# Bivariate Sinh-Normal Distribution and A Related Model

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**Abstract**

Sinh-normal distribution is a symmetric distribution with three parameters. In this paper we introduce bivariate sinh-normal distribution, which has seven parameters. Due to presence of seven parameters it is a very flexible distribution. Different properties of this new distribution has been established. The model can be obtained as a bivariate Gaussian copula also. Therefore, using the Gaussian copula property, several properties of this proposed distribution can be obtained. Maximum likelihood estimators cannot be obtained in closed forms. We propose to use two step estimators based on Copula, which can be obtained in a more convenient manner. One data analysis has been performed to see how the proposed model can be used in practice. Finally, we consider a bivariate model which can be obtained by transforming the sinh-normal distribution and it is a generalization of the bivariate Birnbaum-Saunders distribution. Several properties of the bivariate Birnbaum-Saunders distribution can be obtained as special cases of the proposed generalized bivariate Birnbaum-Saunders distribution.

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# 1 Introduction

Rieck and Nedelman [13], introduced sinh-normal (SHN) distribution, as a three parameter symmetric model with the following cumulative distribution function (CDF)

$$F_Y(y;\alpha,\sigma,\mu) = \Phi(a(y;\alpha,\sigma,\mu)); \quad y \in \mathbb{R}. \tag{1}$$

Here $\Phi(\cdot)$ is the CDF of a standard normal random variable, $\alpha > 0$, $\sigma > 0$, $-\infty < \mu < \infty$, $\sinh(x)$ is the hyperbolic sine function of $x$ where $\sinh(x) = (e^x - e^{-x})/2$, and

$$a(y;\alpha,\sigma,\mu) = \frac{2}{\alpha}\sinh\left(\frac{y-\mu}{\sigma}\right).$$

Note that $a(y;\alpha,\sigma,\mu)$ is an increasing function of $y$, and as $y \to -\infty$, $a(y;\alpha,\sigma,\mu) \to -\infty$, and as $y \to \infty$, $a(y;\alpha,\sigma,\mu) \to \infty$, therefore, $F_Y(\cdot)$ as defined in (1) is a proper distribution function. From now on a three-parameter sinh-normal random variable with CDF (1) will be denoted by SHN$(\alpha,\sigma,\mu)$. It has several important properties, and a brief review of SHN distribution will be provided in Section 2.

The main aim of this paper is to introduce bivariate sinh-normal (BSHN) distribution, which is a natural extension of SHN distribution of one dimension to two dimensions. Recently, Kundu et al. [7] introduced a bivariate Birnbaum-Saunders distribution. It is observed that the bivariate log-Birnbaum-Saunders distribution can be obtained as a special case of the BSHN distribution. The proposed BSHN distribution has seven parameters, and due to presence of two location, two shapes, two scales and one correlation parameters, it is a very flexible model. After proper normalization, as the shape parameters converge to zero, it approaches to a standard bivariate normal distribution.

The probability density function (PDF) of BSHN, can be both unimodal or bimodal depending on the shape parameters. Due to presence of the two shape parameters, it is more flexible than the bivariate normal distribution. The marginals of BSHN are SHN distributions, and the conditional distribution also can be obtained in closed form. The generation from a BSHN distribution is quite straight forward, hence simulation experiments can be performed in a routine matter. The proposed BSHN model can be obtained as a bivariate Gaussian copula, hence different copula properties can be easily incorporated for BSHN distribution. The model has the total positivity of order two ($TP_2$) or reverse rule of order two ($RR_2$) properties depending on the correlation parameter. The maximum likelihood estimators of the unknown parameters cannot be obtained in closed form, as expected. Nonlinear optimization method needs to be used to compute the MLEs. We propose to use two stage estimators as proposed by Joe [6]. One data analysis has been performed for illustrative purposes.

Finally we introduce a new bivariate distribution, which can be obtained by transforming the BSHN random variable. The new distribution is a generalization of the bivariate Birnbaum-Saunders distribution introduced by Kundu et al. [7], and we call it as the bivariate generalized Birnbaum-Saunders distribution. We establish different properties of the generalized Birnbaum-Saunders distribution, and it is observed that several properties of the bivariate Birnbaum-Saunders distribution can be obtained as special cases of the proposed distribution. Finally we indicate multivariate generalization of the proposed models.

Rest of the paper is organized as follows. In Section 2, we provide some preliminaries. The proposed BSHN distribution is defined and its properties are discussed in Section 3. The statistical inference of the unknown parameters are discussed in Section 4. The analysis of a data set is provided in Section 5. In Section 6, we proposes generalized Birnbaum-Saunders distribution, and finally we conclude the paper in Section 6.

# 2 Preliminaries

## 2.1 Sinh-Normal Distribution

Suppose $Y \sim \text{SHN}(\alpha, \sigma, \mu)$, then the CDF of $Y$ is provided in (1), and the corresponding probability density function (PDF) becomes

$$f_Y(y; \alpha, \sigma, \mu) = \phi(a(y; \alpha, \sigma, \mu))A(y; \alpha, \sigma, \mu); \quad y \in \mathbb{R},$$

here $\phi(\cdot)$ is the PDF of a standard normal random variable, and

$$A(y; \alpha, \sigma, \mu) = \frac{d}{dy} a(y; \alpha, \sigma, \mu) = \frac{2}{\alpha\sigma} \cosh\left(\frac{y-\mu}{\sigma}\right) = \frac{e^{(y-\mu)/\sigma} + e^{-(y-\mu)/\sigma}}{\alpha\sigma}.$$

By simple transformation, it follows that if $Y \sim \text{SHN}(\alpha, \sigma, \mu)$, then

$$Z = \frac{2}{\alpha} \sinh\left(\frac{y-\mu}{\sigma}\right) \sim N(0, 1). \tag{2}$$

Here $N(0, 1)$ denotes a standard normal random variable. From (2), it follows that if $Z \sim N(0, 1)$, then

$$Y = \sigma \text{ arcsinh}\left(\frac{\alpha Z}{2}\right) + \mu \sim \text{SHN}(\alpha, \sigma, \mu), \tag{3}$$

where arcsinh $(x) = \ln(x + \sqrt{x^2 + 1})$. Using (3), random samples from SHN distribution can be easily generated. Moreover, from (3), it follows that if $Y \sim \text{SHN}(\alpha, \sigma, \mu)$, then $cY \sim \text{SHN}(\alpha, |c|\sigma, c\mu)$.

Rieck [12] has established the following properties of a SHN distribution. The PDF of SHN is symmetric about the location parameter $\mu$. The distribution is strongly unimodal for $\alpha \leq 2$, and it is bimodal if $\alpha > 2$. It can be easily seen using L′Hospital's rule, that if $Y \sim \text{SHN}(\alpha, \sigma, \mu)$, then

$$\frac{2(Y-\mu)}{\alpha\sigma} \rightarrow N(0, 1) \quad \text{as} \quad \alpha \rightarrow 0. \tag{4}$$

Another important property of the SHN distribution is that if $Y \sim \text{SHN}(\alpha, e^{\mu}, 2)$, then $Y$ has a log-Birnbaum-Saunders distribution. The moment generating function of a SHN distribution cannot be obtained in closed form.

## 2.2 Birnbaum-Saunders Distribution

Birnbaum and Saunders [1] introduced a two-parameter failure time distribution for fatigue failure due to cyclic loading. The cumulative distribution function (CDF) of a two-parameter Birnbaum-Saunders random variable $T$ for $\alpha > 0$ and $\beta > 0$, can be written as

$$F_T(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{\beta}{t}\right)^{1/2}\right\}\right]. \tag{5}$$

It is an absolute continuous distribution with the PDF

$$f_T(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right], \quad t > 0.$$

Here $\alpha$ is the shape parameter and $\beta$ is the scale parameter. From now on a two-parameter Birnbaum-Saunders random variable with CDF (5) will be denoted by BS$(\alpha,\beta)$. If the following transformation is used, *i.e.*,

$$Z = \frac{1}{\alpha}\left\{\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{\beta}{T}\right)^{1/2}\right\}, \tag{6}$$

then $Z$ follows standard normal distribution. Several properties and several measures of a Birnbaum-Saunders random variable can be obtained using the transformation (6).

## 2.3 Copula

It is well known that the dependence among two or more random variables, say $X_1,\ldots,X_p$ is completely determined by its joint distribution function $F_{X_1,\ldots,X_p}(x_1,\ldots,x_p)$. The idea of separating $F_{X_1,\ldots,X_P}(x_1,\ldots,x_p)$ in two parts - the one which describes the dependence structure, and the other one which describes only the marginal behavior, leads to the concept of copula. A $p$-variate copula defined on $[0,1]^p$ is a multivariate distribution with univariate marginals on $[0,1]$. Let $X_1,\ldots,X_p$ be $p$ random variables with continuous distribution functions $F_{X_1}(\cdot),\ldots,F_{X_p}(\cdot)$, respectively, then according to Sklar's theorem, $F_{X_1,\ldots,X_p}(x_1,\ldots,x_p)$

has a unique copula representation;

$$F_{X_1,\ldots,X_p}(x_1,\ldots,x_p) = C(F_{X_1}(x_1),\ldots,F_{X_p}(x_p)).$$

Moreover,

$$C(u_1,\ldots,u_p) = F_{X_1,\ldots,X_p}(F^{-1}_{X_1}(u_1),\ldots,F^{-1}_{X_p}(u_p)).$$

It is well known that many dependence properties of a multivariate distribution depend only on the corresponding copula. Therefore, many dependence properties of a multivariate distribution can be obtained by studying the corresponding copula.

The bivariate Gaussian copula is defined as follows;

$$C_G(u,v) = \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)} \phi_2(x,y;\rho)dxdy = \Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\rho).$$

Here $\phi(\cdot)$, $\Phi(\cdot)$ are same as defined before, and $\phi_2(u,v;\rho)$ denotes the standard bivariate normal density function;

$$\phi_2(u,v;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)\right\}.$$

The bivariate Gaussian copula density can be obtained as

$$\begin{aligned} c_G(u,v;\rho) &= \frac{\partial^2}{\partial u\partial v}C_G(u,v;\rho) = \frac{\phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\rho)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \\ &= \frac{1}{\sqrt{1-\rho^2}}\exp\left(\frac{2\rho\Phi^{-1}(u)\Phi^{-1}(v)-\rho^2(\Phi^{-1}(u)^2+\Phi^{-1}(v)^2)}{2(1-\rho^2)}\right). \end{aligned}$$

We recall that a non-negative function $g$ defined in $\mathbb{R}^2$ is total positivity of order two, abbreviated by $\text{TP}_2$ if for all $x_1 < x_2$, $y_1 < y_2$, with $x, y \in \mathbb{R}$

$$g(x_1,y_1)g(x_2,y_2) \geq g(x_2,y_1)g(x_1,y_2). \tag{7}$$

If the equality (7) is reversed, it is called reverse rule of order two ($\text{RR}_2$).

The following result will be useful for future development, and it may have some independent interest also.

Result 2: The Gaussian copula density is (a) $\text{TP}_2$ for $0 < \rho < 1$, (b) $\text{RR}_2$ if $-1 < \rho < 0$.

Proof: To prove (a), we need to show that for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$, if $0 < a_1 < a_2 < 1$, and $0 < b_1 < b_2 < 1$, then for $0 < \rho < 1$,

$$c_G(a_1, b_1; \rho) c_G(a_2, b_2; \rho) \geq c_G(a_2, b_1; \rho) c_G(a_1, b_2; \rho). \tag{8}$$

Proving (8) is equivalent to prove that for all $-\infty < x_1 < x_2 < \infty$, $-\infty < y_1 < y_2 < \infty$

$$\phi_2(a_1, b_1; \rho)\phi_2(a_2, b_2; \rho) \geq \phi_2(a_2, b_1; \rho)\phi_2(a_1, b_2; \rho). \tag{9}$$

Now considering all possible six cases, like (i) $x_1 < x_2 < y_1 < y_2$, (ii) $x_1 < y_1 < x_2 < y_2$, etc., (9) can be easily verified for $0 < \rho < 1$. Similarly, (b) can be easily obtained along the same line.

# 3 Bivariate Sinh-Normal Distribution

In this section, we introduce bivariate sinh-normal distribution, and study its different properties. The bivariate random vector $(Y_1, Y_2)$ is said to have bivariate sinh-normal distribution with parameters $\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho$, if the CDF of $Y_1$ and $Y_2$ is

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = \Phi_2(a(y_1; \alpha_1, \sigma_1, \mu_1), a(y_2; \alpha_2, \sigma_2, \mu_2); \rho); \quad y_1, y_2 \in \mathbb{R}^2. \tag{10}$$

Here $\alpha_1 > 0$, $\alpha_2 > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $-1 < \rho < 1$, and $\Phi_2(u, v, \rho)$ is the CDF of a standard bivariate normal vector $(Z_1, Z_2)$, with correlation coefficient $\rho$. The corresponding PDF of $Y_1$ and $Y_2$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \phi_2(a(y_1; \alpha_1, \sigma_1, \mu_1), a(y_2; \alpha_2, \sigma_2, \mu_2); \rho) \prod_{i=1}^{2} A(y_i; \alpha_i, \sigma_i, \mu_i); \quad y_1, y_2 \in \mathbb{R}^2.$$

Here $a(\cdot), A(\cdot)$ and $\phi_2(\cdot)$ are same as defined before. From now on a BSHN random variable with CDF (10) will be denoted by $\text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$.

It is clear that $\mu_1$, $\mu_2$ are the two location parameters, and $\sigma_1$, $\sigma_2$ are the two scale parameters. If $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$, it is called a standard BSHN distribution. A standard BSHN random variable has the PDF which is centered at (0,0), and it is symmetric around (0,0). It is observed that the surface of the PDF can be unimodal, bimodal or multimodal depending on the parameter values.

It is immediate that if $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, and

$$Z_i = a(Y_i; \alpha_i, \sigma_i, \mu_i); \quad i = 1, 2,$$

then $(Z_1, Z_2)$ has a standard bivariate normal distribution with correlation coefficient $\rho$. Moreover, if $(Z_1, Z_2)$ has a standard bivariate normal distribution with correlation coefficient $\rho$, and if

$$Y_i = \sigma_i \text{ arcsinh}\left(\frac{\alpha_i Z_i}{2}\right) + \mu_i; \quad i = 1, 2, \tag{11}$$

then $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$. The representation (11) can be used to generate samples from a BSHN distribution. We present the following simple algorithm to generate samples from BSHN distribution.

Algorithm

- Step 1: Generate independent $U_1$ and $U_2$ from $N(0, 1)$.
- Step 2: Compute

$$\begin{aligned} Z_1 &= \frac{\sqrt{1+\rho} + \sqrt{1-\rho}}{2} U_1 + \frac{\sqrt{1+\rho} - \sqrt{1-\rho}}{2} U_2 \\ Z_2 &= \frac{\sqrt{1+\rho} - \sqrt{1-\rho}}{2} U_1 + \frac{\sqrt{1+\rho} + \sqrt{1-\rho}}{2} U_2. \end{aligned}$$

- Step 3:

$$Y_i = \sigma_i \text{ arcsinh}\left(\frac{\alpha_i Z_i}{2}\right) + \mu_i; \quad i = 1, 2.$$

It is further follows using L'Hospital's rule that as $\alpha_1 \to 0$, $\alpha_2 \to 0$, $\left\{2\left(\frac{Y_1-\mu_1}{\alpha_1\sigma_1}\right), 2\left(\frac{Y_2-\mu_2}{\alpha_2\sigma_2}\right)\right\}$ converges to a standard bivariate normal distribution with correlation coefficient $\rho$.

The following theorem provides the marginal and conditional distributions of the BSHN distribution.

Theorem 3.1: If $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, then

(a) $Y_i \sim \text{SHN}(\alpha_i, \sigma_i, \mu_i)$; $\quad i = 1,2$.

(b) The conditional CDF of $Y_1$, given $Y_2 = y_2$, is given by

$$P(Y_1 \le y_1 | Y_2 = y_2) = \Phi\left\{\frac{a(y : \alpha_1, \sigma_1, \mu_1) - \rho a(y_2; \alpha_2, \sigma_2, \mu_2)}{\sqrt{1-\rho^2}}\right\}$$

(c) The conditional PDF of $Y_1$, given $Y_2 = y_2$ is

$$f_{Y_1|Y_2=y_2}(y_1) = \phi\left\{\frac{a(y; \alpha_1, \sigma_1, \mu_1) - \rho a(y_2; \alpha_2, \sigma_2, \mu_2)}{\sqrt{1-\rho^2}}\right\}\frac{1}{\sqrt{1-\rho^2}}A(y_1; \alpha_1, \sigma_1, \mu_1).$$

(d) The conditional hazard of $Y_1$ at $y_1$, given $Y_2 = y_2$ is an increasing function of $y_1$, for $y_1 > 0$, for all values of $y_2$ and $\rho$.

Proof: The proof of (a) can be obtained from the definition. The proof of (b) can be obtained by taking the transformation

$$u = a(y_1; \alpha_1, \sigma_1, \mu_1) \quad \text{and} \quad v = \frac{u - \rho a(y_2; \alpha_2, \sigma_2, \mu_2)}{\sqrt{1-\rho^2}}.$$

Part (c) can be obtained from (b) by differentiating with respect to $y_1$. To prove (d), note that the conditional hazard of $Y_1$ at $y$, given $Y_2 = y_2$ can be written as

$$h_{Y_1|Y_2=y_2}(y) = \frac{\phi(b(y))}{1-\Phi(b(y))} \times \frac{1}{\sqrt{1-\rho^2}}A(y; \alpha_1, \sigma_1, \mu_1),$$

where

$$b(y) = \frac{a(y; \alpha_1, \sigma_1, \mu_1) - \rho a(y_2; \alpha_2, \sigma_2, \mu_2)}{\sqrt{1-\rho^2}}.$$

Since $b(y)$ is an increasing function of $y$, $A(y;\alpha_1,\sigma_1,\mu_1)$ is an increasing function of $y$ for $y>0$ and the hazard function of a normal distribution is an increasing function, the result follows. ∎

Consider the following random variable $U(y_2)=\{Y_1|Y_2=y_2\}$. If

$$c=-\frac{\rho a(y_2;\alpha_2,\sigma_2,\mu_2)}{\sqrt{1-\rho^2}},$$

then

$$P(U(y_2)\le y)=\Phi(a(y;\alpha_1\sqrt{1-\rho^2},\sigma_1,\mu_1)+c).$$

Therefore, $U(y_2)$ has the following representation

$$U(y_2)=\sigma_1 \text{ arcsinh}\left(\frac{\alpha_1\sqrt{1-\rho^2}V}{2}\right)+\mu_1, \tag{12}$$

where $V\sim N(c,1)$. Using (12), different properties of the conditional distribution can be obtained.

The following theorem provides the symmetric properties of the BSHN random variable.

THEOREM 3.2: If $(Y_1,Y_2)\sim \text{BSHN}(\alpha_1,\sigma_1,\mu_1,\alpha_2,\sigma_2,\mu_2,\rho)$, and

$$\text{sgn}(x)=\begin{cases}1 & \text{if } \; x>0\\ 0 & \text{if } \; x=0\\ -1 & \text{if } \; x<0\end{cases}$$

then

(a) $(c_1Y_1,c_2Y_2)\sim \text{BSHN}(\alpha_1,|c_1|\sigma_1,c_1\mu_1,\alpha_2,|c_2|\sigma_2,c_2\mu_2,\text{sgn}(c_1c_2)\rho)$.

(b) $(Y_1,-Y_2)\sim \text{BSHN}(\alpha_1,\sigma_1,\mu_1,\alpha_2,\sigma_2,-\mu_2,-\rho)$.

(c) $(-Y_1,Y_2)\sim \text{BSHN}(\alpha_1,\sigma_1,-\mu_1,\alpha_2,\sigma_2,\mu_2,-\rho)$.

(d) $(-Y_1,-Y_2)\sim \text{BSHN}(\alpha_1,\sigma_1,-\mu_1,\alpha_2,\sigma_2,-\mu_2,\rho)$.

PROOF: The proof of (a) follows from the joint PDF of $(Y_1,Y_2)$ and using the appropriate transformation. The results of (b), (c) and (d) follow from (a). ∎

Since $F_{Y_i}(y_i) = \Phi(a(y_i; \alpha_i\sigma_i\mu_i))$, for $i = 1, 2$, therefore,

$$\begin{aligned} F_{Y_1,Y_2}(y_1, y_2) &= \Phi_2(a(y_1; \alpha_1, \sigma_1, \mu_1), a(y_2; \alpha_2, \sigma_2, \mu_2); \rho) \\ &= C_G(\Phi^{-1}(\Phi(a(y_1; \alpha_1, \sigma_1, \mu_1))), \Phi^{-1}(\Phi(a(y_2; \alpha_2, \sigma_2, \mu_2))); \rho) \\ &= C_G(F_{Y_1}(y_1), F_{Y_2}(y_2); \rho). \end{aligned}$$

Therefore, BSHN distribution can be obtained as a bivariate Gaussian copula, with marginals as SHN distributions. We immediately have the following results.

THEOREM 3.3: If $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, then for $\rho > 0$, it is $\text{TP}_2$ and for $\rho < 0$, it is $\text{RR}_2$.

PROOF: Since both $\text{TP}_2$ and $\text{RR}_2$ are copula properties, the results immediately follow from Result 2. ∎

THEOREM 3.4: If $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, then for $\rho > 0$ (a) $T_1$ is stochastically increasing in $T_2$ (b) $T_2$ is stochastically increasing in $T_1$, for all values of $\alpha_1$, $\beta_1$, $\alpha_2$ and $\beta_2$.

PROOF: We will prove (a), (b) follows along the same line. Note that the results can be established if we can show that $C_G(u, v; \rho)$ is a concave function in $u$ for fixed $v$ when $\rho > 0$, see Nelsen ([10], page 197). It is equivalent to prove that $\frac{\partial}{\partial u} C_G(u, v; \rho)$ is a decreasing function in $u$ for fixed $v$. Using Meyer [9], we have

$$\frac{\partial}{\partial u} C(u, v; \rho) = \Phi \left( \frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1 - \rho^2}} \right).$$

Clearly, for $\rho > 0$, the right hand side is a decreasing function in $u$ for fixed $v$ and the result follows. ∎

The bivariate hazard rate of $Y_1$ and $Y_2$ is defined as

$$h(y_1, y_2) = \left( -\frac{\partial}{\partial y_1}, -\frac{\partial}{\partial y_2} \right) \ln P(Y_1 > y_1, Y_2 > y_2) = (h_1(y_1, y_2), h_2(y_1, y_2))$$

see Marshall [8]. Moreover this bivariate vector valued function $h(y_1, y_2)$ uniquely determines the probability distribution. We have the following result.

THEOREM 3.4: If $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, then $h_1(y_1, y_2)$ is an increasing function of $y_1$, for $y_1 > 0$, and for all values of $y_2$. Similarly, $h_2(y_1, y_2)$ is an increasing function of $y_2$ for $y_2 > 0$ for all values of $y_1$.

PROOF: Using similar technique as in Gupta and Gupta [3], it can be easily shown that

$$h_1(y_1, y_2) = \left[ \frac{\left\{ 1 - \Phi\left( \frac{a(y_2;\alpha_2,\sigma_2,\mu_2) - \rho a(y_1:\alpha_1,\sigma_1,\mu_1)}{\sqrt{1-\rho^2}} \right) \right\} \phi(a(y_1;\alpha_1,\sigma_1,\mu_1)}{\overline{\Phi}_2(a(y_1;\alpha_1,\sigma_1,\mu_1), a(y_2;\alpha_2,\sigma_2,\mu_2);\rho)} \right] \times A(y_1;\alpha_1,\sigma_1,\mu_1).$$

Since $a(y_1;\alpha_1,\sigma_1,\mu_1)$ is an increasing function of $y_1$, and the hazard gradient of a bivariate normal distribution is an increasing function, see Gupta and Gupta [3], it follows that the function within $[\,\cdot\,]$ is an increasing function of $y_1$. The first part of the result follows by observing the fact that the function $A(y_1;\alpha_1,\sigma_1,\mu_1)$ is an increasing function of $y_1$ for $y_1 > 0$. Similarly, the second part also can be established. ∎

From the Gaussian copula, it follows that if $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, then for all values of $\alpha_1$, $\alpha_2$, $\sigma_1$, $\sigma_2$, $\mu_1$ and $\mu_2$, (a) Kendall's tau and (b) Spearman's rho becomes

$$\tau = \frac{2}{\pi}\sin^{-1}(\rho), \quad \text{and} \quad \rho_S = \frac{6}{\pi}\sin^{-1}\left(\frac{\rho}{2}\right),$$

respectively.

Now we provide the medial correlation of the BSHN distribution using the copula property. If $M_{Y_1}$ and $M_{Y_2}$ denote the medians of $Y_1$ and $Y_2$, respectively, then $M_{Y_1,Y_2}$, the medial correlation coefficient of $Y_1$ and $Y_2$ is

$$M_{Y_1,Y_2} = P[(Y_1 - M_{Y_1})(Y_2 - M_{Y_2}) > 0] - P[(Y_1 - M_{Y_1})(Y_2 - M_{Y_2}) < 0].$$

It has been shown by Nelsen [10], that the medial correlation coefficient is a copula property and $M_{Y_1,Y_2} = 4C\left(\frac{1}{2},\frac{1}{2}\right)$ - 1. Therefore, of $(Y_1, Y_2)$ follows BSHN, then

$$M_{Y_1,Y_2} = 4C_G\left(\frac{1}{2},\frac{1}{2}\right) - 1 = 4\Phi_2(0,0,\rho) - 1 = 4\left(\frac{1}{4} + \frac{\sin^{-1}\rho}{2\pi}\right) - 1 = \frac{2\sin^{-1}\rho}{\pi}.$$

The concept of bivariate tail dependence relates to the amount of dependence in the upper-quadrant (or lower quadrant) tail of bivariate distribution, see Joe ([5], page 33). In terms of original random variables $Y_1$ and $Y_2$, the upper tail dependence is defines as

$$\chi = \lim_{z\to 1} P(Y_2 \geq F_{Y_2}^{-1}(z)|Y_1 \geq F_{Y_1}^{-1}(z)).$$

Intuitively, the upper-tail dependence exists when there is a positive probability that some positive outliers may occur jointly. Since the Gaussian copula is upper tail and lower tail independent, it follows that if $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$, then $Y_1$ and $Y_2$ are upper tail and lower tail independent.

# 4 Inference

## 4.1 Maximum Likelihood Estimators

In this section we discuss the estimation of the unknown parameters based on a sample of size $n$, $\{(y_{11}, y_{21}), \ldots, (y_{n1}, y_{n2})\}$ from $\text{BSHN}(\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$. Let us denote $\theta = (\alpha_1, \sigma_1, \mu_1, \alpha_2, \sigma_2, \mu_2, \rho)$. The log-likelihood function becomes

$$l(\theta) = \sum_{i=1}^{n} \ln \phi_2(a(y_{i1}; \alpha_1, \sigma_1, \mu_1), a(y_{2i}; \alpha_2, \sigma_2, \mu_2); \rho) + \sum_{j=1}^{2}\sum_{i=1}^{n} \ln A(y_{ij}; \alpha_j, \sigma_j, \mu_j). \quad (13)$$

The MLEs of the unknown parameters can be obtained by maximizing (13) with respect to the unknown parameters. It is a seven-dimensional optimization problem. Clearly, they cannot be obtained in explicit forms. To avoid solving the seven dimensional optimization problem, we propose the profile log-likelihood function approach. Since

$$\left\{\left(e^{\frac{Y_1-\mu_1}{\sigma_1}} - e^{-\frac{Y_1-\mu_1}{\sigma_1}}\right), \left(e^{\frac{Y_2-\mu_2}{\sigma_2}} - e^{-\frac{Y_2-\mu_2}{\sigma_2}}\right)\right\} \sim N_2\left\{(0,0), \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2\rho \\ \alpha_1\alpha_2\rho & \alpha_2^2 \end{pmatrix}\right\},$$

for fixed $\sigma_1, \mu_1, \sigma_2, \mu_2$, the MLEs of $\alpha_1$, $\alpha_2$ and $\rho$ are

$$\widehat{\alpha}_1(\sigma_1, \mu_1) = \left( \frac{1}{n} \sum_{i=1}^{n} g^2(y_{i1}; \sigma_1, \mu_1) \right)^{1/2}, \quad \widehat{\alpha}_2(\sigma_2, \mu_2) = \left( \frac{1}{n} \sum_{i=1}^{n} g^2(y_{i2}; \sigma_2, \mu_2) \right)^{1/2}$$

and

$$\widehat{\rho}(\sigma_1, \mu_1, \sigma_2, \mu_2) = \frac{\sum_{i=1}^{n} g(y_{i1}; \sigma_1, \mu_1) g(y_{i2}; \sigma_2, \mu_2)}{\sqrt{\sum_{i=1}^{n} g^2(y_{i1}; \sigma_1, \mu_1)} \sqrt{\sum_{i=1}^{n} g^2(y_{i2}; \sigma_2, \mu_2)}},$$

where

$$g(y; \sigma, \mu) = e^{\frac{y-\mu}{\sigma}} - e^{-\frac{y-\mu}{\sigma}}. \tag{14}$$

Observe that $\widehat{\alpha}_1(\sigma_1, \mu_1)$ is a function of $\sigma_1$ and $\mu_1$, and $\widehat{\alpha}_2(\sigma_2.\mu_2)$ is a function of $\sigma_2$ and $\mu_2$ only. Finally, the MLEs of $\sigma_1$, $\mu_1$, $\sigma_2$ and $\mu_2$ can be obtained by maximizing the profile log-likelihood function

$$l_{profile}(\sigma_1, \mu_2, \sigma_2, \mu_2) = l(\widehat{\alpha}_1(\sigma_1, \mu_1), \sigma_1, \mu_1, \widehat{\alpha}_2(\sigma_2.\mu_2), \sigma_2, \mu_2, \widehat{\rho}(\sigma_1, \mu_1, \sigma_2, \mu_2)), \tag{15}$$

with respect to $\sigma_1$, $\mu_1$, $\sigma_2$ and $\mu_2$. Note that the maximization of (15) cannot be performed explicitly. Newton-Raphson or some iterative procedure is needed to solve this problem. If the MLEs of $\sigma_1$, $\mu_1$, $\sigma_2$ and $\mu_2$ are denoted by $\widehat{\sigma}_1$, $\widehat{\mu}_1$, $\widehat{\sigma}_2$ and $\widehat{\mu}_2$, respectively, then the MLEs of $\alpha_1$, $\alpha_2$ and $\rho$ become

$$\widehat{\alpha}_1 = \widehat{\alpha}_1(\widehat{\sigma}_1, \widehat{\mu}_1), \quad \widehat{\alpha}_2 = \widehat{\alpha}_2(\widehat{\sigma}_2, \widehat{\mu}_2), \quad \widehat{\rho} = \widehat{\rho}(\widehat{\sigma}_1, \widehat{\mu}_1, \widehat{\sigma}_2, \widehat{\mu}_2).$$

The exact distribution of the MLEs cannot be obtained, since BSHN is a regular family, it satisfies all the required conditions for the MLEs to be consistent and asymptotically normally distributed. We have the following results.

Theorem 4.1: If $\widehat{\theta}$ is the MLE of $\theta$, then

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{d} N_7(0, I^{-1}),$$

here $\xrightarrow{d}$ means convergence in distribution, $N_7(0, I^{-1})$ denotes the 7-variate normal distribution with mean vector 0 and the covariance matrix $I^{-1}$. Here the matrix $I$ is the Fisher information matrix.

It may be mentioned that the elements of the Fisher information matrix can be obtained in a routine matter, although they are not in explicit forms. All the elements are in the double integration form and they are not presented here.

## 4.2 Two-Stage Estimators:

In the previous subsection we have seen that the MLEs of unknown parameters can be obtained by maximizing the profile log-likelihood function, and it needs solving a four dimensional optimization problem. To avoid that we propose to use the estimation of the unknown parameters using the copula structure based on two stage estimation method. Using the copula structure, the log-likelihood function can be written as

$$\begin{aligned} l(\theta) &= \sum_{i=1}^{n} \ln c_G(F_{Y_1}(y_{i1};\alpha_1,\sigma_1,\mu_1), F_{Y_2}(y_{i2};\alpha_2,\sigma_2,\mu_2);\rho) \\ &\quad + \sum_{i=1}^{n} \ln f_{Y_1}(y_{i1};\alpha_1,\sigma_1,\mu_1) + \sum_{i=1}^{n} \ln f_{Y_2}(y_{i2};\alpha_2,\sigma_2,\mu_2). \end{aligned}$$

Two step estimation procedure can be incorporated as follows. First compute the estimates of $\alpha_1$, $\sigma_1$ and $\mu_1$, say $\widetilde{\alpha}_1$, $\widetilde{\sigma}_1$ and $\widetilde{\mu}_1$, respectively, by maximizing

$$h_1(\alpha_1,\sigma_1,\mu_1) = \sum_{i=1}^{n} \ln f_{Y_1}(y_{i1};\alpha_1,\sigma_1,\mu_1)$$

with respect to $\alpha_1$, $\sigma_1$ and $\mu_1$. Similarly, obtain the estimates of $\alpha_2$, $\sigma_2$ and $\mu_2$, say $\widetilde{\alpha}_2$, $\widetilde{\sigma}_2$ and $\widetilde{\mu}_2$, respectively, by maximizing

$$h_2(\alpha_2,\sigma_2,\mu_2) = \sum_{i=1}^{n} \ln f_{Y_2}(y_{i2};\alpha_2,\sigma_2,\mu_2)$$

with respect to $\alpha_2$, $\sigma_2$ and $\mu_2$. Finally, obtain the estimate of $\rho$, say $\widetilde{\rho}$, by maximizing $h(\rho)$ with respect to $\rho$, where

$$h(\rho) = \sum_{i=1}^{n} \ln c_G(u_{i1},u_{i2};\rho),$$

and

$$u_{i1} = F_{Y_1}(y_{i1};\widetilde{\alpha}_1,\widetilde{\sigma}_1,\widetilde{\mu}_1) \quad \text{and} \quad u_{i2} = F_{Y_2}(y_{i2};\widetilde{\alpha}_2,\widetilde{\sigma}_2,\widetilde{\mu}_2).$$

Now we indicate how to compute $\widetilde{\alpha}_1$, $\widetilde{\sigma}_1$ and $\widetilde{\mu}_1$. Note that $h_1(\cdot)$ can be written as

$$h_1(\alpha_1, \sigma_1, \mu_1) = \sum_{i=1}^{n} \ln \phi(a(y_{i1}; \alpha_1, \sigma_1, \mu_1)) + \sum_{i=1}^{n} \ln(A(y_{i1}; \alpha_1, \sigma_1, \mu_1)). \tag{16}$$

Hence, for fixed $\sigma_1$ and $\mu_1$, the maximization of (16) with respect to $\alpha_1$ can be obtained at

$$\widetilde{\alpha}_1(\sigma_1, \mu_1) = \left( \frac{1}{n} \sum_{i=1}^{n} g^2(y_{i1}; \sigma_1, \mu_1) \right)^{1/2},$$

where $g(\cdot)$ is same as defined (14). Further $\widetilde{\sigma}_1$ and $\widetilde{\mu}_1$ can be obtained by maximizing

$$h_1(\widetilde{\alpha}_1(\sigma_1, \mu_1), \sigma_1, \mu_1) = \sum_{i=1}^{n} \ln \phi(a(y_{i1}; \widetilde{\alpha}_1(\sigma_1, \mu_1), \sigma_1, \mu_1)) + \sum_{i=1}^{n} \ln(A(y_{i1}; \widetilde{\alpha}_1(\sigma_1, \mu_1), \sigma_1, \mu_1)),$$

with respect to $\sigma_1$ and $\mu_1$. Similarly, we can obtain $\widetilde{\alpha}_2$, $\widetilde{\sigma}_2$ and $\widetilde{\mu}_2$. Finally, obtain the $\widetilde{\rho}$ by maximizing $h(\rho)$ with respect to $\rho$, which is equivalent to maximize

$$\sum_{i=1}^{n} \ln \phi_2(z_{i1}, z_{i2}; \rho),$$

where for $i = 1, \ldots, n$,

$$z_{i1} = \Phi^{-1}(u_{i1}) = a(y_{i1}; \widetilde{\alpha}_1, \widetilde{\sigma}_1, \widetilde{\mu}_1) \quad \text{and} \quad z_{i2} = \Phi^{-1}(u_{i2}) = a(y_{i2}; \widetilde{\alpha}_2, \widetilde{\sigma}_2, \widetilde{\mu}_2).$$

Therefore,

$$\widetilde{\rho} = \frac{\sum_{i=1}^{n} z_{i1} z_{i2}}{\sqrt{\sum_{i=1}^{n} z_{i1}^2} \sqrt{\sum_{i=1}^{n} z_{i2}^2}}.$$

A two-stage estimation process involves solving two independent two-dimensional optimization problems rather than one four dimensional optimization problem. Clearly it saves a significant amount of computational time. The estimators obtained by the two-stage procedure are consistent estimators of the unknown parameters. The asymptotic distribution of the two-stage estimators can be obtained in a routine matter as it has been obtained in Joe [6], and it is not pursued here.

# 5 Data Analysis

This data set represents two different measurements of stiffness, 'Shock' and 'Vibration' of each of 30 boards. The first measurement (Shock) involves sending a shock wave down the board and the second measurement (Vibration) is determined while vibrating the board. The data set was originally from William Galligan, and it is reported in Johnson and Wichern ([4], page 162). We divide all the data points by 1000, just for computational purposes, it is not going to make any difference in the statistical inference.

We fit the SHN distributions to both the marginals and the estimates of the unknown parameters, namely $\alpha_1$, $\sigma_1$, $\mu_1$, $\alpha_2$, $\sigma_2$ and $\mu_2$, are as follows:

$$\widetilde{\alpha}_1 = 0.00327,\ \widetilde{\sigma}_1 = 184.6307,\ \widetilde{\mu}_1 = 1.4660,\ \widetilde{\alpha}_2 = 0.00318,\ \widetilde{\sigma}_2 = 198.9323,\ \widetilde{\mu}_1 = 1.7189.$$

We want to see how good the SHN distribution fits the marginals. The Kolmogorov-Smirnov distance between the empirical distribution function and the fitted distribution function, and the associated $p$ value (reported in brackets) for $Y_1$ and $Y_2$ are 0.1048 (0.8965) and 0.0940 (0.9535), respectively. Therefore, based on the Kolmogorov-Smirnov distances, it is clear that the SHN distribution fits the marginals quite well. Finally, we obtain $\widetilde{\rho} = 0.9238$. The 95% confidence intervals of $\alpha_1$, $\sigma_1$, $\mu_1$, $\alpha_2$, $\sigma_2$, $\mu_2$ and $\rho$ become, (0.00172,0.00482), (166.0629,203.1985), (0.7781,2.1539), (0.00173,0.00463), (181.9448,215.9198), (0.9874,2.4504), (0.8449,1.0000), respectively.

For comparison purposes, we have fitted the bivariate normal distribution to the data set, and the estimates are as follows:

$$\widehat{\mu}_1 = 1.5079, \quad \widehat{\sigma}_1 = 0.2987, \quad \widehat{\mu}_2 = 1.7249, \quad \widehat{\sigma}_2 = 0.3174, \quad \widehat{\rho} = 0.9241.$$

The Kolmogorov-Smirnov distance between the empirical distribution function and the fitted distribution function, and the associated $p$ value (reported in brackets) for $Y_1$ and $Y_2$

are 0.1219 (0.7641) and 0.1003 (0.9233), respectively. Therefore, for the marginals SHN distribution fits better than the normal distribution.

We also perform the bivariate goodness of fit test. In case of bivariate normal distribution the chi-square value is 14.1740, and the associated $p$ value is between 0.95 and 0.99. For BSHN distribution, we transform it to bivariate normal and perform the bivariate normal goodness of fit test. The chi-square value in this case is 14.1123, and the associated $p$ value is also between 0.95 and 0.99. Since for BSHN, the chi-square value is lower, it indicates that BSHN provides marginally a better fit than the bivariate normal distribution.

# 6 Related Model

In this section we consider bivariate generalized Birnbaum-Saunders distribution, which can be obtained from BSHN by transformation. Owen [11], see also Diaz-Garcia and Dominguez-Molina [2], proposed a generalization of the Birnbaum-Saunders distribution, which can be described as follows. A random variable $T$, is said to have a generalized Birnbaum-Saunders distribution with parameters $\alpha > 0$, $\beta > 0$ and $\lambda > 0$, if the CDF of $T$ is

$$F_T(t) = P(T \le t) = \Phi(b(t;\alpha,\beta,\lambda)); \quad t > 0.$$

where

$$b(t;\alpha,\beta,\lambda) = \frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\lambda} - \left(\frac{\beta}{t}\right)^{\lambda}\right\}.$$

The corresponding PDF becomes;

$$f_T(t) = \phi(b(t;\alpha,\beta,\lambda)B(t;\alpha,\beta,\lambda); \quad t > 0,$$

where

$$B(t;\alpha,\beta,\lambda) = \frac{d}{dt}b(t;\alpha,\beta,\lambda) = \frac{\lambda}{\alpha t}\left[\left(\frac{t}{\beta}\right)^{\lambda} + \left(\frac{\beta}{t}\right)^{\lambda}\right].$$

We denote this random variable as $\text{GBS}(\alpha, \beta, \lambda)$. It can be easily seen that if

$$T \sim \text{GBS}(\alpha, \beta, \lambda) \quad \Leftrightarrow \quad \ln T \sim \text{SHN}(\alpha, \lambda^{-1}, \ln \beta).$$

Now we are in a position to introduce the bivariate generalized Birnbaum-Saunders distribution as follows. A bivariate random vector $(T_1, T_2)$ is said to have bivariate generalized Birnbaum-Saunders distribution with parameters $\alpha_1$, $\beta_1$, $\lambda_1$, $\alpha_2$, $\beta_2$, $\lambda_2$ and $\rho$, if

$$(T_1, T_2) = (e^{Y_1}, e^{Y_2}),$$

where $(Y_1, Y_2) \sim \text{BSHN}(\alpha_1, \lambda_1^{-1}, \ln \beta_1, \alpha_2, \lambda_2^{-1}, \ln \beta_2, \rho)$. The joint CDF of $(T_1, T_2)$ becomes

$$P(T_1 \le t_1, T_2 \le t_2) = \Phi_2(b(t_1; \alpha_1, \beta_1, \lambda_1), b(t_2; \alpha_2, \beta_2, \lambda_2); \rho); \quad t_1 > 0, t_2 > 0. \tag{17}$$

A random vector $(T_1, T_2)$ with CDF (17), will be denoted by $\text{BGBS}(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \rho)$. The joint PDF of $(T_1, T_2)$ becomes

$$f_{T_1,T_2}(t_1, t_2) = \phi_2(b(t_1; \alpha_1, \beta_1, \lambda_1), b(t_2; \alpha_2, \beta_2, \lambda_2); \rho) \prod_{i=1}^{2} B(t_i; \alpha_i, \beta_i, \lambda_i); \quad t_1 > 0, t_2 > 0. \tag{18}$$

Note that when $\lambda_1 = \lambda_2 = 1/2$, it matches with the bivariate Birnbaum-Saunders distribution introduced by Kundu et al. [7]. Therefore, most of the results obtained in Kundu et al. [7] can be obtained as special cases of the results of this paper. We present the results without proofs, as they can be obtained along the same line as the proofs provided in Section 3.

Theorem 6.1: If $(T_1, T_2) \sim \text{BGBS}(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \rho)$, then

(a) $T_i \sim \text{GBS}(\alpha_i, \beta_i, \lambda_i); \quad i = 1,2.$

(b) The conditional CDF of $T_1$ given $T_2 = t_2$, is given by

$$P(T_1 \le t_1 | T_2 = t_2) = \Phi\left[\frac{b(t_1; \alpha_1, \beta_1, \lambda_1) - \rho b(t_2; \alpha_2, \beta_2, \lambda_2)}{\sqrt{1-\rho^2}}\right]$$

(c) The conditional PDF of $T_1$, given $T_2 = t_2$ is

$$f_{T_1|T_2=t_2}(t_1) = \phi\left\{\frac{b(t_1; \alpha_1, \beta_1, \lambda_1) - \rho b(t_2; \alpha_2, \beta_2, \lambda_2)}{\sqrt{1-\rho^2}}\right\} \frac{1}{\sqrt{1-\rho^2}} B(t_1; \alpha_1, \beta_1, \lambda_1).$$

THEOREM 6.2: If $(T_1, T_2) \sim \text{BGBS}(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \rho)$, then

(a) For $c_1 > 0, c_2 > 0$, $(c_1T_1, c_2T_2) \sim \text{BGBS}(\alpha_1, c_1\beta_1, \lambda_1, \alpha_2, c_2\beta_2, \lambda_2, \rho)$.

(b) $(T_1, T_2^{-1}) \sim \text{BGBS}(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2^{-1}, \lambda_2, -\rho)$.

(c) $(T_1^{-1}, T_2) \sim \text{BGBS}(\alpha_1, \beta_1^{-1}, \lambda_1, \alpha_2, \beta_2, \lambda_2, -\rho)$.

(d) $(T_1^{-1}, T_2^{-1}) \sim \text{BGBS}(\alpha_1, \beta_1^{-1}, \lambda_1, \alpha_2, \beta_2^{-1}, \lambda_2, \rho)$.

THEOREM 6.3: If $(T_1, T_2) \sim \text{BGBS}(\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \rho)$, then for $\rho > 0$, it is $\text{TP}_2$ and for $\rho < 0$, it is $\text{RR}_2$.

Note that the results of Theorems 3.1, 3.2, and 3.3 of Kundu et al. [7] can be obtained as special cases of the results of Theorems 6.1, 6.2 and 6.3, respectively. Due to copula property, Blomqvist's beta, Kendall's tau and Spearman's rho will be same as the bivariate sinh-normal distribution. The maximum likelihood estimators of the unknown parameters cannot be obtained in closed form. In this case also directly the two stage estimators can be used as before, alternatively, after making the data transformation, the two stage procedure as described in Section 4, can be used quite easily.

# 7 Conclusion

In this paper we consider a new bivariate distribution which is a natural extension of the univariate sinh-normal distribution. The proposed model has seven unknown parameters, and we obtain different properties of the model. It is observed that the model can be obtained as a Gaussian copula, hence several properties of the Gaussian copula can be easily obtained for this model. The maximum likelihood estimators of the unknown parameters cannot be obtained in closed form, and they can be obtained by using profile log-likelihood method. It involves solving a four dimensional optimization problem. To avoid that we propose to use two step estimation procedure using the copula structure. The estimators obtained by two stage procedure can be obtained by solving two independent two-dimensional optimization problems. They are consistent, and asymptotically normally distributed. Computationally they can be obtained in a much easier manner than the MLEs. Further, we have considered bivariate generalized Birnbaum-Saunders distribution, and studied its different properties. It is observed that many properties of the bivariate Birnbaum-Saunders distribution can be obtained as special cases of the proposed model.

It may be mentioned that although in this paper we have mainly concentrated for the bivariate model, the same concept can be extended for any $p$ dimensional model. Several properties can be obtained along the same line as the bivariate model. Clearly, it is more flexible than the multivariate normal model. Different other characteristics are under investigation, it will be reported later.

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