Exact Likelihood Inference for Two Exponential Populations under Jointly Generalized Progressive Hybrid Censoring

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ABSTRACT

The generalized progressive hybrid censoring schemes (GPHCS) have become quite popular in the case when there are very few failures before pre-determined time T in progressive hybrid censoring schemes. Whereas the GPHCS always ensures a fixed number of failures, which makes this scheme very popular. In this paper, we introduce a new joint generalized progressive hybrid censoring scheme (J-GPHCS) for two independent samples from different populations. We place both the samples simultaneously on the life testing experiment. Further, we assume the lifetime of the experimental units, under both samples, to follow exponential distribution with mean θ_1 and θ_2, respectively. The maximum likelihood estimators, of the unknown parameters and their exact distributions, are derived. Asymptotic and bootstrap confidence intervals are also constructed. Further, the Bayesian inference of some unknown parametric functions is considered under a very flexible Inverse-Gamma priors. We also obtain Bayes estimators, and associated credible intervals of the unknown parameters. Extensive simulation studies are performed to investigate the proposed estimators. Finally, the methods are illustrated with the analysis of a real data set.

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KEYWORDS

Exponential Distribution; Generalized Progressive Hybrid Censoring; Joint Censoring Scheme; Maximum Likelihood Estimation; Bayes Estimation

1. Introduction

In some life testing experiments the exact datasets of many observations can not be recorded due to time restrictions, deficiency of the cost, operational adversities etc. Censoring can be useful in such situations to reduce the experimental time or cost of the experiment as required. Therefore, in the reliability theory, many researchers are focused on the censoring schemes and their applications under various problems. The very basic censoring schemes are Type-I censoring or time censoring i.e., the experiment terminates at a pre-fixed time point T and Type-II censoring where experiment terminates at a pre-specified number of failures r. For more details one can see Schneider and Weissfeld [24], Balakrishnan [6], Write et al. [26]. Following these two types of censoring scheme, Epstein [15] introduced the hybrid censoring as combination of both Type-I and Type-II censoring schemes. All the above discussed censoring schemes the removals of the experimental units during the experiment are not allowed. To overcome such a situation different types of censoring schemes have been proposed in the literature such as progressive Type-I, progressive Type-II, progressive hybrid Type-I, generalized progressive censoring (see Balakrishnan and Cramer [7] for details).

In progressive hybrid censoring schemes there may be some cases where very few failures occurred before the pre-determined time point T, see Kundu and Joarder [20]. Recently, Cho et al. [13] proposed a new censoring scheme namely generalized progressive hybrid censoring scheme (GPHCS) which always ensures a fixed number of failures at the end of the experiment. The scheme is described as follows.

Suppose the life-testing experiment starts with n independent and identically distributed (i.i.d.) units $(X_1, X_2, \cdots, X_n)$. Also, assume a pre-fixed time point T, the pre-determined integers k, r such that $1 \leq k < r \leq n$. Further, assume that the number of removals $R_1, R_2, \cdots, R_r$ which satisfies $\sum_{i=1}^{r} R_i = n - r$. Therefore, at the time of the first failure, say $X_{1:r:n}$ has occurred, remove R_1 surviving units randomly from the experiment. Similarly, at the time of the second failure, $X_{2:r:n}$, R_2 units are removed from remaining $(n - R_1 - 2)$ surviving units and such a scheme continues until the terminated time

$$T^* = \max\{X_{k:r:n}, \min\{T, X_{r:r:n}\}\},$$

occurred and at that time remove all the remaining units from the experiment (see Figure 1). Therefore, in this scheme, one of the following types of failure times are observed

$$X_{1:r:n}, X_{2:r:n}, \cdots, X_{k:r:n}, \text{ if } T < X_{k:r:n} < X_{r:r:n} \quad \text{(Case-I)}$$
$$X_{1:r:n}, X_{2:r:n}, \cdots, X_{D:r:n}, \text{ if } X_{k:r:n} < T < X_{r:r:n} \quad \text{(Case-II)}$$
$$X_{1:r:n}, \cdots, X_{k:r:n}, \cdots, X_{r:r:n}, \text{ if } X_{k:r:n} < X_{r:r:n} < T, \quad \text{(Case-III)}$$

where $X_{D:r:n} < T < X_{D+1:r:n}$ and D denotes the number of observed failures until pre-determined time point T. Note that at terminated time T^*, this censoring scheme always provides a minimum of k failures which will be the main advantage of this censoring scheme over the progressive hybrid censoring scheme.

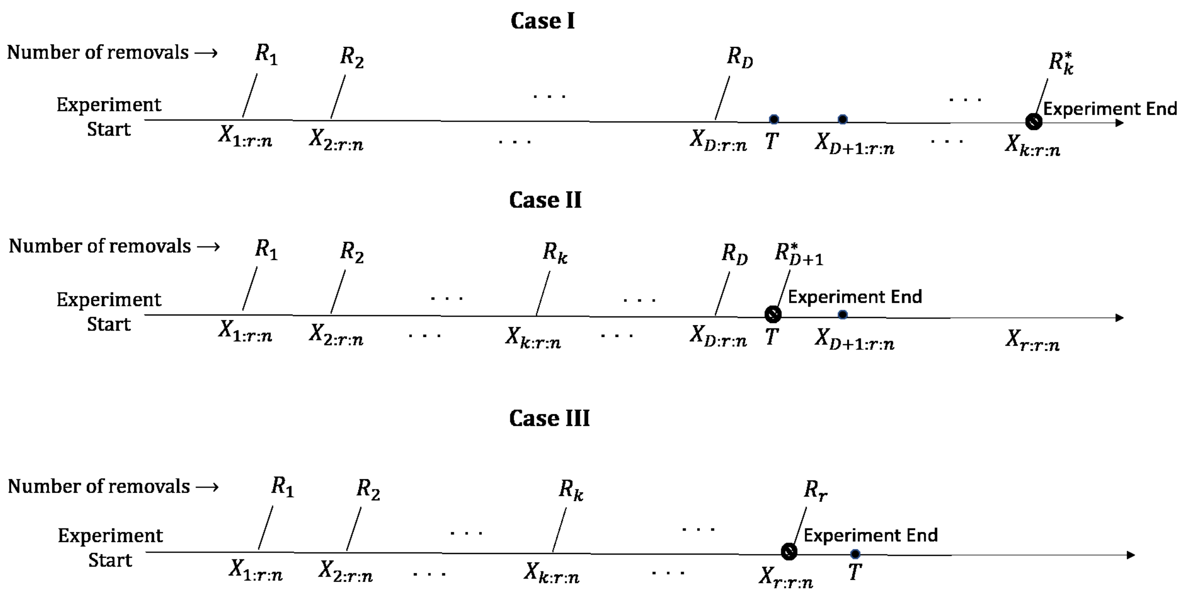

Figure 1.: Schematic representation of generalized progressive hybrid censoring scheme.

In reliability theory, comparative lifetime experiments are needed to study to save experimental costs and to accelerate the execution of the experiments. To handle such problems many reliability studies used one-sample censoring schemes, however, recent studies head for the jointly censoring schemes since the experimenters may need to decide under conducting comparative life tests of units. Undoubtedly, some products can be produced by two different production lines within the same factory. An experimenter may need two independent samples from these two production lines and be placed simultaneously on a life-testing experiment due to cost consideration. For example, comparative lifetimes of two groups in insulating fluids subjected to high voltage stress, comparative lifetimes of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes (See Rasouli and Balakrishnan [21]), comparative lifetimes of the two fabric weaving in the same factory, comparative lifetimes of two simultaneously working rotogravure printing press in the same printery and etc. In these types of experiments, the experimenters can evaluate the individual performances of the devices. However, in the joint structure, the performances of the devices can be assessed in a simultaneous operation.

For more details on joint censoring schemes, Balakrishnan and Rasouli [9], Rasouli and Balakrishnan [21] introduced the joint Type-II censoring and the joint progressive Type-II censoring schemes, respectively. The jointly censored samples are handled in various studies by Ashour and Abo-Kasem [2]-[5], Balakrishnan and Su [11], Balakrishnan et al. [10], Doostparast et al. [14], Abo-Kasem et al. [1], Krishna and Goel [19]. In these studies, mainly Type-II and related censoring schemes are handled. Recently, Su and Zhu [22] studied the joint generalized Type-I hybrid censoring scheme for two exponential populations. Unlike similarities in mathematical structures of joint generalized Type-I hybrid censoring scheme and generalized progressive hybrid censoring scheme, there are significant differences between these censoring schemes. The major difference is considering the progressive censoring schemes i.e., $R_1, R_2, \cdots, R_m$. It is known that there are many situations where progressive censoring schemes are very much useful like the life testing experiments where the lifetime units are allowed to remove from a test during the test time and one can use the removed items in some future experiments. Further, we have shown that the joint generalized Type-I hybrid censoring proposed by Su and Zhu [22] is a special case of our proposed joint censoring scheme. For some particular choice of removals R_i's we can get the joint generalized Type-I hybrid censoring scheme.

In this paper, our main aim is to introduce a new jointly censoring scheme which will be useful over the existing literature for joint censoring schemes due to its flexibility of getting a fixed number of failures in an adequate time range. We mainly consider two exponential populations, with mean θ_1 and θ_2, respectively, under the jointly generalized progressive hybrid censoring scheme (J-GPHCS). We have considered only two different sample sizes of the different populations. This proposed scheme also ensures a pre-fixed number of failures as in single sample GPHCS. For simplicity we have considered only two sample cases but one can extend this methodology for more than two sample cases also.

Rest of the paper is organized as follows. We introduced the modeling of J-GPHCS for two samples in Section 2. Then, we derive the maximum likelihood estimators (MLEs) of the unknown parameters θ_1, θ_2 and their exact distributions are provided in Section 3. The approximate confidence intervals in two different methods namely asymptotic confidence intervals and bootstrap confidence intervals are constructed in Sections 4. We also obtain the Bayesian estimation and also calculate the credible intervals in Section 5. The simulations are provided to evaluate the performances of the estimators in Section 6. We illustrate the proposed methods through the analysis of a real life data set in Section 6, while Section 7 ends with some concluding remarks.

2. Model Description

Suppose the products of a life-testing experiment belong to two independent populations, say Pop-1 and Pop-2. We draw a random sample of size n_1, say Sam-1 from Pop-1 and a random sample of size n_2 from Pop-2 say Sam-2. One can also take more than two different populations. More specifically, suppose the lifetimes of n_1 units of Sam-1 $X_1, X_2, \cdots, X_{n_1}$ are i.i.d. random variables having exponential distribution with mean θ_1 with density and distribution functions as

$$f(x) = \frac{1}{\theta_1} e^{-\frac{x}{\theta_1}} \quad \text{and} \quad F(x) = 1 - e^{-\frac{x}{\theta_1}} \quad \text{for} \quad x > 0, \quad \theta_1 > 0, \tag{1}$$

respectively. Similarly, let the lifetimes of n_2 units of Sam-2, $Y_1, Y_2, \cdots, Y_{n_2}$ be i.i.d. random variables from exponential $Exp(\theta_2)$ population with density and distribution functions as

$$g(y) = \frac{1}{\theta_2} e^{-\frac{y}{\theta_2}} \quad \text{and} \quad G(y) = 1 - e^{-\frac{y}{\theta_2}} \quad \text{for} \quad y > 0, \quad \theta_2 > 0, \tag{2}$$

respectively. Therefore, it is assumed that $w = (W_1, W_2, \cdots, W_n)$ denote the order statistics of the $n = n_1 + n_2$ random variables $(X_1, X_2, \cdots, X_{n_1}, Y_1, Y_2, \cdots, Y_{n_2})$. Then the proposed J-GPHCS for two samples can be described as follows.

Let T be the pre-fixed time point and also fixed two integers r, k such that $1 \leq k < r \leq n$ in advance. Firstly, at the time of the first failure (which may be from either Sam-1 or Sam-2), R_1 surviving units are randomly withdrawn from the remaining $(n-1)$ surviving units. Similarly, at the time of the second failure (which also from either Sam-1 or Sam-2), R_2 units are randomly withdrawn from the remaining $(n - R_1 - 2)$ surviving units, and such a scheme continues until the terminated time

$$T^* = \max\{W_{k:r:n}, \min\{T, W_{r:r:n}\}\},$$

has arrived. Under the J-GPHCS the possible values of T^* would be

$$T^* = \begin{cases} W_{k:r:n} & \text{, if} \quad T < W_{k:r:n} < W_{r:r:n}, \\ T & \text{, if} \quad W_{k:r:n} < T < W_{r:r:n}, \\ W_{r:r:n} & \text{, if} \quad W_{k:r:n} < W_{r:r:n} < T, \end{cases}$$

and the corresponding number of observed failures are

$$m = \begin{cases} k & ,\text{Case-I} \\ D & ,\text{Case-II} \\ r & ,\text{Case-III}, \end{cases}$$

where D denote the number of observed failures until predetermined time point T for Case-II, i.e., $W_{D:r:n} < T < W_{D+1:r:n}$.

Note that the total number of failures m for different cases and the progressive censoring scheme $R_1, R_2, \cdots, R_m$ are pre-specified. Further, suppose $R_i = s_i + q_i$ $i = 1, 2, \cdots, m$ where s_i and q_i denotes the pre-fixed number of units withdrawn at the time of $i-$th failure that belongs to Sam-1 and Sam-2, respectively. Observed that s_i and q_i are unknown and random variables. Therefore, $R = (R_1, R_2, \cdots, R_m)$ can be decomposed as $R = S + T$, where $S = (s_1, s_2, \cdots, s_m)$ and $Q = (q_1, q_2, \cdots, q_m)$. In particular if $m = r$ then the J-GPHCS reduce to a joint Type-II progressive censoring scheme established by Rasouli and Balakrishnan [21]. Let $Z = (Z_1, Z_2, \cdots, Z_m)$ for which Z_i, $(\forall i = 1, 2, \cdots, m)$ takes two values either 1 or 0 depending on W_i from Sam-I or Sam-II, respectively. Then, under J-GPHCS the likelihood function of (W, Z, R) where can be written as

$$L_W(\theta_1, \theta_2) = C \begin{cases} \prod_{i=1}^{k} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{s_i} \left[\bar{G}(w_i)\right]^{q_i} & ,\text{Case-I} \\ \prod_{j=1}^{D} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{s_i} \left[\bar{G}(w_i)\right]^{q_i} \left[\bar{F}(T)\right]^{R_S^*} \left[\bar{G}(T)\right]^{R_Q^*} & ,\text{Case-II} \\ \prod_{j=1}^{r} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{s_i} \left[\bar{G}(w_i)\right]^{q_i} & ,\text{Case-III}, \end{cases}$$

where $\bar{F} = 1 - F$, $\bar{G} = 1 - G$, $D_1 = \sum_{i=1}^{d} z_i$, $D_2 = \sum_{i=1}^{d}(1 - z_i)$, $\sum_{i=1}^{m} s_i + \sum_{i=1}^{m} q_i =$

$\sum_{i=1}^{m} R_i$, $R_S^* = n_1 - D_1 - \sum_{i=1}^{D} s_i$, $R_Q^* = n_2 - D_2 - \sum_{i=1}^{D}(R_i - s_i)$ and constant

$$C = \begin{cases} \prod_{j=1}^{k} \left[\sum_{i=j}^{m}(s_i + z_i)\right]\left[\sum_{i=j}^{m}((R_i - s_i) + (1 - z_i))\right] & \text{, Case-I} \\ \prod_{j=1}^{D} \left[\sum_{i=j}^{m}(s_i + z_i)\right]\left[\sum_{i=j}^{m}((R_i - s_i) + (1 - z_i))\right] & \text{, Case-II} \\ \prod_{j=1}^{r} \left[\sum_{i=j}^{m}(s_i + z_i)\right]\left[\sum_{i=j}^{m}((R_i - s_i) + (1 - z_i))\right] & \text{, Case-III.} \end{cases}$$

On the other, a special case of this model can be obtained by ignoring the removals. If we take $s_i = 0$, $q_i = 0$ for $i = 1, 2, \cdots, m-1$ and $s_m = n_1 - \sum_{i=1}^{m} z_i$, $q_m = n_2 - \sum_{i=1}^{m}(1 - z_i)$ i.e., if we will not remove any surviving items in between the test, only the removals can occur at the stopping time of the test, then we obtain the model proposed by Su and Zhu [22] as the likelihood can be written in the following form

$$L_W(\theta_1, \theta_2) = C \begin{cases} \prod_{i=1}^{k} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_r)\right]^{n_1 - n_{1_k}} \left[\bar{G}(w_r)\right]^{n_2 - n_{2_k}} & \text{, Case-I} \\ \prod_{j=1}^{D} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(T)\right]^{n_1 - n_{1_D}} \left[\bar{G}(T)\right]^{n_2 - n_{2_D}} & \text{, Case-II} \\ \prod_{j=1}^{r} f(w_i)^{z_i} g(w_i)^{1-z_i} \left[\bar{F}(w_i)\right]^{n_1 - n_{1_r}} \left[\bar{G}(w_i)\right]^{n_2 - n_{2_r}} & \text{, Case-III,} \end{cases}$$

where $n_{1_k} = \sum_{i=1}^{k} z_i$, $n_{2_k} = \sum_{i=1}^{k}(1 - z_i)$, $n_{1_D} = \sum_{i=1}^{D} z_i$, $n_{2_D} = \sum_{i=1}^{D}(1 - z_i)$, $n_{1_r} = \sum_{i=1}^{r} z_i$, $n_{2_r} = \sum_{i=1}^{r}(1 - z_i)$ and $C = n_1!n_2!/(n_1 - n_{1_k})!(n_2 - n_{2_k})!$ for the Case-I and also similarly can be obtained for the Case-II and Case-III.

On the other hand, in progressive censoring schemes, the most common censoring scheme $R_1, R_2, \cdots, R_m$ is mostly pre-determined before experiment. However, the amount of units removed from the test may not always be determined in advance, in reliability studies and these amounts can occur as random. Thus, random removals are need to be considered. Therefore, we assume that a random number of units removed at each failure time instead of pre-fixed number of units. For this purpose, we considered the values of R_i following a discrete uniform distribution such that

$$P(R_1 = \varphi_1) = \frac{1}{n - r + 1},$$

where $\varphi = (\varphi_1, \cdots, \varphi_r)$ denotes the value of R and

$$P\left(R_i = \varphi_i \mid R_{i-1} = \varphi_{i-1}, \ldots, R_1 = \varphi_1\right) = \frac{1}{n - r - (\varphi_1 + \cdots + \varphi_{i-1}) + 1}, \tag{3}$$

where $0 \leq \varphi_1 \leq n - r$ and $0 \leq \varphi_i \leq n - r - (\varphi_1 + \varphi_2 + \cdots + \varphi_{i-1})$.Similar procedures were used by various authors, one can see Wu [27], Wu et al. [28], Gunasekera [17]. Since R_i is independent of observations W_i, the likelihood function can be written as

$$L(\theta_1, \theta_2|\mathbf{W}, \mathbf{Z}) = L_W(\theta_1, \theta_2) P\left(R = \varphi\right),$$

where

$$\begin{aligned} P(R = \varphi) = P\left(R_{r-1} = \varphi_{r-1} \mid R_{r-2} = \varphi_{r-2}, \ldots, R_1 = \varphi_1\right) \times \\ \cdots \times P\left(R_2 = \varphi_2 \mid R_1 = \varphi_1\right) P\left(R_1 = \varphi_1\right). \end{aligned}$$

It is clearly seen that $P(R = \varphi)$ does not depend on the parameters θ_1 and θ_2. Thus, the maximum likelihood estimators can be obtained by maximizing equation $L_W(\theta_1, \theta_2)$.

3. Maximum Likelihood Estimation

Suppose $X_1, \cdots, X_{n_1}$ denote the lifetimes of n_1 i.i.d. units from Pop-1, which follows exponential random variables with mean θ_1, denoted by $Exp(\theta_1)$. Similarly, let $Y_1, \cdots, Y_{n_2}$ denote the lifetimes of n_2 i.i.d. units from Pop-2, which follows exponential random variables with mean θ_2, denoted by $Exp(\theta_2)$ Then, by replacing the equations (1) and (2) in $L_W(\theta_1, \theta_2)$, the likelihood function of the observed sample can be obtained as

$$L(\theta_1, \theta_2) \propto \frac{1}{\theta_1^{D_1}} \frac{1}{\theta_2^{D_2}} e^{-\frac{V_1}{\theta_1}} e^{-\frac{V_2}{\theta_2}}, \tag{4}$$

where $D_1 = \sum_{i=1}^{m} z_i$, $D_2 = \sum_{i=1}^{m}(1 - z_i)$ are completely observed failures from Sam-1 and Sam-2, respectively with $D_1 + D_2 = m$. Also

$$V_1 = \begin{cases} \sum_{i=1}^{k} w_i(z_i + s_i) & \text{, Case-I} \\ \sum_{i=1}^{D} w_i(z_i + s_i) + TR_S^* & \text{, Case-II} \\ \sum_{i=1}^{r} w_i(z_i + s_i) & \text{, Case-III,} \end{cases}$$

and

$$V_2 = \begin{cases} \sum_{i=1}^{k} w_i(1 - z_i) + \sum_{i=1}^{k} w_i(R_i - s_i) & \text{, Case-I} \\ \sum_{i=1}^{D} w_i(1 - z_i) + \sum_{i=1}^{D} w_i(R_i - s_i) + TR_Q^* & \text{, Case-II} \\ \sum_{i=1}^{r} w_i(1 - z_i) + \sum_{i=1}^{r} w_i(R_i - s_i) & \text{, Case-III.} \end{cases}$$

The log-likelihood function is obtained as

$$\ell(\theta_1, \theta_2) \propto -D_1 \ln \theta_1 - D_2 \ln \theta_2 - \frac{V_1}{\theta_1} - \frac{V_2}{\theta_2}, \tag{5}$$

and the MLEs of the parameters can be obtained as

$$\hat{\theta}_1 = \frac{V_1}{D_1}, \qquad \hat{\theta}_2 = \frac{V_2}{D_2} = \frac{V_2}{m - D_1}. \tag{6}$$

From the above equation (6), it is clear that the MLEs $\hat{\theta_1}$ and $\hat{\theta_2}$ do not exist if $D_1 = 0$ or m for different cases of the J-GPHCS. Hence, the MLEs in (6) are only conditional MLEs, conditioned on $1 \leq D_1 \leq m - 1$. Therefore, in the next section, we are now interested in deriving the exact conditional distributions of $\hat{\theta_1}$ and $\hat{\theta_2}$, conditioned on $1 \leq D_1 \leq m - 1$.

3.1. Distribution of Estimators

Theorem 3.1. *The probability mass function of $D_1 = \sum_{i=1}^{m} Z_i$ is given by*

$$\begin{aligned} P(D_1 = i) &= P(D_1 = i, T < W_{k:r:n} < W_{r:r:n}) + P(D_1 = i, W_{k:r:n} < T < W_{r:r:n}) \\ &\quad + P(D_1 = i, W_{k:r:n} < W_{r:r:n} < T) \qquad (7) \\ &= \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}}\!\!\cdots\sum \;\sum_{(s_1,\cdots,s_{k-1})\in T_i^{(1)}}\!\!\cdots\sum \prod_{j=1}^{k} P_j^{(1)} \prod_{l=1}^{k-1} P^{(1)}_{s_l|z_l^*;s_{l-1}^*} \\ &+ \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}}\!\!\cdots\sum \;\sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}}\!\!\cdots\sum \prod_{j=1}^{D} P_j^{(2)} \prod_{l=1}^{D-1} P^{(2)}_{s_l|z_l^*;s_{l-1}^*} \\ &+ \sum_{(z_1,\cdots,z_r)\in Q_i^{(3)}}\!\!\cdots\sum \;\sum_{(s_1,\cdots,s_{r-1})\in T_i^{(3)}}\!\!\cdots\sum \prod_{j=1}^{r} P_j^{(3)} \prod_{l=1}^{r-1} P^{(3)}_{s_l|z_l^*;s_{l-1}^*}, \end{aligned}$$

where

$$P_j^{(i_1)} = \frac{(n_1 - d1^{i_1}_{j-1} - \sum_{i=1}^{j-1} s_i)\, z_j + (n_2 - d2^{i_1}_{j-1} - \sum_{i=1}^{j-1}(R_i - s_i))\,(1 - z_j)}{(n_1 - d1^{i_1}_{j-1} - \sum_{i=1}^{j-1} s_i)\theta_2 + (n_2 - d2^{i_1}_{j-1} - \sum_{i=1}^{j-1}(R_i - s_i))\theta_1} \theta_1^{1-z_j}\theta_2^{z_j},$$

and

$$\begin{aligned} P^{(i_1)}_{s_l|z_l^*;s_{l-1}^*} &= P(S_l = s_l | Z_1 = z_1, \cdots, Z_l = z_l; S_1 = s_1, \cdots, S_{l-1} = s_{l-1}) \\ &= \frac{\binom{n_1 - \sum_{i=1}^{l-1} z_i - \sum_{i=1}^{l-1} s_i}{s_l}\binom{n_2 - \sum_{i=1}^{l-1}(1-z_i) - \sum_{i=1}^{l-1}(R_i - s_i)}{q_l}}{\binom{n - l - \sum_{i=1}^{l-1} R_i}{R_l}}. \end{aligned}$$

Also,

$$Q_i^{(i_1)} = \{(z_1, \cdots, z_m) : z_j = 0 \quad or \quad 1 \quad and \quad \sum_{j=1}^{m} z_j = i\},$$

and

$$T_i^{(i_1)} = \{(s_1, \cdots, s_{m-1}) : \sum_{j=1}^{m} s_j = n_1 - i \quad with \quad 0 \le s_m \le R_m\}.$$

Note that $d1^{i_1}_{j-1} = \sum_{i=1}^{j-1} z_i$ *and* $d2^{i_1}_{j-1} = \sum_{i=1}^{j-1}(1-z_i)$ *for* $i_1 = 1, 2, 3$, *corresponding to Case-I, Case-II, and Case-III, respectively.*

Proof: Proof of this theorem is given in Appendix.

Theorem 3.2. *Conditional on* $1 \leq D_1 < m-1$ *the moment generating function (MGF) of* $\hat{\theta}_1$ *is given by*

$$\begin{aligned} M_{\hat{\theta}_1}(t) &= \frac{1}{P(1 \leq D_1 < k-1)} \sum_{i=1}^{k-1} \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \cdots \sum \sum_{(s_1,\cdots,s_{k-1})\in T_i^{(1)}} \cdots \sum \prod_{j=1}^{k} P_j^{(1)} \prod_{l=1}^{k-1} P^{(1)}_{s_l|z_l^*;s_{l-1}^*} \prod_{l_1=1}^{k-1} (1-\alpha^{(1)}_{l_1} t)^{-1} \\ &+ \frac{e^{-(\frac{T\,R_S^*}{\theta_1} - \frac{T\,R_S^* t}{i} + \frac{T\,R_Q^*}{\theta_2})}}{P(1 \leq D_1 < D-1)} \sum_{i=1}^{D-1} \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \cdots \sum \sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}} \cdots \sum \prod_{j=1}^{D} P_j^{(2)} \prod_{l=1}^{D-1} P^{(2)}_{s_l|z_l^*;s_{l-1}^*} \prod_{l_1=1}^{D-1} (1-\alpha^{(2)}_{l_1} t)^{-1} \\ &+ \frac{1}{P(1 \leq D_1 < r-1)} \sum_{i=1}^{r-1} \sum_{(z_1,\cdots,z_r)\in Q_i^{(3)}} \cdots \sum \sum_{(s_1,\cdots,s_{r-1})\in T_i^{(3)}} \cdots \sum \prod_{j=1}^{r} P_j^{(3)} \prod_{l=1}^{r-1} P^{(3)}_{s_l|z_l^*;s_{l-1}^*} \prod_{l_1=1}^{r-1} (1-\alpha^{(3)}_{l_1} t)^{-1}, \end{aligned}$$

where

$$\alpha^{(i_1)}_{l_1} = \frac{(n_1 - d1^{i_1}_{j-1} - \sum_{i=1}^{l_1-1} s_i)\theta_1\theta_2}{\Big((n_1 - d1^{i_1}_{j-1} - \sum_{i=1}^{l_1-1} s_i)\theta_2 + (n_2 - d2^{i_1}_{j-1} - \sum_{i=1}^{l_1-1}(R_i - s_i))\theta_1\Big)}.$$

Proof: Proof of this theorem is given in Appendix.

Theorem 3.3. *Conditioning on* $1 \leq D_1 < m-1$, *the conditional density function of* $\hat{\theta}_1$ *is given by*

$$\begin{aligned} f_{\hat{\theta}_1}(x) &= \frac{1}{P(1 \leq D_1 < k-1)} \sum_{i=1}^{k-1} \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \cdots \sum \sum_{(s_1,\cdots,s_{k-1})\in T_i^{(1)}} \cdots \sum \prod_{j=1}^{k} P_j^{(1)} \prod_{l=1}^{k-1} P^{(1)}_{s_l|z_l^*;s_{l-1}^*} \times g^{(1)}_{Y_1}(y_1) \\ &+ \frac{e^{-(\frac{T\,R_S^*}{\theta_1} - \frac{T\,R_S^* t}{i} + \frac{T\,R_Q^*}{\theta_2})}}{P(1 \leq D_1 < D-1)} \sum_{i=1}^{D-1} \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \cdots \sum \sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}} \cdots \sum \prod_{j=1}^{D} P_j^{(2)} \prod_{l=1}^{D-1} P^{(2)}_{s_l|z_l^*;s_{l-1}^*} \times g^{(2)}_{Y_1}(y_1) \\ &+ \frac{1}{P(1 \leq D_1 < r-1)} \sum_{i=1}^{r-1} \sum_{(z_1,\cdots,z_r)\in Q_i^{(3)}} \cdots \sum \sum_{(s_1,\cdots,s_{r-1})\in T_i^{(3)}} \cdots \sum \prod_{j=1}^{r} P_j^{(3)} \prod_{l=1}^{r-1} P^{(3)}_{s_l|z_l^*;s_{l-1}^*} g^{(3)}_{Y_1}(y_1), \quad (8) \end{aligned}$$

where $Y_1^{(i_1)} \overset{d}{=} \sum_{i=1}^{m} Y_{l_1}^{(i_1)}$ *with* $Y_{l_1}^{(i_1)}$ *is the independent random variables having ex-*

ponential distributions with scale parameters $\alpha_{l_1}^{(i_1)}$, and $g_{Y_1}^{(i_1)}$ is the PDF of $Y_1^{(i_1)}$ for $i_1 = 1, 2, 3$ corresponding to all the three cases in J-GPHCS, respectively.

Proof: This result follows immediately from the MGF of $\hat{\theta}_1$ given in Theorem 3.2. Notice that $(1 - \alpha_{l_1}^{(i_1)} t)^{-1}$ is the MGF of an exponential distribution with scale parameter $\alpha_{l_1}^{(i_1)}$ for $i_1 = 1, 2, 3$.

Similarly, we can find the conditional density function of $\hat{\theta}_2$ by some necessary changes in the expressions for $\hat{\theta}_1$. For example replacing D_1 and θ_1 by D_2 and θ_2, respectively and other variables accordingly changes. For more details one case see Rosouli and Balakrishnan [21], Koley and Kundu [18].

4. Approximate Confidence Intervals

A symmetric approximate confidence interval of each of the parameters θ_i, $i = 1, 2$ can be constructed by using conditional distribution of θ_i as derived in Section 3.1. In this purpose, it is required that for $i = 1, 2$, $F_{\hat{\theta}_i}(x)$ from $f_{\hat{\theta}_i}(x)$ the conditional distribution of θ_i at any arbitrary value x monotonically decreases as θ_i increases. Then, For $0 < \gamma < 1$, the $100(1 - \gamma)$ symmetric approximate confidence interval of θ_i can be constructed using the conditional distribution of θ_i for $i = 1, 2$. Following the approach of Koley and Kundu [18], for $i = 1, 2$, the lower and upper confidence limits of θ_i are obtained by solving the following two equations in θ_i;

$$F_{\hat{\theta}_i}(\hat{\theta}_{i\ observed}) = 1 - \frac{\gamma}{2} \qquad \text{and} \qquad F_{\hat{\theta}_i}(\hat{\theta}_{i\ observed}) = \frac{\gamma}{2}$$

Here, the unknown parameter θ_i is replaced by its estimate. Thus, the confidence interval obtained by solving the above equations is called approximate confidence interval. Various authors such as Balakrishnan et al. [8] Childs et al. [12] used this technique to construct confidence intervals for the parameters. By considering the complexity on proving the monotonicity property and the complicated nature of the conditional CDF $F_{\hat{\theta}_i}(x)$ some numerical methods such as Newton-Rapshon method or Bisection method are needed to obtain these exact confidence intervals.

Alternatively, we consider the approximate confidence intervals for the unknown parameters θ_1 and θ_2 based on the large sample approximations of the maximum likelihood estimators (may be referred as asymptotic theory). For this purpose, we can use the Fisher information matrix, $I(\theta_1, \theta_2)$, to obtain the asymptotic variance of MLEs. The observed Fisher information matrix is given by

$$I(\hat{\theta}_1, \hat{\theta}_2)^{-1} = \begin{pmatrix} -E\left(\frac{\partial^2 l}{\partial \theta_1^2}\right) & 0 \\ 0 & -E\left(\frac{\partial^2 l}{\partial \theta_2^2}\right) \end{pmatrix}^{-1}_{(\theta_1,\theta_2)=(\hat{\theta}_1,\hat{\theta}_2)} = \begin{pmatrix} Var(\hat{\theta}_1) & 0 \\ 0 & Var(\hat{\theta}_2) \end{pmatrix}.$$

From the log-likelihood function in equation (5), we have

$$\frac{\partial^2 \ell}{\partial \theta_1^2} = \frac{D_1}{\theta_1^2} - \frac{2V_1}{\theta_1^3} \quad \text{and} \quad \frac{\partial^2 \ell}{\partial \theta_2^2} = \frac{D_2}{\theta_2^2} - \frac{2V_2}{\theta_2^3}. \tag{9}$$

By replacing $(\theta_1, \theta_2) = (\hat{\theta}_1, \hat{\theta}_2)$ in equation (9) we obtain

$$Var(\hat{\theta}_1) = \frac{V_1^2}{D_1^3} \qquad and \qquad Var(\hat{\theta}_2) = \frac{V_2^2}{D_2^3}.$$

The asymptotic normality of the MLEs tells us that $(\hat{\theta}_1, \hat{\theta}_2)$ is approximately bivariate normal with mean (θ_1, θ_2) and the variance-covariance matrix $I(\theta_1, \theta_2)$. That is, $(\hat{\theta}_1, \hat{\theta}_2) \sim N\big((\theta_1, \theta_2), I(\hat{\theta}_1, \hat{\theta}_2)\big)$. Then, the approximate $100(1-\gamma)\%$ confidence intervals for θ_1 and θ_2 are obtained as

$$\left(\hat{\theta}_1 - z_{\frac{\gamma}{2}} \frac{V_1}{\sqrt{D_1^3}}, \hat{\theta}_1 + z_{\frac{\gamma}{2}} \frac{V_1}{\sqrt{D_1^3}}\right) \quad \text{and} \quad \left(\hat{\theta}_2 - z_{\frac{\gamma}{2}} \frac{V_2}{\sqrt{D_2^3}}, \hat{\theta}_2 + z_{\frac{\gamma}{2}} \frac{V_2}{\sqrt{D_2^3}}\right)$$

where $z_{\frac{\gamma}{2}}$ denotes the upper $\gamma/2$ percentage point of the standard normal distribution.

Further, an approximate $100(1-\gamma)\%$ joint (or simultaneous) confidence set for θ_1 and θ_2 can be obtained as

$$\left\{\left(\hat{\theta}_1 \pm z_{\frac{1-\sqrt{1-\gamma}}{2}} \frac{V_1}{\sqrt{D_1^3}}\right); \left(\hat{\theta}_2 \pm z_{\frac{1-\sqrt{1-\gamma}}{2}} \frac{V_2}{\sqrt{D_2^3}}\right)\right\}$$

4.1. Bootstrap Confidence Intervals

Following asymptotic confidence intervals (ACI) for MLEs, bootstrap confidence intervals are obtained in this subsection. For this purpose, we obtain parametric percentile bootstrap method (boot-p) and studentized bootstrap method (boot-t) to obtain approximate confidence intervals for estimation. The corresponding algorithms which is defined by Efron and Tibshirani [16] can be described as follows.

For boot-p confidence intervals

Step 1: Generate two samples $X = (X_1, \cdots, X_{n_1})$ and $Y = (Y_1, \cdots, Y_{n_2})$ of size n_1 and n_2 from exponential populations, with corresponding scale parameters θ_1 and θ_2, respectively.

Step 2: Evaluate the censored sample $\{w_1, w_2, \cdots, w_m; z_1, z_2, \cdots, z_m\}$ where m is the number of failures for different cases of J-GPHCS. Then, calculate MLEs of $\hat{\theta}_1$ and $\hat{\theta}_2$ by using equation (6).

Step 3: Generate bootstrap samples $(X_1^*, X_2^*, \cdots, X_{n_1}^*)$ and $(Y_1^*, Y_2^*, \cdots, Y_{n_2}^*)$ based on $Exp(\hat{\theta}_1)$ and $Exp(\hat{\theta}_2)$ then compute the bootstrap estimates of θ_1 and θ_2 which is denoted by $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$, respectively.

Step 4: Repeat steps 2-3 for B times and obtain bootstap estimates $\hat{\theta}_{1_j}^*$ and $\hat{\theta}_{2_j}^*$, $j = 1, 2, \cdots, B$.

Step 5: Let $G_1(z) = P(\hat{\theta}_i^* \leq z)$, for $i = 1, 2$, be the cumulative distribution function (CDF) of $\hat{\theta}_i^*$. Define $\hat{\theta}_{i_{Boot-p}} = G_1^{-1}(z)$ for a given z. Then, the approximate $100(1-\gamma)\%$ confidence interval of θ_i, for $i = 1, 2$, is given by

$$\left(\hat{\theta}_{i_{Boot-p}}^{*(\gamma/2)}, \hat{\theta}_{i_{Boot-p}}^{*(1-\gamma/2)}\right).$$

where $\hat{\theta}_{i_{Boot-p}}^{*(\gamma)}$ is the γ percentile of $\hat{\theta}_{i_j}^*$, $i = 1, 2$ and $j = 1, 2, \cdots, B$. Further, using the Bonferroni method, the $100(1-\gamma)\%$ simultaneous Boot-p confidence interval for (θ_1, θ_2) can be obtained as given by

$$\left\{\left(\hat{\theta}_{1_{Boot-p}}^{*(\gamma/4)}, \hat{\theta}_{1_{Boot-p}}^{*(1-\gamma/4)}\right); \left(\hat{\theta}_{2_{Boot-p}}^{*(\gamma/4)}, \hat{\theta}_{2_{Boot-p}}^{*(1-\gamma/4)}\right)\right\}$$

For boot-t confidence intervals, after generating the bootstrap samples and calculating the bootstrap estimates in the first four steps;

Step 1: Compute estimates of $Var(\hat{\theta}_i^*)$ from the observed Fisher information matrix in equation (9).

Step 2: Calculate,

$$T^* = \frac{\hat{\theta}_i^* - \hat{\theta}_i}{\sqrt{Var(\hat{\theta}_i^*)}}, \; i = 1, 2.$$

Step 3: Let $G_2(z) = P(T^* \leq z)$ be the CDF of T^*. From the obtained T^* values, the approximate $100(1-\gamma)\%$ confidence interval of θ_i, for $i = 1, 2$, can be obtained as follows. For a given z, define

$$\hat{\theta}_{iBoot-t} = \hat{\theta}_i + G_2^{-1}(z)\sqrt{Var(\hat{\theta}_i)}.$$

Then, the approximate $100(1-\gamma)\%$ confidence interval of θ_i, for $i = 1, 2$, is given by

$$\left(\hat{\theta}^{*(\gamma/2)}_{i_{Boot-t}}, \hat{\theta}^{*(1-\gamma/2)}_{i_{Boot-t}}\right).$$

Similarly to boot-p method, the simultaneous $100(1-\gamma)\%$ boot-t confidence interval for (θ_1, θ_2) is given by

$$\left\{\left(\hat{\theta}^{*(\gamma/4)}_{1_{Boot-t}}, \hat{\theta}^{*(1-\gamma/4)}_{1_{Boot-t}}\right); \left(\hat{\theta}^{*(\gamma/4)}_{2_{Boot-t}}, \hat{\theta}^{*(1-\gamma/4)}_{2_{Boot-t}}\right)\right\}$$

5. Bayesian Estimation

In this section, we obtained the Bayes' estimators, under the squared error loss function, of the unknown parameters and their corresponding credible intervals based on the J-GPHCS data. Since θ_1 and θ_2 are both unknown, we may choose any specific forms of the priors. Here, we assume independent Inverse Gamma (IG) priors for both the unknow parameters θ_1 and θ_2. Therefore, let us assume that θ_1 and θ_2 have $IG(a_1, b_1)$ and $IG(a_2, b_2)$ priors, respectively, with non-negative hyper-parameters a_1, b_1, a_2, b_2. Hence, the joint prior distribution for θ_1 and θ_2 is given by

$$\pi_0(\theta_1, \theta_2) \propto \theta_1^{-a_1-1} e^{-b_1/\theta_1} \theta_2^{-a_2-1} e^{-b_2/\theta_2},$$

where $\theta_1, \theta_2 > 0$ and $a_1, b_1, a_2, b_2 > 0$. Using the observed censored samples and the prior distributions for the parameters, the joint posterior density function of parameters θ_1 and θ_2 are obtained as

$$\pi(\theta_1, \theta_2|\mathbf{w},\mathbf{z}) \propto \theta_1^{-D_1-a_1-1} \theta_2^{-D_2-a_2-1} e^{-\frac{(V_1+b_1)}{\theta_1}} e^{-\frac{(V_2+b_2)}{\theta_2}}. \tag{10}$$

It is clear that, the joint posterior density of θ_1 and θ_2 are combination of two independent Inverse Gamma distributions such as

$$\pi(\theta_1) \sim \mathrm{IG}\bigg(\sum_{i=1}^{m} z_i + a_1, V_1 + b_1\bigg)$$

$$\pi(\theta_2) \sim \mathrm{IG}\bigg(\sum_{i=1}^{m}(1 - z_i) + a_2, V_2 + b_2\bigg).$$

Under the squared error loss function, the Bayes' estimators of θ_1 and θ_2, denoted by $\hat{\theta}_1^B$ and $\hat{\theta}_2^B$, are obtained as

$$\hat{\theta}_1^B = \frac{V_1 + b_1}{\sum_{i=1}^m z_i + a_1 - 1} \qquad \text{and} \qquad \hat{\theta}_2^B = \frac{V_2 + b_2}{\sum_{i=1}^m (1 - z_i) + a_2 - 1}, \tag{11}$$

where $\sum_{i=1}^m z_i + a_1 > 1$ and $\sum_{i=1}^m (1 - z_i) + a_2 > 1$. The $\hat{\theta}_1^B$ and $\hat{\theta}_2^B$ are the unique Bayesian estimators of θ_1 and θ_2 under the squared error loss function and so they are admissible (Shafay et al. [25]). In particular, if the hyper-parameters $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$, the Bayes' estimators of θ_1, θ_2 are equal to their corresponding MLEs. The joint posterior distribution of θ_1 and θ_2 is used to provide Bayesian credible intervals (BCI) for the Bayes' estimation. For this purpose, let us assume

$$\xi_1 = \frac{2(V_1 + b_1)}{\theta_1} \quad \text{and} \quad \xi_2 = \frac{2(V_2 + b_2)}{\theta_2}.$$

The pivots ξ_1 and ξ_2 follows χ^2 distribution with degrees of freedom $2\big(D_1 + a_1 - 1\big)$ and $2\big(D_2 + a_2 - 1\big)$, respectively. Thus, the $100(1-\gamma)\%$ confidence intervals for θ_1 and θ_2 can be obtained as

$$\left[\frac{2(V_1 + b_1)}{\chi^2_{2(D_1+a_1),1-\gamma/2}}, \frac{2(V_1 + b_1)}{\chi^2_{2(D_1+a_1),\gamma/2}}\right] \quad \text{and} \quad \left[\frac{2(V_2 + b_2)}{\chi^2_{2(D_2+a_2),1-\gamma/2}}, \frac{2(V_2 + b_2)}{\chi^2_{2(D_2+a_2),\gamma/2}}\right],$$

respectively, where $\chi^2_{u,\gamma}$ is the γ percentage point of the χ^2_u distribution with degrees of freedom u. Note that gamma distribution can be used to construct the credible intervals if $2\big(D_1 + a_1\big)$ and $2\big(D_2 + a_2\big)$ are not integers. We can also obtain the $100(1-\gamma)\%$ simultaneous HPD credible interval for (θ_1, θ_2) from the joint posterior density functions as

$$\left\{\left[\frac{2(V_1 + b_1)}{\chi^2_{2(D_1+a_1),\frac{1+\sqrt{(1-\gamma)}}{2}}}, \frac{2(V_1 + b_1)}{\chi^2_{2(D_1+a_1),\frac{1+\sqrt{(1+\gamma)}}{2}}}\right]; \left[\frac{2(V_2 + b_2)}{\chi^2_{2(D_2+a_2),\frac{1+\sqrt{(1-\gamma)}}{2}}}, \frac{2(V_2 + b_2)}{\chi^2_{2(D_2+a_2),\frac{1+\sqrt{(1+\gamma)}}{2}}}\right]\right\}$$

6. Simulation Study

It is clear that numerous censoring schemes can be constructed for the proposed J-GPHCS since the presence of guaranteed failure number k, pre-determined time T, and failure number r. These pre-fixed values have unlimited combinations between them and also the arbitrary actual parameter values have a worthwhile effect on these values. In this section, we provided simulation studies by considering arbitrary censoring schemes to illustrate the theoretical outcomes. We constructed these censoring schemes to observe the three cases of the censoring schemes at each values of the parameters. We also avoided fast failure options since the increase in unobserved sample size affects the confidence intervals of bootstrap samples.

We considered two sets of actual values of the parameters such as $(\theta_1, \theta_2) = (1.5, 1.5)$ and $(0.75, 1.25)$ to compare the performances of the estimators under equal and unequal parameter values. We generated random samples with different sample sizes such as $(n_1, n_2) = (12, 16)$, $(20, 20)$, $(28, 24)$, and $(35, 40)$. Then, we considered three different censoring schemes (CS) as

- I $\rightarrow$ Providing at least 50% failures, that is $k = (0.50) \times N$ where $(N = n_1 + n_2)$ and we determined $r = (0.70) \times N$ to provide 70% observed sample.
- II $\rightarrow$ Providing at least 50% failures, that is $k = (0.50) \times N$ and we determined $r = (0.90) \times N$ to provide 90% observed sample.
- III $\rightarrow$ Providing at least 80% failures, that is $k = (0.80) \times N$ and we determined $r = (0.90) \times N$ to provide 90% observed sample.

Then, we generate removals $R_1, R_2, \cdots, R_m$ for each CS from the discrete uniform distribution by using Equation 3. Further, the different pre-fixed time points are determined by considering the ranges of generated samples in each case of the parameters. In the first set of parameter we considered two values of pre-fixed time point as $T = (2, 4)$. Then, we take $T = (2, 3)$ for the second set of the actual parameter values. In Bayesian estimation procedure, we considered informative hyper-parameters. In this purpose, we determined the values of hyper-parameters from Inverse Gamma distributions as providing the actual values of the parameters. We used $a_1 = a_2 = 2$, $b_1 = b_2 = 1.5$ for $(\theta_1, \theta_2) = (1.5, 1.5)$ and $a_1 = a_2 = 3$, $b_1 = 1.5$, $b_2 = 2.5$ for $(\theta_1, \theta_2) = (0.75, 1.25)$. Thus, we run the simulations for $3,000$ replications and $1,000$ bootstrap sample in each iteration. Then, we evaluated the estimations with their biases and mean squared error (MSE). We also compared the performances of the approximate confidence intervals with their average length (AL) and the coverage probabilities (CP). The corresponding results of these simulation schemes are reported in Tables 1-4. We further presented the comparative plots of the MSEs of the estimates in Figures 2-3. The simulation results can be summarized as follows:

- It is observed that the MSEs and ALs decrease in parallel to increasing sample sizes, as expected. Especially in small sample sizes, The Bayes estimates shows better performance and the differences between the MLEs decrease with increasing sample sizes.

- We see that this joint censoring scheme is very sensitive in the case of fast failure. We constructed simulation schemes by considering this observation. Especially, the boot-t confidence intervals have larger intervals in the fast failure cases. Among the three censoring plan that we handled, the estimations of the Case-III show better performances than estimations of the Cases II and Case-I, respectively. Similarly, estimations of the Case-II performes better than Case-I's.
- The HPD credible intervals of the Bayes' estimates are more robust than the ACIs of the MLEs. In terms of the coverage probabilities, the HPD CIs are obtained more close than ACI to their actual values 0.95. The boot-p CIs mostly have smaller lengths among the CIs of the MLEs.
- It is clear that the emergence of the Case-I occurs fast failure in the experiment if k is relatively small since it has terminated at k-th ($T < k < r$) failure. We constructed the simulation tables for the purpose of comparing the performances of the estimators in different cases. In Table I ($T = 2$), we mostly observed the Case-I and the Case-II and III are mostly observed in Table II ($T = 4$). It is seen that performances of the estimations are better in Table 2 since more observed samples are provided in its simulation plans.
- The estimation methods provided consistent results in both cases of equal and different actual parameter values.

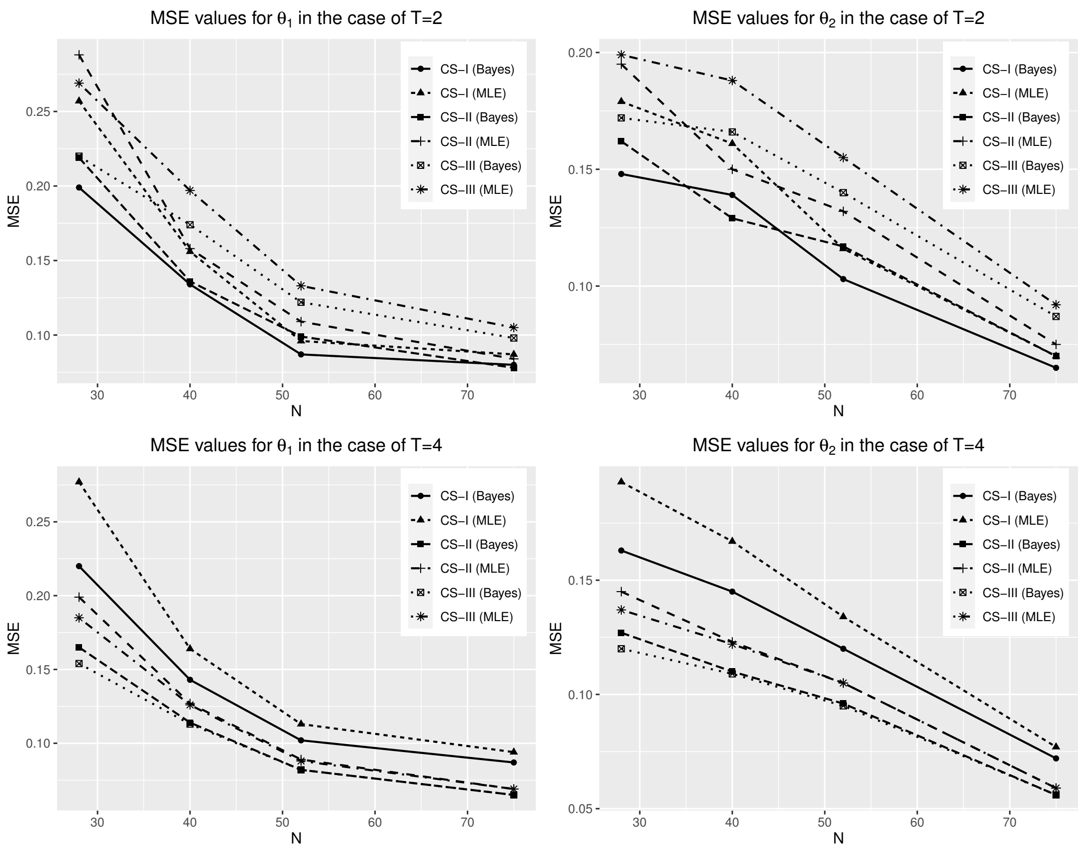

Figure 2.: MSE values of the estimations in the case of $(\theta_1, \theta_2) = (1.5, 1.5)$.

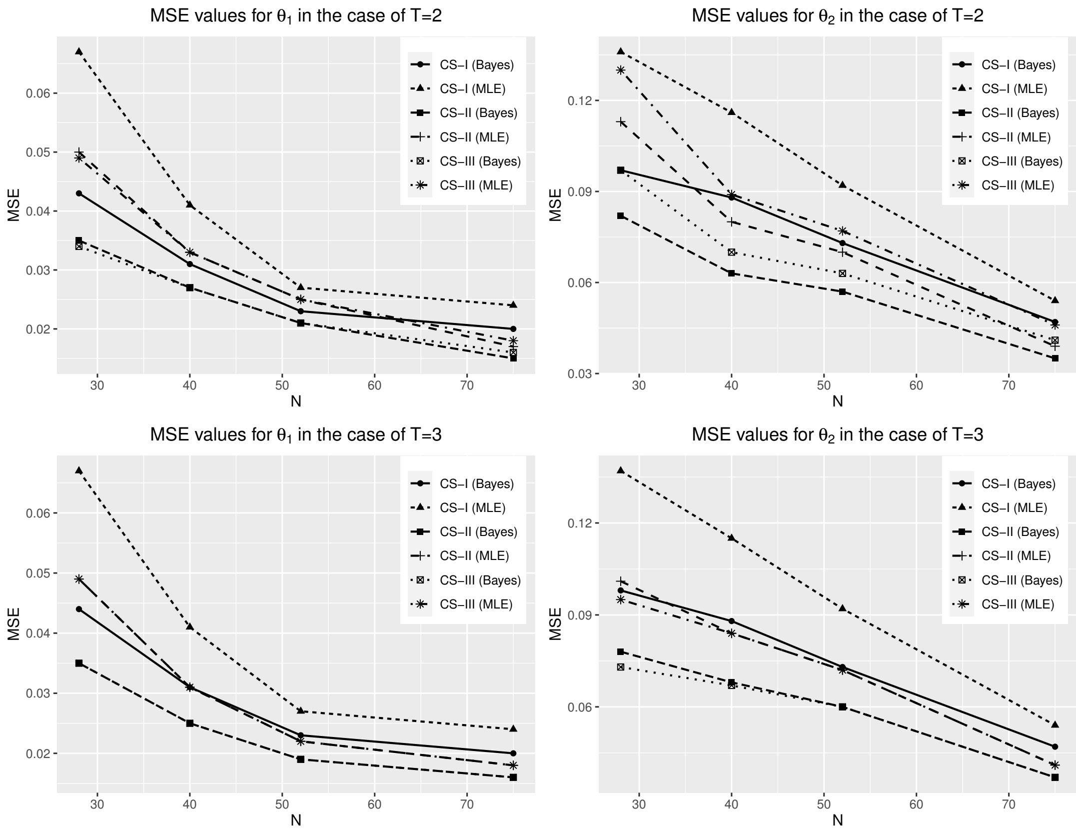

Figure 3.: MSE values of the estimations in the case of $(\theta_1, \theta_2) = (0.75, 1.25)$.

Table 1.: The average estimates (AE) of the parameters with their mean squared errors (MSE), average lengths (AL) of the ACI, boot-p, boot-t CIs and HPD credible intervals with their coverage probabilities (CP).

$\theta_1 = 1.5$, $\theta_2 = 1.5$ and $T = 2$														
			AE		MSE		AL				CP			
n_1, n_2	CS	Parameter	MLE	Bayes	MLE	Bayes	ACI	Boot-p	Boot-t	HPD	ACI	Boot-p	Boot-t	HPD
12,16	I	θ_1	1.477	1.476	0.257	0.199	2.054	1.898	2.567	1.995	0.888	0.916	0.915	0.958
		θ_2	1.473	1.474	0.179	0.148	1.761	1.590	2.040	1.728	0.909	0.929	0.914	0.961
	II	θ_1	1.485	1.478	0.288	0.219	2.040	2.061	2.503	1.970	0.897	0.920	0.917	0.953
		θ_2	1.466	1.464	0.195	0.162	1.721	1.667	2.035	1.686	0.902	0.920	0.914	0.948
	III	θ_1	1.474	1.474	0.269	0.220	1.858	1.783	2.828	1.818	0.865	0.836	0.904	0.924
		θ_2	1.465	1.466	0.199	0.172	1.590	1.566	2.122	1.568	0.860	0.819	0.910	0.917
20,20	I	θ_1	1.473	1.474	0.156	0.134	1.572	1.493	1.810	1.550	0.911	0.926	0.924	0.949
		θ_2	1.472	1.473	0.161	0.139	1.570	1.489	1.809	1.549	0.906	0.929	0.911	0.945
	II	θ_1	1.434	1.435	0.158	0.136	1.497	1.434	1.754	1.477	0.892	0.903	0.927	0.940
		θ_2	1.437	1.438	0.150	0.129	1.501	1.437	1.757	1.481	0.890	0.904	0.933	0.943
	III	θ_1	1.470	1.471	0.197	0.174	1.447	1.475	1.992	1.431	0.853	0.826	0.904	0.890
		θ_2	1.472	1.473	0.188	0.166	1.450	1.480	1.990	1.434	0.852	0.833	0.918	0.902
28,24	I	θ_1	1.468	1.469	0.096	0.087	1.311	1.179	1.384	1.300	0.933	0.944	0.916	0.959
		θ_2	1.473	1.474	0.116	0.103	1.421	1.290	1.534	1.406	0.923	0.938	0.917	0.954
	II	θ_1	1.420	1.423	0.109	0.099	1.246	1.170	1.409	1.236	0.895	0.886	0.932	0.930
		θ_2	1.429	1.431	0.132	0.117	1.359	1.298	1.572	1.346	0.887	0.892	0.928	0.927
	III	θ_1	1.480	1.480	0.133	0.122	1.224	1.263	1.558	1.214	0.866	0.831	0.902	0.895
		θ_2	1.481	1.482	0.155	0.140	1.326	1.358	1.744	1.314	0.862	0.836	0.902	0.897
35,40	I	θ_1	1.485	1.485	0.087	0.080	1.183	1.120	1.268	1.175	0.933	0.945	0.919	0.958
		θ_2	1.475	1.476	0.070	0.065	1.098	1.029	1.153	1.092	0.938	0.948	0.921	0.959
	II	θ_1	1.418	1.420	0.084	0.078	1.108	1.005	1.204	1.102	0.892	0.864	0.932	0.930
		θ_2	1.406	1.408	0.075	0.070	1.027	0.925	1.104	1.022	0.889	0.845	0.933	0.931
	III	θ_1	1.486	1.486	0.105	0.098	1.104	1.116	1.374	1.097	0.875	0.838	0.886	0.896
		θ_2	1.475	1.475	0.092	0.087	1.025	1.063	1.275	1.020	0.873	0.836	0.887	0.891

Table 2.: The average estimates (AE) of the parameters with their mean squared errors (MSE), average lengths (AL) of the ACI, boot-p, boot-t CIs and HPD credible intervals with their coverage probabilities (CP).

$\theta_1 = 1.5$, $\theta_2 = 1.5$ and $T = 4$														
			AE		MSE		AL				CP			
n_1, n_2	CS	Parameter	MLE	Bayes	MLE	Bayes	ACI	Boot-p	Boot-t	HPD	ACI	Boot-p	Boot-t	HPD
12,16	I	θ_1	1.512	1.510	0.277	0.220	2.045	2.023	2.534	1.991	0.894	0.919	0.947	0.954
		θ_2	1.504	1.503	0.193	0.163	1.752	1.735	2.053	1.721	0.912	0.931	0.949	0.954
	II	θ_1	1.491	1.491	0.199	0.165	1.779	1.741	2.059	1.746	0.906	0.924	0.929	0.954
		θ_2	1.485	1.485	0.145	0.127	1.530	1.485	1.692	1.511	0.914	0.929	0.932	0.952
	III	θ_1	1.475	1.476	0.185	0.154	1.756	1.616	2.085	1.726	0.905	0.927	0.912	0.955
		θ_2	1.469	1.471	0.137	0.120	1.513	1.374	1.681	1.495	0.912	0.931	0.905	0.953
20,20	I	θ_1	1.500	1.500	0.164	0.143	1.579	1.567	1.796	1.558	0.918	0.932	0.946	0.947
		θ_2	1.498	1.498	0.167	0.145	1.577	1.563	1.792	1.556	0.915	0.931	0.941	0.945
	II	θ_1	1.489	1.489	0.127	0.114	1.383	1.367	1.533	1.370	0.915	0.928	0.933	0.946
		θ_2	1.498	1.497	0.123	0.110	1.391	1.376	1.542	1.377	0.928	0.937	0.936	0.956
	III	θ_1	1.488	1.488	0.126	0.113	1.382	1.337	1.535	1.369	0.915	0.931	0.923	0.947
		θ_2	1.497	1.497	0.122	0.109	1.390	1.345	1.544	1.376	0.928	0.941	0.921	0.956
28,24	I	θ_1	1.504	1.503	0.113	0.102	1.326	1.313	1.449	1.314	0.936	0.941	0.949	0.951
		θ_2	1.508	1.507	0.134	0.120	1.434	1.423	1.592	1.419	0.924	0.936	0.949	0.946
	II	θ_1	1.490	1.490	0.089	0.082	1.166	1.145	1.247	1.158	0.921	0.929	0.929	0.943
		θ_2	1.495	1.495	0.105	0.096	1.264	1.251	1.379	1.254	0.918	0.923	0.932	0.943
	III	θ_1	1.489	1.489	0.088	0.082	1.165	1.128	1.245	1.157	0.921	0.931	0.920	0.943
		θ_2	1.494	1.495	0.105	0.095	1.263	1.234	1.378	1.253	0.918	0.926	0.923	0.944
35,40	I	θ_1	1.511	1.511	0.094	0.087	1.195	1.186	1.282	1.186	0.935	0.938	0.948	0.955
		θ_2	1.500	1.500	0.077	0.072	1.107	1.100	1.176	1.101	0.939	0.945	0.949	0.953
	II	θ_1	1.497	1.497	0.069	0.065	1.044	1.010	1.087	1.039	0.934	0.940	0.929	0.951
		θ_2	1.490	1.490	0.059	0.056	0.973	0.935	1.000	0.969	0.941	0.941	0.931	0.954
	III	θ_1	1.497	1.497	0.069	0.065	1.044	1.006	1.087	1.039	0.934	0.940	0.925	0.951
		θ_2	1.490	1.490	0.059	0.056	0.973	0.932	1.000	0.969	0.940	0.942	0.929	0.954

Table 3.: The average estimates (AE) of the parameters with their mean squared errors (MSE), average lengths (AL) of the ACI, boot-p, boot-t CIs and HPD credible intervals with their coverage probabilities (CP).

$\theta_1 = 0.75$, $\theta_2 = 1.25$ and $T = 2$														
			AE		MSE		AL				CP			
n_1, n_2	CS	Parameter	MLE	Bayes	MLE	Bayes	ACI	Boot-p	Boot-t	HPD	ACI	Boot-p	Boot-t	HPD
12,16	I	θ_1	0.755	0.753	0.067	0.043	1.013	1.004	1.257	0.935	0.903	0.928	0.939	0.963
		θ_2	1.255	1.253	0.136	0.097	1.475	1.427	1.716	1.388	0.909	0.929	0.929	0.966
	II	θ_1	0.745	0.744	0.050	0.035	0.896	0.883	1.036	0.841	0.905	0.923	0.935	0.963
		θ_2	1.218	1.219	0.113	0.082	1.348	1.294	1.475	1.276	0.913	0.920	0.938	0.959
	III	θ_1	0.728	0.731	0.049	0.034	0.870	0.867	1.430	0.822	0.897	0.925	0.928	0.960
		θ_2	1.150	1.162	0.130	0.097	1.246	1.127	1.524	1.196	0.864	0.825	0.918	0.936
20,20	I	θ_1	0.749	0.749	0.041	0.031	0.785	0.781	0.893	0.748	0.913	0.928	0.943	0.953
		θ_2	1.250	1.250	0.116	0.088	1.323	1.304	1.504	1.261	0.913	0.926	0.936	0.956
	II	θ_1	0.739	0.739	0.033	0.027	0.691	0.703	0.818	0.666	0.904	0.927	0.945	0.945
		θ_2	1.211	1.213	0.080	0.063	1.179	1.100	1.278	1.133	0.917	0.933	0.939	0.963
	III	θ_1	0.733	0.734	0.033	0.027	0.684	0.699	0.927	0.660	0.897	0.924	0.940	0.941
		θ_2	1.183	1.189	0.089	0.070	1.145	1.029	1.309	1.105	0.900	0.908	0.931	0.949
28,24	I	θ_1	0.752	0.751	0.027	0.023	0.660	0.654	0.721	0.639	0.930	0.939	0.950	0.958
		θ_2	1.257	1.256	0.092	0.073	1.200	1.184	1.332	1.154	0.923	0.931	0.943	0.953
	II	θ_1	0.735	0.736	0.025	0.021	0.579	0.583	0.654	0.564	0.905	0.924	0.928	0.935
		θ_2	1.208	1.210	0.070	0.057	1.069	0.982	1.139	1.035	0.911	0.921	0.925	0.953
	III	θ_1	0.731	0.732	0.025	0.021	0.575	0.580	0.705	0.561	0.902	0.923	0.927	0.931
		θ_2	1.187	1.192	0.077	0.063	1.046	0.938	1.169	1.015	0.895	0.900	0.915	0.943
35,40	I	θ_1	0.759	0.758	0.024	0.020	0.599	0.595	0.642	0.582	0.939	0.943	0.950	0.956
		θ_2	1.247	1.248	0.054	0.047	0.923	0.915	0.981	0.902	0.938	0.941	0.949	0.957
	II	θ_1	0.739	0.740	0.017	0.015	0.521	0.503	0.556	0.510	0.927	0.938	0.931	0.954
		θ_2	1.180	1.183	0.039	0.035	0.807	0.697	0.802	0.794	0.919	0.888	0.936	0.958
	III	θ_1	0.736	0.737	0.018	0.016	0.518	0.508	0.580	0.508	0.924	0.940	0.928	0.951
		θ_2	1.168	1.172	0.046	0.041	0.797	0.693	0.834	0.784	0.901	0.851	0.930	0.939

Table 4.: The average estimates (AE) of the parameters with their mean squared errors (MSE), average lengths (AL) of the ACI, boot-p, boot-t CIs and HPD credible intervals with their coverage probabilities (CP).

$\theta_1 = 0.75$, $\theta_2 = 1.25$ and $T = 3$														
			AE		MSE		AL				CP			
n_1, n_2	CS	Parameter	MLE	Bayes	MLE	Bayes	ACI	Boot-p	Boot-t	HPD	ACI	Boot-p	Boot-t	HPD
12,16	I	θ_1	0.756	0.754	0.067	0.044	1.013	1.007	1.250	0.935	0.903	0.928	0.944	0.963
		θ_2	1.255	1.254	0.137	0.098	1.474	1.454	1.731	1.387	0.909	0.928	0.948	0.964
	II	θ_1	0.747	0.747	0.049	0.035	0.880	0.866	1.030	0.829	0.904	0.926	0.942	0.963
		θ_2	1.239	1.239	0.101	0.078	1.280	1.233	1.407	1.223	0.917	0.928	0.933	0.959
	III	θ_1	0.745	0.746	0.049	0.035	0.877	0.862	1.071	0.827	0.903	0.929	0.933	0.963
		θ_2	1.229	1.231	0.095	0.073	1.269	1.122	1.383	1.214	0.916	0.931	0.895	0.960
20,20	I	θ_1	0.749	0.749	0.041	0.031	0.785	0.781	0.890	0.749	0.913	0.928	0.946	0.953
		θ_2	1.250	1.250	0.115	0.088	1.323	1.309	1.507	1.261	0.913	0.926	0.943	0.956
	II	θ_1	0.747	0.747	0.031	0.025	0.690	0.698	0.788	0.665	0.917	0.933	0.949	0.953
		θ_2	1.249	1.249	0.084	0.068	1.161	1.131	1.264	1.119	0.928	0.940	0.931	0.963
	III	θ_1	0.746	0.747	0.031	0.025	0.690	0.696	0.791	0.665	0.917	0.934	0.948	0.953
		θ_2	1.249	1.249	0.084	0.067	1.160	1.107	1.259	1.118	0.928	0.942	0.920	0.963
28,24	I	θ_1	0.752	0.751	0.027	0.023	0.660	0.656	0.722	0.639	0.930	0.939	0.950	0.958
		θ_2	1.257	1.256	0.092	0.073	1.200	1.190	1.336	1.153	0.923	0.930	0.946	0.953
	II	θ_1	0.745	0.745	0.022	0.019	0.580	0.585	0.634	0.566	0.924	0.937	0.947	0.951
		θ_2	1.252	1.252	0.072	0.060	1.058	1.029	1.132	1.026	0.925	0.932	0.926	0.955
	III	θ_1	0.745	0.745	0.022	0.019	0.580	0.584	0.635	0.566	0.924	0.937	0.947	0.951
		θ_2	1.252	1.252	0.072	0.060	1.058	1.020	1.129	1.026	0.925	0.933	0.918	0.955
35,40	I	θ_1	0.759	0.758	0.024	0.020	0.599	0.595	0.641	0.582	0.939	0.943	0.951	0.956
		θ_2	1.247	1.248	0.054	0.047	0.923	0.916	0.982	0.902	0.938	0.940	0.952	0.957
	II	θ_1	0.752	0.752	0.018	0.016	0.524	0.522	0.557	0.513	0.937	0.942	0.944	0.949
		θ_2	1.243	1.243	0.041	0.037	0.811	0.775	0.828	0.796	0.939	0.947	0.924	0.961
	III	θ_1	0.752	0.752	0.018	0.016	0.524	0.522	0.557	0.513	0.937	0.942	0.944	0.949
		θ_2	1.243	1.243	0.041	0.037	0.811	0.773	0.827	0.796	0.939	0.947	0.923	0.961

7. Data Analysis

In this section, we illustrated the theoretical outcomes on a numerical example. We used Nelson's data set (Nelson, [23]) which correspond to breakdown in minutes of an insulating fluid subjected to high voltage stress. This data set is also used by Abo-Kasem et al. [1] for jointly Type-I progressive hybrid censored exponential populations and by Krishna and Goel [19] for jointly Type-II censored Lindley populations, respectively. This data set of failure times are denoted by X and Y, respectively. The failure times with sample sized $n_1 = n_2 = 10$ are given in the following

Sam-1 (X): 1.99 0.64 2.15 1.08 2.57 0.93 4.75 0.82 2.06 0.49,

Sam-2 (Y): 8.11 3.17 5.55 0.80 0.20 1.13 6.63 1.08 2.44 0.78.

We consider different J-GPHCS plans as given in the following:

CS-I: $k = 12$, $r = 14$, $T = 3.5$, $R = (0_{(13)}, 6_{(1)})$.
CS-II: $k = 12$, $r = 16$, $T = 3$, $R = (0_{(15)}, 4_{(1)})$.
CS-III: $k = 14$, $r = 16$, $T = 4$, $R = (1_{(4)}, 0_{(12)})$.
CS-IV: $k = 14$, $r = 18$, $T = 5$, $R = (1_{(2)}, 0_{(16)})$.

where $R = (0_{(13)}, 6_{(1)})$ denotes the removals in progressive censoring schemes. For example; $(0_{(13)}, 6_{(1)})$ denotes never withdraws in first 13 failures and 6 withdraws in $r-$th failure. Based on the proposed censoring schemes, joint censored data is given in Tables 5-8. In complete cases, parameter estimates of θ_1 and θ_2 are 1.748 and 2.989, respectively. Therefore, we used informative hyper-parameter values for Bayesian estimation and credible intervals as $a_1 = a_2 = 2$, $b_1 = 1.75$ and $b_2 = 3$. Based on the J-GPHCS which are given above, we obtained the estimates with their standard deviations and lengths of approximate confidence intervals. All results are reported in Table 9.

In this data example, sample sizes are smaller than the sizes are taken for simulation. As we expect, the Bayes' estimates have smaller standard deviations and biases than MLEs. We see that the performances of the estimations and the corresponding credible intervals are not affected by only one failure point (k, r or T). The combinations of all pre-fixed values of the k, r, T and removal numbers, R_i $(i = 1, 2, \cdots, r)$, are effective on the performances of the estimates. It is observed that boot-t confidence intervals show the worst performance if the removals are determined in the last failures. Further, high standard deviations on MLEs affect the boot-t credible lengths. In this example, the forth censoring scheme is more suitable for better estimation performances among the four censoring schemes. We also performed simultaneous confidence intervals along with the separate intervals for the real data set and the results are reported in Table 10. The lengths of the intervals are larger in simultaneously confidence intervals. Therefore, we suggest using the approximate confidence intervals separately for each parameter.

Table 5.: The jointly generalized progressive hybrid censored data based on CS-I.

R	0	0	0	0	0	0	0	0	0	0	0	0	0	6
s	0	0	0	0	0	0	0	0	0	0	0	0	0	2
q	0	0	0	0	0	0	0	0	0	0	0	0	0	4
w	0.20	0.49	0.64	0.78	0.8	0.82	0.93	1.08	1.08	1.13	1.99	2.06	2.15	2.44
z	0	1	1	0	0	1	1	1	0	0	1	1	1	0

Table 6.: The jointly generalized progressive hybrid censored data based on CS-II.

R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
w	0.20	0.49	0.64	0.78	0.8	0.82	0.93	1.08	1.08	1.13	1.99	2.06	2.15	2.44	2.57	3.17
z	0	1	1	0	0	1	1	1	0	0	1	1	1	0	1	0

Table 7.: The jointly generalized progressive hybrid censored data based on CS-III.

R	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
s	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
q	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
w	0.20	0.49	0.64	0.78	0.80	0.82	0.93	1.08	1.08	2.15	2.44	2.57	3.17	4.75	6.63	8.11
z	0	1	1	0	0	1	1	1	0	1	0	1	0	1	0	0

Table 8.: The jointly generalized progressive hybrid censored data based on CS-IV.

R	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
s	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
w	0.20	0.49	0.64	0.78	0.82	0.93	1.08	1.08	1.13	1.99	2.06	2.44	2.57	3.17	4.75	5.55	6.63	8.11
z	0	1	1	0	1	1	1	0	0	1	1	0	1	0	1	0	0	0

Table 9.: Estimates for real data example with their standard deviations and lengths of approximate confidence intervals.

	CS-I		CS-II		CS-III		CS-IV	
	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
MLE	1.8800	2.6983	1.7478	3.0717	2.1263	2.2262	1.4256	3.2788
	(0.6647)	(1.1016)	(0.5826)	(1.2540)	(0.7517)	(0.7871)	(0.4752)	(1.1592)
Bayes	1.8656	2.7414	1.7480	3.0614	2.0844	2.3122	1.4580	3.2478
	(0.4553)	(0.6258)	(0.4181)	(0.6613)	(0.4813)	(0.5069)	(0.3818)	(0.6007)
ACI	2.6055	4.3182	2.2837	4.9156	2.9468	3.0854	1.8627	4.5440
Boot-p	2.5537	3.9874	2.4529	4.5969	2.7379	3.1044	1.8584	3.9263
Boot-t	3.1072	6.0027	3.1425	7.7469	3.6964	4.3245	2.4676	5.5040
HPD	2.5185	4.2256	2.2328	4.7188	2.8140	3.1215	1.8624	4.3846

8. Concluding Remarks

In this work, we have developed a newly joint generalized progressive hybrid censoring scheme (J-GPHCS) for two exponential populations. From the development of the model, it is clear that the experimenter always gets a pre-fixed number of failures at the end of the experiment. The principle behind the proposed J-GPHCS modeling approach is simple and appealing, possibly compared to the other existing approaches in the literature. We have considered the method of maximum likelihood estimator (MLE) and Bayes estimator for the model parameters. The exact distributions of the MLEs and the corresponding asymptotic, bootstrap and Bayesian credible intervals are constructed. Simulations and the real data studies show that theoretical findings perform well. In this plan, the censoring scheme and the removal times should have to be determined attentively. It is observed that the performances of the estimations are quite sensitive according to the location of the pre-fixed k, r, and T values in the experiment. We handled many possible outcomes of these censoring schemes and observed that the both estimation methods give satisfactory results with small MSE values. Consequently, following the previous jointly censoring schemes, the J-GPHCS provide experimenters a certain number of failures and their lifetimes in the case of

Table 10.: Confidence intervals (CI) and simultaneously confidence intervals (SCI) for the parameter estimations of the real data example for the CS-I (first line) and CS-III (second line).

	CI of θ_1	CI of θ_2	SCI of (θ_1, θ_2)
ACI	(0.5773, 3.1827)	(0.5393, 4.8574)	(0.3935, 3.3665); (0.2347, 5.1620)
	(0.6529, 3.5996)	(0.6836, 3.7689)	(0.4450, 3.8075); (0.4659, 3.9866)
Boot-p	(0.8308, 3.3845)	(1.0428, 5.0302)	(0.7694, 3.6727); (0.9458, 5.6384)
	(0.9063, 3.6442)	(0.8568, 3.9612)	(0.7749, 3.9058); (0.7935, 4.1730)
Boot-t	(1.1233, 4.2305)	(1.5608, 7.5635)	(1.0880, 4.7824); (1.4860, 8.4595)
	(1.2919, 4.9883)	(1.3302, 5.6547)	(1.1911, 5.7873); (1.2242, 6.2428)
HPD	(0.9827, 3.5013)	(1.3305, 5.5561)	(0.9146, 3.9166); (1.2299, 6.3246)
	(1.0981 3.9121)	(1.2180, 4.3396)	(1.0220, 4.3762); (1.1336, 4.8544)

few failures occurring before a pre-determined time. That is, this censoring scheme provides to observe r failures with willingness to accept minimum k failures for two exponential populations. One interesting topic in this direction will be to develop exact inference for the likelihood ratio tests for the joint censoring scheme recently proposed by Zhu et al. [29]. Work on this direction for this jointly GPHCS is currently under progress, and we hope to report these findings in near future.

9. Appendix

Now to compute each of the probability in the right hand side of equation (7), in Theorem 3.1, separately from the below Lemmas and finally add them together to get the proof of Theorem 3.1.

Lemma 9.1. *The probability mass function of* $D_1 = \sum_{i=1}^{k} Z_i$ *if* $T < W_{k:r:n} < W_{r:r:n}$ *is given by*

$$P(D_1 = i) \quad = \quad \sum \cdots \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \sum \cdots \sum_{(s_1,\cdots,s_{k-1})\in T_i^{(1)}} \prod_{j=1}^{k} P_j^{(1)} \prod_{l=1}^{k-1} P^{(1)}_{s_l|z_l^*;s_{l-1}^*}, \tag{12}$$

where

$$P_j^{(1)} = \frac{(n_1 - d1_{j-1}^1 - \sum_{i=1}^{j-1} s_i)\, z_j + (n_2 - d2_{j-1}^1 - \sum_{i=1}^{j-1}(R_i - s_i))\,(1 - z_j)}{(n_1 - d1_{j-1}^1 - \sum_{i=1}^{j-1} s_i)\theta_2 + (n_2 - d2_{j-1}^1 - \sum_{i=1}^{j-1}(R_i - s_i))\theta_1} \theta_1^{1-z_j}\theta_2^{z_j},$$

and

$$\begin{aligned} P^{(1)}_{s_l|z_l^*;s_{l-1}^*} &= P(S_l = s_l | Z_1 = z_1, \cdots, Z_l = z_l; S_1 = s_1, \cdots, S_{l-1} = s_{l-1}) \\ &= \frac{\binom{n_1 - \sum_{i=1}^{l-1} z_i - \sum_{i=1}^{l-1} s_i}{s_l}\binom{n_2 - \sum_{i=1}^{l-1}(1-z_i) - \sum_{i=1}^{l-1}(R_i - s_i)}{q_l}}{\binom{n-l-\sum_{i=1}^{l-1} R_i}{R_l}}. \end{aligned}$$

Proof:

$$\begin{aligned} P(D_1 = i) &= P(D_1 = i, T < W_{k:r:n} < W_{r:r:n}) \\ &= \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \cdots \sum P(Z_1 = z_1, \cdots, Z_k = z_k) \\ &= \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \cdots \sum \sum_{(s_1,\cdots,s_{k-1})\in T_i^{(1)}} \cdots \sum P(Z_1 = z_1, \cdots, Z_k = z_k; S_1 = s_1, \cdots, S_{k-1} = s_{k-1}) \\ &= \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \cdots \sum \sum_{(s_1,\cdots,s_{k-1})\in T_i^{(1)}} \cdots \sum P(Z_1 = z_1) P(S_1 = s_1 | Z_1 = z_1) \\ &\quad \times P(Z_2 = z_2) P(Z_1 = z_1; S_1 = s_1) \times \cdots \times \\ &\quad P(Z_r = z_r) P(Z_1 = z_1, \cdots, Z_l = z_l; S_1 = s_1, \cdots, S_{l-1} = s_{l-1}). \\ &= \sum_{(z_1,\cdots,z_k)\in Q_i^{(1)}} \cdots \sum \sum_{(s_1,\cdots,s_{-1})\in T_i^{(1)}} \cdots \sum \prod_{j=1}^{k} P_j^{(1)} \prod_{l=1}^{k-1} P^{(1)}_{s_l|z_l^*;s_{l-1}^*} \end{aligned}$$

For details one can see the results by Balakrishnan and Rasouli [9]. Therefore, in the similar direction we can write the next two Lemmas for Case-II and Case-III below.

Lemma 9.2. *The probability mass function of $D_1 = \sum_{i=1}^{D} Z_i$ if $W_{k:r:n} < T < W_{r:r:n}$ is given by*

$$P(D_1 = i) = \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \cdots \sum \sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}} \cdots \sum \prod_{j=1}^{D} P_j^{(2)} \prod_{l=1}^{D-1} P^{(2)}_{s_l|z_l^*;s_{l-1}^*}, \qquad (13)$$

where

$$P_j^{(2)} = \frac{(n_1 - d1_{j-1}^2 - \sum_{i=1}^{j-1} s_i)\, z_j + (n_2 - d2_{j-1}^2 - \sum_{i=1}^{j-1}(R_i - s_i))\,(1 - z_j)}{(n_1 - d1_{j-1}^2 - \sum_{i=1}^{j-1} s_i)\theta_2 + (n_2 - d2_{j-1}^2 - \sum_{i=1}^{j-1}(R_i - s_i))\theta_1} \theta_1^{1-z_j} \theta_2^{z_j},$$

and

$$\begin{aligned} P^{(1)}_{s_l|z_l^*;s_{l-1}^*} &= P(S_l = s_l | Z_1 = z_1, \cdots, Z_l = z_l; S_1 = s_1, \cdots, S_{l-1} = s_{l-1}) \\ &= \frac{\binom{n_1 - \sum_{i=1}^{l-1} z_i - \sum_{i=1}^{l-1} s_i}{s_l}\binom{n_2 - \sum_{i=1}^{l-1}(1-z_i) - \sum_{i=1}^{l-1}(R_i - s_i)}{q_l}}{\binom{n-l-\sum_{i=1}^{l-1} R_i}{R_l}}. \end{aligned}$$

Proof: The proof is similar to Lemma 9.3.

Lemma 9.3. *The probability mass function of* $D_1 = \sum_{i=1}^{r} Z_i$ *if* $W_{k:r:n} < W_{r:r:n} < T$ *is given by*

$$P(D_1 = i) = \sum_{(z_1,\cdots,z_r) \in Q_i^{(3)}} \cdots \sum \sum_{(s_1,\cdots,s_{r-1}) \in T_i^{(3)}} \cdots \sum \prod_{j=1}^{r} P_j^{(3)} \prod_{l=1}^{r-1} P^{(3)}_{s_l|z_l^*;s_{l-1}^*}, \qquad (14)$$

where

$$P_j^{(3)} = \frac{(n_1 - d1_{j-1}^3 - \sum_{i=1}^{j-1} s_i)\, z_j + (n_2 - d2_{j-1}^3 - \sum_{i=1}^{j-1}(R_i - s_i))\,(1 - z_j)}{(n_1 - d1_{j-1}^3 - \sum_{i=1}^{j-1} s_i)\theta_2 + (n_2 - d2_{j-1}^3 - \sum_{i=1}^{j-1}(R_i - s_i))\theta_1} \theta_1^{1-z_j} \theta_2^{z_j},$$

and

$$\begin{aligned} P^{(3)}_{s_l|z_l^*;s_{l-1}^*} &= P(S_l = s_l | Z_1 = z_1, \cdots, Z_l = z_l; S_1 = s_1, \cdots, S_{l-1} = s_{l-1}) \\ &= \frac{\binom{n_1 - \sum_{i=1}^{l-1} z_i - \sum_{i=1}^{l-1} s_i}{s_l}\binom{n_2 - \sum_{i=1}^{l-1}(1-z_i) - \sum_{i=1}^{l-1}(R_i - s_i)}{q_l}}{\binom{n-l-\sum_{i=1}^{l-1} R_i}{R_l}}. \end{aligned}$$

Proof: The proof is similar to Lemma 9.3.

Proof of Theorem 3.2: For proof of this Theorem 3.2, we have to compute each part of the right hand side of the above theorem separately. Here, we only calculate the

Case-II part and other two cases are similar of this proof. For more details see Rasouli and Balakrishnan [21].

The basic idea of the proof is we first, conditioning on the value of D_1 for Case-II where $1 \leq D_1 \leq D-1$ and then on conditioning on $\mathbf{Z} = (Z_1, \cdots, Z_D)$ and , $\mathbf{S} = (S_1, \cdots, S_D)$, we obtain

$$\begin{aligned}
M_{\hat{\theta}_1}(t) &= E(e^{t\,\hat{\theta}_1} | 1 \leq D_1 \leq D-1) \\
&= \sum_{i=1}^{D} E(e^{t\,\hat{\theta}_1} | D_1 = i) P(D_1 = i | 1 \leq D_1 \leq D-1) \\
&= \sum_{i=1}^{D-1} \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \!\!\cdots \sum E(e^{t\,\hat{\theta}_1} | D_1 = i, \mathbf{Z} = \mathbf{z}) P(\mathbf{Z} = \mathbf{z} | D_1 = i) P(D_1 = i | 1 \leq D_1 \leq D-1) \\
&= \sum_{i=1}^{D-1} \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \!\!\cdots \sum \; \sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}} \!\!\cdots \sum E(e^{t\,\hat{\theta}_1} | D_1 = i, \mathbf{Z} = \mathbf{z}, \mathbf{S} = \mathbf{s}) \\
&\quad \times P(\mathbf{Z} = \mathbf{z}, \mathbf{S} = \mathbf{s} | D_1 = i) P(D_1 = i | 1 \leq D_1 \leq D-1) \\
&= \frac{1}{P(1 \leq D_1 \leq D-1)} \sum_{i=1}^{D-1} \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \!\!\cdots \sum \; \sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}} \!\!\cdots \sum \frac{1}{\theta_1^i \theta_2^{D-i}} \\
&\quad \times \int_0^\infty \cdots \int_{w_{D-1}}^\infty exp\left(t\frac{\sum_{j=1}^D w_j(z_j + s_j) + T\,R_S^*}{i} - \frac{\sum_{j=1}^D w_j(z_j + s_j) + T\,R_S^*}{\theta_1} \right) \\
&\quad \times exp\left(-\frac{\sum_{j=1}^D w_j(1 - z_j) + \sum_{j=1}^D (R_j - s_j) + T\,R_Q^*}{\theta_2} \right) \, dw_D \cdots dw_1.
\end{aligned}$$

This expression can be simplified as

$$
\begin{aligned}
M_{\hat{\theta}_1}(t) \;=\; & \frac{e^{-\left(\frac{T\,R_S^*}{\theta_1}-\frac{T\,R_S^* t}{i}+\frac{T\,R_Q^*}{\theta_2}\right)}}{P(1 \le D_1 < D-1)} \sum_{i=1}^{D-1} \sum_{(z_1,\cdots,z_D)\in Q_i^{(2)}} \!\!\!\cdots\! \sum \; \sum_{(s_1,\cdots,s_{D-1})\in T_i^{(2)}} \!\!\!\cdots\! \sum \frac{1}{\theta_1^i}\frac{1}{\theta_2^{D-i}} \\
& \times \int_0^\infty exp\left\{-\left(\frac{w_1(z_1+s_1)}{\theta_1}+\frac{w_1((1-z_1)+(R_1-s_1))}{\theta_2}-\frac{w_1(z_1+s_1)}{i}t\right)\right\} \\
& \times \int_{w_1}^\infty exp\left\{-\left(\frac{w_2(z_2+s_2)}{\theta_1}+\frac{w_2((1-z_2)+(R_2-s_2))}{\theta_2}-\frac{w_2(z_2+s_2)}{i}t\right)\right\} \\
& \times \cdots \\
& \times \int_{w_{D-1}}^\infty exp\left\{-\left(\frac{w_D(z_D+s_D)}{\theta_1}+\frac{w_D((1-z_D)+(R_D-s_D))}{\theta_2}-\frac{w_D(z_D+s_D)}{i}t\right)\right\} \\
& \times dw_D \cdots dw_1.
\end{aligned}
$$

Therefore, by the method mentioned in Rasouli and Balakrishnan [21] and Koley and Kundu [18] we can separately calculate the three cases and add them to get the above theorem.

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