The Bivariate Generalized Linear Failure Rate Distribution and its Multivariate Extension

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Abstract

The two-parameter linear failure rate distribution has been used quite successfully to analyze lifetime data. Recently, a new three-parameter distribution, known as the generalized linear failure rate distribution has been introduced by exponentiating the linear failure rate distribution. The generalized linear failure rate distribution is a very flexible lifetime distribution, and the probability density function of the generalized linear failure rate distribution can take different shapes. Its hazard function also can be increasing, decreasing and bathtub shaped. The main aim of this paper is to introduce a bivariate generalized linear failure rate distribution, whose marginals are generalized linear failure rate distributions. It is obtained using the same approach as the Marshall-Olkin bivariate exponential distribution. Different properties of this new distribution are established. The bivariate generalized linear failure rate distribution has five parameters and the maximum likelihood estimators are obtained using the EM algorithm. A data set is analyzed for illustrative purposes. Finally, some generalizations to the multivariate case are proposed.

Key Words and Phrases Marshall-Olkin copula; Maximum likelihood estimator; failure rate; EM algorithm; Fisher information matrix.

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1 Introduction

The two-parameter linear failure rate (LFR) distribution, whose hazard function is monotonically increasing in a linear fashion, has been used quite successfully to analyze lifetime data. For some basic properties and for different estimation procedures of the parameters of the LFR distribution, the readers are referred to Bain (1974), Pandey *et al.* (1993), Sen and Bhattacharyya (1995), Lin *et al.* (2003, 2006) and the references cited therein.

Recently, Sarhan and Kundu (2009) introduced a three-parameter generalized linear failure rate (GLFR) distribution by exponentiating the LFR distribution as was done for the exponentiated Weibull distribution by Mudholkar *et al.* (1995). The exponentiation introduces an extra shape parameter in the model, which may bring more flexibility in the shape of the probability density function (PDF) and hazard function. Several properties of this new distribution are established. It is observed that several known distributions like exponential, Rayleigh and LFR distributions can be obtained as special cases of the GLFR distribution.

The aim of this paper is to introduce a new bivariate generalized linear failure rate (BGLFR) distribution, whose marginals are GLFR distributions. This new five-parameter BGLFR distribution is obtained using a similar method as was used for the Marshall-Olkin bivariate exponential model, Marshall and Olkin (1969). The proposed BGLFR distribution is constructed from three independent GLFR distributions using a maximization process. Creating a bivariate distribution with given marginals using this technique is nothing new. Alternatively, the same BGLFR distribution can be obtained by coupling the GLFR marginals with the Marshall-Olkin copula (Nelsen, 1999). This new distribution is a singular distribution, and it can be used quite conveniently if there are ties in the data. The joint cumulative distribution function (CDF) can be expressed as a mixture of an absolute continuous distribution function and a singular distribution function. The joint probability density function (PDF) of the BGLFR distribution can take different shapes and the cumulative distribution function can be expressed in a compact form. The BGLFR distribution can be applied to a maintenance model or a stress model as introduced by Kundu and Gupta (2009).

Several dependency properties of this new distribution are investigated, which will be useful for data analysis purposes. The BGLFR copula has a total positivity of order two (TP_2) property. Each component is stochastically increasing with respect to the other. This implies that the correlation is always non-negative and the two variables are positively quadrant dependent. Moreover, the correlation between the two variables varies between 0 and 1. Kendall's tau index can be calculated using the copula property and can be positive.The population version of the medial correlation coefficient as defined by Blomqvist (1950) is always non-negative. The bivariate tail dependence is always positive.

The BGLFR distribution has five parameters, and their estimation is an important problem in practice. The usual maximum likelihood estimators can be obtained by solving five non-linear equations in five unknowns directly, which is not a trivial issue. To avoid difficult computation we treat this problem as a missing value problem and use the EM algorithm, which can be implemented more conveniently than the direct maximization process. Another advantage of the EM algorithm is that it can be used to obtain the observed Fisher information matrix, which is helpful for constructing the asymptotic confidence intervals for the parameters.

Alternatively, it is possible to obtain approximate maximum likelihood estimators by estimating the marginals first and then estimating the dependence parameter through copula function, as suggested by Joe (1997, chapter 10), which have the same rate of convergence as the maximum likelihood estimators. It is computationally less involved compared to the MLE calculations. This approach is not pursued here. Analysis of a data set is presented for illustrative purposes. The poroposed model provides a better fit than the Marshall-Olkin bivariate exponential model or the recently proposed bivariate generalized exponential model (Kundu and Gupta, 2009).

Although in this paper we mainly discussed the BGLFR, many of our results can be easily extended to the multivariate case. Moreover, the LFR distribution is a proportional reversed hazard model, and our method may be used to introduce other bivariate proportional reversed hazard models.

The rest of the paper is organized as follows. We briefly introduce the GLFR distribution in Section 2. In Section 3 we introduce the BGLFR distribution and study its different properties. The EM algorithm is described in Section 4, and analysis of a data set is presented in Section 5. We discuss the multivariate generalization in Section 6, and finally conclude the paper in Section 7.

2 Generalized Linear Failure Rate Distribution

A random variable X has a linear failure rate distribution with parameters $\beta \geq 0$ and $\gamma \geq 0$ (such that $\beta + \gamma > 0$), if X has the following distribution function;

$$F_{LFR}(x; \beta, \gamma) = 1 - \exp\left\{-\beta x - \frac{\gamma}{2}x^2\right\}, \tag{1}$$

for $x > 0$. The exponential distribution with mean $1/\beta$ (ED(β)) and the Rayleigh distribution with parameter γ (RD(γ)) can be obtained as special cases from the LFR distribution. The PDF of the LFR distribution can be decreasing or unimodal, but the failure rate function is either increasing or constant only (Sen and Bhattacharyya, 1995).

Sarhan and Kundu (2009) introduced the GLFR distribution by exponentiating the LFR distribution function as follows. A random variable X is said to have a GLFR distribution with parameters $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ (GLFR(α, β, γ)), if it has the CDF

$$F_{GLFR}(x; \alpha, \beta, \gamma) = \left(1 - \exp\left\{-\beta x - \frac{\gamma}{2}x^2\right\}\right)^{\alpha}, \tag{2}$$

for $x > 0$. The corresponding PDF has the form;

$$f_{GLFR}(x; \alpha, \beta, \gamma) = \alpha(\beta + \gamma x)e^{-\left(\beta x + \frac{\gamma}{2}x^2\right)}\left(1 - \exp\left\{-\beta x - \frac{\gamma}{2}x^2\right\}\right)^{\alpha - 1}, \tag{3}$$

for $x > 0$. The PDF of the GLFR distribution is either decreasing or unimodal, and it can have constant, increasing, decreasing or bathtub shaped hazard function. It is immediate from (2) that if α is an integer, then the CDF of GLFR(α, β, γ) represents the CDF of the maximum of a simple random sample of size α, from the LFR distribution. Therefore, when α is an integer, GLFR provides the distribution function of a parallel system when each component has the LFR distribution.

The mean or the other moments cannot be obtained in explicit form, but can be written in terms of infinite series (Sarhan and Kundu, 2009). However, because of the closed form CDF, the median or other percentile points can be obtained explicitly. Because of the exponentiated nature of the CDF, the GLFR distribution is closed under maximum, *i.e.* if $X_1, \cdots, X_n$ are independently distributed such that X_i follows the GLFR$(\alpha_i, \beta, \gamma)$ distribution, for $i = 1, \cdots, n$, then $\max\{X_1, \cdots, X_n\}$ is GLFR $\left(\sum_{i=1}^{n} \alpha_i, \beta, \gamma\right)$. Moreover, if R is the stress-strength parameter, *i.e.* $R = P(X_1 < X_2)$ where X_1 and X_2 are as defined above, then

$$R = P(X_1 < X_2) = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

For order statistics, moments of order statistics, characterization, and for an estimation procedure, the readers are referred to Sarhan and Kundu (2009).

3 Bivariate Generalized Failure Rate Distribution

In this section we introduce the BGLFR distribution using a method similar to that which was used by Marshall and Olkin (1969) to define the Marshall-Olkin bivariate exponential (MOBE) distribution.

Suppose U_1, U_2 and U_3 are three independent random variables such that $U_i \sim$ GLFR $(\alpha_i,\ \beta, \gamma)$ for $i = 1, 2$ and 3. Define

$$X_1 = \max\{U_1, U_3) \qquad \text{and} \qquad X_2 = \max\{U_2, U_3\}. \tag{4}$$

Then we say that the bivariate vector (X_1, X_2) has a bivariate GLFR (BGLFR) distribution, with parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$ and we denote it by BGLFR $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$. The following interpretations can be provided for BGLFR model.

Competing risks model: Assume a system has two components, labeled 1 and 2 and the survival time of component i is denoted by X_i, $i = 1, 2$. It is considered that there are three independent causes of failures, which may affect the system. Only component 1 can fail due to cause 1, and similarly only component 2 can fail due to cause 2, while both the components fail at the same time due to cause 3. Let U_i be the lifetime of cause i, $i = 1, 2, 3$. If U_1, U_2, U_3 follow GLFR distribution, then $(X_1,\ X_2)$ follows BGLFR model.

Shock model: Suppose there are three independent sources of shocks, say 1, 2, and 3. Suppose these shocks are affecting a system with two components, say 1 and 2. It is assumed that, the shock from source 1 reaches the system destroys component 1 immediately, the shock from source 2 reaches the system destroys component 2 immediately while if the shock from source 3 hits the system destroys both the components immediately. Let U_i denote the inter-arrival times between the shocks in source i, i=1,2,3, which follow the distribution GLFRD. If X_1, X_2 denote the survival times of the components, then (X_1, X_2) follows BGLFR model.

If $(X_1, X_2) \sim$ BGLFR $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$, then the corresponding CDF, PDF and the marginals are provided in the following theorem. The proofs are not difficult and therefore are omitted.

THEOREM 3.1: Suppose $(X_1, X_2) \sim$ BGLFR $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$. Then

(a) The joint CDF of (X_1, X_2) can be written as

$$F_{X_1,X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) = \prod_{i=1}^{3} F_{GLFR}(x_i; \alpha_i,\ \beta,\ \gamma), \tag{5}$$

where $x_3 = \min\{x_1, x_2\}$.

(b) The joint PDF of (X_1, X_2) can be written as

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if} \quad 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & \text{if} \quad 0 < x_2 < x_1 < \infty \\ f_0(x) & \text{if} \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{6}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_{GLFR}(x_1; \alpha_1 + \alpha_3, \beta, \gamma)\ f_{GLFR}(x_2; \alpha_2, \beta, \gamma) \\ f_2(x_1, x_2) &= f_{GLFR}(x_1; \alpha_1, \beta, \gamma)\ f_{GLFR}(x_2; \alpha_2 + \alpha_3, \beta, \gamma) \\ f_0(x) &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\ f_{GLFR}(x; \alpha_1 + \alpha_2 + \alpha_3, \beta, \gamma) \end{aligned}$$

(c) The marginal distributions of X_1 and X_2 are GLFR$(\alpha_1{+}\alpha_3, \beta, \gamma)$ and GLFR$(\alpha_2{+}\alpha_3, \beta, \gamma)$ respectively.

The joint distribution function of X_1 and X_2 has a singular part along the line $x_1 = x_2$, with weight $\frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}$, and has an absolute continuous part on $0 < x_1 \neq x_2 < \infty$ with weight $\frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}$. In writing the joint PDF, it is understood that the first two parts are the joint PDF with respect to two dimensional Lebesgue meausre, whereas the third part is the PDF with respect to one dimensional Lebesgue measure along the line $x_1 = x_2$.

This is similar to the Marshall-Olkin bivariate exponential model or bivariate generalized exponential model.

For fixed α_1, α_2, β and γ, as α_3 varies from 0 to ∞, the correlation between X_1 and X_2 varies between 0 and 1. This is because, if $\alpha_3 = 0$, then X_1 and X_2 become independent, and when α_3 tends to infinity, then U_3 tends to infinity with probability 1. Thus $U_3 > U_1$ and $U_3 > U_2$ with probability 1. Therefore, $X_1 = X_2$ with probability 1, as α_3 tends to infinity. The joint survival function and the conditional distributions can be easily obtained. Surface plots of the absolutely continuous part of the joint PDF of (X_1, X_2) are provided in Figure 1. The joint PDF can take various shapes depending on the parameter values.

Interestingly, the BGLFR distribution can be obtained by using the Marshall-Olkin (MO) copula with the marginals as the GLFR distributions. To every bivariate distribution function F_{X_1,X_2} with continuous marginals F_{X_1} and F_{X_2} corresponds a unique bivariate distribution function with uniform margins $C : [0,1]^2 \rightarrow [0,1]$ called a copula, such that $F_{X_1,X_2}(x_1, x_2) = C\{F_{X_1}(x_1), F_{X_2}(x_2)\}$ holds for all $(x_1, x_2) \in \Re^2$ (Nelsen, 1999). The MO copula is

$$C_{\theta_1,\theta_2}(u_1, u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} \min\{u_1^{\theta_1}, u_2^{\theta_2}\}, \tag{7}$$

for $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. Using $u_i = F_{X_i}(x_i)$ where X_i is GLFR$(\alpha_i + \alpha_3, \beta, \gamma)$ and $\theta_i = \alpha_3/(\alpha_i + \alpha_3)$, i=1,2,3, gives the same joint distribution function F_{X_1,X_2} as (5).

Generating values from a BGLFR distribution is straightforward. First, we can generate values for three independent GLFR random variables and then use (4) to generate (X_1, X_2). Alternatively, we can generate (u_1, u_2) from the copula C_{θ_1,θ_2}, and then use the inversion formula to obtain (X_1, X_2). If (X_1, X_2) follows the BGLFR$(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$, then $\max\{X_1, X_2\}$ follows the GLFR$(\alpha_1 + \alpha_2 + \alpha_3, \beta, \gamma)$ distribution. The stress strength parameter of (X_1, X_2) is

$$P(X_1 < X_2) = P(U_1 < U_3 < U_2) + P(U_3 < U_1 < U_2) = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}. \tag{8}$$

Now we will provide several dependency results between the two variables. Lehmann (1966) defined two random variables X_1 and X_2 to be positive quadrant dependent (PQD) if for all x_1 and x_2,

$$P(X_1 \le x_1, X_2 \le x_2) \ge P(X_1 \le x_1) P(X_2 \le x_2). \tag{9}$$

Intuitively, X_1 and X_2 are PQD if the probability that they are simultaneously small or simultaneously large is at least as great as it would be if they were independent. PQD is a copula property and (9) can be written equivalently as

$$C(u_1, u_2) \ge u_1 \; u_2, \qquad \text{for all} \;\; u_1, u_2 \in [0,1]^2. \tag{10}$$

This condition is satisfied by the MO copula. Therefore, if (X_1, X_2) follow the BGLFR distribution, then they are PQD. Because X_1 and X_2 are PQD, for every pair of increasing functions $g_1(\cdot)$ and $g_2(\cdot)$ (Barlow and Proschan, 1981) the following relation is satisfied

$$\text{Cov}\{g_1(X_1), g_2(X_2)\} \ge 0 \tag{11}$$

Moreover, it can also be verified that X_1 is stochastically increasing in X_2, and similarly X_2 is also stochastically increasing in X_1.

A non-negative function g defined on $\Re^2$ is total positivity of order two, abbreviated by TP_2, if for all $x_1 < x_2$ and $y_1 < y_2$,

$$g(x_1, y_1) g(x_2, y_2) \ge g(x_2, y_1) g(x_1, y_2). \tag{12}$$

The MO copula satisfies this condition. Therefore (Nelsen, 1999), if (X_1, X_2) follows the BGLFR distribution, then X_1 and X_2 are left corner set decreasing, *i.e.* $P(X_1 \le x_1, X_2 \le x_2 | X_1 \le x_1', X_2 \le x_2')$ is non-decreasing in x_1' and in x_2', for all x_1 and x_2.

The copula provides a natural way to measure the dependence between two random variables. Now we provide some measures of dependence namely the Kendall's tau and the medial correlation. We further study the dependence of extreme events.

Kendall's tau is defined as the probability of concordance minus the probability of discordance between two pairs of random vectors (X_1, X_2) and (Y_1, Y_2),

$$\tau = P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P[(X_1 - Y_1)(X_2 - Y_2) < 0]. \tag{13}$$

where (X_1, X_2) and (Y_1, Y_2) are independent and identically distributed random vectors. Nelsen (1999) has shown that Kendall's tau index is also a copula property. Moreover, the MO copula has Kendall's tau as $\dfrac{\theta_1\theta_2}{\theta_1 - \theta_1\theta_2 + \theta_2}$. So, if $(X_1, X_2) \sim \text{BGLFR}(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$, the Kendall's tau index between X_1 and X_2 is

$$\tau_{X_1,X_2} = \frac{\theta_1\theta_2}{\theta_1 - \theta_1\theta_2 + \theta_2} = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}. \tag{14}$$

For fixed α_1 and α_2, as α_3 varies from 0 to ∞, τ_{X_1,X_2} varies between 0 and 1.

Blomqvist (1950) defined the median correlation coefficient, $M_{X_1X_2}$, between two continuous random variables X_1 and X_2 as follows. If M_{X_1} and M_{X_2} denote the median of X_1 and X_2 respectively, then

$$M_{X_1X_2} = P[(X_1 - M_{X_1})(X_2 - M_{X_2}) > 0] - P[(X_1 - M_{X_1})(X_2 - M_{X_2}) < 0]. \tag{15}$$

Domma (2009) observed that Blomqvist's medial correlation coefficient is a copula property and it can be verified that

$$M_{X_1X_2} = 4F_{X_1,X_2}(M_{X_1}, M_{X_2}) - 1 = 4C_{\theta_1,\theta_2}\left(\frac{1}{2}, \frac{1}{2}\right) - 1. \tag{16}$$

Therefore, if $(X_1, X_2) \sim \text{BGLFR}(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$, then $M_{X_1X_2}$ is

$$M_{X_1X_2} = \begin{cases} \left(\frac{1}{2}\right)^{2-\theta_2} & \text{if} \quad \theta_1 > \theta_2 \\ \left(\frac{1}{2}\right)^{2-\theta_1} & \text{if} \quad \theta_1 < \theta_2. \end{cases} \tag{17}$$

where as above $\theta_i = \alpha_3/(\alpha_3 + \alpha_i)$ The minimum and the maximum values of $M_{X_1X_2}$ are $1/4$ and $1/2$ respectively.

The bivariate tail dependence measures the amount of dependence in the upper quadrant (or lower quadrant) tail of a bivariate distribution (Joe, 1997). For bivariate random vectors (X_1, X_2), the upper tail dependence (if it exists) is defined as follows

$$\lambda_U = \lim_{z \to 1^-} P(X_2 > F_{X_2}^{-1}(z) |\ X_1 > F_{X_1}^{-1}(z))\,. \tag{18}$$

Intuitively, the upper tail dependence exists when there is a positive probability that some positive outliers may occur jointly. If $\lambda_U \in (0, 1]$, then X_1 and X_2 are said to be asymptotically dependent, if $\lambda_U = 0$, then they are asymptotically independent. Similarly, the lower tail dependence parameter λ_L (if it exists) is defined as follows

$$\lambda_L = \lim_{z \to 0^+} P(X_2 \leq F_{X_2}^{-1}(z) |\ X_1 \leq F_{X_1}^{-1}(z)). \tag{19}$$

These parameters are non-parametric and both depend only on the copula C of X_1 and X_2 as follows:

$$\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t,t)}{1 - t} \qquad \text{and} \qquad \lambda_L = \lim_{t \to 0^+} \frac{C(t,t)}{t}. \tag{20}$$

If (X_1, X_2) follows BGLFR$(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$, then

$$\lambda_U = \begin{cases} \theta_1 & \text{if} \quad \theta_1 < \theta_2 \\ \theta_2 & \text{if} \quad \theta_2 < \theta_1, \end{cases} \tag{21}$$

and $\lambda_L = 0$.

4 Estimation

In this section we consider the estimation of the unknown parameters of the BGLFR model. It is assumed that we have a sample of size n, of the form

$$\{(x_{11}, x_{12}), \cdots, (x_{n1}, x_{n2})\} \tag{22}$$

from BGLFR$(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$ and our problem is to estimate $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ from the given sample. First we obtain the MLEs of the unknown parameters. Since the computation of the MLEs is computationally quite involved, we propose alternative estimators, which can be obtained in a more convenient manner.

For further development we use the following notations;

$$I_1 = \{i; x_{i1} < x_{i2}\}, \quad I_2 = \{i; x_{i1} > x_{i2}\}, \quad I_0 = \{i; x_{i1} = x_{i2} = x_i\}, \quad I = I_0 \cup I_1 \cup I_2$$

and

$$n_0 = |I_0|, \quad n_1 = |I_1|, \quad n_2 = |I_2|.$$

Based on the sample (22) mentioned above, the log-likelihood function of the observed data can be written as;

$$l(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma) = \sum_{i \in I_1} \ln f_1(x_{i1}, x_{i2}) + \sum_{i \in I_2} \ln f_2(x_{i1}, x_{i2}) + \sum_{i \in I_0} \ln f_0(x_i, x_i). \tag{23}$$

Therefore, the MLEs of the unknown parameters can be obtained by maximizing (23) with respect to the unknown parameters. It is clearly a five dimensional optimization problem. We need to solve five non-linear equations simultaneously to compute the MLEs, which may not very simple. To avoid that we propose to use the expectation maximization (EM) algorithm to compute the MLEs in this case.

It may be noted that if instead of (X_1, X_2), we had observed U_1, U_2 and U_3, the MLEs of $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ can be obtained by solving a two dimensional optimization process, which is clearly much convenient than solving a five dimensional optimization process. Due to this reason, we treat this problem as a missing value problem. It is assumed that for the bivariate random vector (X_1, X_2), there is an associated random vector (λ_1, λ_2) as follows;

$$\Lambda_1 = \begin{cases} 1 & \text{if} \quad U_1 > U_3 \\ \\ 3 & \text{if} \quad U_1 < U_3 \end{cases} \quad \text{and} \quad \Lambda_2 = \begin{cases} 2 & \text{if} \quad U_2 > U_3 \\ \\ 3 & \text{if} \quad U_2 < U_3. \end{cases} \tag{24}$$

Therefore, if $X_1 = X_2$, then clearly $\lambda_1 = \lambda_2 = 3$. But if $X_1 < X_2$ or $X_1 > X_2$, the corresponding (Λ_1, Λ_2) is missing. If $(X_1, X_2) \in I_1$ then the possible values of (Λ_1, Λ_2) are

(3,2) or (1,2) and if $(X_1, X_2) \in I_2$ then the possible values of (Λ_1, Λ_2) are (1,3) or (1,2). It implies that if $(X_1, X_2) \in I_1$, then Λ_2 is known, but Λ_1 is unknown, and if $(X_1, X_2) \in I_2$, then Λ_1 is known, but Λ_2 is unknown. The following Table 1 provides the all possible orders of U_i's, the associated (X_1, X_2), (Λ_1, Λ_2) values and the corresponding probabilities, which will be useful for further development.

Table 1: All possible orders of U_i's, the associated (X_1, X_2), (Λ_1, Λ_2) values and the corresponding probabilities.

Case	Possible Order	X_1	X_2	Λ_1	Λ_2	Prob	Set
1	$U_1 < U_2 < U_3$	U_3	U_3	3	3	$\dfrac{\alpha_2\alpha_3}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)}$	I_0
2	$U_2 < U_1 < U_3$	U_3	U_3	3	3	$\dfrac{\alpha_1\alpha_3}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)}$	I_0
3	$U_1 < U_3 < U_2$	U_3	U_2	3	2	$\dfrac{\alpha_2\alpha_3}{(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}$	I_1
4	$U_3 < U_1 < U_2$	U_1	U_2	1	2	$\dfrac{\alpha_1\alpha_2}{(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}$	I_1
5	$U_2 < U_3 < U_1$	U_1	U_3	1	3	$\dfrac{\alpha_1\alpha_3}{(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}$	I_2
6	$U_3 < U_2 < U_1$	U_1	U_2	1	2	$\dfrac{\alpha_1\alpha_2}{(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}$	I_2

Now we are in a position to provide the EM algorithm. In the 'E' step of the EM algorithm the observations belong to I_0, we treat them as complete observations. If the observation belongs to either I_1 OR I_2, we treat it as the missing observation. If $(x_1, x_2) \in I_1$, we form the 'pseudo observation' by fractioning (x_1, x_2) to two partially complete 'pseudo observation' of the form $(x_1, x_2, u_1(\theta))$ and $(x_1, x_2, u_2(\theta))$ respectively. Here θ is the parameter vector, *i.e.* $\theta = (\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$ and the fractional mass $u_1(\theta)$ and $u_2(\theta)$ assigned to the 'pseudo observation' is the conditional probability, that Λ_1 takes the values 3 or 1 respectively given $X_1 < X_2$. It is clear from Table 1 that

$$u_1(\theta) = P(\Lambda_1 = 3|X_1 < X_2) = \frac{\alpha_3}{\alpha_1 + \alpha_3}, \quad u_2(\theta) = P(\Lambda_1 = 1|X_1 < X_2) = \frac{\alpha_1}{\alpha_1 + \alpha_3}. \tag{25}$$

Similarly, if $(x_1, x_2) \in I_2$, we form the 'pseudo observation' of the form $(x_1, x_2, w_1(\theta))$ and $(x_1, x_2, w_2(\theta))$. Here the fractional mass $w_1(\theta)$ or $w_2(\theta)$ assigned to the 'pseudo observation', is the conditional probability that the random variable Λ_2 takes the values 3 or 2 respectively, given $X_1 > X_2$. Again from Table 1 it is clear that

$$w_1(\theta) = P(\Lambda_2 = 3|X_1 > X_2) = \frac{\alpha_3}{\alpha_2 + \alpha_3}, \quad w_2(\theta) = P(\Lambda_2 = 2|X_1 > X_2) = \frac{\alpha_2}{\alpha_2 + \alpha_3}. \tag{26}$$

From now on for brevity, we write $u_1(\theta), u_2(\theta), w_1(\theta), w_2(\theta)$ as u_1, u_2, w_1, w_2 respectively.

Now we are in a position to provide the 'E'-step of the EM algorithm. We will be using the following notation; $\theta_i = (\alpha_i, \beta, \gamma)$; $i = 1, 2, 3$. Also, $f(\cdot; \theta_i)$ and $F(\cdot; \theta_i)$ denote respectively the PDF and CDF of the GLFR$(\alpha_i, \beta, \gamma)$ for $i = 1, 2, 3$. The log-likelihood function of the 'pseudo data' ('E' - step) can be written as

$$\begin{aligned} l_{pseudo}(\theta) \;\; = \;\; & \sum_{i \in I_0} \ln f(x_i; \theta_3) + \sum_{i \in I_0} \ln F(x_i; \theta_1) + \sum_{i \in I_0} \ln F(x_i; \theta_2) + \\ & u_1 \left(\sum_{i \in I_1} \ln f(x_{i1}; \theta_3) + \sum_{i \in I_1} \ln f(x_{i2}; \theta_2) + \sum_{i \in I_1} \ln F(x_{i1}; \theta_1) \right) \\ & u_2 \left(\sum_{i \in I_1} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_1} \ln f(x_{i2}; \theta_2) + \sum_{i \in I_1} \ln F(x_{i1}; \theta_3) \right) \\ & w_1 \left(\sum_{i \in I_2} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_2} \ln f(x_{i2}; \theta_3) + \sum_{i \in I_2} \ln F(x_{i2}; \theta_2) \right) \end{aligned}$$

$$
\begin{aligned}
& w_2\left(\sum_{i\in I_2}\ln f(x_{i1};\theta_1)+\sum_{i\in I_2}\ln f(x_{i2};\theta_2)+\sum_{i\in I_2}\ln F(x_{i2};\theta_3)\right)\\
= \;& l_1(\theta_1)+l_2(\theta_2)+l_3(\theta_3), \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
l_1(\theta_1) &= \sum_{i\in I_0}\ln F(x_i;\theta_1)+u_1\sum_{i\in I_1}\ln F(x_{i1};\theta_1)+u_2\sum_{i\in I_1}\ln f(x_{i1};\theta_1)+\sum_{i\in I_2}\ln f(x_{i1};\theta_1)\\
l_2(\theta_2) &= \sum_{i\in I_0}\ln F(x_i;\theta_2)+w_1\sum_{i\in I_2}\ln F(x_{i2};\theta_2)+w_2\sum_{i\in I_2}\ln f(x_{i2};\theta_2)+\sum_{i\in I_1}\ln f(x_{i2};\theta_2)\\
l_3(\theta_3) &= \sum_{i\in I_0}\ln f(x_i;\theta_3)+u_1\sum_{i\in I_1}\ln f(x_{i1};\theta_3)+u_2\sum_{i\in I_1}\ln F(x_{i1};\theta_3)+\\
&\quad w_1\sum_{i\in I_2}\ln f(x_{i2};\theta_3)+w_2\sum_{i\in I_1}\ln F(x_{i2};\theta_3).
\end{aligned}
$$

Now at the 'M'-step we need to maximize (27) with respect to unknown parameters. For fixed β and γ, the maximization of $l_{pseudo}(\theta)$ with respect to α_1, α_2 and α_3 can be obtained by maximizing $l_1(\theta_1)$, $l_2(\theta_2)$, and $l_3(\theta_3)$ with respect to α_1, α_2 and α_3 respectively. If we denote them as $\widetilde{\alpha}_1(\beta,\gamma)$, $\widetilde{\alpha}_2(\beta,\gamma)$ and $\widetilde{\alpha}_3(\beta,\gamma)$ respectively, then

$$
\widetilde{\alpha}_1(\beta,\gamma)=\frac{u_2n_1+n_2}{\sum_{i\in I_0}a(x_i;\beta,\gamma)+\sum_{i\in I_1}a(x_{i1};\beta,\gamma)+\sum_{i\in I_2}a(x_{i1};\beta,\gamma)} \qquad (28)
$$

$$
\widetilde{\alpha}_2(\beta,\gamma)=\frac{w_2n_2+n_1}{\sum_{i\in I_0}a(x_i;\beta,\gamma)+\sum_{i\in I_1}a(x_{i2};\beta,\gamma)+\sum_{i\in I_2}a(x_{i2};\beta,\gamma)} \qquad (29)
$$

$$
\widetilde{\alpha}_3(\beta,\gamma)=\frac{n_0+u_1n_1+w_1n_2+n_1}{\sum_{i\in I_0}a(x_i;\beta,\gamma)+\sum_{i\in I_1}a(x_{i1};\beta,\gamma)+\sum_{i\in I_2}a(x_{i2};\beta,\gamma)}. \qquad (30)
$$

where

$$
a(x;\beta,\gamma)=\ln\left[1-exp(-\beta x-\frac{\gamma}{2}x^2)\right]
$$

Finally the maximization of $l_{pseudo}(\theta)$ with respect to θ, can be obtained by maximizing $l_{pseudo}(\widetilde{\alpha}_1(\beta,\gamma),\widetilde{\alpha}_2(\beta,\gamma),\widetilde{\alpha}_3(\beta,\gamma),\beta,\gamma)$, the pseudo profile log-likelihood function of β and γ. If $\widetilde{\beta}$ and $\widetilde{\gamma}$ maximize the pseudo profile log-likelihood function, then $\widetilde{\alpha}_1(\widetilde{\beta},\widetilde{\gamma})$, $\widetilde{\alpha}_2(\widetilde{\beta},\widetilde{\gamma})$, $\widetilde{\alpha}_3(\widetilde{\beta},\widetilde{\gamma})$, $\widetilde{\beta}$, $\widetilde{\gamma}$ become the next iterate of the EM algorithm. We propose to use the following algorithm to compute the MLEs of the unknown parameters by EM algorithm;

Algorithm:

STEP 1: Take some initial guess value of θ, say $\theta^{(0)} = (\alpha_1^{(0)}, \alpha_2^{(0)}, \alpha_3^{(0)}, \beta^{(0)}, \gamma^{(0)})$

STEP 2: Compute $u_1(\theta^{(0)})$, $u_2(\theta^{(0)})$, $w_1(\theta^{(0)})$ and $w_2(\theta^{(0)})$.

STEP 3: For given $u_1(\theta^{(0)})$, $u_2(\theta^{(0)})$, $w_1(\theta^{(0)})$ and $w_2(\theta^{(0)})$, maximize the pseudo log-likelihood function $l_{pseudo}(\widetilde{\alpha}_1(\beta,\gamma), \widetilde{\alpha}_2(\beta,\gamma), \widetilde{\alpha}_3(\beta,\gamma), \beta, \gamma)$ with respect β and γ, say $\beta^{(1)}$ and $\gamma^{(1)}$ respectively.

STEP 3: Obtain $\alpha_1^{(1)} = \widetilde{\alpha}_1(\beta^{(1)}, \gamma^{(1)})$, $\alpha_2^{(1)} = \widetilde{\alpha}_2(\beta^{(1)}, \gamma^{(1)})$ and $\alpha_3^{(1)} = \widetilde{\alpha}_3(\beta^{(1)}, \gamma^{(1)})$, and therefore $\theta^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \beta^{(1)}, \gamma^{(1)})$

STEP 4: Replace $\theta^{(0)}$ by $\theta^{(1)}$ and go back to Step 1 and continue the process unless convergence takes place.

5 DATA ANALYSIS

In this section we present the analysis of one data set mainly to illustrate how the proposed model and the EM algorithm work in practice.

UEFA CHAMPION'S LEAGUE DATA: The data set has been obtained from Meintanis (2007). The data set is presented in Table 2. It represents the soccer data where at least one goal is scored by the home team and at least goal is scored directly from a penalty kick, foul kick or any other direct kick (all of them will be called as *kick* goal) by any team has been considered. Here X_1 and X_2 represent the time in minutes of the first *kick* goal scored by any team and X_2 represents the first goal of any type scored by the home team. Clearly all possibilities are open, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = Y$ (say).

Meintanis (2007) analyzed this data set using Marshall-Olkin bivariate exponential model. Kundu and Gupta (2009) re-analyzed the same data set using bivariate generalized exponential model. It is observed that the bivariate generalized exponential distribution provides a

Table 2: UEFA Champion's League Data

2005-2006	X_1	X_2	2004-2005	X_1	X_2
Lyon-Real Madrid	26	20	Internazionale-Bremen	34	34
Milan-Fenerbahce	63	18	Real Madrid-Roma	53	39
Chelsea-Anderlecht	19	19	Man. United-Fenerbahce	54	7
Club Brugge-Juventus	66	85	Bayern-Ajax	51	28
Fenerbahce-PSV	40	40	Moscow-PSG	76	64
Internazionale-Rangers	49	49	Barcelona-Shakhtar	64	15
Panathinaikos-Bremen	8	8	Leverkusen-Roma	26	48
Ajax-Arsenal	69	71	Arsenal-Panathinaikos	16	16
Man. United-Benfica	39	39	Dynamo Kyiv-Real Madrid	44	13
Real Madrid-Rosenborg	82	48	Man. United-Sparta	25	14
Villarreal-Benfica	72	72	Bayern-M. TelAviv	55	11
Juventus-Bayern	66	62	Bremen-Internazionale	49	49
Club Brugge-Rapid	25	9	Anderlecht-Valencia	24	24
Olympiacos-Lyon	41	3	Panathinaikos-PSV	44	30
Internazionale-Porto	16	75	Arsenal-Rosenborg	42	3
Schalke-PSV	18	18	Liverpool-Olympiacos	27	47
Barcelona-Bremen	22	14	M. Tel-Aviv-Juventus	28	28
Milan-Schalke	42	42	Bremen-Panathinaikos	2	2
Rapid-Juventus	36	52			

better fit than the Marshall-Olkin bivariate exponential model. It has been shown by Kundu and Gupta (2009) using the scaled TTT transform of Aarset (1987), that both the marginals (X_1 and X_2) have increasing empirical hazard rates. It has prompted us to use the BGLFR distribution to analyze this model.

Before trying to analyze the data using BGLFR model, we first fit the GLFR model to X_1 and X_2 separately. The MLEs of the parameters (β, γ, α) of the corresponding GLFR distribution for X_1 and X_2 are (5.1828×10^{-3}, 9.3294×10^{-4}, 1.3031) and (0.0194, 5.6825 $\times \times 10^{-4}$, 1.1433) and the corresponding log-likelihood values are -162.676 and -162.938 respectively. Since both exponential and generalized exponential distributions are special cases of the GLFR distribution, we perform the following two testing of hypotheses:

Problem 1: H_{01}: $\gamma = 0$, $\alpha = 1$ (exponential) vs H_1: $\gamma > 0$, $\alpha > 0$ (GLFR).

Problem 2: H_{02}: $\gamma = 0$ (generalized exponential) vs H_1: $\gamma > 0$ (GLFR).

The log-likelihood values ($\mathcal{L}$), the likelihood ratio test statistic (Λ), the MLEs of each model, and the associated p values are presented in Table 3. Based on the p values it is clear that: (1) GLFR distribution provides a significantly better fit for both X_1 and X_2 compared to the exponential; (2) GLFR distribution provides a significantly better fit for X_1 compared to the generalized exponential distribution; (3) GLFR distribution provides a better fit for X_2 than the generalized exponential distributions. Finally using the EM algorithm we obtain

Table 3: The MLEs and the values of $\mathcal{L}$, Λ, and the p values of X_1 and X_2.

Null	X_1 MLEs	$\mathcal{L}$	Λ	p-value	X_2 MLEs	$\mathcal{L}$	Λ	p-value
H_{01}	$\widehat{\beta} = 0.024$	-174.304	23.257	< 0.0001	$\widehat{\beta} = 0.0304$	-166.219	6.562	0.038
H_{02}	$\widehat{\beta} = 0.0449$ $\widehat{\alpha} = 3.1193$	-168.815	6.279	0.012	$\widehat{\beta} = 0.0413$ $\widehat{\alpha} = 1.6776$	-163.937	1.998	0.157

the MLEs of α_1, α_2, α_3, β and γ as (0.492, 0.166, 0.411, 2.013×10^{-4}, 8.051×10^{-4}).

In order to investigate if the BGLFR distribution provides a better fit to data set 1, than the MO model and the BVGE model, we use the Akaike Information Criterion (AIC), see Akaike (1969), Bayesian Information Criterion (BIC), see Schwartz (1978), and also likelihood ratio test (LRT). Since the MO model cannot be obtained as a special case of the BGLFR distribution we cannot use the LRT test directly to compare the MO model and the BGLFR model. It is natural to use AIC or BIC in this case. On the other hand since BVGE distribution can be obtained as a special case of the BGLFR model, the LRT also can be used in testing between BVGE and BGLFR models.

In the enclosed Table 4 we provide the MLEs of the unknown parameters of the MO and the BVGE models. We have also enclosed the AIC and BIC values for model selection purposes.

Table 4: The MLEs and the values of $\mathcal{L}$, Λ, AIC and BIC.

Model	MLEs	$\mathcal{L}$	AIC	BIC
MO	$\widehat{\lambda}_1 = 0.012,\ \widehat{\lambda}_2 = 0.014,\ \widehat{\lambda}_3 = 0.022$	-339.006	684.012	-344.423
BVGE	$\widehat{\alpha}_1 = 1.351,\ \widehat{\alpha}_2 = 0.465,\ \widehat{\alpha}_3 = 1.153$ $\widehat{\beta} = 0.039$	-296.935	601.870	-304.157
BGLFR	$\widehat{\alpha}_1 = 0.492,\ \widehat{\alpha}_2 = 0.166,\ \widehat{\alpha}_3 = 0.411$ $\widehat{\beta} = 2.013 \times 10^{-4},\ \widehat{\gamma} = 8.051 \times 10^{-4}$	-293.379	596.757	-302.406

It is clear that between MO model and BGLFR model, clearly BGLFR model is preferable, based on both AIC and BIC values. Now to choose between BVGE and BGLFR, based on AIC, BGLFR is preferable, BIC suggests BVGE model. If we perform the LRT test, while the null hypothesis is BVGE model and the alternative is BGLFR model, the test statistic is 6.73 with the $0.025 < p < 0.05$. Since p value is not very high, we prefer BGLFR than BVGE for analyzing this data set.

6 Multivariate Generalized Linear Failure Rate Distribution

In this section we are in a position to define the m-variate generalized linear failure rate distribution and provide some of its properties. It may be mentioned that recently Franco and

Vivo (2009), provided a multivariate extension of Sarhan-Balakrishnan bivariate distribution and studied its several properties.

Suppose $U_1, \cdots, U_{m+1}$ are $m+1$ independent random variables such that $U_i \sim$ GLFR(α_i, β, γ) for $i = 1, \cdots, m+1$. Define

$$X_j = \max\{U_j, U_{m+1}\},\ j = 1, 2, \cdots, m\,,$$

then we say that $\boldsymbol{X} = (X_1, \cdots, X_m)$ is a m-variate GLFR with parameters (α_1, $\cdots, \alpha_{m+1}$, β, γ), and it will be denoted by MGLFR($m, \alpha_1, \cdots, \alpha_{m+1}, \beta, \gamma$). The joint CDF of $\boldsymbol{X}$ can be easily obtained as follows;

THEOREM 6.1: If $\boldsymbol{X} = (X_1, \cdots, X_m) \sim$ MGLFR($m, \alpha_1, \cdots, \alpha_{m+1}, \beta, \gamma$), then the joint CDF of $\boldsymbol{X}$ for $x_1 > 0, \cdots, x_m > 0$ is

$$F_X(\boldsymbol{x}) = \prod_{i=1}^{m+1} F_{GLFR}(x_i; \alpha_i, \beta, \gamma), \tag{31}$$

where $\boldsymbol{x} = (x_1, \cdots, x_m)$ and $x_{m+1} = \min\{x_1, \cdots, x_m\}$.

Along the same line as the bivariate GLFR distribution, the multivariate GLFR distribution (31) also can be obtained from the m-variate Marshall-Olkin copula with the marginals as the GLFR distributions. In this case (31) can be obtained from the following MO copula

$$C_\theta(u_1, \cdots, u_m) = u_1^{1-\theta_1} \cdots u_m^{1-\theta_m} \min\{u_1^{\theta_1} \cdots u_m^{\theta_m}\}, \tag{32}$$

here $\theta = (\theta_1, \cdots, \theta_m)$, and

$$\theta_1 = \frac{\alpha_{m+1}}{\alpha_1 + \alpha_{m+1}}, \cdots, \theta_m = \frac{\alpha_{m+1}}{\alpha_m + \alpha_{m+1}}.$$

For $m > 1$, the MGLFR distribution function can also be written as

$$F_X(\boldsymbol{x}) = pF_a(\boldsymbol{x}) + (1-p)F_s(\boldsymbol{x}), \tag{33}$$

here $0 < p < 1$, F_a and F_s denote the absolute continuous and singular part of F respectively. The corresponding PDF of $\boldsymbol{X}$ also can be written as

$$f_X(\boldsymbol{x}) = p f_a(\boldsymbol{x}) + (1-p) f_s(\boldsymbol{x}). \qquad (34)$$

In writing (34) it needs to be understood that f_a is the PDF with respect to m-dimensional Lebesgue measure, and f_s also can be further decomposed and they are PDFs with respect to $1, \cdots, (m-1)$ dimensional Lebesgue measures. It is not difficult to obtain the explicit expressions of F_s and f_s for the general m, but they are quite tedious, and they are not pursued here. We provide the explicit expression of f_a and p in the appendix.

Now we provide the distribution functions of the marginals, conditionals and the extreme order statistics of the MGLFR distribution.

THEOREM 6.2: If $\boldsymbol{X} = (X_1, \cdots, X_m) \sim \text{MGLFR}(m, \alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \beta, \gamma)$, then

(a) $X_1 \sim \text{GLFR}(\alpha_1 + \alpha_{m+1}, \beta, \gamma), \cdots, X_m \sim \text{GLFR}(\alpha_m + \alpha_{m+1}, \beta, \gamma)$.

(b) For $2 \leq s \leq m$, $(X_1, \cdots, X_s) \sim \text{MGLFR}(s, \alpha_1, \cdots, \alpha_s, \alpha_{m+1}, \beta, \gamma)$

(c) The conditional distribution of $(X_1, \cdots, X_s)$, given $\{X_{s+1} \leq x_{s+1}, \cdots, X_m \leq x_m\}$ is

$$P(X_1 \leq x_1, \cdots, X_s \leq x_s | X_{s+1} \leq x_{s+1}, \cdots, X_m \leq x_m) =$$
$$\left[\prod_{j=1}^{s} F_{GLFR}(x_j, \alpha_j, \beta, \gamma)\right] \begin{cases} 1 & \text{if } z = v \\ F_{GLFR}(z, \alpha_{m+1}, \beta, \gamma) F_{GLFR}(v, \alpha_{m+1}, \beta, \gamma) & \text{if } z < v, \end{cases}$$

where $z = \min\{x_1, \cdots, x_s\}$ and $v = \min\{x_{s+1}, \cdots, x_m\}$.

(d) If $T_m = \max\{X_1, \cdots, X_m\}$, then

$$F_{T_n}(t) = P(T_n \leq t) = F_{GLFR}(t, \alpha_1 + \cdots + \alpha_{m+1}, \beta, \gamma)$$

(e) If $T_1 = \min\{X_1, \cdots, X_m\}$, then

$$F_{T_1}(t) = P(T_1 \leq t) = F_{GLFR}(t, \alpha_{m+1}, \beta, \gamma) \times \left(1 - \prod_{i=1}^{m}(1 - F_{GLFR}(t, \alpha_i, \beta, \gamma))\right)$$

PROOF: The proofs of (a), (b), (c) and (d) are quite simple and are not provided here.

(e) Note that

$$F_{T_1}(t) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{I_k \in S_k} F_{I_k}(t, \cdots, t),$$

where $I_k = (i_1, \cdots, i_k),\ 1 \leq i_1 \neq \cdots \neq i_k \leq m$, is a k-dimensional subset and S_k is the set of all ordered k-dimensional subsets of $\{1, \cdots, m\}$. Further

$$F_{I_k}(t, \cdots, t) = P(X_{i_1} \leq t, \cdots, X_{i_k} \leq t).$$

Therefore, using part (b),

$$F_{T_1}(t) = F_{GLFR}(t, \alpha_{m+1}, \beta, \gamma) \times \sum_{I_k \in S_k} F_{GLFR}(t, \alpha_{i_1} + \cdots + \alpha_{i_k}, \beta, \gamma)$$

Now using the fact

$$\sum_{k=1}^{m} (-1)^{k-1} \sum_{I_k \in S_k} F_{GLFR}(t, \alpha_{i_1} + \cdots + \alpha_{i_k}, \beta, \gamma) = 1 - \prod_{i=1}^{m} (1 - F_{GLFR}(t, \alpha_i, \beta, \gamma))$$

the result follows. ∎

7 CONCLUSIONS

In this paper we have introduced the bivariate generalized linear failure rate distribution whose marginals are generalized linear failure rate distributions. The proposed bivariate distribution is a singular distribution, and it can be used quite effectively instead of Marshall-Olkin bivariate exponential model, or the bivariate generalized exponential model when there are ties in the data. Several properties of this new distribution have been established, and also we proposed to use the EM algorithm to compute the maximum likelihood estimators.

Further we have proposed its multivariate generalization. Several properties have been discussed. It can be obtained by using the multivariate Marshall-Olkin copula coupled with generalized linear failure rate marginals. It may be mentioned that the EM algorithm along the same line as the bivariate case may be developed. Alternatively, using the copula structure, other estimators as proposed by Kim *et al.* (2006) may be used and their properties can be established. The work is in progress, it will be reported later.

Appendix

In this appendix we provide the explicit expression of f_a and p of (34) for general m. Let $k \in \{1, \cdots, m\}$ be the number of the different components of $\boldsymbol{x} = (x_1, \cdots, x_m)$, *i.e.* when $k = 1$, all x_i's are equal, and all x_i's are different when $k = m$. Then $\boldsymbol{x}$ belongs to the set where F_X is absolutely continuous if and only if $k = m$. For each $\boldsymbol{x}$ with $k = m$, there exists a permutation $P_m = (i_1, \cdots, i_m)$, such that $x_{i_1} < \cdots < x_{i_m}$, and let us define the following function

$$f_{P_m}(\boldsymbol{x}) = f_{GLFR}(x_{i_1}, \alpha_{i_1} + \alpha_{m+1}, \beta, \gamma) f_{GLFR}(x_{i_2}, \alpha_{i_2}, \beta, \gamma) \cdots f_{GLFR}(x_{i_m}, \alpha_{i_m}, \beta, \gamma) \tag{35}$$

Differentiating (33) with respect to $x_1, \cdots, x_m$, we obtain

$$\frac{\partial^m F_X(x_1, \cdots, x_m)}{\partial x_1 \cdots \partial x_m} = p f_a(\boldsymbol{x}) = f_{P_m}(\boldsymbol{x}),$$

for $P_m = (i_1, \cdots, i_m)$, such that $x_{i_1} < \cdots < x_{i_m}$, and f_a is the joint density function of the absolute continuous part as mentioned before. Moreover, p may be obtained

$$\begin{aligned} p &= p \int_{\Re^m} f_a(\boldsymbol{x}) dx_1 \cdots dx_m = \sum_{P_m} \int_{x_{i_m}=0}^{\infty} \int_{x_{i_{m-1}}=0}^{x_{i_m}} \cdots \int_{x_{i_1}=0}^{x_{i_2}} f_{P_m}(\boldsymbol{x}) dx_{i_1} \cdots dx_{i_m} \\ &= \sum_{P_m} \frac{\alpha_{i_2}}{\alpha_{i_1} + \alpha_{i_2} + \alpha_{m+1}} \times \frac{\alpha_{i_3}}{\alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} + \alpha_{m+1}} \times \cdots \times \frac{\alpha_{i_m}}{\alpha_{i_1} + \cdots + \alpha_{i_m} + \alpha_{m+1}}. \end{aligned}$$

Therefore,

$$f_a(\boldsymbol{x}) = \frac{1}{p} f_{P_n}(\boldsymbol{x}).$$

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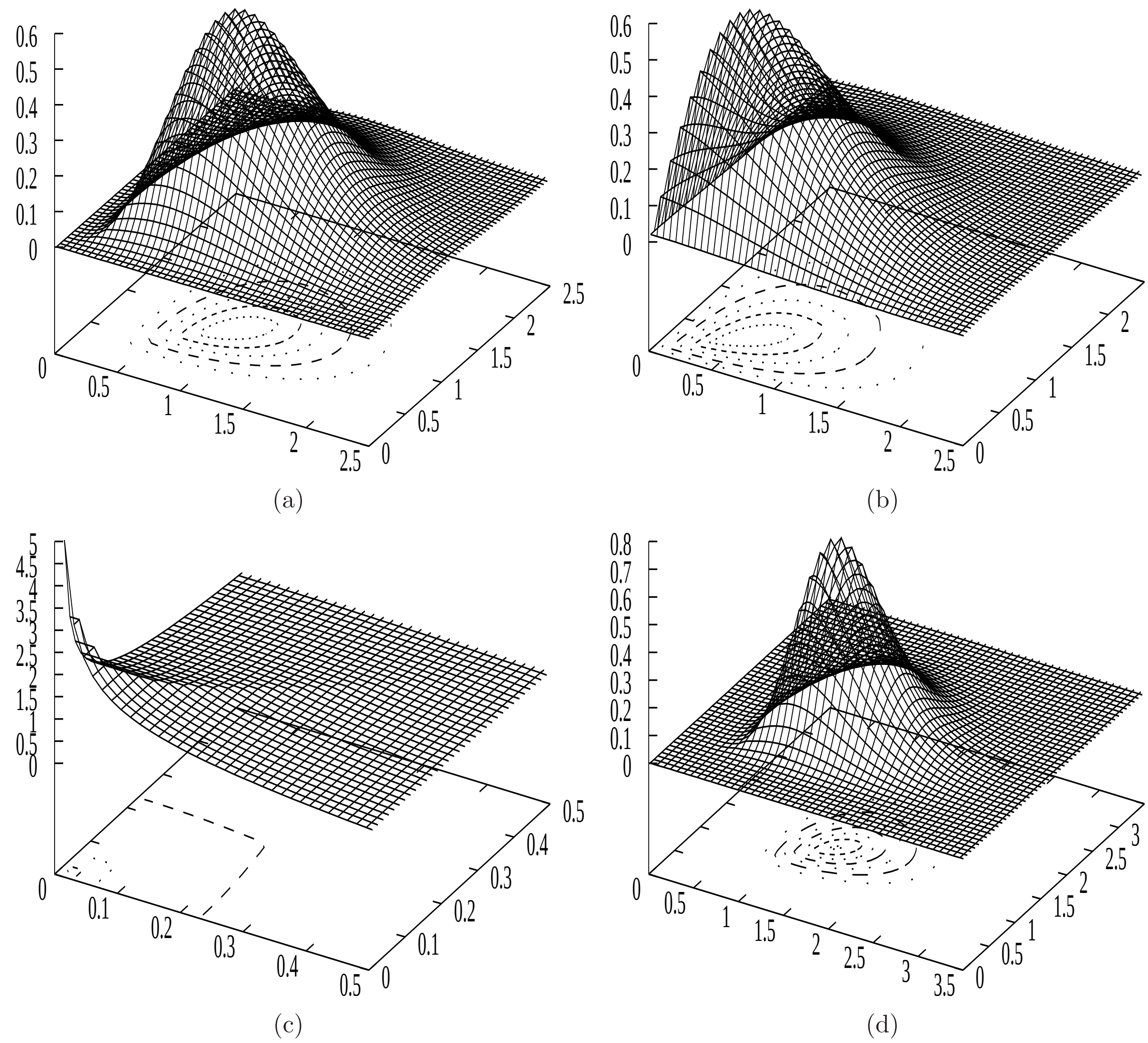

Figure 1: Surface and contour plots of the absolute continuous part of the joint PDF of the BGLFR model, for different values of $(\alpha_1, \alpha_2, \alpha_3)$. We have assumed $\beta = \gamma = 1$ in all the cases. (a) (2, 2, 2) (b) (1, 1, 1) (c) (0.5, 0.5, 0.5) (d) (5, 5, 5).