The Bivariate Generalized Linear Failure Rate Distribution and its Multivariate Extension

Ammar M. Sarhan∗†, David C. Hamilton†, Bruce Smith† & Debasis Kundu‡

Abstract

The two-parameter linear failure rate distribution has been used quite successfully to analyze lifetime data. Recently, a new three-parameter distribution, known as the generalized linear failure rate distribution has been introduced by exponentiating the linear failure rate distribution. The generalized linear failure rate distribution is a very flexible lifetime distribution, and the probability density function of the generalized linear failure rate distribution can take different shapes. Its hazard function also can be increasing, decreasing and bathtub shaped. The main aim of this paper is to introduce a bivariate generalized linear failure rate distribution, whose marginals are generalized linear failure rate distributions. It is obtained using the same approach as the Marshall-Olkin bivariate exponential distribution. Different properties of this new distribution are established. The bivariate generalized linear failure rate distribution has five parameters and the maximum likelihood estimators are obtained using the EM algorithm. A data set is analyzed for illustrative purposes. Finally, some generalizations to the multivariate case are proposed.

Key Words and Phrases Marshall-Olkin copula; Maximum likelihood estimator; failure rate; EM algorithm; Fisher information matrix.

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1 INTRODUCTION

The two-parameter linear failure rate (LFR) distribution, whose hazard function is monotonically increasing in a linear fashion, has been used quite successfully to analyze lifetime data. For some basic properties and for different estimation procedures of the parameters of the LFR distribution, the readers are referred to Bain (1974), Pandey et al. (1993), Sen and Bhattacharyya (1995), Lin et al. (2003, 2006) and the references cited therein.

Recently, Sarhan and Kundu (2009) introduced a three-parameter generalized linear failure rate (GLFR) distribution by exponentiating the LFR distribution as was done for the exponentiated Weibull distribution by Mudholkar et al. (1995). The exponentiation introduces an extra shape parameter in the model, which may bring more flexibility in the shape of the probability density function (PDF) and hazard function. Several properties of this new distribution are established. It is observed that several known distributions like exponential, Rayleigh and LFR distributions can be obtained as special cases of the GLFR distribution.

The aim of this paper is to introduce a new bivariate generalized linear failure rate (BGLFR) distribution, whose marginals are GLFR distributions. This new five-parameter BGLFR distribution is obtained using a similar method as was used for the Marshall-Olkin bivariate exponential model, Marshall and Olkin (1969). The proposed BGLFR distribution is constructed from three independent GLFR distributions using a maximization process. Creating a bivariate distribution with given marginals using this technique is nothing new. Alternatively, the same BGLFR distribution can be obtained by coupling the GLFR marginals with the Marshall-Olkin copula (Nelsen, 1999). This new distribution is a singular distribution, and it can be used quite conveniently if there are ties in the data. The joint cumulative distribution function (CDF) can be expressed as a mixture of an absolute continuous distribution function and a singular distribution function. The joint probability
density function (PDF) of the BGLFR distribution can take different shapes and the cumulative distribution function can be expressed in a compact form. The BGLFR distribution can be applied to a maintenance model or a stress model as introduced by Kundu and Gupta (2009).

Several dependency properties of this new distribution are investigated, which will be useful for data analysis purposes. The BGLFR copula has a total positivity of order two (TP$_2$) property. Each component is stochastically increasing with respect to the other. This implies that the correlation is always non-negative and the two variables are positively quadrant dependent. Moreover, the correlation between the two variables varies between 0 and 1. Kendall’s tau index can be calculated using the copula property and can be positive. The population version of the medial correlation coefficient as defined by Blomqvist (1950) is always non-negative. The bivariate tail dependence is always positive.

The BGLFR distribution has five parameters, and their estimation is an important problem in practice. The usual maximum likelihood estimators can be obtained by solving five non-linear equations in five unknowns directly, which is not a trivial issue. To avoid difficult computation we treat this problem as a missing value problem and use the EM algorithm, which can be implemented more conveniently than the direct maximization process. Another advantage of the EM algorithm is that it can be used to obtain the observed Fisher information matrix, which is helpful for constructing the asymptotic confidence intervals for the parameters.

Alternatively, it is possible to obtain approximate maximum likelihood estimators by estimating the marginals first and then estimating the dependence parameter through copula function, as suggested by Joe (1997, chapter 10), which have the same rate of convergence as the maximum likelihood estimators. It is computationally less involved compared to the MLE calculations. This approach is not pursued here. Analysis of a data set is presented
for illustrative purposes. The proposed model provides a better fit than the Marshall-Olkin bivariate exponential model or the recently proposed bivariate generalized exponential model (Kundu and Gupta, 2009).

Although in this paper we mainly discussed the BGLFR, many of our results can be easily extended to the multivariate case. Moreover, the LFR distribution is a proportional reversed hazard model, and our method may be used to introduce other bivariate proportional reversed hazard models.

The rest of the paper is organized as follows. We briefly introduce the GLFR distribution in Section 2. In Section 3 we introduce the BGLFR distribution and study its different properties. The EM algorithm is described in Section 4, and analysis of a data set is presented in Section 5. We discuss the multivariate generalization in Section 6, and finally conclude the paper in Section 7.

2 Generalized Linear Failure Rate Distribution

A random variable $X$ has a linear failure rate distribution with parameters $\beta \geq 0$ and $\gamma \geq 0$ (such that $\beta + \gamma > 0$), if $X$ has the following distribution function:

$$F_{LFR}(x; \beta, \gamma) = 1 - \exp \left\{ -\beta x - \frac{\gamma}{2} x^2 \right\},$$

(1)

for $x > 0$. The exponential distribution with mean $1/\beta$ (ED($\beta$)) and the Rayleigh distribution with parameter $\gamma$ (RD($\gamma$)) can be obtained as special cases from the LFR distribution. The PDF of the LFR distribution can be decreasing or unimodal, but the failure rate function is either increasing or constant only (Sen and Bhattacharyya, 1995).

Sarhan and Kundu (2009) introduced the GLFR distribution by exponentiating the LFR distribution function as follows. A random variable $X$ is said to have a GLFR distribution
with parameters $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ (GLFR$(\alpha, \beta, \gamma)$), if it has the CDF
\[ F_{GLFR}(x; \alpha, \beta, \gamma) = \left(1 - \exp\left\{-\beta x - \frac{\gamma}{2} x^2\right\}\right)^\alpha, \tag{2} \]
for $x > 0$. The corresponding PDF has the form;
\[ f_{GLFR}(x; \alpha, \beta, \gamma) = \frac{\alpha}{\alpha + 1}(\beta + \gamma x) e^{-\left(\beta x + \frac{\gamma}{2} x^2\right)} \left(1 - \exp\left\{-\beta x - \frac{\gamma}{2} x^2\right\}\right)^{\alpha - 1}, \tag{3} \]
for $x > 0$. The PDF of the GLFR distribution is either decreasing or unimodal, and it can have constant, increasing, decreasing or bathtub shaped hazard function. It is immediate from (2) that if $\alpha$ is an integer, then the CDF of GLFR$(\alpha, \beta, \gamma)$ represents the CDF of the maximum of a simple random sample of size $\alpha$, from the LFR distribution. Therefore, when $\alpha$ is an integer, GLFR provides the distribution function of a parallel system when each component has the LFR distribution.

The mean or the other moments cannot be obtained in explicit form, but can be written in terms of infinite series (Sarhan and Kundu, 2009). However, because of the closed form CDF, the median or other percentile points can be obtained explicitly. Because of the exponentiated nature of the CDF, the GLFR distribution is closed under maximum, \textit{i.e.} if $X_1, \cdots, X_n$ are independently distributed such that $X_i$ follows the GLFR$(\alpha_i, \beta, \gamma)$ distribution, for $i = 1, \cdots, n$, then $\max\{X_1, \cdots, X_n\}$ is GLFR $\left(\sum_{i=1}^{n} \alpha_i, \beta, \gamma\right)$. Moreover, if $R$ is the stress-strength parameter, \textit{i.e.} $R = P(X_1 < X_2)$ where $X_1$ and $X_2$ are as defined above, then
\[
R = P(X_1 < X_2) = \frac{\alpha_1}{\alpha_1 + \alpha_2}.
\]
For order statistics, moments of order statistics, characterization, and for an estimation procedure, the readers are referred to Sarhan and Kundu (2009).
3 BIVARIATE GENERALIZED FAILURE RATE DISTRIBUTION

In this section we introduce the BGLFR distribution using a method similar to that which was used by Marshall and Olkin (1969) to define the Marshall-Olkin bivariate exponential (MOBE) distribution.

Suppose $U_1, U_2$ and $U_3$ are three independent random variables such that $U_i \sim \text{GLFR} (\alpha_i, \beta, \gamma)$ for $i = 1, 2$ and 3. Define

$$X_1 = \max\{U_1, U_3\} \quad \text{and} \quad X_2 = \max\{U_2, U_3\}. \quad (4)$$

Then we say that the bivariate vector $(X_1, X_2)$ has a bivariate GLFR (BGLFR) distribution, with parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$ and we denote it by BGLFR $(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$. The following interpretations can be provided for BGLFR model.

**Competing risks model:** Assume a system has two components, labeled 1 and 2 and the survival time of component $i$ is denoted by $X_i, i = 1, 2$. It is considered that there are three independent causes of failures, which may affect the system. Only component 1 can fail due to cause 1, and similarly only component 2 can fail due to cause 2, while both the components fail at the same time due to cause 3. Let $U_i$ be the lifetime of cause $i, i = 1, 2, 3$. If $U_1, U_2, U_3$ follow GLFR distribution, then $(X_1, X_2)$ follows BGLFR model.

**Shock model:** Suppose there are three independent sources of shocks, say 1, 2, and 3. Suppose these shocks are affecting a system with two components, say 1 and 2. It is assumed that, the shock from source 1 reaches the system destroys component 1 immediately, the shock from source 2 reaches the system destroys component 2 immediately while if the shock from source 3 hits the system destroys both the components immediately. Let $U_i$ denote the inter-arrival times between the shocks in source $i, i=1,2,3$, which follow the distribution GLFRD. If $X_1, X_2$ denote the survival times of the components, then $(X_1, X_2)$ follows BGLFR model.
If \((X_1, X_2) \sim \text{BGLFR} (\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)\), then the corresponding CDF, PDF and the marginals are provided in the following theorem. The proofs are not difficult and therefore are omitted.

**Theorem 3.1:** Suppose \((X_1, X_2) \sim \text{BGLFR} (\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)\). Then

(a) The joint CDF of \((X_1, X_2)\) can be written as

\[
F_{X_1,X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) = \prod_{i=1}^{3} F_{\text{GLFR}}(x_i; \alpha_i, \beta, \gamma),
\]

where \(x_3 = \min\{x_1, x_2\}\).

(b) The joint PDF of \((X_1, X_2)\) can be written as

\[
f_{X_1,X_2}(x_1, x_2) = \begin{cases} 
  f_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 < \infty \\
  f_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 < \infty \\
  f_0(x) & \text{if } 0 < x_1 = x_2 = x < \infty,
\end{cases}
\]

where

\[
\begin{align*}
  f_1(x_1, x_2) &= f_{\text{GLFR}}(x_1; \alpha_1 + \alpha_3, \beta, \gamma) f_{\text{GLFR}}(x_2; \alpha_2, \beta, \gamma) \\
  f_2(x_1, x_2) &= f_{\text{GLFR}}(x_1; \alpha_1, \beta, \gamma) f_{\text{GLFR}}(x_2; \alpha_2 + \alpha_3, \beta, \gamma) \\
  f_0(x) &= \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{\text{GLFR}}(x; \alpha_1 + \alpha_2 + \alpha_3, \beta, \gamma)
\end{align*}
\]

(c) The marginal distributions of \(X_1\) and \(X_2\) are \(\text{GLFR}(\alpha_1+\alpha_3, \beta, \gamma)\) and \(\text{GLFR}(\alpha_2+\alpha_3, \beta, \gamma)\) respectively.

The joint distribution function of \(X_1\) and \(X_2\) has a singular part along the line \(x_1 = x_2\), with weight \(\frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\), and has an absolute continuous part on \(0 < x_1 \neq x_2 < \infty\) with weight \(\frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}\). In writing the joint PDF, it is understood that the first two parts are the joint PDF with respect to two dimensional Lebesgue measure, whereas the third part is the PDF with respect to one dimensional Lebesgue measure along the line \(x_1 = x_2\).
This is similar to the Marshall-Olkin bivariate exponential model or bivariate generalized
exponential model.

For fixed \( \alpha_1, \alpha_2, \beta \) and \( \gamma \), as \( \alpha_3 \) varies from 0 to \( \infty \), the correlation between \( X_1 \) and \( X_2 \) varies between 0 and 1. This is because, if \( \alpha_3 = 0 \), then \( X_1 \) and \( X_2 \) become independent, and when \( \alpha_3 \) tends to infinity, then \( U_3 \) tends to infinity with probability 1. Thus \( U_3 > U_1 \) and \( U_3 > U_2 \) with probability 1. Therefore, \( X_1 = X_2 \) with probability 1, as \( \alpha_3 \) tends to infinity. The joint survival function and the conditional distributions can be easily obtained. Surface plots of the absolutely continuous part of the joint PDF of \((X_1, X_2)\) are provided in Figure 1. The joint PDF can take various shapes depending on the parameter values.

Interestingly, the BGLFR distribution can be obtained by using the Marshall-Olkin (MO) copula with the marginals as the GLFR distributions. To every bivariate distribution function \( F_{X_1, X_2} \) with continuous marginals \( F_{X_1} \) and \( F_{X_2} \) corresponds a unique bivariate distribution function with uniform margins \( C : [0,1]^2 \rightarrow [0,1] \) called a copula, such that \( F_{X_1, X_2}(x_1, x_2) = C\{F_{X_1}(x_1), F_{X_2}(x_2)\} \) holds for all \((x_1, x_2) \in \mathbb{R}^2\) (Nelsen, 1999). The MO copula is

\[
C_{\theta_1, \theta_2}(u_1, u_2) = u_1^{1-\theta_1}u_2^{1-\theta_2} \min\{u_1^{\theta_1}, u_2^{\theta_2}\},
\]

for \( 0 < \theta_1 < 1 \) and \( 0 < \theta_2 < 1 \). Using \( u_i = F_{X_i}(x_i) \) where \( X_i \) is GLFR(\( \alpha_i + \alpha_3, \beta, \gamma \)) and \( \theta_i = \alpha_3/(\alpha_i + \alpha_3) \), \( i=1,2,3 \), gives the same joint distribution function \( F_{X_1, X_2} \) as (5).

Generating values from a BGLFR distribution is straightforward. First, we can generate values for three independent GLFR random variables and then use (4) to generate \((X_1, X_2)\). Alternatively, we can generate \((u_1, u_2)\) from the copula \( C_{\theta_1, \theta_2} \), and then use the inversion formula to obtain \((X_1, X_2)\). If \((X_1, X_2)\) follows the BGLFR(\( \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \)), then \( \max\{X_1, X_2\} \) follows the GLFR(\( \alpha_1 + \alpha_2 + \alpha_3, \beta, \gamma \)) distribution. The stress strength parameter of \((X_1, X_2)\) is

\[
P(X_1 < X_2) = P(U_1 < U_3 < U_2) + P(U_3 < U_1 < U_2) = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}.
\]
Now we will provide several dependency results between the two variables. Lehmann (1966) defined two random variables $X_1$ and $X_2$ to be positive quadrant dependent (PQD) if for all $x_1$ and $x_2$,

$$P(X_1 \leq x_1, X_2 \leq x_2) \geq P(X_1 \leq x_1)P(X_2 \leq x_2).$$  \hspace{1cm} (9)

Intuitively, $X_1$ and $X_2$ are PQD if the probability that they are simultaneously small or simultaneously large is at least as great as it would be if they were independent. PQD is a copula property and (9) can be written equivalently as

$$C(u_1, u_2) \geq u_1 u_2, \text{ for all } u_1, u_2 \in [0,1]^2.$$  \hspace{1cm} (10)

This condition is satisfied by the MO copula. Therefore, if $(X_1, X_2)$ follow the BGLFR distribution, then they are PQD. Because $X_1$ and $X_2$ are PQD, for every pair of increasing functions $g_1(\cdot)$ and $g_2(\cdot)$ (Barlow and Proschan, 1981) the following relation is satisfied

$$\text{Cov}\{g_1(X_1), g_2(X_2)\} \geq 0$$  \hspace{1cm} (11)

Moreover, it can also be verified that $X_1$ is stochastically increasing in $X_2$, and similarly $X_2$ is also stochastically increasing in $X_1$. A non-negative function $g$ defined on $\mathbb{R}^2$ is total positivity of order two, abbreviated by $\text{TP}_2$, if for all $x_1 < x_2$ and $y_1 < y_2$,

$$g(x_1, y_1)g(x_2, y_2) \geq g(x_2, y_1)g(x_1, y_2).$$  \hspace{1cm} (12)

The MO copula satisfies this condition. Therefore (Nelsen, 1999), if $(X_1, X_2)$ follows the BGLFR distribution, then $X_1$ and $X_2$ are left corner set decreasing, i.e. $P(X_1 \leq x_1, X_2 \leq x_2 | X_1 \leq x'_1, X_2 \leq x'_2)$ is non-decreasing in $x'_1$ and in $x'_2$, for all $x_1$ and $x_2$.

The copula provides a natural way to measure the dependence between two random variables. Now we provide some measures of dependence namely the Kendall's tau and the medial correlation. We further study the dependence of extreme events.
Kendall’s tau is defined as the probability of concordance minus the probability of discordance between two pairs of random vectors \((X_1, X_2)\) and \((Y_1, Y_2)\),

\[
\tau = P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P[(X_1 - Y_1)(X_2 - Y_2) < 0].
\]  

(13)

where \((X_1, X_2)\) and \((Y_1, Y_2)\) are independent and identically distributed random vectors. Nelsen (1999) has shown that Kendall’s tau index is also a copula property. Moreover, the MO copula has Kendall’s tau as

\[
\theta_1 \theta_2 \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}.
\]

For fixed \(\alpha_1\) and \(\alpha_2\), as \(\alpha_3\) varies from 0 to \(\infty\), \(\tau_{X_1,X_2}\) varies between 0 and 1.

Blomqvist (1950) defined the median correlation coefficient, \(M_{X_1,X_2}\), between two continuous random variables \(X_1\) and \(X_2\) as follows. If \(M_{X_1}\) and \(M_{X_2}\) denote the median of \(X_1\) and \(X_2\) respectively, then

\[
M_{X_1,X_2} = P[(X_1 - M_{X_1})(X_2 - M_{X_2}) > 0] - P[(X_1 - M_{X_1})(X_2 - M_{X_2}) < 0].
\]

(15)

Domma (2009) observed that Blomqvist’s medial correlation coefficient is a copula property and it can be verified that

\[
M_{X_1,X_2} = 4F_{X_1,X_2}(M_{X_1}, M_{X_2}) - 1 = 4C_{\theta_1,\theta_2} \left( \frac{1}{2}, \frac{1}{2} \right) - 1.
\]

(16)

Therefore, if \((X_1, X_2) \sim \text{BGLFR}(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)\), then \(M_{X_1,X_2}\) is

\[
M_{X_1,X_2} = \begin{cases} 
\left( \frac{1}{2} \right)^{2-\theta_2} & \text{if } \theta_1 > \theta_2 \\
\left( \frac{1}{2} \right)^{2-\theta_1} & \text{if } \theta_1 < \theta_2.
\end{cases}
\]

(17)

where as above \(\theta_i = \alpha_3/(\alpha_3 + \alpha_i)\) The minimum and the maximum values of \(M_{X_1,X_2}\) are 1/4 and 1/2 respectively.
The bivariate tail dependence measures the amount of dependence in the upper quadrant (or lower quadrant) tail of a bivariate distribution (Joe, 1997). For bivariate random vectors \((X_1, X_2)\), the upper tail dependence (if it exists) is defined as follows

\[
\lambda_U = \lim_{z \to 1^-} P(X_2 > F_{X_2}^{-1}(z) \mid X_1 > F_{X_1}^{-1}(z)).
\] (18)

Intuitively, the upper tail dependence exists when there is a positive probability that some positive outliers may occur jointly. If \(\lambda_U \in (0, 1]\), then \(X_1\) and \(X_2\) are said to be asymptotically dependent, if \(\lambda_U = 0\), then they are asymptotically independent. Similarly, the lower tail dependence parameter \(\lambda_L\) (if it exists) is defined as follows

\[
\lambda_L = \lim_{z \to 0^+} P(X_2 \leq F_{X_2}^{-1}(z) \mid X_1 \leq F_{X_1}^{-1}(z)).
\] (19)

These parameters are non-parametric and both depend only on the copula \(C\) of \(X_1\) and \(X_2\) as follows:

\[
\lambda_U = 2 - \lim_{t \to 1^-} \frac{1 - C(t, t)}{1 - t} \quad \text{and} \quad \lambda_L = \lim_{t \to 0^+} \frac{C(t, t)}{t}.
\] (20)

If \((X_1, X_2)\) follows BGLFR\((\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)\), then

\[
\lambda_U = \begin{cases} 
\theta_1 & \text{if} \quad \theta_1 < \theta_2 \\
\theta_2 & \text{if} \quad \theta_2 < \theta_1,
\end{cases}
\] (21)

and \(\lambda_L = 0\).

4 Estimation

In this section we consider the estimation of the unknown parameters of the BGLFR model. It is assumed that we have a sample of size \(n\), of the form

\[\{(x_{11}, x_{12}), \ldots, (x_{n1}, x_{n2})\}\] (22)

from BGLFR\((\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)\) and our problem is to estimate \(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\) from the given sample. First we obtain the MLEs of the unknown parameters. Since the computation of
the MLEs is computationally quite involved, we propose alternative estimators, which can be obtained in a more convenient manner.

For further development we use the following notations:

\[ I_1 = \{ i; x_{i1} < x_{i2} \}, \quad I_2 = \{ i; x_{i1} > x_{i2} \}, \quad I_0 = \{ i; x_{i1} = x_{i2} = x_i \}, \quad I = I_0 \cup I_1 \cup I_2 \]

and

\[ n_0 = |I_0|, \quad n_1 = |I_1|, \quad n_2 = |I_2|. \]

Based on the sample (22) mentioned above, the log-likelihood function of the observed data can be written as;

\[ l(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma) = \sum_{i \in I_1} \ln f_1(x_{i1}, x_{i2}) + \sum_{i \in I_2} \ln f_2(x_{i1}, x_{i2}) + \sum_{i \in I_0} \ln f_0(x_i, x_i). \] (23)

Therefore, the MLEs of the unknown parameters can be obtained by maximizing (23) with respect to the unknown parameters. It is clearly a five dimensional optimization problem. We need to solve five non-linear equations simultaneously to compute the MLEs, which may not very simple. To avoid that we propose to use the expectation maximization (EM) algorithm to compute the MLEs in this case.

It may be noted that if instead of \((X_1, X_2)\), we had observed \(U_1, U_2\) and \(U_3\), the MLEs of \(\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\) can be obtained by solving a two dimensional optimization process, which is clearly much convenient than solving a five dimensional optimization process. Due to this reason, we treat this problem as a missing value problem. It is assumed that for the bivariate random vector \((X_1, X_2)\), there is an associated random vector \((\lambda_1, \lambda_2)\) as follows;

\[
\Lambda_1 = \begin{cases} 
1 & \text{if } U_1 > U_3 \\
3 & \text{if } U_1 < U_3 
\end{cases} \quad \text{and} \quad \Lambda_2 = \begin{cases} 
2 & \text{if } U_2 > U_3 \\
3 & \text{if } U_2 < U_3. 
\end{cases}
\] (24)

Therefore, if \(X_1 = X_2\), then clearly \(\lambda_1 = \lambda_2 = 3\). But if \(X_1 < X_2\) or \(X_1 > X_2\), the corresponding \((\Lambda_1, \Lambda_2)\) is missing. If \((X_1, X_2) \in I_1\) then the possible values of \((\Lambda_1, \Lambda_2)\) are
(3,2) or (1,2) and if \((X_1, X_2) \in I_2\) then the possible values of \((\Lambda_1, \Lambda_2)\) are (1,3) or (1,2). It implies that if \((X_1, X_2) \in I_1\), then \(\Lambda_2\) is known, but \(\Lambda_1\) is unknown, and if \((X_1, X_2) \in I_2\), then \(\Lambda_1\) is known, but \(\Lambda_2\) is unknown. The following Table 1 provides the all possible orders of \(U_i\)'s, the associated \((X_1, X_2)\), \((\Lambda_1, \Lambda_2)\) values and the corresponding probabilities, which will be useful for further development.

Table 1: All possible orders of \(U_i\)'s, the associated \((X_1, X_2)\), \((\Lambda_1, \Lambda_2)\) values and the corresponding probabilities.

<table>
<thead>
<tr>
<th>Case</th>
<th>Possible Order</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(\Lambda_1)</th>
<th>(\Lambda_2)</th>
<th>Prob</th>
<th>Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(U_1 &lt; U_2 &lt; U_3)</td>
<td>(U_3)</td>
<td>(U_3)</td>
<td>3</td>
<td>3</td>
<td>(\frac{\alpha_2 \alpha_3}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3)})</td>
<td>(I_0)</td>
</tr>
<tr>
<td>2</td>
<td>(U_2 &lt; U_1 &lt; U_3)</td>
<td>(U_3)</td>
<td>(U_3)</td>
<td>3</td>
<td>3</td>
<td>(\frac{\alpha_1 \alpha_3}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3)})</td>
<td>(I_0)</td>
</tr>
<tr>
<td>3</td>
<td>(U_1 &lt; U_3 &lt; U_2)</td>
<td>(U_3)</td>
<td>(U_2)</td>
<td>3</td>
<td>2</td>
<td>(\frac{\alpha_2 \alpha_3}{(\alpha_1 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)})</td>
<td>(I_1)</td>
</tr>
<tr>
<td>4</td>
<td>(U_3 &lt; U_1 &lt; U_2)</td>
<td>(U_1)</td>
<td>(U_2)</td>
<td>1</td>
<td>2</td>
<td>(\frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)})</td>
<td>(I_1)</td>
</tr>
<tr>
<td>5</td>
<td>(U_2 &lt; U_3 &lt; U_1)</td>
<td>(U_1)</td>
<td>(U_3)</td>
<td>1</td>
<td>3</td>
<td>(\frac{\alpha_1 \alpha_3}{(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)})</td>
<td>(I_2)</td>
</tr>
<tr>
<td>6</td>
<td>(U_3 &lt; U_2 &lt; U_1)</td>
<td>(U_1)</td>
<td>(U_2)</td>
<td>1</td>
<td>2</td>
<td>(\frac{\alpha_1 \alpha_2}{(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3)})</td>
<td>(I_2)</td>
</tr>
</tbody>
</table>

Now we are in a position to provide the EM algorithm. In the ‘E’ step of the EM algorithm the observations belong to \(I_0\), we treat them as complete observations. If the observation
belongs to either $I_1$ OR $I_2$, we treat it as the missing observation. If $(x_1, x_2) \in I_1$, we form the ‘pseudo observation’ by fractioning $(x_1, x_2)$ to two partially complete ‘pseudo observation’ of the form $(x_1, x_2, u_1(\theta))$ and $(x_1, x_2, u_2(\theta))$ respectively. Here $\theta$ is the parameter vector, i.e. $\theta = (\alpha_1, \alpha_2, \alpha_3, \beta, \gamma)$ and the fractional mass $u_1(\theta)$ and $u_2(\theta)$ assigned to the ‘pseudo observation’ is the conditional probability, that $\Lambda_1$ takes the values 3 or 1 respectively given $X_1 < X_2$. It is clear from Table 1 that

$$u_1(\theta) = P(\Lambda_1 = 3|X_1 < X_2) = \frac{\alpha_3}{\alpha_1 + \alpha_3}, \quad u_2(\theta) = P(\Lambda_1 = 1|X_1 < X_2) = \frac{\alpha_1}{\alpha_1 + \alpha_3}. \quad (25)$$

Similarly, if $(x_1, x_2) \in I_2$, we form the ‘pseudo observation’ of the form $(x_1, x_2, w_1(\theta))$ and $(x_1, x_2, w_2(\theta))$. Here the fractional mass $w_1(\theta)$ or $w_2(\theta)$ assigned to the ‘pseudo observation’, is the conditional probability that the random variable $\Lambda_2$ takes the values 3 or 2 respectively, given $X_1 > X_2$. Again from Table 1 it is clear that

$$w_1(\theta) = P(\Lambda_2 = 3|X_1 > X_2) = \frac{\alpha_3}{\alpha_2 + \alpha_3}, \quad w_2(\theta) = P(\Lambda_2 = 2|X_1 > X_2) = \frac{\alpha_2}{\alpha_2 + \alpha_3}. \quad (26)$$

From now on for brevity, we write $u_1(\theta), u_2(\theta), w_1(\theta), w_2(\theta)$ as $u_1, u_2, w_1, w_2$ respectively.

Now we are in a position to provide the ‘E’-step of the EM algorithm. We will be using the following notation; $\theta_i = (\alpha_i, \beta, \gamma); \ i = 1, 2, 3$. Also, $f(\cdot; \theta_i)$ and $F(\cdot; \theta_i)$ denote respectively the PDF and CDF of the GLFR($\alpha_i, \beta, \gamma$) for $i = 1, 2, 3$. The log-likelihood function of the ‘pseudo data’ (‘E’-step) can be written as

$$l_{pseudo}(\theta) = \sum_{i \in I_0} \ln f(x_i; \theta_3) + \sum_{i \in I_0} \ln F(x_i; \theta_1) + \sum_{i \in I_0} \ln F(x_i; \theta_2) +$$

$$u_1 \left( \sum_{i \in I_1} \ln f(x_{i1}; \theta_3) + \sum_{i \in I_1} \ln f(x_{i2}; \theta_2) + \sum_{i \in I_1} \ln F(x_{i1}; \theta_1) \right)$$

$$u_2 \left( \sum_{i \in I_1} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_1} \ln f(x_{i2}; \theta_2) + \sum_{i \in I_1} \ln F(x_{i1}; \theta_3) \right)$$

$$w_1 \left( \sum_{i \in I_2} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_2} \ln f(x_{i2}; \theta_3) + \sum_{i \in I_2} \ln F(x_{i2}; \theta_2) \right)$$

$$w_2 \left( \sum_{i \in I_2} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_2} \ln f(x_{i2}; \theta_3) + \sum_{i \in I_2} \ln F(x_{i2}; \theta_2) \right)$$

14
\[
\begin{align*}
\sum_{i \in I_2} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_2} \ln f(x_{i2}; \theta_2) + \sum_{i \in I_2} \ln F(x_{i2}; \theta_3)
\end{align*}
\]

where

\[
\begin{align*}
l_1(\theta_1) &= \sum_{i \in I_0} \ln F(x_i; \theta_1) + u_1 \sum_{i \in I_1} \ln F(x_{i1}; \theta_1) + u_2 \sum_{i \in I_2} \ln f(x_{i1}; \theta_1) + \sum_{i \in I_2} \ln f(x_{i1}; \theta_1) \\
l_2(\theta_2) &= \sum_{i \in I_0} \ln F(x_i; \theta_2) + u_1 \sum_{i \in I_1} \ln F(x_{i2}; \theta_2) + u_2 \sum_{i \in I_2} \ln f(x_{i2}; \theta_2) + \sum_{i \in I_2} \ln f(x_{i2}; \theta_2) \\
l_3(\theta_3) &= \sum_{i \in I_0} \ln f(x_i; \theta_3) + u_1 \sum_{i \in I_1} \ln f(x_{i1}; \theta_3) + u_2 \sum_{i \in I_2} \ln F(x_{i1}; \theta_3) + \\
&\quad w_1 \sum_{i \in I_2} \ln f(x_{i2}; \theta_3) + w_2 \sum_{i \in I_1} \ln F(x_{i2}; \theta_3).
\end{align*}
\]

Now at the 'M'-step we need to maximize (27) with respect to unknown parameters. For fixed \(\beta\) and \(\gamma\), the maximization of \(l_{\text{pseudo}}(\theta)\) with respect to \(\alpha_1, \alpha_2\) and \(\alpha_3\) can be obtained by maximizing \(l_1(\theta_1), l_2(\theta_2), \text{and} \ l_3(\theta_3)\) with respect to \(\alpha_1, \alpha_2\) and \(\alpha_3\) respectively. If we denote them as \(\tilde{\alpha}_1(\beta, \gamma)\), \(\tilde{\alpha}_2(\beta, \gamma)\) and \(\tilde{\alpha}_3(\beta, \gamma)\) respectively, then

\[
\begin{align*}
\tilde{\alpha}_1(\beta, \gamma) &= \frac{u_2n_1 + n_2}{\sum_{i \in I_0} a(x_i; \beta, \gamma) + \sum_{i \in I_1} a(x_{i1}; \beta, \gamma) + \sum_{i \in I_2} a(x_{i1}; \beta, \gamma)} \\
\tilde{\alpha}_2(\beta, \gamma) &= \frac{w_2n_2 + n_1}{\sum_{i \in I_0} a(x_i; \beta, \gamma) + \sum_{i \in I_1} a(x_{i2}; \beta, \gamma) + \sum_{i \in I_2} a(x_{i2}; \beta, \gamma)} \\
\tilde{\alpha}_3(\beta, \gamma) &= \frac{n_0 + u_1n_1 + w_1n_2 + n_1}{\sum_{i \in I_0} a(x_i; \beta, \gamma) + \sum_{i \in I_1} a(x_{i1}; \beta, \gamma) + \sum_{i \in I_2} a(x_{i2}; \beta, \gamma)}.
\end{align*}
\]

Finally the maximization of \(l_{\text{pseudo}}(\theta)\) with respect to \(\theta\), can be obtained by maximizing \(l_{\text{pseudo}}(\tilde{\alpha}_1(\beta, \gamma), \tilde{\alpha}_2(\beta, \gamma), \tilde{\alpha}_3(\beta, \gamma), \beta, \gamma)\), the pseudo profile log-likelihood function of \(\beta\) and \(\gamma\). If \(\tilde{\beta}\) and \(\tilde{\gamma}\) maximize the pseudo profile log-likelihood function, then \(\tilde{\alpha}_1(\tilde{\beta}, \tilde{\gamma}), \tilde{\alpha}_2(\tilde{\beta}, \tilde{\gamma}), \tilde{\alpha}_3(\tilde{\beta}, \tilde{\gamma})\), \(\tilde{\beta}, \tilde{\gamma}\) become the next iterate of the EM algorithm. We propose to use the following algorithm to compute the MLEs of the unknown parameters by EM algorithm;

**Algorithm:**

\[
\begin{align*}
\text{Algorithm:}
\end{align*}
\]
STEP 1: Take some initial guess value of \( \theta \), say \( \theta^{(0)} = (\alpha^{(0)}_1, \alpha^{(0)}_2, \alpha^{(0)}_3, \beta^{(0)}, \gamma^{(0)}) \)

STEP 2: Compute \( u_1(\theta^{(0)}), w_2(\theta^{(0)}), w_1(\theta^{(0)}) \) and \( w_2(\theta^{(0)}) \).

STEP 3: For given \( u_1(\theta^{(0)}), u_2(\theta^{(0)}), w_1(\theta^{(0)}) \) and \( w_2(\theta^{(0)}) \), maximize the pseudo log-likelihood function \( l_{pseudo}(\tilde{\alpha}_1(\beta, \gamma), \tilde{\alpha}_2(\beta, \gamma), \tilde{\alpha}_3(\beta, \gamma), \beta, \gamma) \) with respect \( \beta \) and \( \gamma \), say \( \beta^{(1)} \) and \( \gamma^{(1)} \) respectively.

STEP 3: Obtain \( \alpha_1^{(1)} = \tilde{\alpha}_1(\beta^{(1)}, \gamma^{(1)}) \), \( \alpha_2^{(1)} = \tilde{\alpha}_2(\beta^{(1)}, \gamma^{(1)}) \) and \( \alpha_3^{(1)} = \tilde{\alpha}_3(\beta^{(1)}, \gamma^{(1)}) \), and therefore \( \theta^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}, \beta^{(1)}, \gamma^{(1)}) \)

STEP 4: Replace \( \theta^{(0)} \) by \( \theta^{(1)} \) and go back to Step 1 and continue the process unless convergence takes place.

5 Data Analysis

In this section we present the analysis of one data set mainly to illustrate how the proposed model and the EM algorithm work in practice.

UEFA Champion’s League Data: The data set has been obtained from Meintanis (2007). The data set is presented in Table 2. It represents the soccer data where at least one goal is scored by the home team and at least goal is scored directly from a penalty kick, foul kick or any other direct kick (all of them will be called as kick goal) by any team has been considered. Here \( X_1 \) and \( X_2 \) represent the time in minutes of the first kick goal scored by any team and \( X_2 \) represents the first goal of any type scored by the home team. Clearly all possibilities are open, for example \( X_1 < X_2 \) or \( X_1 > X_2 \) or \( X_1 = X_2 = Y \) (say).

Meintanis (2007) analyzed this data set using Marshall-Olkin bivariate exponential model. Kundu and Gupta (2009) re-analyzed the same data set using bivariate generalized exponential model. It is observed that the bivariate generalized exponential distribution provides a
Table 2: UEFA Champion’s League Data

<table>
<thead>
<tr>
<th>2005-2006</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>2004-2005</th>
<th>$X_1$</th>
<th>$X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lyon-Real Madrid</td>
<td>26</td>
<td>20</td>
<td>Internazionale-Bremen</td>
<td>34</td>
<td>34</td>
</tr>
<tr>
<td>Milan-Fenerbahce</td>
<td>63</td>
<td>18</td>
<td>Real Madrid-Roma</td>
<td>53</td>
<td>39</td>
</tr>
<tr>
<td>Chelsea-Anderlecht</td>
<td>19</td>
<td>19</td>
<td>Man. United-Fenerbahce</td>
<td>54</td>
<td>7</td>
</tr>
<tr>
<td>Club Brugge-Juventus</td>
<td>66</td>
<td>85</td>
<td>Bayern-Ajax</td>
<td>51</td>
<td>28</td>
</tr>
<tr>
<td>Fenerbahce-PSV</td>
<td>40</td>
<td>40</td>
<td>Moscow-PSG</td>
<td>76</td>
<td>64</td>
</tr>
<tr>
<td>Internazionale-Rangers</td>
<td>49</td>
<td>49</td>
<td>Barcelona-Shakhtar</td>
<td>64</td>
<td>15</td>
</tr>
<tr>
<td>Panathinaikos-Bremen</td>
<td>8</td>
<td>8</td>
<td>Leverkusen-Roma</td>
<td>26</td>
<td>48</td>
</tr>
<tr>
<td>Ajax-Arsenal</td>
<td>69</td>
<td>71</td>
<td>Arsenal-Panathinaikos</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Real Madrid-Rosenborg</td>
<td>82</td>
<td>48</td>
<td>Man. United-Sparta</td>
<td>25</td>
<td>14</td>
</tr>
<tr>
<td>Villarreal-Benfica</td>
<td>72</td>
<td>72</td>
<td>Bayern-M. TelAviv</td>
<td>55</td>
<td>11</td>
</tr>
<tr>
<td>Juventus-Bayern</td>
<td>66</td>
<td>62</td>
<td>Bremen-Internazionale</td>
<td>49</td>
<td>49</td>
</tr>
<tr>
<td>Club Brugge-Rapid</td>
<td>25</td>
<td>9</td>
<td>Anderlecht-Valencia</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>Olympiacos-Lyon</td>
<td>41</td>
<td>3</td>
<td>Panathinaikos-PSV</td>
<td>44</td>
<td>30</td>
</tr>
<tr>
<td>Internazionale-Porto</td>
<td>16</td>
<td>75</td>
<td>Arsenal-Rosenborg</td>
<td>42</td>
<td>3</td>
</tr>
<tr>
<td>Schalke-PSV</td>
<td>18</td>
<td>18</td>
<td>Liverpool-Olympiacos</td>
<td>27</td>
<td>47</td>
</tr>
<tr>
<td>Barcelona-Bremen</td>
<td>22</td>
<td>14</td>
<td>M. Tel-Aviv-Juventus</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td>Milan-Schalke</td>
<td>42</td>
<td>42</td>
<td>Bremen-Panathinaikos</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Rapid-Juventus</td>
<td>36</td>
<td>52</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

better fit than the Marshall-Olkin bivariate exponential model. It has been shown by Kundu and Gupta (2009) using the scaled TTT transform of Aarset (1987), that both the marginals ($X_1$ and $X_2$) have increasing empirical hazard rates. It has prompted us to use the BGLFR distribution to analyze this model.

Before trying to analyze the data using BGLFR model, we first fit the GLFR model to $X_1$ and $X_2$ separately. The MLEs of the parameters ($\beta, \gamma, \alpha$) of the corresponding GLFR distribution for $X_1$ and $X_2$ are ($5.1828 \times 10^{-3}, 9.3294 \times 10^{-4}, 1.3031$) and ($0.0194, 5.6825 \times 10^{-4}, 1.1433$) and the corresponding log-likelihood values are -162.676 and -162.938 respectively. Since both exponential and generalized exponential distributions are special cases of the GLFR distribution, we perform the following two testing of hypotheses:
Problem 1: $H_{01}: \gamma = 0, \alpha = 1$ (exponential) vs $H_{1}: \gamma > 0, \alpha > 0$ (GLFR).

Problem 2: $H_{02}: \gamma = 0$ (generalized exponential) vs $H_{1}: \gamma > 0$ (GLFR).

The log-likelihood values ($L$), the likelihood ratio test statistic ($\Lambda$), the MLEs of each model, and the associated $p$ values are presented in Table 3. Based on the $p$ values it is clear that: (1) GLFR distribution provides a significantly better fit for both $X_1$ and $X_2$ compared to the exponential; (2) GLFR distribution provides a significantly better fit for $X_1$ compared to the generalized exponential distribution; (3) GLFR distribution provides a better fit for $X_2$ than the generalized exponential distributions. Finally using the EM algorithm we obtain

Table 3: The MLEs and the values of $L$, $\Lambda$, and the $p$ values of $X_1$ and $X_2$.  

<table>
<thead>
<tr>
<th>Null</th>
<th>$X_1$ MLEs</th>
<th>$L$</th>
<th>$\Lambda$</th>
<th>$p$-value</th>
<th>$X_2$ MLEs</th>
<th>$L$</th>
<th>$\Lambda$</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{01}$</td>
<td>$\hat{\beta} = 0.024$</td>
<td>-174.304</td>
<td>23.257</td>
<td>$&lt; 0.0001$</td>
<td>$\hat{\beta} = 0.0304$</td>
<td>-166.219</td>
<td>6.562</td>
<td>0.038</td>
</tr>
<tr>
<td>$H_{02}$</td>
<td>$\hat{\beta} = 0.0449$</td>
<td>$\hat{\alpha} = 3.1193$</td>
<td>-168.815</td>
<td>6.279</td>
<td>0.012</td>
<td>$\hat{\beta} = 0.0413$</td>
<td>$\hat{\alpha} = 1.6776$</td>
<td>-163.937</td>
</tr>
</tbody>
</table>

the MLEs of $\alpha_1, \alpha_2, \alpha_3, \beta$ and $\gamma$ as $(0.492, 0.166, 0.411, 2.013 \times 10^{-4}, 8.051 \times 10^{-4})$.

In order to investigate if the BGLFR distribution provides a better fit to data set 1, than the MO model and the BVGE model, we use the Akaike Information Criterion (AIC), see Akaike (1969), Bayesian Information Criterion (BIC), see Schwartz (1978), and also likelihood ratio test (LRT). Since the MO model cannot be obtained as a special case of the BGLFR distribution we cannot use the LRT test directly to compare the MO model and the BGLFR model. It is natural to use AIC or BIC in this case. On the other hand since BVGE distribution can be obtained as a special case of the BGLFR model, the LRT also
can be used in testing between BVGE and BGLFR models.

In the enclosed Table 4 we provide the MLEs of the unknown parameters of the MO and the BVGE models. We have also enclosed the AIC and BIC values for model selection purposes.

<table>
<thead>
<tr>
<th>Model</th>
<th>MLEs</th>
<th>$\mathcal{L}$</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>MO</td>
<td>$\hat{\lambda}_1 = 0.012, \hat{\lambda}_2 = 0.014, \hat{\lambda}_3 = 0.022$</td>
<td>-339.006</td>
<td>684.012</td>
<td>-344.423</td>
</tr>
<tr>
<td>BVGE</td>
<td>$\hat{\alpha}_1 = 1.351, \hat{\alpha}_2 = 0.465, \hat{\alpha}_3 = 1.153, \hat{\beta} = 0.039$</td>
<td>-296.935</td>
<td>601.870</td>
<td>-304.157</td>
</tr>
<tr>
<td>BGLFR</td>
<td>$\hat{\alpha}_1 = 0.492, \hat{\alpha}_2 = 0.166, \hat{\alpha}_3 = 0.411, \hat{\beta} = 2.013 \times 10^{-4}, \hat{\gamma} = 8.051 \times 10^{-4}$</td>
<td>-293.379</td>
<td>596.757</td>
<td>-302.406</td>
</tr>
</tbody>
</table>

It is clear that between MO model and BGLFR model, clearly BGLFR model is preferable, based on both AIC and BIC values. Now to choose between BVGE and BGLFR, based on AIC, BGLFR is preferable, BIC suggests BVGE model. If we perform the LRT test, while the null hypothesis is BVGE model and the alternative is BGLFR model, the test statistic is 6.73 with the $0.025 < p < 0.05$. Since $p$ value is not very high, we prefer BGLFR than BVGE for analyzing this data set.

6 MULTIVARIATE GENERALIZED LINEAR FAILURE RATE DISTRIBUTION

In this section we are in a position to define the $m$-variate generalized linear failure rate distribution and provide some of its properties. It may be mentioned that recently Franco and
Vivo (2009), provided a multivariate extension of Sarhan-Balakrishnan bivariate distribution and studied its several properties.

Suppose $U_1, \ldots, U_{m+1}$ are $m+1$ independent random variables such that $U_i \sim \text{GLFR}(\alpha_i, \beta, \gamma)$ for $i = 1, \ldots, m+1$. Define

$$X_j = \max\{U_j, U_{m+1}\}, j = 1, 2, \ldots, m,$$

then we say that $\mathbf{X} = (X_1, \ldots, X_m)$ is a $m$-variate GLFR with parameters $(\alpha_1, \ldots, \alpha_{m+1}, \beta, \gamma)$, and it will be denoted by MGLFR$(m, \alpha_1, \ldots, \alpha_{m+1}, \beta, \gamma)$. The joint CDF of $\mathbf{X}$ can be easily obtained as follows;

**Theorem 6.1:** If $\mathbf{X} = (X_1, \ldots, X_m) \sim \text{MGLFR}(m, \alpha_1, \ldots, \alpha_{m+1}, \beta, \gamma)$, then the joint CDF of $\mathbf{X}$ for $x_1 > 0, \ldots, x_m > 0$ is

$$F_X(\mathbf{x}) = \prod_{i=1}^{m+1} F_{\text{GLFR}}(x_i; \alpha_i, \beta, \gamma),$$

(31)

where $\mathbf{x} = (x_1, \ldots, x_m)$ and $x_{m+1} = \min\{x_1, \ldots, x_m\}$.

Along the same line as the bivariate GLFR distribution, the multivariate GLFR distribution (31) also can be obtained from the $m$-variate Marshall-Olkin copula with the marginals as the GLFR distributions. In this case (31) can be obtained from the following MO copula

$$C_{\theta}(u_1, \ldots, u_m) = u_1^{1-\theta_1} \cdots u_m^{1-\theta_m} \min\{u_1^{\theta_1} \cdots u_m^{\theta_m}\},$$

(32)

here $\theta = (\theta_1, \ldots, \theta_m)$, and

$$\theta_1 = \frac{\alpha_{m+1}}{\alpha_1 + \alpha_{m+1}}, \ldots, \theta_m = \frac{\alpha_{m+1}}{\alpha_m + \alpha_{m+1}}.$$

For $m > 1$, the MGLFR distribution function can also be written as

$$F_X(\mathbf{x}) = pF_a(\mathbf{x}) + (1 - p)F_z(\mathbf{x}),$$

(33)
here $0 < p < 1$, $F_a$ and $F_s$ denote the absolute continuous and singular part of $F$ respectively. The corresponding PDF of $X$ also can be written as

$$f_X(x) = pf_a(x) + (1 - p)f_s(x). \quad (34)$$

In writing (34) it needs to be understood that $f_a$ is the PDF with respect to $m$-dimensional Lebesgue measure, and $f_s$ also can be further decomposed and they are PDFs with respect to $1, \cdots, (m - 1)$ dimensional Lebesgue measures. It is not difficult to obtain the explicit expressions of $F_s$ and $f_s$ for the general $m$, but they are quite tedious, and they are not pursued here. We provide the explicit expression of $f_a$ and $p$ in the appendix.

Now we provide the distribution functions of the marginals, conditionals and the extreme order statistics of the MGLFR distribution.

**Theorem 6.2:** If $X = (X_1, \cdots, X_m) \sim \text{MGLFR}(m, \alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \beta, \gamma)$, then

(a) $X_1 \sim \text{GLFR}(\alpha_1 + \alpha_{m+1}, \beta, \gamma)$, $\cdots$, $X_m \sim \text{GLFR}(\alpha_m + \alpha_{m+1}, \beta, \gamma)$.

(b) For $2 \leq s \leq m$, $(X_1, \cdots, X_s) \sim \text{MGLFR}(s, \alpha_1, \cdots, \alpha_s, \alpha_{m+1}, \beta, \gamma)$

(c) The conditional distribution of $(X_1, \cdots, X_s)$, given $\{X_{s+1} \leq x_{s+1}, \cdots, X_m \leq x_m\}$ is

$$P(X_1 \leq x_1, \cdots, X_s \leq x_s | X_{s+1} \leq x_{s+1}, \cdots, X_m \leq x_m) = \left\{ \begin{array}{ll}
\prod_{j=1}^s F_{\text{GLFR}}(x_j, \alpha_j, \beta, \gamma) & \text{if } z = v \\
F_{\text{GLFR}}(z, \alpha_{m+1}, \beta, \gamma)F_{\text{GLFR}}(v, \alpha_{m+1}, \beta, \gamma) & \text{if } z < v,
\end{array} \right.$$

where $z = \min\{x_1, \cdots, x_s\}$ and $v = \min\{x_{s+1}, \cdots, x_m\}$.

(d) If $T_m = \max\{X_1, \cdots, X_m\}$, then

$$F_{T_m}(t) = P(T_m \leq t) = F_{\text{GLFR}}(t, \alpha_1 + \cdots + \alpha_{m+1}, \beta, \gamma)$$

(e) If $T_1 = \min\{X_1, \cdots, X_m\}$, then

$$F_{T_1}(t) = P(T_1 \leq t) = F_{\text{GLFR}}(t, \alpha_{m+1}, \beta, \gamma) \times \left(1 - \prod_{i=1}^m (1 - F_{\text{GLFR}}(t, \alpha_i, \beta, \gamma))\right)$$
Proof: The proofs of (a), (b), (c) and (d) are quite simple and are not provided here.

(e) Note that
\[ F_{T_1}(t) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{I_k \in S_k} F_{I_k}(t, \ldots, t), \]
where \( I_k = (i_1, \ldots, i_k), \) \( 1 \leq i_1 \neq \cdots \neq i_k \leq m, \) is a \( k \)-dimensional subset and \( S_k \) is the set of all ordered \( k \)-dimensional subsets of \( \{1, \cdots, m\} \). Further
\[ F_{I_k}(t, \ldots, t) = P(X_{i_1} \leq t, \ldots, X_{i_k} \leq t). \]

Therefore, using part (b),
\[ F_{T_1}(t) = F_{GLFR}(t, \alpha_{m+1}, \beta, \gamma) \times \sum_{I_k \in S_k} F_{GLFR}(t, \alpha_{i_1} + \cdots + \alpha_{i_k}, \beta, \gamma) \]
Now using the fact
\[ \sum_{k=1}^{m} (-1)^{k-1} \sum_{I_k \in S_k} F_{GLFR}(t, \alpha_{i_1} + \cdots + \alpha_{i_k}, \beta, \gamma) = 1 - \prod_{i=1}^{m} (1 - F_{GLFR}(t, \alpha_i, \beta, \gamma)) \]
the result follows.

7 Conclusions

In this paper we have introduced the bivariate generalized linear failure rate distribution whose marginals are generalized linear failure rate distributions. The proposed bivariate distribution is a singular distribution, and it can be used quite effectively instead of Marshall-Olkin bivariate exponential model, or the bivariate generalized exponential model when there are ties in the data. Several properties of this new distribution have been established, and also we proposed to use the EM algorithm to compute the maximum likelihood estimators.

Further we have proposed its multivariate generalization. Several properties have been discussed. It can be obtained by using the multivariate Marshall-Olkin copula coupled with generalized linear failure rate marginals. It may be mentioned that the EM algorithm
along the same line as the bivariate case may be developed. Alternatively, using the copula structure, other estimators as proposed by Kim et al. (2006) may be used and their properties can be established. The work is in progress, it will be reported later.

**APPENDIX**

In this appendix we provide the explicit expression of \( f_a \) and \( p \) of (34) for general \( m \). Let \( k \in \{1, \ldots, m\} \) be the number of the different components of \( x = (x_1, \ldots, x_m) \), i.e. when \( k = 1 \), all \( x_i \)'s are equal, and all \( x_i \)'s are different when \( k = m \). Then \( x \) belongs to the set where \( F_X \) is absolutely continuous if and only if \( k = m \). For each \( x \) with \( k = m \), there exists a permutation \( P_m = (i_1, \ldots, i_m) \), such that \( x_{i_1} \prec \cdots \prec x_{i_m} \), and let us define the following function

\[
\phi^{p_m}(x) = \phi_{GLFR}(x_{i_1}, \alpha_{i_1} + \alpha_{m+1}, \beta, \gamma) \phi_{GLFR}(x_{i_2}, \alpha_{i_2}, \beta, \gamma) \cdots \phi_{GLFR}(x_{i_m}, \alpha_{i_m}, \beta, \gamma) \quad (35)
\]

Differentiating (33) with respect to \( x_1, \ldots, x_m \), we obtain

\[
\frac{\partial^m F_X(x_1, \ldots, x_m)}{\partial x_1 \cdots \partial x_m} = pf_a(x) = \phi_{P_m}(x),
\]

for \( P_m = (i_1, \ldots, i_m) \), such that \( x_{i_1} \prec \cdots \prec x_{i_m} \), and \( f_a \) is the joint density function of the absolute continuous part as mentioned before. Moreover, \( p \) may be obtained

\[
p = p \int_{\mathbb{R}^m} f_a(x) dx_1 \cdots dx_m = \sum_{P_m} \int_{x_{i_m}=0}^{\infty} \int_{x_{i_{m-1}}=0}^{x_{i_m}} \cdots \int_{x_{i_1}=0}^{\infty} f_{P_m}(x) dx_{i_1} \cdots dx_{i_m}
\]

\[
= \sum_{P_m} \frac{\alpha_{i_2}}{\alpha_{i_1} + \alpha_{i_2} + \alpha_{m+1}} \times \frac{\alpha_{i_3}}{\alpha_{i_2} + \alpha_{i_3} + \alpha_{m+1}} \times \cdots \times \frac{\alpha_{i_m}}{\alpha_{i_1} + \cdots + \alpha_{i_m} + \alpha_{m+1}}.
\]

Therefore,

\[
f_a(x) = \frac{1}{p} \phi_{P_n}(x).
\]

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References


Figure 1: Surface and contour plots of the absolute continuous part of the joint PDF of the BGLFR model, for different values of $(\alpha_1, \alpha_2, \alpha_3)$. We have assumed $\beta = \gamma = 1$ in all the cases. (a) (2, 2, 2) (b) (1, 1, 1) (c) (0.5, 0.5, 0.5) (d) (5, 5, 5).