

POWER NORMAL DISTRIBUTION

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Abstract

Recently Gupta and Gupta [10] proposed the power normal distribution for which normal distribution is a special case. The power normal distribution is a skewed distribution, whose support is the whole real line. Our main aim of this paper is to consider bivariate power normal distribution, whose marginals are power normal distributions. We obtain the proposed bivariate power normal distribution from Clayton copula, and by making a suitable transformation in both the marginals. Lindley-Singpurwalla distribution also can be used to obtain the same distribution. Different properties of this new distribution have been investigated in details. Two different estimators are proposed. One data analysis has been performed for illustrative purposes. Finally we propose some generalizations to multivariate case also along the same line and discuss some of its properties.

KEY WORDS AND PHRASES: Clayton copula; maximum likelihood estimator; failure rate; approximate maximum likelihood estimator.

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1 INTRODUCTION

The skew-normal distribution proposed by Azzalini [3] has received a considerable attention in the recent years. Due to its flexibility, it has been used quite extensively for analyzing skewed data on the entire real line. It has a nice physical interpretation also, as a hidden truncation model, see Arnold and Beaver [2]. Several generalizations namely to other distributions and also to multivariate cases have been proposed and investigated in details.

Although, it is well known that for skew normal distribution the maximum likelihood estimators (MLEs) are consistent and asymptotically normally distributed, the probability of non-existence of the MLEs is quite high, particularly, if the sample size is small. By extensive simulation studies, it has been observed by Gupta and Gupta [10] that even for moderate skewness, one needs a very large sample size to achieve a reasonable estimate of the skewness parameter.

Due to this reason, Gupta and Gupta [10] proposed an alternative skewed model for which normal distribution becomes a special case and named it a power normal distribution. The power normal distribution is defined as follows: The random variable X is said to have a power normal distribution if it has the cumulative distribution function (CDF);

$$F_X(x; \alpha) = P(X \leq x) = (\Phi(x))^\alpha; \quad -\infty < x < \infty. \quad (1)$$

Here $\alpha > 0$, and $\Phi(\cdot)$ denoted the CDF of a standard normal distribution. Different properties of the power normal distribution have been discussed by Gupta and Gupta [10].

The aim of this paper is to introduce a bivariate power normal distribution (BPN) whose marginals are power normal distributions. This new distribution has been obtained from Clayton copula coupled with power normal marginals. Many properties of Clayton copula can be used in establishing different properties of the proposed bivariate power normal distribution. It can also be obtained by making a suitable transformation from the well known

Lindley-Singpurwalla distribution, see [23]. Multivariate generalization and the generalization for the other distribution functions are also possible along the same line.

It can be easily seen that the bivariate power normal distribution is an absolute continuous distribution. The joint probability density function (PDF) can take different shapes depending on the values of the parameters. The joint PDF can be expressed in terms of $\Phi(\cdot)$ function. The conditional PDF can be easily obtained from the joint and marginal distribution functions. It has been shown that although the power normal distribution has an increasing hazard rate but the bivariate hazard rate in the sense of Johnson and Kotz [19] is a decreasing function.

We study different dependency properties using the properties of Clayton copula. It is observed that the two variables (marginals) are positive quadrant dependent. It automatically implies that Kendall's τ , Spearman's τ , Gini's γ and Blomqvist's β are all non-negative. We provide all these concordance measures. It has been shown that one variable is stochastically increasing with respect to the other. It can be verified that Clayton copula satisfies the total positivity of order two (TP_2) property in the sense of Karlin [15]. Therefore, if a bivariate random vector has a BPN distribution, it satisfies the left corner set property also.

Estimation of the unknown parameters is an important problem for any inferential procedure. We propose to use the maximum likelihood method to estimate the unknown parameters. The maximum likelihood estimators (MLEs) can be obtained by solving a two dimensional optimization process. The expected Fisher information matrix has also been provided which can be used for testing of hypothesis or for constructing confidence intervals. Further we propose to use a two-step procedure as suggested by Joe [18], mainly to avoid the two dimensional optimization process, and we provide the asymptotic distribution of the proposed estimators. One data set has been analyzed for illustrative purposes. Finally we provide the multivariate power normal distribution and study some of its properties.

Rest of the paper is organized as follows. In Section 2, we briefly describe the power normal distribution. The bivariate power normal distribution is introduced in Section 3. Different properties of the proposed bivariate power normal are discussed in Section 4. In Section 5, we provide the two estimation procedures. Analysis of one data set is presented in Section 6. In Section 7 we provide the multivariate power normal distribution. Finally we conclude the paper in Section 8. The elements of the Fisher information matrix and the elements of the asymptotic variance covariance matrix of the proposed estimators are provided in the Appendix.

2 POWER NORMAL DISTRIBUTION: A BRIEF REVIEW

Power Normal distribution was proposed by Gupta and Gupta [10], as an alternative to the Azzalini's skew normal distribution. A random variable X is said to have a power normal distribution with parameter $\alpha > 0$, if X has the CDF (1), and therefore, it has the probability density function (PDF);

$$f_X(x; \alpha) = \alpha (\Phi(x))^{\alpha-1} \phi(x); \quad -\infty < x < \infty. \quad (2)$$

It is clear that the PDF (2) is a weighted normal PDF with weight function $(\Phi(x))^{\alpha-1}$. For $\alpha = 1$, it coincides with the standard normal density function. The power normal density function is a unimodal density function, which is skewed to the right if $\alpha > 1$ and to the left if $\alpha < 1$. It has some nice physical interpretations when α is an integer. It can be observed as the lifetime of a parallel system, see Gupta and Gupta [10] for a detailed discussions on this issue.

Note that a class of distribution functions $\{F(\cdot; \alpha); \alpha > 0\}$ is said to be a proportional reversed hazard rate family, if

$$F(x; \alpha) = (F_0(x))^\alpha; \quad -\infty < x < \infty. \quad (3)$$

Here $F_0(\cdot)$ is known as the baseline distribution function. Proportional reversed hazard rate model was introduced as an alternative to the celebrated proportional hazard rate model of Cox. Proportional reversed hazard rate model was originally introduced by Lehmann [20], in a testing of hypothesis problem, and it is known as Lehmann alternatives also. Power normal distribution can be seen as a member of the proportional reversed hazard rate family. Many general properties of the proportional reversed hazard rate models can be easily translated for the power normal distribution.

It can be easily seen that the power normal distribution has an increasing hazard rate function and hence a decreasing mean residual life for all $\alpha > 0$. It has likelihood ratio ordering, and hence it has hazard rate ordering and mean residual life ordering. Furthermore, unlike the skew normal distribution, the maximum likelihood estimator of the power normal distribution always exist. Therefore, for data analysis purposes power normal distribution can be used more effectively than the skew normal distribution.

3 BIVARIATE POWER NORMAL DISTRIBUTION

In this section first we introduce the bivariate power normal distribution and discuss its different properties. It may be mentioned that every bivariate distribution function, F_{X_1, X_2} with continuous marginals F_{X_1} and F_{X_2} , corresponds a unique function $C : [0, 1]^2 \rightarrow [0, 1]$, called a copula such that

$$F_{X_1, X_2}(x_1, x_2) = C \{F_{X_1}(x_1), F_{X_2}(x_2)\}, \quad \text{for } (x_1, x_2) \in (-\infty, \infty) \times (-\infty, \infty). \quad (4)$$

Conversely, it is possible to construct a bivariate distribution function having the desired marginal distributions and a chosen description structure, *i.e.* copula. Let us consider the following copula

$$C_\alpha(u, v) = \left(u^{-1/\alpha} + v^{-1/\alpha} - 1\right)^{-\alpha}; \quad 0 < u, v < 1, \quad (5)$$

known as Clayton copula, see for example Nelsen [28]. The following properties of the random vector (U, V) with the joint CDF (5) can be easily established, and therefore the proofs are omitted.

THEOREM 3.1: If (U, V) has the joint CDF (5) for $\alpha > 0$, then

(i) $U, V \sim \text{uniform}[0, 1]$.

(ii) The joint PDF of (U, V) for $0 \leq u, v \leq 1$ is

$$f_{U,V}(u, v) = \frac{(\alpha + 1)}{\alpha} \times \frac{1}{(uv)^{\frac{1+\alpha}{\alpha}}} \times \frac{1}{(u^{-1/\alpha} + v^{-1/\alpha} - 1)^{\alpha+2}}.$$

(iii) The joint survival function of (U, V) for $0 < u, v < 1$ is

$$S_{U,V}(u, v) = P(U \geq u, V \geq v) = 1 - u - v + (u^{-1/\alpha} + v^{-1/\alpha} - 1)^{-\alpha}.$$

(iv) $U|V = v$ has the CDF

$$P(U \leq u|V = v) = \frac{1}{v^{(\alpha+1)/\alpha}} \times \frac{1}{(u^{-1/\alpha} + v^{-1/\alpha} - 1)^{\alpha+1}}.$$

Now we would like to define a bivariate power normal distribution using this Clayton copula so that the marginals are univariate power normal distributions. Consider the following random variables for $\beta_1 > 0$ and $\beta_2 > 0$;

$$X_1 = \Phi^{-1}(U^{1/\beta_1}) \quad \text{and} \quad X_2 = \Phi^{-1}(V^{1/\beta_2}). \quad (6)$$

The joint CDF of X_1 and X_2 becomes;

$$\begin{aligned} F_{X_1, X_2}(x_1, x_2) &= P\{U \leq \Phi^{\beta_1}(x_1), V \leq \Phi^{\beta_2}(x_2)\} = C_\alpha(\Phi^{\beta_1}(x_1), \Phi^{\beta_2}(x_2)) \\ &= \left(\Phi^{-\beta_1/\alpha}(x_1) + \Phi^{-\beta_2/\alpha}(x_2) - 1\right)^{-\alpha}. \end{aligned} \quad (7)$$

Now we have the following definition;

DEFINITION: The bivariate random variables (X_1, X_2) is said to have a bivariate power normal distribution, denoted by $\text{BPN}(\alpha, \beta_1, \beta_2)$, if for $\beta_1 > 0$ and $\beta_2 > 0$, (X_1, X_2) has the joint CDF (7).

The joint PDF of (X_1, X_2) for $x_1 > 0$ and $x_2 > 0$ can be easily seen as

$$f_{X_1, X_2}(x_1, x_2) = \frac{c\phi(x_1)\phi(x_2) [\Phi(x_1)]^{-\beta_1/\alpha-1} [\Phi(x_2)]^{-\beta_2/\alpha-1}}{[[\Phi(x_1)]^{-\beta_1/\alpha} + [\Phi(x_2)]^{-\beta_2/\alpha} - 1]^{\alpha+2}}, \quad (8)$$

here $c = (\alpha + 1)\beta_1\beta_2/\alpha$.

It may be mentioned that the above bivariate power normal distribution can also be obtained by using a suitable transformation from the famous Lindley-Singpurwalla exchangeable distribution. It may be recalled that a bivariate random vector (V_1, V_2) is said to have Lindley-Singpurwalla exchangeable distribution, if the joint PDF of (V_1, V_2) is

$$f_{V_1, V_2}(v_1, v_2) = \frac{\alpha(\alpha + 1)}{(1 + v_1 + v_2)^{\alpha+2}}; \quad v_1 > 0, v_2 > 0, \quad (9)$$

and 0 otherwise. It can be obtained as a frailty model, and also for some other interpretations and properties, see Lindley and Singpurwalla [23]. We make the following transformation

$$V_i = (\Phi(X_i))^{-\beta_i/\alpha} - 1; \quad i = 1, 2.$$

Here the bivariate random vector (V_1, V_2) has the joint PDF (9). Then (X_1, X_2) has the joint PDF (8).

The surface plots of the joint PDF of (X_1, X_2) for different values of α , β_1 and β_2 are provided in Figure 1. From the Figure 1, it is quite apparent that it can take different shapes, and it is unimodal for different values of the parameters.

The following result provides the marginal distribution function, joint CDF, survival function, and the conditional CDF of the bivariate power normal distribution.

THEOREM 3.2: If (X_1, X_2) follows $\text{BPN}(\alpha, \beta_1, \beta_2)$, then

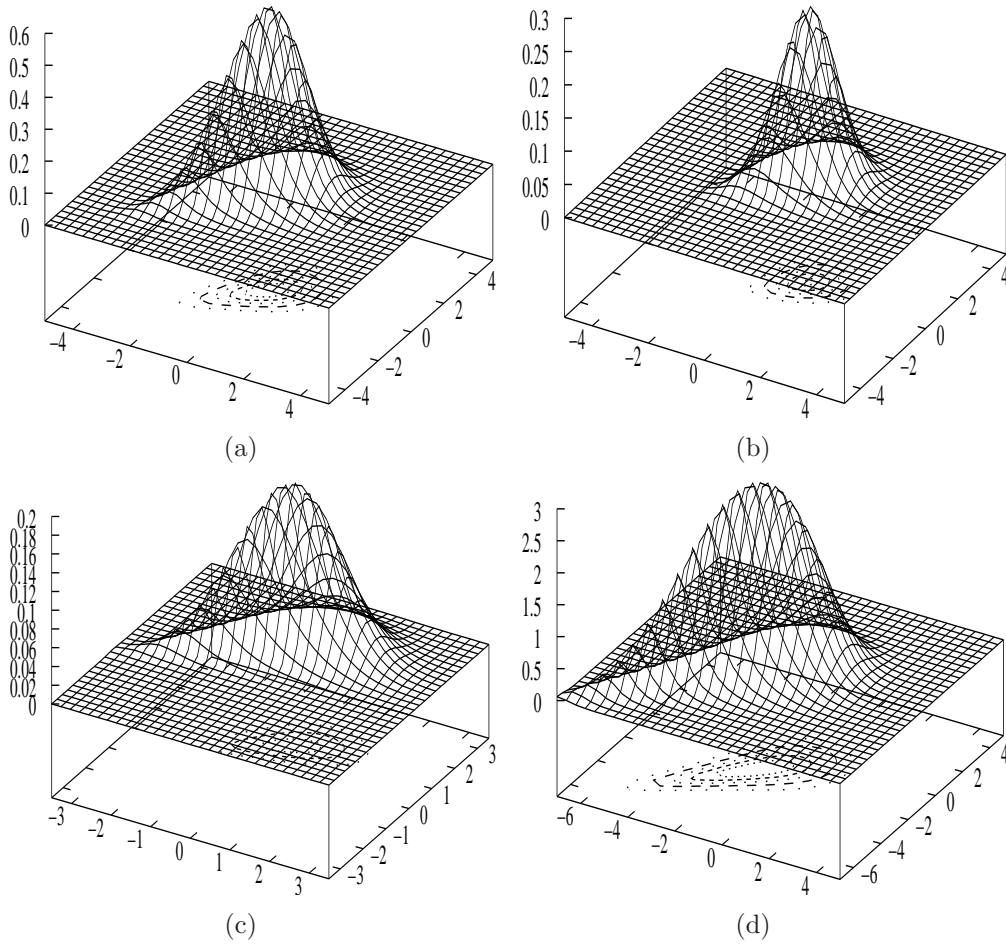


Figure 1: Contour plots of the joint PDF of $\text{BPN}(\alpha, \alpha_1, \alpha_2)$ for different values of $(\alpha, \alpha_1, \alpha_2)$:
(a) (1.0, 1.0, 1.0) (b) (2.0, 2.0, 2.0) (c) (1.0, 1.0, 5.0) (d) (0.5, 0.25, 0.25)

(i) $X_1 \sim \text{PN}(\beta_1)$ and $X_2 \sim \text{PN}(\beta_2)$.

(ii) The joint CDF of (X_1, X_2) is

$$F_{X_1, X_2}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) = \left[[\Phi(x_1)]^{-\beta_1/\alpha} + [\Phi(x_2)]^{-\beta_2/\alpha} - 1 \right]^{-\alpha}. \quad (10)$$

(iii) The joint survival function of (X_1, X_2) is

$$\begin{aligned} S_{X_1, X_2}(x_1, x_2) &= P(X_1 \geq x_1, X_2 \geq x_2) \\ &= 1 - [\Phi(x_1)]^{\beta_1} - [\Phi(x_2)]^{\beta_2} + \left[[\Phi(x_1)]^{-\beta_1/\alpha} + [\Phi(x_2)]^{-\beta_2/\alpha} - 1 \right]^{-\alpha}. \end{aligned} \quad (11)$$

(iv) The conditional CDF of X_1 given $X_2 = x_2$ is

$$P(X_1 \leq x_1 | X_2 = x_2) = \frac{[\Phi(x_2)]^{-\beta_2(\alpha+1)/\alpha}}{[\Phi(x_1)]^{-\beta_1/\alpha} + [\Phi(x_2)]^{-\beta_2/\alpha} - 1]^{\alpha+1}}. \quad (12)$$

4 DIFFERENT PROPERTIES

4.1 DEPENDENCY PROPERTIES AND DEPENDENCY MEASURES

It is well known that the dependency between the two random variables X_1 and X_2 is completely described by the joint distribution function $F_{X_1, X_2}(x_1, x_2)$. It is well known from Sklar's theorem, see Nelsen [28], that there exists a unique copula namely

$$C(u, v) = F_{X_1, X_2}(F_{X_1}^{-1}(u), F_{X_2}^{-1}(v)) \quad (13)$$

for a given joint distribution function $F_{X_1, X_2}(x_1, x_2)$. Many dependency properties of the copula function are true for the corresponding joint distribution function also. We have the following result for the bivariate power normal distribution.

RESULT 1: Let $(X_1, X_2) \sim \text{BPN}(\alpha, \beta_1, \beta_2)$, then X_2 is stochastically increasing in X_1 and vice versa.

PROOF: Using the copula function it follows that X_2 is stochastically increasing in X_1 , if and only if for any v , in $[0, 1]$, $C(u, v)$ is a concave function of u , see Nelsen [28]. In case of Clayton copula note that for $\theta = \frac{1}{\alpha}$,

$$\frac{\partial^2 C(u, v)}{\partial u^2} = -(1 + \theta) \frac{u^{\theta-1}(v^{-\theta} - 1)}{(1 + u^\theta(v^{-\theta} - 1))^{\frac{1+2\theta}{\theta}}} \leq 0.$$

Therefore, the result follows.

Let us recall that a non-negative function g defined on \mathcal{R}^2 is totally positive of order 2, abbreviated by TP_2 , if for all $x_{11} \leq x_{12}$, $x_{21} \leq x_{22}$,

$$g(x_{11}, x_{21})g(x_{12}, x_{22}) \geq g(x_{12}, x_{21})g(x_{11}, x_{22}). \quad (14)$$

If the inequality in (14) is reversed, then g is said to have reverse rule of order two, and it is denoted by RR_2 . We have the following result on bivariate power normal distribution, which follows from the copula property.

RESULT 2: If $(X_1, X_2) \sim \text{BPN}(\alpha, \beta_1, \beta_2)$, then (X_1, X_2) has the TP_2 property.

Now we discuss some dependency measures. The Kendall's τ defined as the probability of concordance minus the probability of discordance of two pairs (X_1, X_2) and (Y_1, Y_2) of random vectors, having the same joint distribution function, is

$$\tau = P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P[(X_1 - Y_1)(X_2 - Y_2) < 0]. \quad (15)$$

In case of bivariate power normal distribution, we have the following result, which also mainly follows from the copula property. Therefore, the detailed proof is avoided.

RESULT 3: Let $(X_1, X_2) \sim \text{BPN}(\alpha, \beta_1, \beta_2)$, then the Kendall's τ index is given by $\frac{1}{1 + 2\alpha}$.

The population version of the medial correlation coefficient for a pair (X_1, X_2) of continuous random variables was defined by Blomqvist [7]. If M_{X_1} and M_{X_2} denote the medians of X_1 and X_2 , respectively, then $M_{X_1 X_2}$, the medial correlation of X_1 and X_2 is

$$M_{X_1 X_2} = P[(X_1 - M_{X_1})(X_2 - M_{X_2}) > 0] - P[(X_1 - M_{X_1})(X_2 - M_{X_2}) < 0].$$

It has been shown by Nelsen [28] that the median correlation coefficient is also a copula property, and $M_{X_1 X_2} = 4C\left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore, for bivariate power normal distribution the medial correlation coefficient between X_1 and X_2 is $4\left(2^{(\alpha+1)/\alpha} - 1\right)^{-\alpha}$.

The concept of bivariate tail dependence relates to the amount of dependence in the upper quadrant (or lower quadrant) tail of a bivariate distribution, see Joe [17] (page 33). In terms of the original random variables X_1 and X_2 , the upper tail dependence is defined as

$$\chi = \lim_{z \rightarrow 1} P(X_2 \geq F_{X_2}^{-1}(z) | X_1 \geq F_{X_1}^{-1}(z)).$$

Intuitively, the upper tail dependence exists, when there is a positive probability that some positive outliers may occur jointly. If $\chi \in (0, 1]$, then X_1 and X_2 are said to be asymptotically dependent, and if $\chi = 0$, they are asymptotically independent. Coles *et al.* [8] showed using the copula function that

$$\chi = \lim_{u \rightarrow 1} \frac{1 - 2u + C(u, u)}{1 - u} = \lim_{u \rightarrow 1} \left\{ 2 - \frac{\log C(u, u)}{\log u} \right\}.$$

In case of bivariate power normal distribution, it can be shown that $\chi = 0$, *i.e.* X_1 and X_2 are asymptotically independent.

4.2 BIVARIATE HAZARD RATE

In this subsection we provide the bivariate hazard rate of the bivariate power normal distribution. There are several ways of defining the bivariate hazard rates. Basu [4] first defined the bivariate hazard function of an absolute continuous bivariate distribution by simply extending the one-dimensional definition to two-dimension, *i.e.*

$$h_B(x_1, x_2) = \frac{f_{X_1, X_2}(x_1, x_2)}{S_{X_1, X_2}(x_1, x_2)}. \quad (16)$$

Unfortunately, the above definition of the hazard function does not uniquely define the joint probability density function. The joint bivariate hazard rate in the sense of Johnson and Kotz [19] is defined as follows;

$$h(x_1, x_2) = \left(-\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2} \right) \ln S_{X_1, X_2}(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)). \quad (17)$$

It is well known that the bivariate hazard function $h(x_1, x_2)$ uniquely determines the joint PDF, see Marshall [24]. It may be noted that the hazard function of the power normal distribution is an increasing function, see Gupta and Gupta [10]. We have the following result.

THEOREM 4.1: If $(X_1, X_2) \sim \text{BPN}(\alpha, \beta_1, \beta_2)$, then for fixed x_2 (x_1), $h_1(x_1, x_2)$ ($h_2(x_1, x_2)$), is a decreasing function of x_1 (x_2).

PROOF: We will show the result for $h_1(x_1, x_2)$, the result for $h_2(x_1, x_2)$ will follow accordingly.

Let us use the following notation;

$$v = (\Phi(x_1))^{-\beta_1/\alpha}, \quad c_1 = 1 - (\Phi(x_2))^{\beta_2}, \quad c_2 = (\Phi(x_2))^{-\beta_2/\alpha} - 1.$$

Therefore,

$$h_1(x_1, x_2) = -\frac{\partial}{\partial x_1} \ln S_{X_1, X_2}(x_1, x_2) = -\left\{ \frac{\partial}{\partial v} \ln [c_1 - v^{-\alpha} + (v + c_2)^{-\alpha}] \right\} \times \frac{\partial v}{\partial x_1}.$$

Let us denote;

$$g(v) = \frac{\partial}{\partial v} \ln [c_1 - v^{-\alpha} + (v + c_2)^{-\alpha}].$$

Note that to prove for fixed x_2 , $h_1(x_1, x_2)$ is a decreasing function of x_1 , it is enough to prove that both $g(v)$ and $\frac{\partial v}{\partial x_1}$ are increasing functions of x_1 . After simplification $g(v)$ can be written as

$$g(v) = \frac{\left(1 + \frac{c_2}{v}\right)^{1+\alpha} - 1}{(v + c_2) \left[c_1(v + c_2)^\alpha - \left(1 + \frac{c_2}{v}\right)^{1+\alpha} + 1 \right]}.$$

Since the numerator is a decreasing function and the denominator is an increasing function of v , $g(v)$ is a decreasing function of v , *i.e.* $\frac{\partial g(v)}{\partial v} < 0$. Now

$$\frac{\partial v}{\partial x_1} = -\frac{\beta_1}{\alpha} (\Phi(x_1))^{-\beta_1/\alpha - 1} \phi(x_1) = -\frac{\beta_1}{\alpha} (\Phi(x_1))^{-\beta_1/\alpha} \times \frac{\phi(x_1)}{\Phi(x_1)}. \quad (18)$$

It is clear that $\frac{\partial v}{\partial x_1} < 0$. Since the reversed hazard function of a standard normal distribution, $\frac{\phi(x_1)}{\Phi(x_1)}$, is a decreasing function of x_1 , Block *et al.* [6], $\frac{\partial v}{\partial x_1}$ is an increasing function of x_1 .

Hence, $\frac{\partial^2 v}{\partial x_1^2} > 0$. Further

$$\frac{\partial g(v)}{\partial x_1} = \frac{\partial g(v)}{\partial v} \times \frac{\partial v}{\partial x_1} > 0.$$

Therefore, both $g(v)$ and $\frac{\partial v}{\partial x_1}$ are increasing functions of x_1 , hence the result follows. ■

4.3 MISCELLANEOUS PROPERTIES

In this section we provide the CDF and PDF of the maximum and minimum of the bivariate power normal distribution which may be useful for modeling series and parallel system. Moreover they are quite useful in competing and complementary risks analysis, see for example Basu and Ghosh [5]. We have the following result;

RESULT 4: If $(X_1, X_2) \sim \text{BPN}(\alpha, \beta_1, \beta_2)$, then

(i) the CDF and PDF of $\max\{X_1, X_2\}$ is

$$F_{\max\{X_1, X_2\}}(x) = P(X_1 \leq x, X_2 \leq x) = \left[[\Phi(x)]^{-\beta_1/\alpha} + [\Phi(x)]^{-\beta_2/\alpha} - 1 \right]^{-\alpha}$$

and

$$f_{\max\{X_1, X_2\}}(x) = \frac{\phi(x)(\beta_1[\Phi(x)]^{-\beta_1/\alpha-1} + \beta_2[\Phi(x)]^{-\beta_2/\alpha-1})}{\left[[\Phi(x)]^{-\beta_1/\alpha} + [\Phi(x)]^{-\beta_2/\alpha} - 1 \right]^{\alpha+1}}.$$

(ii) the CDF and PDF of $\min\{X_1, X_2\}$ is

$$F_{\min\{X_1, X_2\}}(x) = P(\min\{X_1, X_2\} \leq x) = [\Phi(x)]^{\beta_1} + [\Phi(x)]^{\beta_2} - \left\{ \Phi(x)^{-\beta_1/\alpha} + \Phi(x)^{-\beta_2/\alpha} - 1 \right\}^{-\alpha}$$

and

$$f_{\min\{X_1, X_2\}}(x) = \beta_1 \phi(x) \Phi(x)^{\beta_1-1} + \beta_2 \phi(x) \Phi(x)^{\beta_2-1} - f_{\max\{X_1, X_2\}}(x).$$

5 STATISTICAL INFERENCE

In this section we will discuss the statistical inference of the unknown parameters of the bivariate power normal distribution. It is assumed that we have a sample of size n , from $\text{BPN}(\alpha, \beta_1, \beta_2)$ and based on the random sample $\{(x_{11}, x_{12}), \dots, (x_{n1}, x_{n2})\}$, we would like to estimate and draw inference about the unknown parameters.

We make a simple transformation of β_1 and β_2 , as follows: $\beta_1 = \alpha\alpha_1$, and $\beta_2 = \alpha\alpha_2$. First we obtain the MLEs of α , α_1 and α_2 . Then using the invariance property of the MLEs

we can easily obtain the MLEs of β_1 and β_2 . Based on the observation, first we write the log-likelihood function as a function of α , α_1 and α_2 as follows, see (8);

$$l(\alpha, \alpha_1, \alpha_2) = n \ln \alpha + n \ln(\alpha + 1) + n \ln \alpha_1 + n \ln \alpha_2 - \alpha_1 \sum_{i=1}^n \ln \Phi(x_{i1}) - \alpha_2 \sum_{i=1}^n \ln \Phi(x_{i2}) - (\alpha + 2) \sum_{i=1}^n \ln \left((\Phi(x_{i1}))^{-\alpha_1} + (\Phi(x_{i2}))^{-\alpha_2} - 1 \right). \quad (19)$$

The maximum likelihood estimators (MLEs) of the unknown parameters can be obtained by maximizing (19) with respect to the unknown parameters. For fixed α_1 and α_2 , the MLE of α can be obtained as;

$$\hat{\alpha}(\alpha_1, \alpha_2) = \frac{(2 - A) + \sqrt{4 + A^2}}{2A}, \quad (20)$$

where

$$A = \frac{1}{n} \sum_{i=1}^n \ln \left([\Phi(x_{i1})]^{-\alpha_1} + [\Phi(x_{i2})]^{-\alpha_2} - 1 \right).$$

The MLEs of α_1 and α_2 can be obtained by maximizing $l(\hat{\alpha}(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ with respect to α_1 and α_2 . They cannot be obtained in explicit forms, they have to be obtained numerically.

The bivariate power normal distribution is a regular family and the MLEs of the unknown parameters are asymptotically normally distributed. We have the following result:

$$\sqrt{n}(\hat{\alpha}_1 - \alpha_1, \hat{\alpha}_2 - \alpha_2, \hat{\alpha} - \alpha) \xrightarrow{d} N_3(0, I^{-1}). \quad (21)$$

Here I is the expected Fisher information matrix. The explicit expressions of the Fisher information matrix are provided in the appendix.

Once $\hat{\alpha}$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are obtained, we can easily obtain the MLEs of β_1 , β_2 , say $\hat{\beta}_1$ and $\hat{\beta}_2$ respectively. Using (21), and using the transformation method, see Lehmann [22] (page 126), we can easily obtain the following result:

$$\sqrt{n}(\hat{\beta}_1 - \beta_1, \hat{\beta}_2 - \beta_2, \hat{\alpha} - \alpha) \xrightarrow{d} N_3(0, J^T I^{-1} J). \quad (22)$$

Here the Jacobian matrix J is as follows;

$$J = \begin{bmatrix} \alpha & 0 & \beta_1/\alpha \\ 0 & \alpha & \beta_2/\alpha \\ -\alpha/\beta_1 & -\alpha/\beta_2 & 1 \end{bmatrix},$$

and the elements of the matrix I are expressed in terms of α , β_1 and β_2 .

Alternatively, to avoid the two dimensional maximization process, we propose to use the following two step process, as suggested by Joe [18], see also Kim *et al.* [16]. This may be very useful in the higher dimensional case. In this two step process, in the first step the two marginal distributions are estimated separately.

In the first stage, based on the marginals $\{x_{ij}\}, i = 1, \dots, n$, for $j = 1, 2$, the MLEs of β_1 and β_2 are

$$\tilde{\beta}_1 = -\frac{n}{\sum_{i=1}^n \ln \Phi(x_{i1})} \quad \text{and} \quad \tilde{\beta}_2 = -\frac{n}{\sum_{i=1}^n \ln \Phi(x_{i2})} \quad (23)$$

respectively, see Gupta and Gupta [10].

In the second stage α is estimated by substituting $\hat{\beta}_1$ and $\hat{\beta}_2$ for β_1 and β_2 respectively, and then maximize the resulting log-likelihood function. The estimate of α can be obtained as the maximizer of

$$\sum_{i=1}^n \ln \left[c\{[\Phi(x_{i1})]^{\tilde{\beta}_1}, [\Phi(x_{i2})]^{\tilde{\beta}_2}; \alpha\} \right], \quad (24)$$

where

$$c(u, v; \alpha) \propto u^{-(1+1/\alpha)} v^{-(1+1/\alpha)} \left(u^{-1/\alpha} + v^{-1/\alpha} - 1 \right)^{-(2+\alpha)}. \quad (25)$$

Therefore, in this case the estimate of α can be obtained as a one dimensional optimization process. We will denote this as $\tilde{\alpha}$. Using the result of Joe [18], we can directly obtain the asymptotic distribution of $(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\alpha})$, and it can be expressed as follows;

$$\sqrt{n}(\tilde{\beta}_1 - \beta_1, \tilde{\beta}_2 - \beta_2, \tilde{\alpha} - \alpha) \xrightarrow{d} N_3(0, V). \quad (26)$$

The elements of the matrix V are provided in the Appendix.

Table 1: Two different stiffness measurements of 30 boards.

No.	Shock	Vibration	No.	Shock	Vibration	No.	Shock	Vibration
1	1889	1651	11	2403	2048	21	2119	1700
2	1645	1627	12	1976	1916	22	1712	1713
3	1943	1685	13	2104	1820	23	2983	2794
4	1745	1600	14	1710	1591	24	2046	1907
5	1840	1841	15	1867	1685	25	1859	1649
6	1954	2149	16	1325	1170	26	1419	1371
7	1828	1634	17	1725	1594	27	2276	2189
8	1899	1614	18	1633	1513	28	2061	1867
9	1856	1493	19	1727	1412	29	2168	1896
10	1655	1675	20	2326	2301	30	1490	1382

6 DATA ANALYSIS

For illustrative purposes we have analyzed one data set using the bivariate power normal model. The data set represents the two different measurements of stiffness, ‘Shock’ and ‘Vibration’ of each of 30 boards. The first measurement (Shock) involves sending a shock wave down the board and the second measurement (Vibration) is determined while vibrating the board. The data set was originally from William Galligan, and it has been reported in Johnson and Wichern [14]. For convenience, the data set is presented in Table 1.

We provide the scaled-TTT plots of the marginals as proposed by Aarset [1] in Figure 2. It provides an indication about the shape of the hazard function of the underlying distribution. Since for both the marginals the scaled-TTT plots are concave, it indicates that the marginal hazard functions are increasing. Therefore, bivariate power normal distribution can be used to analyze this data set.

Before analyzing we scale the data by subtracting the means and dividing by the corresponding standard deviations. We obtain the maximum likelihood estimators of β_1 , β_2 and

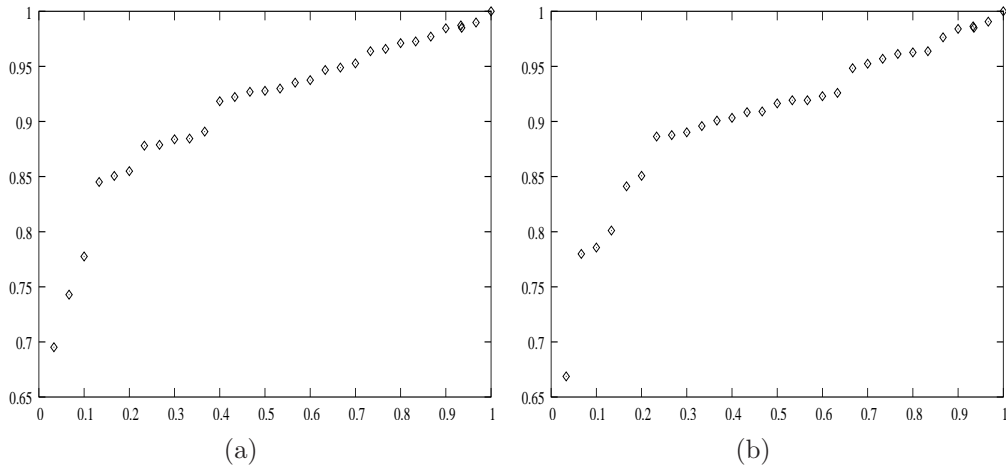


Figure 2: Scaled-TTT plots of the marginals: (a) Shock (b) Vibration

α as 1.0534, 0.9133 and 0.2512 respectively. The associated 95% non-parametric bootstrap confidence intervals are (0.8547, 1.2521), (0.7022, 1.1044) and (0.1914, 0.3110). The approximate maximum likelihood estimators of β_1 and β_2 are 1.0423 and 1.0468 respectively. Using these estimates we obtain the profile log-likelihood of α , as given in (24). It is plotted in Figure 3 and we get the estimate of α as 0.2473. Therefore, the approximate maximum likelihood estimators of β_1 , β_2 and α are 1.0423, 1.0468 and 0.2473 respectively. The associated 95% non-parametric bootstrap confidence intervals are (0.8408, 1.2438), (0.8470, 1.2466) and (0.1819, 0.3127) respectively. In Table 2 we provide the goodness of fit of the fitted power normal distributions to the marginals when the parameters are estimates using maximum likelihood estimators and also by approximate maximum likelihood estimators. We report the associated p values also in brackets. From the K-S distances and the corresponding p values it is clear that power normal distribution fits the marginals quite well. Moreover the maximum likelihood estimators and the approximate maximum likelihood estimators behave quite similarly.

Table 2: Kolmogorov-Smirnov(KS) distances and the associated p values of the marginals

	X_1	X_2
MLE	0.1298 (0.7178)	0.1368 (0.6923)
AMLE	0.1265 (0.7230)	0.1461 (0.6835)

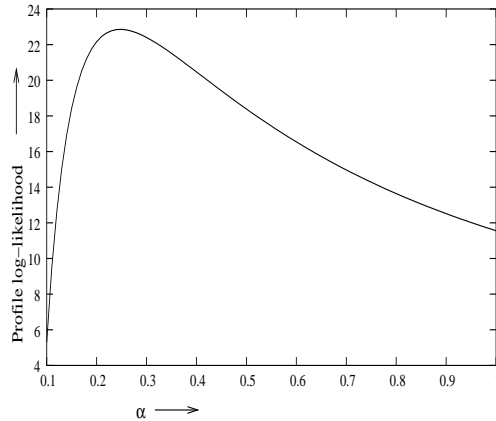


Figure 3: Profile log-likelihood function of α .

7 MULTIVARIATE POWER NORMAL DISTRIBUTION

In this section we provide the multivariate power normal distribution, using the same procedure as before. First let us consider the p -variate Clayton copula as follows;

$$C_{\alpha,p}(u_1, \dots, u_p) = \left(u_1^{-1/\alpha} + \dots + u_p^{-1/\alpha} - (p-1) \right)^{-\alpha}; \quad 0 < u_1, \dots, u_p < 1. \quad (27)$$

As in Theorem 3.1, we have the following multivariate results;

THEOREM 7.1: If (U_1, \dots, U_p) has the joint CDF (27) for $\alpha > 0$, then

(i) $U_1, \dots, U_p \sim \text{uniform } [0, 1]$.

(ii) The joint PDF of (U_1, \dots, U_p) for $0 < u_1, \dots, u_p < 1$, is

$$f_{U_1, \dots, U_p}(u_1, \dots, u_p) = \prod_{i=1}^p \frac{\alpha + i - 1}{u_i^{(1+\alpha)/\alpha}} \times \frac{\alpha^{-p}}{(u_1^{-1/\alpha} + \dots + u_p^{-1/\alpha} - (p-1))^{\alpha+p}}. \quad (28)$$

(iii) The joint CDF of U_1, \dots, U_q , for $q < p$, and for $0 < u_1, \dots, u_q < 1$, is

$$P(U_1 \leq u_1, \dots, U_q \leq u_q) = F_{U_1, \dots, U_q}(u_1, \dots, u_q) = \left(u_1^{-1/\alpha} + \dots + u_q^{-1/\alpha} - (p-1) \right)^{-\alpha}. \quad (29)$$

(iv) The joint survival function of U_1, \dots, U_p for $0 < u_1 < 1, \dots, 0 < u_p < 1$, is

$$P(U_1 > u_1, \dots, U_p > u_p) = 1 - \sum_{i=1}^p F_{U_i}(u_i) - \sum_{1 \leq i \neq j \leq p} F_{U_i U_j}(u_i, u_j) + \dots + (-1)^p F_{U_1 \dots U_p}(u_1, \dots, u_p). \quad (30)$$

(v) The conditional PDF and CDF of (U_{q+1}, \dots, U_p) given $U_1 = u_1, \dots, U_q = u_q$ is

$$f_{U_{q+1}, \dots, U_p | U_1 = u_1, \dots, U_q = u_q}(u_{q+1}, \dots, u_p) = \prod_{i=q+1}^p \frac{\alpha + i - 1}{\alpha u_i^{(1+\alpha)/\alpha}} \times \frac{\left(u_1^{-1/\alpha} + \dots + u_q^{-1/\alpha} - (q-1) \right)^{\alpha+q}}{\left(u_1^{-1/\alpha} + \dots + u_p^{-1/\alpha} - (p-1) \right)^{\alpha+p}}. \quad (31)$$

and

$$F_{U_{q+1}, \dots, U_p | U_1 = u_1, \dots, U_q = u_q}(u_{q+1}, \dots, u_p) = \frac{\left(u_1^{-1/\alpha} + \dots + u_q^{-1/\alpha} - (q-1) \right)^{\alpha+q}}{\left(u_1^{-1/\alpha} + \dots + u_p^{-1/\alpha} - (p-1) \right)^{\alpha+p}}. \quad (32)$$

respectively.

PROOF: The proofs can be obtained by routine calculations, and hence they are avoided.

Now we are ready to define the multivariate power normal distribution similarly as the bivariate power normal distribution. For $\beta_1 > 0, \dots, \beta_p > 0$, consider the random variables;

$$X_1 = \Phi^{-1}(U_1^{1/(\beta_1)}), \quad \dots, \quad X_p = \Phi^{-1}(U_p^{1/(\beta_p)}). \quad (33)$$

Therefore, the joint CDF of X_1, \dots, X_p becomes

$$\begin{aligned} F_{X_1, \dots, X_p}(x_1, \dots, x_p) &= P \left\{ U_1 \leq \Phi^{\beta_1}(x_1), \dots, U_p \leq \Phi^{\beta_p}(x_p) \right\} = C_{\alpha, p}(\Phi^{\beta_1}(x_1), \dots, \Phi^{\beta_p}(x_p)) \\ &= \left(\Phi^{-\beta_1/\alpha} + \dots + \Phi^{-\beta_p/\alpha} - (p-1) \right)^{-\alpha} \end{aligned} \quad (34)$$

The joint PDF of (X_1, \dots, X_p) is

$$f_{X_1, \dots, X_p}(x_1, \dots, x_p) = c \times \frac{(\Phi(x_1))^{-\beta_1/\alpha-1} \dots (\Phi(x_p))^{-\beta_p/\alpha-1} \phi(x_1) \dots \phi(x_p)}{\left[(\Phi(x_1))^{-\beta_1/\alpha} \dots (\Phi(x_p))^{-\beta_p/\alpha} - (p-1) \right]^{\alpha+p}}, \quad (35)$$

where $c = \prod_{i=1}^p \beta_i \times \frac{\Gamma(\alpha + p)}{\alpha^p \Gamma(\alpha)}$.

The random variables (X_1, \dots, X_p) is said to have a p -variate power normal distribution with parameters $\beta_1 > 0, \dots, \beta_p > 0, \alpha > 0$, and it will be denoted by $\text{MPN}(\beta_1, \dots, \beta_p, \alpha)$, if it has the joint PDF (35).

It may be mentioned that the multivariate power normal distribution as defined in (35) can be obtained from the multivariate generalization of the Lindley-Singpurwalla bivariate model, also known as Lomax model, see for example Nayak [27]. The random variables (V_1, \dots, V_p) is said to have a multivariate Lomax distribution if it has the following joint PDF:

$$f_{V_1, \dots, V_p}(v_1, \dots, v_p) = \frac{c_p}{(v_1 + \dots + v_p + 1)^{\alpha+p}}, \quad (36)$$

where $c_p = \frac{\Gamma(\alpha + p)}{\Gamma(p)}$. Consider the following transformation

$$V_i = (\Phi(X_i))^{-\beta_i/\alpha} - 1$$

for $i = 1, \dots, p$, then (X_1, \dots, X_p) has the joint PDF (35). Now we have the following results.

THEOREM 7.2: If $(X_1, \dots, X_p) \sim \text{MPN}(\beta_1, \dots, \beta_p, \alpha)$, then

(i) $q < p$, $(X_1, \dots, X_q) \sim \text{MPN}(\beta_1, \dots, \beta_q, \alpha)$.

(ii) The joint survival function of (X_1, \dots, X_p) is

$$\begin{aligned} S_{X_1, \dots, X_p}(x_1, \dots, x_p) &= P(X_1 > x_1, \dots, X_p > x_p) \\ &= 1 - \sum_{i=1}^p (\Phi(x_i))^{\beta_i} + \sum_{1 \leq i \neq j \leq p} \left((\Phi(x_i))^{-\beta_i/\alpha} + (\Phi(x_j))^{-\beta_j/\alpha} - 1 \right)^{-\alpha} + \dots \\ &\quad \dots + (-1)^p \left[(\Phi(x_1))^{-\beta_1/\alpha} + \dots + (\Phi(x_p))^{-\beta_p/\alpha} - (p-1) \right]^{-\alpha}. \end{aligned}$$

PROOF: The proof of (i) can be obtained by taking $x_i \rightarrow \infty$ for $i = q+1, \dots, p$, and the proof of (ii) can be obtained using part (iii) of Theorem 7.1.

8 CONCLUSIONS

In this paper we have considered the bivariate power normal distribution whose marginals are power normal distributions. We have established several properties of the proposed bivariate distribution using Copula property. We have proposed a multivariate generalization of the corresponding bivariate distribution. It is observed that several properties of the bivariate power normal distribution can be extended to the multivariate case also. We have considered two different estimators and it is observed that the two estimators behave quite similarly. In our data analysis we have observed that the MLEs exist, although due to complicated nature of the likelihood function we could not prove it theoretically the existence and the uniqueness of the MLEs. It may be mentioned that since the MLE of the unknown parameter of a univariate power normal distribution function always exists, it is possible to get some reasonable estimators of the unknown parameters in the bivariate case in all cases.

It may be mentioned that although we have provided the bivariate power normal distribution but our method can be applied for a much larger class of distribution functions, namely the proportional reversed hazard class. It is important to establish the properties in the general case. Moreover, in the literature several other power normal distributions have been introduced by several researchers, see for example Goto and Hamasaki [11] and Isogai [12]. Establishing the similarities between different power normal distributions are of interest. More work is needed along these directions.

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APPENDIX A: EXPECTED FISHER INFORMATION MATRIX

In this Appendix we provide the exact expressions of all the elements of the expected Fisher information matrix. We use the following notations;

$$I = - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where

$$\begin{aligned} a_{11} &= E \left(\frac{\partial^2 l}{\partial \alpha_1^2} \right) & a_{22} &= E \left(\frac{\partial^2 l}{\partial \alpha_2^2} \right), & a_{33} &= E \left(\frac{\partial^2 l}{\partial \alpha^2} \right), & a_{12} &= a_{21} = E \left(\frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_2} \right) \\ a_{13} &= a_{31} = E \left(\frac{\partial^2 l}{\partial \alpha \partial \alpha_1} \right) & a_{23} &= a_{32} = E \left(\frac{\partial^2 l}{\partial \alpha \partial \alpha_2} \right) \end{aligned}$$

and $l = l(\alpha, \alpha_1, \alpha_2)$ as defined in (19).

$$\begin{aligned} a_{11} &= \frac{n}{\alpha_1^2} \left[1 + \frac{2(\alpha + 1)}{(\alpha + 3)\alpha^2} \right], & a_{22} &= \frac{n}{\alpha_2^2} \left[1 + \frac{2(\alpha + 1)}{(\alpha + 3)\alpha^2} \right], & a_{33} &= \frac{n}{\alpha^2} + \frac{n}{(1 + \alpha)^2}, \\ a_{13} &= a_{31} = -n \frac{(\alpha + 1)}{\alpha \alpha_1}, & a_{23} &= a_{32} = -n \frac{(\alpha + 1)}{\alpha \alpha_2}, \\ a_{12} &= a_{21} = -n \alpha (\alpha + 1) (\alpha + 2) \int_1^\infty \int_1^\infty \frac{u_1 u_2 \ln u_1 \ln u_2}{(u_1 + u_2 - 1)^{\alpha+4}} du_1 du_2. \end{aligned}$$

We will show how to obtain a_{22} , the others can be obtained along the same line. Note that

$$E \left(\frac{\partial^2 l}{\partial \alpha_1^2} \right) = -\frac{n}{\alpha_1^2} - \frac{n(\alpha + 2)}{\alpha_1^2} E \left[\frac{(\ln \Phi(X_1)^{-\alpha_1})^2 \Phi(X_1)^{-\alpha_1} (\Phi(X_2)^{-\alpha_2} - 1)}{((\Phi(X_1))^{-\alpha_1} + (\Phi(X_2))^{-\alpha_2} - 1)^2} \right].$$

Now consider

$$\begin{aligned} E \left[\frac{(\ln \Phi(X_1)^{-\alpha_1})^2 \Phi(X_1)^{-\alpha_1} \Phi(X_2)^{-\alpha_2}}{((\Phi(X_1))^{-\alpha_1} + (\Phi(X_2))^{-\alpha_2} - 1)^2} \right] &= \\ \alpha(\alpha + 1)\alpha_1\alpha_2 \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{(\ln \Phi(x_1)^{-\alpha_1})^2 \Phi(x_1)^{-2\alpha_1-1} \Phi(x_2)^{-2\alpha_2-1} \phi(x_1)\phi(x_2)}{((\Phi(x_1))^{-\alpha_1} + (\Phi(x_2))^{-\alpha_2} - 1)^{\alpha+4}} dx_1 dx_2. \end{aligned} \quad (37)$$

After making the transformation $u_1 = \Phi(x_1)^{-\alpha_1}$, and $u_2 = \Phi(x_2)^{-\alpha_2}$, the right hand side of (37) becomes;

$$\begin{aligned} \alpha(\alpha + 1) \int_1^\infty \int_1^\infty \frac{(\ln u_1)^2 u_1 u_2}{(u_1 + u_2 - 1)^{\alpha+4}} du_1 du_2 &= \frac{\alpha(\alpha + 1)}{\alpha + 3} \int_1^\infty \frac{(\ln u_1)^2}{u_1^{\alpha+2}} du_1 \\ &+ \frac{\alpha(\alpha + 1)}{(\alpha + 2)(\alpha + 3)} \int_1^\infty \frac{(\ln u_1)^2}{u_1^{\alpha+1}} du_1. \end{aligned} \quad (38)$$

By making the transformation $z = \ln u_1$, the right hand side of (38) can be obtained as

$$\frac{\alpha(\alpha+1)}{\alpha+3} \int_0^\infty z^2 e^{-(\alpha+1)z} dz + \frac{\alpha(\alpha+1)}{(\alpha+2)(\alpha+3)} \int_0^\infty z^2 e^{-\alpha z} dz = \frac{2\alpha}{(\alpha+3)(\alpha+1)^2} + \frac{2(\alpha+1)}{(\alpha+2)(\alpha+3)\alpha^2}.$$

Along the same line we can obtain

$$E \left[\frac{(\ln \Phi(X_1))^{-\alpha_1} \Phi(X_1)^{-\alpha_1}}{((\Phi(X_1))^{-\alpha_1} + (\Phi(X_2))^{-\alpha_2} - 1)^2} \right] = \frac{2\alpha}{(\alpha+3)(\alpha+1)^2}. \quad (39)$$

Therefore, $E \left(\frac{\partial^2 l}{\partial \alpha_1^2} \right) = -\frac{1}{\alpha_1^2} \left[1 + \frac{2(\alpha+1)}{(\alpha+3)\alpha^2} \right].$ ■

THE ELEMENTS OF THE MATRIX V

The 3×3 matrix V can be expressed as follows, see Joe [18],

$$V = (-D^{-1})M(-D^{-1})^T.$$

Here the 3×3 matrix D can be written as follows;

$$-D = \begin{bmatrix} n/\beta_1^2 & 0 & 0 \\ 0 & n/\beta_2^2 & 0 \\ a_1 & a_2 & a_3 \end{bmatrix},$$

a_1, a_2, a_3 respectively are the (1,3)th, (2,3)th, and (3,3)th elements of $J^{-1}I(J^{-1})^T$ respectively. The 3×3 matrix M is as follows;

$$M = E(gg^T),$$

where $g^T = (g_1, g_2, g_3)$, and

$$g_1 = (n/\beta_1) + n \ln \Phi(X_1), \quad g_2 = (n/\beta_2) + n \ln \Phi(X_2)$$

$$g_3 = -\frac{n}{\alpha(\alpha+1)} + \frac{n}{\alpha^2} [\beta_1 \ln(\Phi(X_1)) + \beta_2 \ln(\Phi(X_2))] - n \ln((\Phi(X_1))^{-\beta_1/\alpha} + (\Phi(X_2))^{-\beta_2/\alpha} - 1) \\ + \frac{n(\alpha+2)}{\alpha^2} \times \frac{\beta_1(\Phi(X_1))^{-\beta_1/\alpha-1} + \beta_2(\Phi(X_2))^{-\beta_2/\alpha}}{((\Phi(X_1))^{-\beta_1/\alpha} + (\Phi(X_2))^{-\beta_2/\alpha} - 1)}.$$

Here $(X_1, X_2) \sim \text{BPN}(\beta_1, \beta_2, \alpha)$, and the elements of g can be obtained in terms of integration.

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