.017
16.	0.897	2.032	17.	0.690	0.138	18.	1.883	0.398
19.	0.960	0.257	20.	0.561	0.573	21.	5.370	0.325
22.	0.167	0.260	23.	13.602	0.364	24.	3.922	0.938
25.	0.132	0.547	26.	0.603	0.102	27.	0.226	0.481
28.	0.143	0.779	29.	0.643	0.071	30.	0.349	1.586

function. But Figure 2 shows that it is an unimodal function, and we obtain the estimate of α_0 as $\widehat{\alpha}_0 = 0.7185$. We obtain the bootstrap confidence intervals of α_1, α_2, θ and α_0 as (0.3938,0.8224), (0.5273,1.5055), (0.9328,2.7110), (0.3434,1.0936) respectively. We also

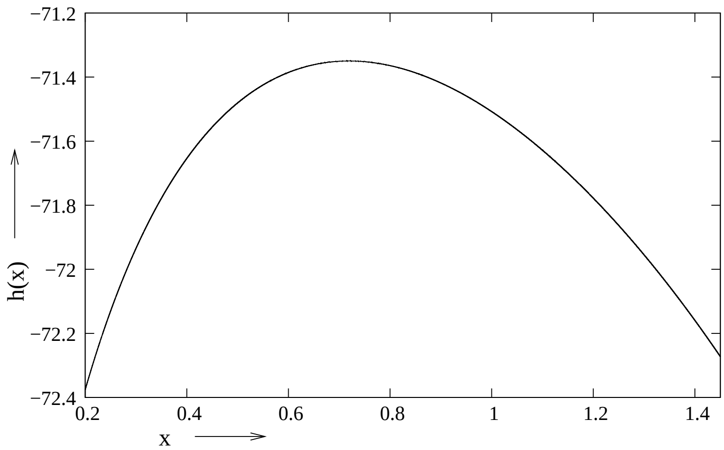

Figure 2: The plot of $h(x)$ for the simulated data set.

compute the MLEs of the unknown parameters by direct maximization of the log-likelihood function, and the estimates are as follows:

$$\widehat{\alpha}_1 = 0.6333, \quad \widehat{\alpha}_2 = 1.0304, \quad \widehat{\theta} = 1.7864, \quad \widehat{\alpha}_0 = 0.7545$$

Table 2: Two different stiffness measurements of 30 boards

No.	Shock	Vibration	No.	Shock	Vibration	No.	Shock	Vibration
1.	1889	1651	2.	2403	2048	3.	2119	1700
4.	1645	1627	5.	1976	1916	6.	1712	1713
7.	1943	1685	8.	2104	1820	9.	2983	2794
10.	1745	1600	11.	1710	1591	12.	2046	1907
13.	1840	1841	14.	1867	1685	15.	1859	1649
16.	1954	2149	17.	1325	1170	18.	1419	1371
19.	1828	1634	20.	1725	1594	21.	2276	2189
22.	1899	1614	23.	1633	1513	24.	2061	1867
25.	1856	1493	26.	1727	1412	27.	2168	1896
28.	1655	1675	29.	2326	2301	30.	1490	1382

The log-likelihood values based on two-stage estimators and the MLEs are -71.353 and -71.349 respectively. It shows that the two stage estimators are very close to the corresponding MLEs.

Stiffness Data

This data set represents the two different measurements of stiffness, 'Shock' and 'Vibration' of each of 30 boards. The first measurement (Shock) involves sending a shock wave down the board and the second measurement (Vibration) is determined while vibrating the board. The data set was originally from William Galligan, and it has been reported in Johnson and Wichern (1992), and for convenience it is presented in Table 2.

We fit the SNBP model to this data set, and the two-step procedure provides the estimates of α_1, α_2 and θ as

$$\widehat{\alpha}_1 = 0.0321, \quad \widehat{\alpha}_2 = 0.0292, \quad \widehat{\theta} = 18.3438.$$

With the above estimated values of α_1, α_2 and θ, $h(x)$ is provided in Figure 3. From the unimodal $h(x)$, we obtain the estimate of α_0 as $\widehat{\alpha}_0 = 0.0154$, and the associated log-likelihood value is -96.5098. The 95% confidence intervals of α_1, α_2, θ and α_0 become (0.0196,0.0446), (0.0181,0.0403), (13.5964,23.0912), (0.0123,0.0185) respectively.

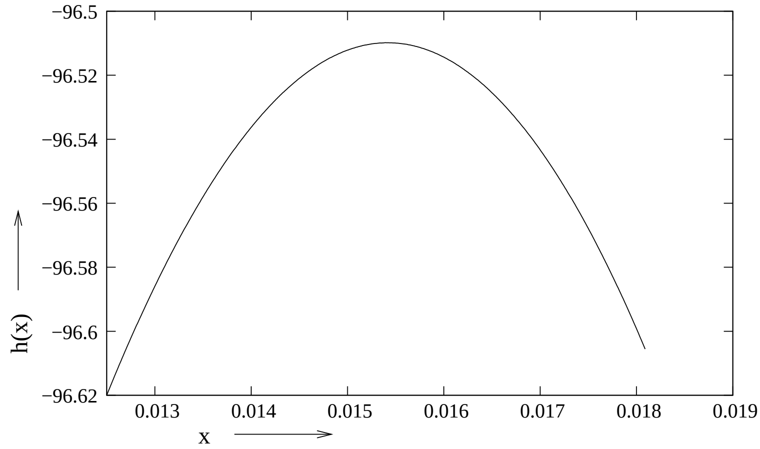

Figure 3: The plot of $h(x)$ for stiffness data set.

We perform the test (30), and obtain the log-likelihood value under the null hypothesis as -96.951, hence the test statistic is 0.884. The associate p value lies between 0.4 and 0.7. Therefore, we cannot reject the null hypothesis. AIC or BIC also suggests that the independent Pareto marginals as the preferred model than the SNBP model.

5 Competing Risks

5.1 Model Assumptions and Estimation

In many life testing experiments, often the failure of items may be associated to more than one cause of failures. In such a case an investigator is often interested with the assessment of a specific risk in presence of other risk factors. In the Statistical literature it is known as the competing risk model. In analyzing the competing risks model, ideally the data consist of a failure time and an indicator denoting the cause of failure. The 'risk factors' in some sense competing with each other for the failure of the experimental unit. An extensive amount of work has been done in the Statistical literature on competing risk model, interested readers may have a look at the excellent monograph by Crowder (2001) in this respect.

A typical competing risk data will be of the form (T, Δ), here T denotes the failure time and Δ denotes the cause of failure. Here $T \geq 0$ is a non-negative random variable and Δ can take values $1 \cdots, m$, where m denotes the total number of cause of failures. To analyze competing risks data, the cause of failure may be assumed to be independent or dependent. Cox (1959) proposed the latent failure time models, and in that case T can be written as $T = \min\{X_1, \cdots, X_m\}$, where $X_1, \cdots, X_m$ denote the latent failure times of m different causes. It is well known that there is an identifiability issue associated with the competing risk model related to the dependence and independence of the failure time models. It has been observed, see for example Crowder (2001), that based on the available competing risk data, it is not possible to test the independence of the latent failure time distributions. Although, in many practical situations, it may not be reasonable to assume that the latent failure times to be independent.

For simplicity, we assume $m = 2$. In case of $m = 2$, an extensive amount of work has been done assuming different bivariate distributions on (X_1, X_2). It is also observed by Basu and Ghosh (1981) that in some cases all the parameters may not be identifiable. In this case it is assumed that (X_1, X_2) has a SNBP$(\alpha_0, \alpha_1, \alpha_2, \theta)$ distribution. Based on the above assumption, the joint pdf of (T, Δ) is

$$
\begin{aligned}
f_{T,\Delta}(t,1) &= f_{X_1}(t) \times P(X_2 \geq t | X_1 = t) \\
&= \frac{\alpha_1 \theta}{(1+\alpha_1 t)^{\theta+1}} \times \left(p_1 \frac{1}{(1+\beta_1 t)^{\theta+1}} + (1-p_1) \left\{ \frac{\beta_1 t \theta}{(1+t\beta_1)^{1+\theta}} + \frac{1}{(1+t\beta_1)^{\theta}} \right\} \right), \\
f_{T,\Delta}(t,2) &= f_{X_2}(t) \times P(X_1 \geq t | X_2 = t) \\
&= \frac{\alpha_2 \theta}{(1+\alpha_2 t)^{\theta+1}} \times \left(p_2 \frac{1}{(1+\beta_2 t)^{\theta+1}} + (1-p_2) \left\{ \frac{\beta_2 t \theta}{(1+t\beta_2)^{1+\theta}} + \frac{1}{(1+t\beta_2)^{\theta}} \right\} \right),
\end{aligned}
$$

here

$$
p_1 = \frac{\theta \alpha_0 \alpha_1 t + \alpha_1 \alpha_2 (1+\theta) - \alpha_0}{\alpha_1 (1+\theta)(\alpha_2 + \alpha_0 t)}, \qquad \beta_1 = \frac{\alpha_2 + \alpha_0 t}{1 + \alpha_1 t}
$$

$$p_2 = \frac{\theta\alpha_0\alpha_2 t + \alpha_1\alpha_2(1+\theta) - \alpha_0}{\alpha_2(1+\theta)(\alpha_1+\alpha_0 t)}, \qquad \beta_2 = \frac{\alpha_1+\alpha_0 t}{1+\alpha_2 t}.$$

Following the same approach as in Basu and Ghosh (1978), it is observed that all the four parameters are identifiable.

Now we would like to consider the maximum likelihood estimation of the unknown parameters, based on the available data. In this paper we consider three possible types of observations which are available to us, namely (i) $(t,1)$, (ii) $(t,2)$ and (iii) $(t,0)$. Here (i) and (ii) mean that the failure has taken place at the time point t due to cause 1 and cause 2 respectively. The observation $(t,0)$ means that the failure has not taken place at t, but it has been censored at t.

It is assumed that we have the following observations;

$$\mathcal{D} = \{(t_i,\delta_i); i = 1,\cdots,n\}. \tag{34}$$

If we denote

$$I_0\{i;\delta_i = 0\}, \qquad I_1 = \{i;\delta_i = 1\}, \qquad I_2 = \{i;\delta_i = 2\},$$

the likelihood function of the observation becomes;

$$l(\mathcal{D}|\alpha_0,\alpha_1,\alpha_2,\theta) = \prod_{i\in I_0} S_T(t_i) \times \prod_{i\in I_1} f_{T,\Delta}(t_i,1) \times \prod_{i\in I_2} f_{T,\Delta}(t_i,2), \tag{35}$$

where

$$S_T(t) = P(\min\{X_1,X_2\} > t) = \frac{1}{(1+\alpha_1 t+\alpha_2 t+\alpha_0 t^2)^{\theta}}; \quad t > 0.$$

Therefore, the MLEs of the unknown parameters can be obtained by maximizing (35) with respect to the unknown parameters. They cannot be obtained explicitly, we need to use some optimization algorithm to solve this maximization problem.

5.2 Data Example

In this section we have taken one real-life data set from Lawless (1982), and will use the SNLP model to analyze the data. The data set consists of failure times or censoring times for 36 appliances subjected to an automated life test. Failures are mainly classified into 18 different modes, though among 33 observed failures only 7 modes are present and only model 6 and 9 appear more than once. We are mainly interested about the failure mode 9. The data consist of two causes of failures, $\delta = 1$ (failure mode 9), $\delta = 2$ (all other failure modes) and $\delta = 0$ indicates that the data are censored at that time point. The data are given below:

Data Set: (11,2), (35,2), (49,2), (170,2), (329,2), (381,2), (708,2) (958,2), (1062,2), (1167,1), (1594,2), (1925,1), (1990,1), (2223,1), (2327,2), (2400,1), (2451,2), (2471,1), (2551,1), (2565,0), (2568,1), (2694,1), (2702,2), (2761,2), (2831,2), (3034,1), (3059,2), (3112,1), (3214,1), (3478,1), (3504,1), (4329,1), (6367,0), (6976,1), (7846,1), (13403,0).

In this case $n = 36$, $|I_0| = 3$, $|I_1| = 17$, $|I_2| = 16$. The MLEs of the unknown parameters are as follows $\widehat{\alpha}_0 = 0.000074$, $\widehat{\alpha}_1 = 0.000340$, $\widehat{\alpha}_2 = 0.00454$, $\widehat{\theta} = 0.43103$ with the log-likelihood value = -276.757. We also fit the LSBP model and obtain the MLEs as $\widehat{\alpha}_1 = 0.00019$, $\widehat{\alpha}_2 = 0.00272$, $\widehat{\theta} = 0.46688$ and the log-likelihood value = -276.929. Therefore, in this case if we want to test the hypothesis

$$H_0 : \text{LNBP}, \quad vs. \quad H_1 : \text{SNBP}$$

we cannot reject the null hypothesis. Similarly AIC or BIC also indicate that LNBP as the preferred model.

6 Some Possible Extensions

Now we will discuss some of the generalizations of the proposed model. First note that although SNBP model has been defined so that its marginals are Pareto distribution, but using the copula structure as it has been mentioned in Section 2, it can be seen that a more general class of distributions can be easily obtained. The following definition will be useful for further development.

Definition: A class of distribution functions $F(\cdot;\theta);\theta > 0$, is called a proportional hazard class if

$$S(x;\theta) = (S_0(x))^{\theta}; \quad x \geq 0,$$

where $S(x;\theta) = 1 - F(x;\theta)$, is the survival function of $F(\cdot;\theta)$ and $S_0(x) = 1 - F_0(x)$ is the survival function of the base distribution function $F_0(\cdot)$.

Note that they are called the proportional hazard class of distributions as the hazard function of any member of this family is proportional to the hazard function of the base distribution function. For example (i) exponential distribution ($S_0(x) = e^{-x}$), (ii) Weibull distribution ($S_0(x) = e^{-x^{\alpha}}; \alpha > 0$), (iii) Burr Type XII distribution ($S_0(x) = (1+x^{\alpha})^{-1}; \alpha > 0$) and (iv) Pareto distribution ($S_0(x) = (1+\alpha x)^{-1}; \alpha > 0$) are proportional hazard class of distributions.

Now more general class of distributions can be obtained by replacing $S_{X_1}(\cdot)$ and $S_{X_2}(\cdot)$ as proportional hazard class of distribution functions. Suppose for $x > 0$,

$$S_{X_1}(x;\theta) = (S_1(x))^{\theta} \quad \text{and} \quad S_{X_2}(x;\theta) = (S_2(x))^{\theta},$$

where $S_1(\cdot)$ and $S_2(\cdot)$ are two base distribution functions of the family $S_{X_1}(\cdot;\theta)$ and $S_{X_2}(\cdot;\theta)$

respectively. Now consider the following survival function of X_1 and X_2

$$S_{X_1,X_2}(x_1,x_2) = \left((S_1(x_1))^{-1} + (S_2(x_2))^{-1} - 1 + \delta((S_1(x_1))^{-1} - 1)((S_2(x_2))^{-1} - 1)\right)^{-\theta}, \tag{36}$$

and it will have marginals with survival functions $(S_1(x))^\theta$ and $(S_2(x))^\theta$ respectively.

Bivariate exponential distribution: If we replace the marginals as exponential distributions then we get a new bivariate exponential distribution which has the following bivariate survival function:

$$S_{X_1,X_2}(x_1,x_2) = \left[e^{\lambda_1 x_1} + e^{\lambda_2 x_2} - 1 + \delta\left(e^{\lambda_1 x_1} - 1\right)\left(e^{\lambda_2 x_2} - 1\right)\right]^{-\theta}; \quad x_1 > 0, x_2 > 0, \tag{37}$$

for $\lambda_1 > 0, \lambda_2 > 0, \theta > 0$. This new bivariate exponential distribution has exponential marginals, and due to presence of four parameters it is expected that it will be more flexible than several three-parameter bivariate exponential distributions. Another important point should be point out that although most of the available bivariate exponential distributions available in the literature can have only positive correlation between the two variables, the proposed bivariate exponential distribution can have a wide range of correlation (both positive and negative) between X_1 and X_2. Therefore, it is expected that the proposed model will be more useful for data analysis purposes than the existing bivariate exponential models.

Bivariate Weibull distribution: If we substitute the marginals as the Weibull marginals we obtain a new bivariate Weibull distribution with survival function

$$S_{X_1,X_2}(x_1,x_2) = \left[e^{\lambda_1 x_1^{\alpha_1}} + e^{\lambda_2 x_2^{\alpha_2}} - 1 + \delta\left(e^{\lambda_1 x_1^{\alpha_1}} - 1\right)\left(e^{\lambda_2 x_2^{\alpha_2}} - 1\right)\right]^{-\theta}, \tag{38}$$

for $\lambda_1 > 0, \lambda_2 > 0, \alpha_1 > 0, \alpha_2 > 0, \theta > 0$. It is expected that the above five-parameter bivariate Weibull distribution will be more flexible than the four-parameter Marshall-Olkin bivariate Weibull distribution.

Bivariate Burr Type XII distribution A new bivariate Burr Type XII distribution can be defined with survival function

$$S_{X_1,X_2}(x_1,x_2) = \Big[(1+x_1^{\alpha_1})^{\beta_1} + (1+x_2^{\alpha_2})^{\beta_2} - 1 + \delta\left((1+x_1^{\alpha_1})^{\beta_1} - 1\right)\left((1+x_2^{\alpha_2})^{\beta_2} - 1\right)\Big]^{-\theta}; \quad x_1 > 0, x_2 > 0, \tag{39}$$

for $\alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0, \theta > 0$.

Due to copula structure, many dependency properties and dependency measures of these new bivariate distribution functions will be same as those of SNBP distribution. Many other new properties may be explored. Moreover, in all these cases two-step procedure can be used to compute estimators of the unknown parameters, and their properties can be established.

7 Conclusions

In this paper we re-visited the bivariate Pareto model proposed by Sankaran and Nair (1993). We have established several new properties of this distribution. We have shown that the SNBP model can be obtained using a survival copula, and that helps in establishing several dependency properties of this model. The MLEs of the unknown parameters cannot be obtained in explicit forms, and non-linear optimization problem needs to be solved to compute the MLEs. This model can be used quite effectively for modeling competing risk data also. We have provided several generalizations of the proposed model using copula structure, which are expected to be more flexible than the existing models. More work is needed to establish different properties of these new distribution.

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