

# EFFICIENT ALGORITHM FOR ESTIMATING THE PARAMETERS OF CHIRP SIGNAL

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**ABSTRACT.** Chirp signals play an important role in the statistical signal processing. Recently Kundu and Nandi [8] derived the asymptotic properties of the least squares estimators of the unknown parameters of the chirp signals model in presence of stationary noise. Unfortunately they did not discuss any estimation procedures. In this article we propose a computationally efficient algorithm for estimating different parameters of a chirp signal in presence of stationary noise. From proper initial guesses, the proposed algorithm produces efficient estimators in a fixed number of iterations. We suggest how to obtain the proper initial guesses also. The proposed estimators are consistent and asymptotically equivalent to least squares estimators of the corresponding parameters. We perform some simulation experiments to see the effectiveness of the proposed method, and it is observed that the proposed estimators perform very well. For illustrative purposes, we have performed the data analysis of a simulated data set. Finally, we propose some generalization in the conclusions.

**Key Words and Phrases:** Chirp signals; least squares estimators; strong consistency, asymptotic distribution; linear processes; iteration.

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## 1. INTRODUCTION

In this paper we consider the following chirp signal model in presence of additive noise:

$$(1) \quad y(n) = A_0 \cos(\alpha_0 n + \beta_0 n^2) + B_0 \sin(\alpha_0 n + \beta_0 n^2) + X(n); \quad n = 1, \dots, N,$$

where  $A_0, B_0$  are non zero amplitudes, with restriction  $A_0^2 + B_0^2 \leq M$ , for some constant  $M$ . The frequency and frequency rate,  $\alpha_0, \beta_0$ , respectively, lie strictly between 0 and  $\pi$ .  $X(n)$  is a stationary noise sequence, and it has the following form;

$$(2) \quad X(n) = \sum_{j=-\infty}^{\infty} a(j)\varepsilon(n-j) \quad , \quad \sum_{j=-\infty}^{\infty} |a(j)| < \infty.$$

Here,  $\{\varepsilon(n)\}$  is a sequence of independent and identically distributed (*i.i.d.*) random variables with zero mean, variance  $\sigma^2$  and with finite fourth moment. Given a sample of size  $N$ , the problem is to estimate the unknown amplitudes  $A_0, B_0$ , the frequency  $\alpha_0$  and the frequency rate  $\beta_0$ .

A chirp signal occurs quite naturally in different areas of science and technology. It is a signal where frequency changes over time and this property of the signal has been exploited quite effectively, to measure the distance of an object from a source. This model can be found in different sonar, radar, communications problems, see for example Abatzoglou [1], Kumaresan and Verma [6], Djuric and Kay [3], Gini *et al.* [5], Nandi and Kundu [9], Kundu and Nandi [8] and the references cited therein.

In practice, the parameters like amplitudes, frequency, frequency rate are unknown, and one tries to find efficient estimators of these unknown parameters, having some desired statistical properties. Recently, Kundu and Nandi [8] derived the asymptotic properties of the least squares estimators (LSEs), and it is observed that the LSEs are consistent and asymptotically normally distributed. It is further observed that the LSEs are efficient, with the convergence rates of the amplitude, frequency and frequency rate are  $O_p(N^{-1/2})$ ,  $O_p(N^{-3/2})$  and  $O_p(N^{-5/2})$  respectively. Here  $O_p(N^{-\delta})$

means  $N^\delta O_p(N^{-\delta})$  is bounded in probability. Note that a sequence of random variables  $\{X_n\}$  is said to be bounded in probability, if for any  $\epsilon > 0$ , there exists  $m, M$ , such that  $P(m < X_n < M) \geq 1 - \epsilon$ .

Unfortunately, Kundu and Nandi [8] did not discuss any estimation procedure of the LSEs. It is clear that they have to be obtained by some iterative procedure like Newton-Raphson method or its variant, see for example Tjalling [14]. Although, the initial guesses and convergence of the iterative procedure seem to be important issues. It is well known, see Rice and Rosenblatt [11], that even in the simple sinusoidal model, finding the LSEs is not an trivial issue. The problem is due to the fact that the least squares surface has several local minima, and therefore, if the initial estimators are not properly chosen, any iterative procedure will converge to a local minimum rather than the global minimum. Several methods have been suggested in the literature to find the efficient frequency estimators, see for example a very recent article by Kundu *et al.* [7] in this respect.

The main aim of this paper is to find estimators of the amplitudes, frequency and frequency rate efficiently, which have the same rates of convergence as the corresponding LSEs. It may be observed that the model (1) can be seen as a non-linear regression model, with  $A_0, B_0$  as linear parameters, and  $\alpha_0, \beta_0$  as non-linear parameters. If we can find efficient estimators of the non-linear parameters  $\alpha_0$  and  $\beta_0$ , then efficient estimators of the linear parameters can be obtained by simple linear regression technique, see for example Richards [12]. Because of this reason, in this paper we mainly concentrate on estimating the non-linear parameters efficiently.

We propose an iterative procedure which has been applied to find efficient estimators of the frequency and frequency rate. It is observed that if we start the initial guesses of  $\alpha_0$  and  $\beta_0$  with convergence rates  $O_p(N^{-1})$  and  $O_p(N^{-2})$  respectively, then after four iterations the algorithm produces an estimate of  $\alpha_0$  with convergence rate  $O_p(N^{-3/2})$ , and an estimate of  $\beta_0$  with convergence rate  $O_p(N^{-5/2})$ . Therefore, it is

clear that the proposed algorithm produces estimates which have the same rates of convergence as the LSEs. Moreover, it is known that the algorithm stops after finite number of iterations.

We perform some simulation experiments, to see the effectiveness of the proposed method for different sample sizes and for different error variances. It is observed that the algorithm works very well. The mean squared errors (MSEs) of the proposed estimators are very close to the corresponding MSEs of the LSEs, and both are very close to the corresponding asymptotic variance of the LSEs. Therefore, the proposed method can be used very effectively instead of the LSEs. For illustrative purposes, we have analyzed one simulated data, and the performance is very satisfactory.

The rest of the paper is organized as follows. We provide the properties of the LSEs in Section 2. In Section 3, we present the proposed algorithm and provide the theoretical justification of the algorithm. The simulation results and the analysis of a simulated data have been presented in Section 4. Conclusions appear in Section 5. All the proofs are presented in the Appendix.

## 2. EXISTING RESULTS

In this section we present briefly the properties of the LSEs for ready reference. The LSEs of the unknown parameters of the model (1) can be obtained by minimizing  $S(\Theta)$  with respect to  $\Theta = (A, B, \alpha, \beta)$ , where

$$\begin{aligned} S(\Theta) &= \sum_{n=1}^N (y(n) - A \cos(\alpha n + \beta n^2) - B \sin(\alpha n + \beta n^2))^2 \\ &= \left[ \mathbf{Y} - \mathbf{W}(\alpha, \beta) \begin{bmatrix} A \\ B \end{bmatrix} \right]^T \left[ \mathbf{Y} - \mathbf{W}(\alpha, \beta) \begin{bmatrix} A \\ B \end{bmatrix} \right]. \end{aligned}$$

Here  $\mathbf{Y} = (y(1), \dots, y(N))^T$ , is the  $N \times 1$  data vector and  $\mathbf{W}(\boldsymbol{\theta})$  is the  $N \times 2$  matrix of the following form;

$$(3) \quad \mathbf{W}(\boldsymbol{\theta}) = \begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ \cos(2\alpha + 4\beta) & \sin(2\alpha + 4\beta) \\ \vdots & \vdots \\ \cos(N\alpha + N^2\beta) & \sin(N\alpha + N^2\beta) \end{bmatrix}.$$

Note that, if  $\alpha$  and  $\beta$  are known the LSEs of  $A_0$  and  $B_0$  can be obtained as  $\widehat{A}(\boldsymbol{\theta})$  and  $\widehat{B}(\boldsymbol{\theta})$  respectively, where  $\boldsymbol{\theta} = (\alpha, \beta)$ ,

$$(4) \quad \left( \widehat{A}(\boldsymbol{\theta}), \widehat{B}(\boldsymbol{\theta}) \right)^T = \left( \mathbf{W}^T(\boldsymbol{\theta}) \mathbf{W}(\boldsymbol{\theta}) \right)^{-1} \mathbf{W}^T(\boldsymbol{\theta}) \mathbf{Y},$$

Therefore, the LSEs of  $\alpha_0$  and  $\beta_0$  can be obtained first by minimizing  $Q(\alpha, \beta)$  with respect to  $\alpha$  and  $\beta$ , where

$$(5) \quad Q(\alpha, \beta) = S(\widehat{A}(\boldsymbol{\theta}), \widehat{B}(\boldsymbol{\theta}), \alpha, \beta) = \mathbf{Y}^T \mathbf{W}(\boldsymbol{\theta}) \left( \mathbf{W}^T(\boldsymbol{\theta}) \mathbf{W}(\boldsymbol{\theta}) \right)^{-1} \mathbf{W}^T(\boldsymbol{\theta}) \mathbf{Y}.$$

Once the LSEs of  $\alpha_0$  and  $\beta_0$ , say  $\widehat{\alpha}$  and  $\widehat{\beta}$  are obtained the LSEs of  $A_0$  and  $B_0$  can be easily obtained as  $\widehat{A}(\widehat{\alpha}, \widehat{\beta})$  and  $\widehat{B}(\widehat{\alpha}, \widehat{\beta})$  respectively, see for example Richards [12].

Kundu and Nandi [8] derived the properties of the LSEs and it is as follows. The LSEs of the unknown parameters of the model (1) are consistent to the corresponding parameters and they have the following asymptotic distribution

$$\left( N^{1/2}(\widehat{A} - A_0), N^{1/2}(\widehat{B} - B_0), N^{3/2}(\widehat{\alpha} - \alpha_0), N^{5/2}(\widehat{\beta} - \beta_0) \right) \xrightarrow{d} N_4(\mathbf{0}, 4c\sigma^2 \boldsymbol{\Sigma}),$$

where  $c = \sum_{j=-\infty}^{\infty} a(j)^2$ . The symbol ' $\xrightarrow{d}$ ' means convergence in distribution,  $N_4(\mathbf{0}, 4c\sigma^2 \boldsymbol{\Sigma})$  means a 4-variate normal distribution with mean vector  $\mathbf{0}$ , the dispersion matrix  $4c\sigma^2 \boldsymbol{\Sigma}$  and

$$(6) \quad \boldsymbol{\Sigma} = \frac{1}{A_0^2 + B_0^2} \begin{bmatrix} \frac{1}{2}(A_0^2 + 9B_0^2) & -4A_0B_0 & 18B_0 & -15B_0 \\ -4A_0B_0 & \frac{1}{2}(9A_0^2 + B_0^2) & -18A_0 & 15A_0 \\ 18B_0 & -18A_0 & 96 & -90 \\ -15B_0 & 15A_0 & -90 & 90 \end{bmatrix}.$$

In the next section we provide the algorithm which produces consistent estimators of  $\alpha_0$  and  $\beta_0$  which have the same rates of convergence as the LSEs.

## 3. PROPOSED ALGORITHM

The aim of this algorithm is to find the estimates of the frequency and frequency rate with the same rates of convergence as the LSEs. We will first deduce a method for improving the speed of convergence of general consistent estimators of  $\alpha_0$  and  $\beta_0$ : this method is the crucial tool for implementing the main algorithm developed in this paper. If  $\tilde{\alpha}$  is an estimator of  $\alpha_0$ , such that  $\tilde{\alpha} - \alpha_0 = O_p(N^{(-1-\delta_1)})$ , for some  $\delta_1 > 0$ , and  $\tilde{\beta}$  is an estimator of  $\beta_0$ , such that  $\tilde{\beta} - \beta_0 = O_p(N^{(-2-\delta_2)})$ , for some  $\delta_2 > 0$ , then an improved estimator of  $\alpha_0$  can be obtained as

$$(7) \quad \tilde{\tilde{\alpha}} = \tilde{\alpha} + \frac{48}{N^2} \text{Im} \left( \frac{P_N^\alpha}{Q_N} \right)$$

with

$$(8) \quad P_N^\alpha = \sum_{n=1}^N y(n) \left( n - \frac{N}{2} \right) e^{-i(\tilde{\alpha}n + \tilde{\beta}n^2)} \quad \text{and} \quad Q_N = \sum_{n=1}^N y(n) e^{-i(\tilde{\alpha}n + \tilde{\beta}n^2)}.$$

Here  $\text{Im}(\cdot)$  denotes the imaginary part of the complex argument. Similarly, an improved estimator of  $\beta_0$  can be obtained as

$$(9) \quad \tilde{\tilde{\beta}} = \tilde{\beta} + \frac{45}{N^4} \text{Im} \left( \frac{P_N^\beta}{Q_N} \right),$$

with

$$(10) \quad P_N^\beta = \sum_{n=1}^N y(n) \left( n^2 - \frac{N^2}{3} \right) e^{-i(\tilde{\alpha}n + \tilde{\beta}n^2)}$$

and  $Q_N$  is same as defined in (8).

The following two theorems provide the justification for the improved estimators, whose proofs will be provided in the appendix.

**THEOREM 1:** If  $\tilde{\alpha} - \alpha_0 = O_p(N^{(-1-\delta_1)})$  for  $\delta_1 > 0$  then

- (a)  $\tilde{\tilde{\alpha}} - \alpha_0 = O_p(N^{(-1-2\delta_1)})$  if  $\delta_1 \leq \frac{1}{4}$
- (b)  $N^{\frac{3}{2}}(\tilde{\tilde{\alpha}} - \alpha_0) \xrightarrow{d} N(0, \sigma_1^2)$  if  $\delta_1 > \frac{1}{4}$ ,

where  $\sigma_1^2 = \frac{384c\sigma^2}{A_0^2 + B_0^2}$ , the asymptotic variance of the LSE of  $\alpha_0$ .

**THEOREM 2:** If  $\tilde{\beta} - \beta_0 = O_p(N^{(-2-\delta_2)})$  for  $\delta_2 > 0$  then

- (a)  $\tilde{\beta} - \beta_0 = O_p(N^{(-2-2\delta_2)})$  if  $\delta_2 \leq \frac{1}{4}$   
 (b)  $N^{\frac{5}{2}}(\tilde{\beta} - \beta_0) \xrightarrow{d} N(0, \sigma_2^2)$  if  $\delta_2 > \frac{1}{4}$

where  $\sigma_2^2 = \frac{360c\sigma^2}{A_0^2 + B_0^2}$ , the asymptotic variance of the LSE of  $\beta_0$ .

**THEOREM 3:** If  $\tilde{\alpha} - \alpha_0 = O_p(N^{(-1-\delta_1)})$  and  $\tilde{\beta} - \beta_0 = O_p(N^{(-2-\delta_2)})$ , then for  $\delta_1 > \frac{1}{4}$  and for  $\delta_2 > \frac{1}{4}$ ,  $\text{Cov}(N^{\frac{3}{2}}(\tilde{\alpha} - \alpha_0), N^{\frac{5}{2}}(\tilde{\beta} - \beta_0)) \longrightarrow \frac{360c\sigma^2}{A_0^2 + B_0^2}$ .

Now we will show that starting from initial guesses  $\tilde{\alpha}, \tilde{\beta}$ , with convergence rates  $\tilde{\alpha} - \alpha_0 = O_p(N^{-1})$  and  $\tilde{\beta} - \beta_0 = O_p(N^{-2})$  respectively, how the above procedure can be used to obtain efficient estimators. It may be noted that finding initial guesses with the above convergence rates are not very difficult. From Rice and Rosenblatt [11] it follows that if one can show,  $\tilde{\alpha}, \tilde{\beta}$ , the minimizer of  $Q(\alpha, \beta)$  are such that,  $N(\tilde{\alpha} - \alpha_0) \rightarrow 0$  *a.s.*,  $N^2(\tilde{\beta} - \beta_0) \rightarrow 0$  *a.s.*, then one has the existence of such an initial guess when searched over the grids  $\left(\frac{\pi j}{N}, \frac{\pi k}{N^2}\right)$ ,  $j = 1, \dots, N$ , and  $k = 1, \dots, N^2$ . Let  $\Theta_0 = (A_0, B_0, \alpha_0, \beta_0)$  be the true parameter value and  $\mathbf{D}$  is a diagonal matrix, such that  $\mathbf{D} = \text{diag}\left(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{1}{N\sqrt{N}}, \frac{1}{N^2\sqrt{N}}\right)$ . Note that, see Kundu and Nandi [8],

$$(11) \quad -S'(\Theta_0)\mathbf{D} = (\tilde{\Theta} - \Theta_0)\mathbf{D}^{-1}[\mathbf{D}S''(\bar{\Theta})\mathbf{D}],$$

where  $S'(\cdot)$  and  $S''(\cdot)$  are the derivative vector and the Hessian matrix of  $S(\cdot)$ , respectively, and  $\bar{\Theta}$  is a point on line joining  $\tilde{\Theta}$  and  $\Theta_0$ . It has been shown in Kundu and Nandi [8] that

$$(12) \quad \mathbf{D}S''(\bar{\Theta})\mathbf{D} \longrightarrow \lim_{N \rightarrow \infty} \mathbf{D}S''(\Theta_0)\mathbf{D} \quad \textit{a.s.},$$

which is a positive definite matrix. Then (11) gives

$$(13) \quad (\tilde{\Theta} - \Theta_0)(\sqrt{N}\mathbf{D})^{-1} = \left[-\frac{1}{\sqrt{N}}S'(\Theta_0)\mathbf{D}\right] [\mathbf{D}S''(\bar{\Theta})\mathbf{D}]^{-1}.$$

Also  $\frac{-S'(\Theta_0)\mathbf{D}}{\sqrt{N}} \rightarrow 0$  *a.s.* implies  $(\tilde{\Theta} - \Theta_0)(\sqrt{N}\mathbf{D})^{-1} \rightarrow 0$  *a.s.* which means

$$N(\tilde{\alpha} - \alpha_0) \rightarrow 0 \textit{ a.s.}, N^2(\tilde{\beta} - \beta_0) \rightarrow 0 \textit{ a.s.}$$

The main idea of the proposed algorithm is not to use the whole sample size at the beginning, as it was first suggested by Bai *et al.* [2]. We will use part of the sample at the beginning and gradually proceed towards the complete sample. The algorithm can be described as follows. We will denote the estimates of  $\alpha_0$  and  $\beta_0$  obtained at the  $i$ -th iteration as  $\tilde{\alpha}^{(i)}$  and  $\tilde{\beta}^{(i)}$  respectively.

ALGORITHM:

Step 1: Choose  $N_1 = N^{8/9}$ . Therefore,  $\tilde{\alpha}^{(0)} - \alpha_0 = O_p(N^{-1}) = O_p(N_1^{-1-1/8})$ , and  $\tilde{\beta}^{(0)} - \beta_0 = O_p(N^{-2}) = O_p(N_1^{-2-1/4})$ . Perform step (7) and (9). Therefore, after the 1-st iteration we have

$$\tilde{\alpha}^{(1)} - \alpha_0 = O_p(N_1^{-1-1/4}) = O_p(N^{-10/9}), \quad \text{and} \quad \tilde{\beta}^{(1)} - \beta_0 = O_p(N_1^{-2-\frac{1}{2}}) = O_p(N^{-20/9}).$$

Step 2: Choose  $N_2 = N^{80/81}$ . Therefore,  $\tilde{\alpha}^{(1)} - \alpha_0 = O_p(N_2^{-1-1/8})$ , and  $\tilde{\beta}^{(1)} - \beta_0 = O_p(N_2^{-2-1/4})$ . Perform step (7) and (9). Therefore, after the 2-nd iteration we have

$$\tilde{\alpha}^{(2)} - \alpha_0 = O_p(N_2^{-1-1/4}) = O_p(N^{-100/81}), \quad \text{and} \quad \tilde{\beta}^{(2)} - \beta_0 = O_p(N_2^{-2-1/2}) = O_p(N^{-200/81}).$$

Step 3: Choose  $N_3 = N$ . Therefore,  $\tilde{\alpha}^{(2)} - \alpha_0 = O_p(N_3^{-1-19/81})$ , and  $\tilde{\beta}^{(2)} - \beta_0 = O_p(N_3^{-2-38/81})$ . Perform step (7) and (9). Therefore, after the 3-rd iteration we have

$$\tilde{\alpha}^{(3)} - \alpha_0 = O_p(N^{-1-38/81}), \quad \text{and} \quad \tilde{\beta}^{(3)} - \beta_0 = O_p(N^{-2-76/81}).$$

Step 4: Choose  $N_4 = N$ , and perform step (7) and (9). Now we obtain the required convergence rates, *i.e.*

$$\tilde{\alpha}^{(4)} - \alpha_0 = O_p(N^{-3/2}), \quad \text{and} \quad \tilde{\beta}^{(4)} - \beta_0 = O_p(N^{-5/2}).$$

#### 4. SIMULATION AND DATA ANALYSIS

4.1. SIMULATION RESULTS: In this section we present some simulation results for different sample sizes and for different error variances to show how the proposed



TABLE 1. Result for model with i.i.d. error

sample size		$\sigma^2=0.05$		$\sigma^2=0.5$	
N=50	PARA	1.75	1.05	1.75	1.05
	ASYV	( 0.1920000E-04)	( 0.7200000E-08)	( 0.1920000E-03)	( 0.7200000E-07)
	MEAN (Algo)	1.749280	1.050021	1.754682	1.049943
	MSE (Algo)	( 0.3999998E-03)	( 0.1587378E-06)	( 0.6944025E-03)	( 0.1154047E-06)
	MEAN (LSE)	1.749617	1.050007	1.748845	1.050022
	MSE (LSE)	( 0.1766448E-04)	( 0.6171723E-08)	( 0.1816612E-03)	( 0.6310458E-07)
N=100	PARA	1.75	1.05	1.75	1.05
	ASYV	( 0.2400000E-05)	( 0.2250000E-09)	( 0.2400000E-04)	( 0.2250000E-08)
	MEAN (Algo)	1.751141	1.050000	1.750380	1.050003
	MSE (Algo)	( 0.1595991E-03)	( 0.9806420E-08)	( 0.1851986E-03)	( 0.9809600E-08)
	MEAN (LSE)	1.750060	1.049997	1.750345	1.049996
	MSE (LSE)	( 0.2423173E-05)	( 0.2181697E-09)	( 0.2432559E-04)	( 0.2156012E-08)
N=200	PARA	1.75	1.05	1.75	1.05
	ASYV	( 0.3000000E-06)	( 0.7031250E-11)	( 0.3000000E-05)	( 0.7031250E-10)
	MEAN (Algo)	1.750400	1.050000	1.750630	1.050005
	MSE (Algo)	( 0.3999993E-05)	( 0.5969992E-09)	( 0.2389996E-04)	( 0.5991395E-09)
	MEAN (LSE)	1.750051	1.049993	1.750236	1.049993
	MSE (LSE)	( 0.1248708E-05)	( 0.2637675E-10)	( 0.1766828E-04)	( 0.3679320E-09)

method behaves in practice. We consider the following model

$$(14) \quad y(n) = 2.0 \cos(1.75n + 1.05n^2) + 2.0 \sin(1.75n + 1.05n^2) + X(n).$$

We have considered (i)  $X(n) = \varepsilon(n)$  (*i.i.d.* errors) and (ii)  $X(n) = \varepsilon(n) + 0.5\varepsilon(n-1)$  (stationary errors), where  $\varepsilon(n)$ 's are *i.i.d.* normal random variables with mean 0 and variance  $\sigma^2$ . We have taken  $N = 50, 100$  and  $200$ ,  $\sigma^2 = 0.05$  and  $0.5$ . In each case we have obtained the initial guesses as has been suggested in Section 3. For each  $N$  and  $\sigma^2$ , we compute the average estimates of  $\alpha_0$  and  $\beta_0$ , and the associate MSEs based on 1000 replications. The results are reported in Table 1 and Table 2. For comparison purposes we have also computed the LSEs and the corresponding asymptotic variances.

From the results presented in Tables 1 and 2, it is clear that the performances of the estimators obtained by the proposed algorithm are quite satisfactory in comparison to the corresponding performances of the LSEs. The average MSEs of the proposed estimators are very close to the asymptotic variance of the corresponding LSEs. The performances are quite good even with moderate sample sizes. It is clear that even with the four steps of iterations, we are able to achieve the same accuracy of the LSEs.

TABLE 2. Result for model with Stationary error

sample size		$\sigma^2=0.05$		$\sigma^2=0.5$	
N=50	PARA	1.75	1.05	1.75	1.05
	ASYV	( 0.2400000E-04)	( 0.9000000E-08)	( 0.2400000E-03)	( 0.9000000E-07)
	MEAN (Algo)	1.755243	1.049993	1.754883	1.049946
	MSE (Algo)	( 0.3999998E-03)	( 0.4903598E-07)	( 0.7968027E-03)	( 0.1484981E-06)
	MEAN (LSE)	1.749538	1.050008	1.748546	1.050027
	MSE (LSE)	( 0.2336846E-04)	( 0.8118942E-08)	( 0.2410938E-03)	( 0.8371937E-07)
N=100	PARA	1.75	1.05	1.75	1.05
	ASYV	( 0.3000000E-05)	( 0.2812500E-09)	( 0.3000000E-04)	( 0.2812500E-08)
	MEAN (Algo)	1.750381	1.050007	1.749740	1.050009
	MSE (Algo)	( 0.1699989E-03)	( 0.9807608E-08)	( 0.1931984E-03)	( 0.9812850E-08)
	MEAN (LSE)	1.750003	1.049998	1.750168	1.049998
	MSE (LSE)	( 0.2838691E-05)	( 0.2514815E-09)	( 0.3133409E-04)	( 0.2738235E-08)
N=200	PARA	1.75	1.05	1.75	1.05
	ASYV	( 0.3750000E-06)	( 0.8789063E-11)	( 0.3750000E-05)	( 0.8789063E-10)
	MEAN (Algo)	1.750420	1.050001	1.750710	1.050004
	MSE (Algo)	( 0.4999991E-05)	( 0.5993741E-09)	( 0.2749995E-04)	( 0.5977831E-09)
	MEAN (LSE)	1.750030	1.049993	1.750358	1.049993
	MSE (LSE)	( 0.1477499E-05)	( 0.3188963E-10)	( 0.2170591E-04)	( 0.4467878E-09)

4.2. DATA ANALYSIS. In this subsection we present the analysis of a simulated data set mainly for illustrative purposes. We have generated a sample of size 100 from the following chirp model

$$(15) \quad y(n) = 1.5 \cos(1.0n + 1.0n^2) + 1.5 \sin(1.0n + 1.0n^2) + X(n)$$

here  $X(n)$  is a sequence of *i.i.d.* Gaussian random variables with mean zero and variance 0.5. It is presented in Figure 1. We also present the least squares surface  $Q(\boldsymbol{\theta})$  in Figure 2 as defined in (5). Since  $Q(\boldsymbol{\theta})$  has only one trough, the initial guesses can be obtained as suggested in Section 3.

Using the algorithm, we obtained the estimates of  $A_0$ ,  $B_0$ ,  $\alpha_0$ ,  $\beta_0$  as 0.9386, 1.7922, 1.000000 and 1.000009 respectively. The associated 95% confidence intervals are obtained using bootstrap approach, and they are (0.0372, 1.8402), (1.2369, 2.3475), (0.9998, 1.0002) and (0.9999, 1.0001) respectively. The residuals are reported in Figure 3. From the run test it follows that the residuals are random, as expected.

## 5. CONCLUSION

In this paper we propose an efficient algorithm to estimate the nonlinear parameters of one dimensional chirp signal in presence of stationary noise. The main advantage

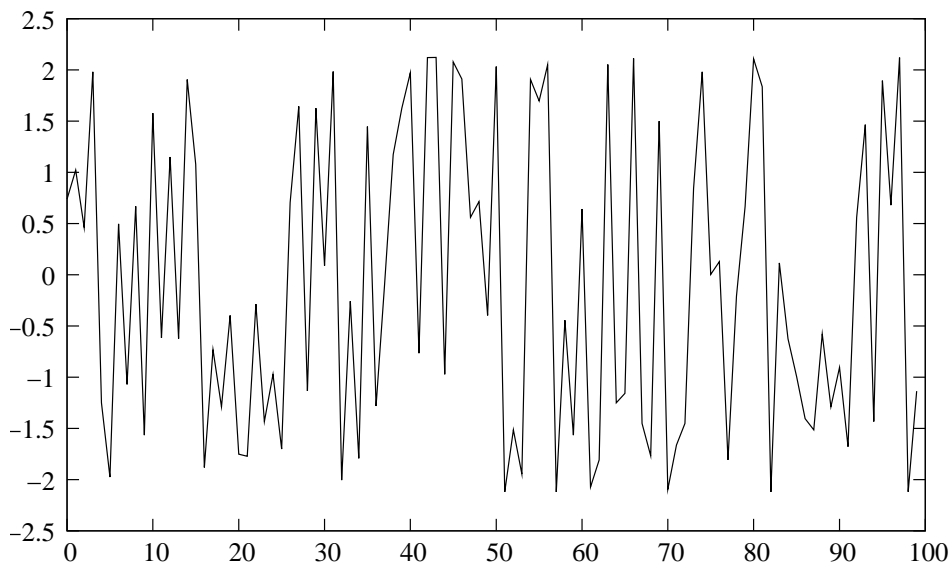


FIGURE 1. Simulated data from the Chirp Model (15) of size 100

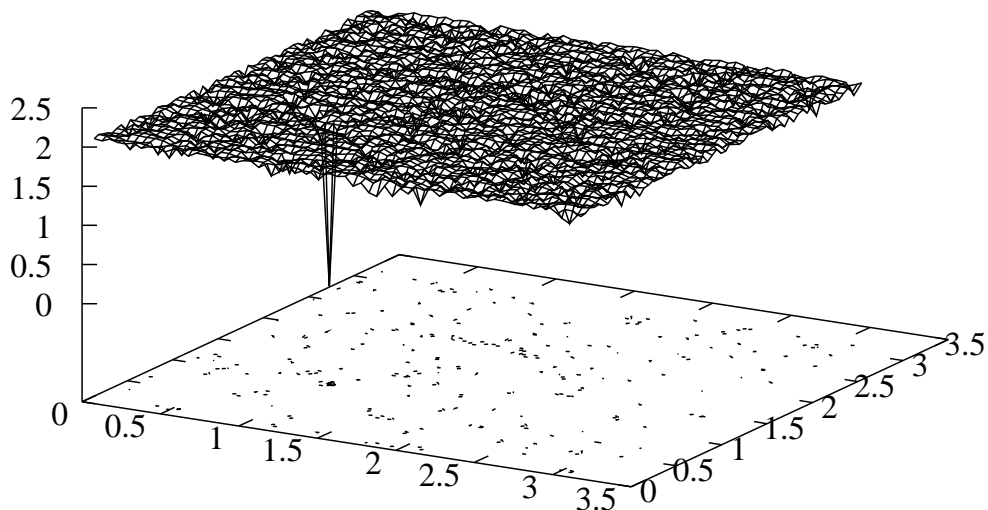


FIGURE 2. Least squares surface plot

of the proposed algorithm is that starting from a reasonable starting values for both the frequency and frequency rate, in four steps only, it produces estimates which are asymptotically equivalent to the LSEs. We have also proposed how to obtain the reasonable initial guesses from the least squares surfaces. We have performed an extensive simulation experiments and it is observed that the proposed algorithm is working very well, and it is producing estimates which are quite close to the LSEs.

Although, in this paper we have considered the estimation procedure of the ordinary chirp model in presence of stationary noise, the method may be extended for

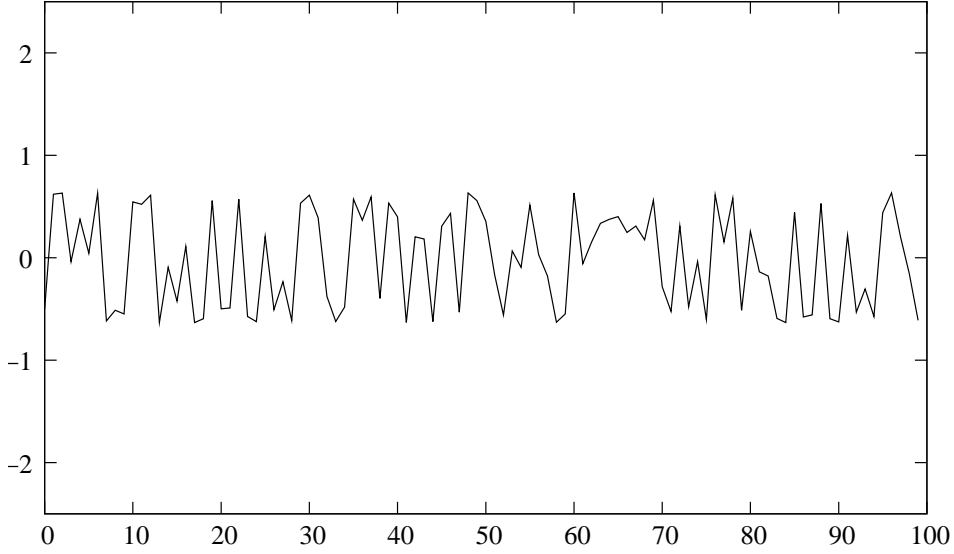


FIGURE 3. Residuals plot

generalized chirp signal model also as it was proposed by Djuric and Kay [3]. The work is in progress, it will be reported later.

#### ACKNOWLEDGEMENT

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#### APPENDIX

We need the following lemma for proving the theorems. Using Vinogradov's [15] result one can prove it.

**Lemma 1.** *If  $(\theta_1, \theta_2)$  in  $(0, \pi) \times (0, \pi)$ ,  $t = 0, 1, 2$  then except for countable number of points the followings are true.*

(i)

$$(16) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \cos(\theta_1 n + \theta_2 n^2) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \sin(\theta_1 n + \theta_2 n^2) = 0.$$

(ii)

$$(17) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{t+1}} \sum_{n=1}^N n^t \cos^2(\theta_1 n + \theta_2 n^2) = \frac{1}{2(t+1)}$$

$$(18) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{t+1}} \sum_{n=1}^N n^t \sin^2(\theta_1 n + \theta_2 n^2) = \frac{1}{2(t+1)}.$$

$$(19) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{t+1}} \sum_{n=1}^N n^t \sin(\theta_1 n + \theta_2 n^2) \cos(\theta_1 n + \theta_2 n^2) = 0.$$

$$(20) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N-|h|} n^t \cos(\theta_1 n + \theta_2 n^2) \cos(\theta_1(n+h) + \theta_2(n+h)^2) = \begin{cases} 0 & \text{for } h \neq 0 \\ \text{and} \\ \frac{1}{2(t+1)} & \text{for } h=0 \end{cases},$$

PROOF OF THEOREM 1:

Note that,

$$(21) \quad \begin{aligned} Q_N &= \sum_{t=1}^N y(t) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} \\ &= \sum_{t=1}^N \left( \frac{A_0 - iB_0}{2} \right) e^{i[(\alpha_0 - \tilde{\alpha})t + (\beta_0 - \tilde{\beta})t^2]} + \sum_{t=1}^N \left( \frac{A_0 + iB_0}{2} \right) e^{-i[(\alpha_0 + \tilde{\alpha})t + (\beta_0 + \tilde{\beta})t^2]} \\ &\quad + \sum_{t=1}^N X(t) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} \end{aligned}$$

and

$$(22) \quad \begin{aligned} P_N^\alpha &= \sum_{t=1}^N y(t) \left( t - \frac{N}{2} \right) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} \\ &= \sum_{t=1}^N X(t) \left( t - \frac{N}{2} \right) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} + \sum_{t=1}^N \left( \frac{A_0 - iB_0}{2} \right) \left( t - \frac{N}{2} \right) e^{i[(\alpha_0 - \tilde{\alpha})t + (\beta_0 - \tilde{\beta})t^2]} \\ &\quad + \sum_{t=1}^N \left( \frac{A_0 + iB_0}{2} \right) \left( t - \frac{N}{2} \right) e^{-i[(\alpha_0 + \tilde{\alpha})t + (\beta_0 + \tilde{\beta})t^2]} \end{aligned}$$

To get 2nd term in equation (21) we use the lemma 1 and get

$$\sum_{t=1}^N e^{-i[(\alpha_0 + \tilde{\alpha})t + (\beta_0 + \tilde{\beta})t^2]} = o_p(N)$$

Using lemma 1 and bivariate Taylor series expansion the 1st term in equation (21) becomes

$$\begin{aligned}
& \sum_{t=1}^N e^{i[(\alpha_0 - \tilde{\alpha})t + (\beta_0 - \tilde{\beta})t^2]} \\
&= \sum_{t=1}^N \left[ e^{i[t(\alpha_0 - \alpha_0) + t^2(\beta_0 - \beta_0)]} + i(\alpha_0 - \tilde{\alpha})t + i(\beta_0 - \tilde{\beta})t^2 \right. \\
&\quad + \frac{i^2(\alpha_0 - \tilde{\alpha})^2 t^2}{2!} e^{i[t(\alpha_0 - \alpha^*) + t^2(\beta_0 - \beta^*)]} + \frac{i^2(\beta_0 - \tilde{\beta})^2 t^4}{2!} e^{i[t(\alpha_0 - \alpha^*) + t^2(\beta_0 - \beta^*)]} \\
&\quad \left. + \frac{2i^2(\alpha_0 - \tilde{\alpha})(\beta_0 - \tilde{\beta})t^3}{2!} e^{i[t(\alpha_0 - \alpha^*) + t^2(\beta_0 - \beta^*)]} \right] \\
&= O(N) + O_p(N^{-1-\delta_1})O(N^2) + O_p(N^{-2-\delta_2})O(N^3) + \frac{1}{2}O_p(N^{-2-2\delta_1})O_p(N^3) \\
&\quad + O_p(N^{-1-\delta_1})O_p(N^{-2-\delta_2})O_p(N^4) + \frac{1}{2}O_p(N^{-4-2\delta_2})O_p(N^5) \\
&= O_p(N) + O_p(N^{1-\delta_1}) + O_p(N^{1-\delta_2}) + \frac{1}{2}O_p(N^{1-2\delta_1}) + O_p(N^{1-\delta_1-\delta_2}) + \frac{1}{2}O_p(N^{1-2\delta_2}) \\
&= O_p(N) + O_p(N^{1-\min(\delta_1, \delta_2)}) + O_p(N^{1-2\min(\delta_1, \delta_2)})
\end{aligned}$$

where  $(\alpha^*, \beta^*)$  is a point on the line joining  $(\alpha_0, \beta_0)$  and  $(\tilde{\alpha}, \tilde{\beta})$

To calculate the 3rd term in equation (21) we need the following two observations.

$$(i) \quad \frac{1}{N^{n+\frac{1}{2}}} \sum_{t=1}^N t^n [\sin(\alpha_0 t + \beta_0 t^2)] X(t) \quad \& \quad \frac{1}{N^{n+\frac{1}{2}}} \sum_{t=1}^N t^n [\cos(\alpha_0 t + \beta_0 t^2)] X(t)$$

satisfy conditions for Central Limit Theorem (CLT), see Fuller [4], when  $\alpha_0, \beta_0$  are the true value as in the model.

$$(ii) \quad \sup_{\alpha, \beta} \left| \frac{1}{N^{n+1}} \sum_{t=1}^N t^n X(t) e^{i[\alpha t + \beta t^2]} \right| \rightarrow 0 \text{ a.s.. It can be proved along the same line as Kundu and Nandi [8].}$$

Now we choose  $L$  large such that  $1 - L \min(\delta_1, \delta_2) < 0$ . Then using again bivariate Taylor series expansion the 3rd term in equation (21) becomes

$$\begin{aligned}
& \sum_{t=1}^N X(t) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} = \sum_{t=1}^N X(t) \left[ e^{-i(t\alpha_0 + t^2\beta_0)} \right. \\
&\quad + \sum_{l=1}^{L-1} \frac{(-i(\tilde{\alpha} - \alpha_0)t)^l}{l!} e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{(-i(\tilde{\beta} - \beta_0)t^2)^l}{l!} e^{-i(t\alpha_0 + t^2\beta_0)} \\
&\quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{(i(\tilde{\alpha} - \alpha_0)t)^k (i(\tilde{\beta} - \beta_0)t^2)^{l-k}}{k!(l-k)!} e^{-i(t\alpha_0 + t^2\beta_0)} \\
&\quad \left. + \sum_{k=0}^L \frac{(i(\tilde{\alpha} - \alpha_0)t)^k (i(\tilde{\beta} - \beta_0)t^2)^{L-k}}{k!(L-k)!} e^{-i(t\alpha^* + t^2\beta^*)} \right]
\end{aligned}$$

$$\begin{aligned}
&= O_p(N^{\frac{1}{2}}) + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-l-l\delta_1}) O_p(N^{l+\frac{1}{2}}) + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-2l-l\delta_2}) O_p(N^{2l+\frac{1}{2}}) \\
&\quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{1}{k!(l-k)!} O_p(N^{-k-k\delta_1}) O_p(N^{-2(l-k)-(l-k)\delta_2}) O_p(N^{k+2(l-k)+\frac{1}{2}}) \\
&\quad + \sum_{k=0}^L \frac{1}{k!(L-k)!} O_p(N^{-k-k\delta_1}) O_p(N^{-2(L-k)-(L-k)\delta_2}) O_p(N^{k+2(L-k)+1}) \\
&= O_p(N^{\frac{1}{2}}) + \sum_{l=1}^{L-1} \frac{1}{l!} [O_p(N^{\frac{1}{2}-l\delta_1}) + O_p(N^{\frac{1}{2}-l\delta_2})] + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{1}{k!(l-k)!} [O_p(N^{\frac{1}{2}-k\delta_1-(l-k)\delta_2})] \\
&\quad + \sum_{k=0}^L \frac{1}{k!(L-k)!} [O_p(N^{1-k\delta_1-(L-k)\delta_2})] \\
&= O_p(N^{\frac{1}{2}}) + \sum_{l=1}^{L-1} \sum_{k=0}^l \frac{1}{k!(l-k)!} [O_p(N^{\frac{1}{2}-k\delta_1-(l-k)\delta_2})] + \sum_{k=0}^L \frac{1}{k!(L-k)!} [O_p(N^{1-k\delta_1-(L-k)\delta_2})] \\
&= O_p(N^{\frac{1}{2}}) + \sum_{l=1}^{L-1} O_p(N^{\frac{1}{2}}) \frac{1}{l!} (N^{-\delta_1} + N^{-\delta_2})^l + O_p(N) \frac{1}{L!} (N^{-\delta_1} + N^{-\delta_2})^L \\
&= O_p(N^{\frac{1}{2}}) + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{\frac{1}{2}-l\min(\delta_1, \delta_2)}) + \frac{1}{L!} O_p(N^{1-L\min(\delta_1, \delta_2)}) \\
&= O_p(N^{\frac{1}{2}}) + O_p(N^{\frac{1}{2}}) \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-l\min(\delta_1, \delta_2)}) + O_p(1) = O_p(N^{\frac{1}{2}})
\end{aligned}$$

Then equation (21) becomes

$$\begin{aligned}
Q_N &= \left(\frac{A_0 - iB_0}{2}\right) [O_p(N) + O_p(N^{1-\delta_1}) + O_p(N^{1-2\delta_1})] + \left(\frac{A_0 + iB_0}{2}\right) O_p(N) + O_p(N^{\frac{1}{2}}) \\
&= O_p(N) \left(\frac{A_0 - iB_0}{2}\right)
\end{aligned}$$

Bivariate Taylor series gives 1st term in equation (22) as

$$\begin{aligned}
\sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} &= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) \left[ e^{-i(t\alpha_0 + t^2\beta_0)} \right. \\
&\quad + \sum_{l=1}^{L-1} \frac{(-i(\tilde{\alpha} - \alpha_0)t)^l}{l!} e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{(-i(\tilde{\beta} - \beta_0)t^2)^l}{l!} e^{-i(t\alpha_0 + t^2\beta_0)} \\
&\quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{(i(\tilde{\alpha} - \alpha_0)t)^k (i(\tilde{\beta} - \beta_0)t^2)^{l-k}}{k!(l-k)!} e^{-i(t\alpha_0 + t^2\beta_0)} \\
&\quad \left. + \sum_{k=0}^L \frac{(i(\tilde{\alpha} - \alpha_0)t)^k (i(\tilde{\beta} - \beta_0)t^2)^{L-k}}{k!(L-k)!} e^{-i(t\alpha_0 + t^2\beta_0)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} \\
&\quad + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-l-l\delta_1}) O_p(N^{l+\frac{3}{2}}) + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-2l-l\delta_2}) O_p(N^{2l+\frac{3}{2}}) \\
&\quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{1}{k!(l-k)!} O_p(N^{-k-k\delta_1}) O_p(N^{-2(l-k)-(l-k)\delta_2}) O_p(N^{k+2(l-k)+\frac{3}{2}}) \\
&\quad + \sum_{k=0}^L \frac{1}{k!(L-k)!} O_p(N^{-k-k\delta_1}) O_p(N^{-2(L-k)-(L-k)\delta_2}) O_p(N^{k+2(L-k)+2}) \\
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{1}{l!} [O_p(N^{\frac{3}{2}-l\delta_1}) + O_p(N^{\frac{3}{2}-l\delta_2})] \\
&\quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{1}{k!(l-k)!} [O_p(N^{\frac{3}{2}-k\delta_1-(l-k)\delta_2})] + \sum_{k=0}^L \frac{1}{k!(L-k)!} [O_p(N^{2-k\delta_1-(L-k)\delta_2})] \\
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \sum_{k=0}^l \frac{1}{k!(l-k)!} [O_p(N^{\frac{3}{2}-k\delta_1-(l-k)\delta_2})] \\
&\quad + \sum_{k=0}^L \frac{1}{k!(L-k)!} [O_p(N^{2-k\delta_1-(L-k)\delta_2})] \\
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} O_p(N^{\frac{3}{2}}) \frac{1}{l!} (N^{-\delta_1} + N^{-\delta_2})^l + O_p(N^2) \frac{1}{L!} (N^{-\delta_1} + N^{-\delta_2})^L \\
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{\frac{3}{2}-l\min(\delta_1, \delta_2)}) + \frac{1}{L!} O_p(N^{2-L\min(\delta_1, \delta_2)}) \\
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + O_p(N^{\frac{3}{2}}) \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-l\min(\delta_1, \delta_2)}) + O_p(N) \\
&= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + o_p(N^{\frac{3}{2}})
\end{aligned}$$

Again using Taylor series we get 2nd term in equation (22) as

$$\begin{aligned}
&\sum_{t=1}^N \left(t - \frac{N}{2}\right) e^{i[(\alpha_0 - \tilde{\alpha})t + (\beta_0 - \tilde{\beta})t^2]} \\
&= \sum_{t=1}^N \left(t - \frac{N}{2}\right) \left[ e^{i[t(\alpha_0 - \alpha_0) + t^2(\beta_0 - \beta_0)]} + i(\alpha_0 - \tilde{\alpha})t + i(\beta_0 - \tilde{\beta})t^2 \right. \\
&\quad + \frac{i^2(\alpha_0 - \tilde{\alpha})^2 t^2}{2!} e^{i[t(\alpha_0 - \alpha^*) + t^2(\beta_0 - \beta^*)]} + \frac{i^2(\beta_0 - \tilde{\beta})^2 t^4}{2!} e^{i[t(\alpha_0 - \alpha^*) + t^2(\beta_0 - \beta^*)]} \\
&\quad \left. + \frac{2i^2(\alpha_0 - \tilde{\alpha})(\beta_0 - \tilde{\beta})t^3}{2!} e^{i[t(\alpha_0 - \alpha^*) + t^2(\beta_0 - \beta^*)]} \right]
\end{aligned}$$



$$\begin{aligned}
&= O(N) + i(\alpha_0 - \tilde{\alpha})O(N^3) + i(\alpha_0 - \tilde{\alpha})O_p(N^{-1-\delta_1})O_p(N^4) \\
&\quad + i(\alpha_0 - \tilde{\alpha})(\beta_0 - \tilde{\beta})O(N^5) \\
&\quad + i(\beta_0 - \tilde{\beta})O(N^4) + i(\beta_0 - \tilde{\beta})O_p(N^{-2-\delta_2})O_p(N^6) \\
&= O_p(N) + i(\alpha_0 - \tilde{\alpha})[O_p(N^3) + O_p(N^{3-\delta_1}) + O_p(N^{3-\delta_2})] \\
&\quad + O_p(N^{2-\delta_2}) + O_p(N^{2-2\delta_2}) \\
&= o_p(N^2) + i(\alpha_0 - \tilde{\alpha})[O_p(N^3) + O_p(N^{3-\delta_1}) + O_p(N^{3-\delta_2})]
\end{aligned}$$

For 3rd term in equation (22) we calculate

$$\sum_{t=1}^N \left(t - \frac{N}{2}\right) e^{-i[(\alpha_0 + \tilde{\alpha})t + (\beta_0 + \tilde{\beta})t^2]} = \sum_{t=1}^N \left(t - \frac{N}{2}\right) O_p(1) = O_p(N)$$

Then equation (22) becomes

$$\begin{aligned}
P_N^\alpha &= \left(\frac{A_0 - iB_0}{2}\right) i(\alpha_0 - \tilde{\alpha}) [O_p(N^3) + O_p(N^{3-\delta_1}) + O_p(N^{3-2\delta_1})] + \left(\frac{A_0 + iB_0}{2}\right) O_p(N) \\
&\quad + o_p(N^2) + o_p(N^{\frac{3}{2}}) + \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}
\end{aligned}$$

$$= \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(\alpha_0 t + \beta_0 t^2)} + \left(\frac{A_0 - iB_0}{2}\right) i(\alpha_0 - \tilde{\alpha}) [O_p(N^3) + O_p(N^{3-\delta_1})]$$

$$\tilde{\tilde{\alpha}} = \tilde{\alpha} + \frac{48}{N^2} \text{Im} \left[ \frac{P_N^\alpha}{Q_N} \right]$$

$$\tilde{\tilde{\alpha}} = \tilde{\alpha} +$$

$$\frac{48}{N^2} \text{Im} \left[ \frac{\left(\frac{A_0 - iB_0}{2}\right) i(\alpha_0 - \tilde{\alpha}) [O_p(N^3) + O_p(N^{3-\delta_1})] + \sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}}{O_p(N) \left(\frac{A_0 - iB_0}{2}\right)} \right]$$

$$\tilde{\tilde{\alpha}} = \tilde{\alpha} + (\alpha_0 - \tilde{\alpha}) + (\alpha_0 - \tilde{\alpha}) O_p(N^{-\delta_1}) + \frac{48}{N^2} \text{Im} \left[ \frac{\sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}}{O_p(N) \left(\frac{A_0 - iB_0}{2}\right)} \right]$$

Now using Lemma 1 the variance of

$$Z_1 = \frac{48}{N^{\frac{1}{2}}} \text{Im} \left[ \frac{\sum_{t=1}^N X(t) \left(t - \frac{N}{2}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}}{O_p(N) \left(\frac{A_0 - iB_0}{2}\right)} \right]$$

is asymptotically same as asymptotic variance of LSE of  $\alpha_0$ . So  $\tilde{\alpha} - \alpha_0 = O_p(N^{-(1-\delta_1)})$  where  $\delta_1 \in (0, \frac{1}{2})$  then  $\tilde{\alpha} - \alpha_0 = O_p(N^{-(1-2\delta_1)})$  if  $\delta_1 \leq \frac{1}{4}$   $N^{\frac{3}{2}}(\tilde{\alpha} - \alpha_0) \rightarrow N(0, \sigma_1^2)$  if  $\delta_1 > \frac{1}{4}$  by CLT of stochastic process in Fuller [4] where  $\sigma_1^2 = \frac{384\sigma^2c}{A_0^2+B_0^2}$  is asymptotic variance of LSE of  $\alpha_0$  where  $c = \sum_{j=-\infty}^{\infty} a(j)^2$ . ■

PROOF OF THEOREM 2:

(23)

$$\begin{aligned} P_N^\beta &= \sum_{t=1}^N y(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} \\ &= \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} + \sum_{t=1}^N \left(\frac{A_0 - iB_0}{2}\right) \left(t^2 - \frac{N^2}{3}\right) e^{i[(\alpha_0 - \tilde{\alpha})t + (\beta_0 - \tilde{\beta})t^2]} \\ &\quad + \sum_{t=1}^N \left(\frac{A_0 + iB_0}{2}\right) \left(t^2 - \frac{N^2}{3}\right) e^{-i[(\alpha_0 + \tilde{\alpha})t + (\beta_0 + \tilde{\beta})t^2]} \end{aligned}$$

Taylor series expansion gives 2nd term in equation (23) as

$$\begin{aligned} &\sum_{t=1}^N \left(t^2 - \frac{N^2}{3}\right) e^{i[(\alpha_0 - \tilde{\alpha})t + (\beta_0 - \tilde{\beta})t^2]} \\ &= \sum_{t=1}^N \left(t^2 - \frac{N^2}{3}\right) \left[ e^{i[t(\alpha_0 - \alpha_0) + t^2(\beta_0 - \beta_0)]} + i(\alpha_0 - \tilde{\alpha})t + i(\beta_0 - \tilde{\beta})t^2 \right. \\ &\quad \left. + \frac{i^2(\alpha_0 - \tilde{\alpha})^2 t^2}{2!} e^{i[t(\alpha_0 - \alpha_0) + t^2(\beta_0 - \beta_0)]} + \frac{i^2(\beta_0 - \tilde{\beta})^2 t^4}{2!} e^{i[t(\alpha_0 - \alpha_0) + t^2(\beta_0 - \beta_0)]} \right. \\ &\quad \left. + \frac{2i^2(\alpha_0 - \tilde{\alpha})(\beta_0 - \tilde{\beta})t^3}{2!} e^{i[t(\alpha_0 - \alpha_0) + t^2(\beta_0 - \beta_0)]} \right] \\ &= O(N^2) + i(\beta_0 - \tilde{\beta})O(N^5) + i(\beta_0 - \tilde{\beta})O_p(N^{-2-\delta_2})O_p(N^7) + i(\alpha_0 - \tilde{\alpha})(\beta_0 - \tilde{\beta})O(N^6) \\ &\quad + i(\alpha_0 - \tilde{\alpha})O(N^4) + i(\alpha_0 - \tilde{\alpha})O_p(N^{-1-\delta_1})O_p(N^5) \\ &= O_p(N^2) + i(\beta_0 - \tilde{\beta})[O_p(N^5) + O_p(N^{5-\delta_2}) + O_p(N^{5-\delta_1})] \\ &\quad + O_p(N^{3-\delta_1}) + O_p(N^{3-2\delta_1}) \\ &= o_p(N^3) + i(\beta_0 - \tilde{\beta})[O_p(N^5) + O_p(N^{5-\delta_2}) + O_p(N^{5-\delta_1})] \end{aligned}$$

Expanding using bivariate Taylor series 1st term in equation (23) becomes

$$\begin{aligned}
& \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\tilde{\alpha}t + \tilde{\beta}t^2)} = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) \left[ e^{-i(t\alpha_0 + t^2\beta_0)} \right. \\
& \quad + \sum_{l=1}^{L-1} \frac{(-i(\tilde{\alpha} - \alpha_0)t)^l}{l!} e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{(-i(\tilde{\beta} - \beta_0)t^2)^l}{l!} e^{-i(t\alpha_0 + t^2\beta_0)} \\
& \quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{(i(\tilde{\alpha} - \alpha_0)t)^k (i(\tilde{\beta} - \beta_0)t^2)^{l-k}}{k!(l-k)!} e^{-i(t\alpha_0 + t^2\beta_0)} \\
& \quad \left. + \sum_{k=0}^L \frac{(i(\tilde{\alpha} - \alpha_0)t)^k (i(\tilde{\beta} - \beta_0)t^2)^{L-k}}{k!(L-k)!} e^{-i(t\alpha_0 + t^2\beta_0)} \right] \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} \\
& \quad + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-l-l\delta_1}) O_p(N^{l+\frac{5}{2}}) + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-2l-l\delta_2}) O_p(N^{2l+\frac{5}{2}}) \\
& \quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{1}{k!(l-k)!} O_p(N^{-k-k\delta_1}) O_p(N^{-2(l-k)-(l-k)\delta_2}) O_p(N^{k+2(l-k)+\frac{5}{2}}) \\
& \quad + \sum_{k=0}^L \frac{1}{k!(L-k)!} O_p(N^{-k-k\delta_1}) O_p(N^{-2(L-k)-(L-k)\delta_2}) O_p(N^{k+2(L-k)+3}) \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{1}{l!} [O_p(N^{\frac{5}{2}-l\delta_1}) + O_p(N^{\frac{5}{2}-l\delta_2})] \\
& \quad + \sum_{l=1}^{L-1} \sum_{k=1}^l \frac{1}{k!(l-k)!} [O_p(N^{\frac{5}{2}-k\delta_1-(l-k)\delta_2})] + \sum_{k=0}^L \frac{1}{k!(L-k)!} [O_p(N^{3-k\delta_1-(L-k)\delta_2})] \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \sum_{k=0}^l \frac{1}{k!(l-k)!} [O_p(N^{\frac{5}{2}-k\delta_1-(l-k)\delta_2})] \\
& \quad + \sum_{k=0}^L \frac{1}{k!(L-k)!} [O_p(N^{3-k\delta_1-(L-k)\delta_2})] \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} O_p(N^{\frac{5}{2}}) \frac{1}{l!} (N^{-\delta_1} + N^{-\delta_2})^l + O_p(N^2) \frac{1}{L!} (N^{-\delta_1} + N^{-\delta_2})^L \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{\frac{5}{2}-l\min(\delta_1, \delta_2)}) + \frac{1}{L!} O_p(N^{2-L\min(\delta_1, \delta_2)}) \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + O_p(N^{\frac{5}{2}}) \sum_{l=1}^{L-1} \frac{1}{l!} O_p(N^{-l\min(\delta_1, \delta_2)}) + O_p(N) \\
& = \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(t\alpha_0 + t^2\beta_0)} + o_p(N^{\frac{5}{2}})
\end{aligned}$$

To get 3rd term in equation (23) we calculate

$$\sum_{t=1}^N \left(t^2 - \frac{N^2}{3}\right) e^{-i[(\alpha_0 + \tilde{\alpha})t + (\beta_0 + \tilde{\beta})t^2]} = \sum_{t=1}^N \left(t^2 - \frac{N^2}{3}\right) O_p(1) = O_p(N^2)$$

Then equation (23) becomes

$$P_N^\beta = \left(\frac{A_0 - iB_0}{2}\right) i(\beta_0 - \tilde{\beta}) [O_p(N^5) + O_p(N^{5-\delta_2}) + O_p(N^{5-2\delta_2})] + \left(\frac{A_0 + iB_0}{2}\right) O_p(N^2)$$

$$+ o_p(N^3) + o_p(N^{\frac{5}{2}}) + \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}$$

$$= \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\alpha_0 t + \beta_0 t^2)} + \left(\frac{A_0 - iB_0}{2}\right) i(\beta_0 - \tilde{\beta}) [O_p(N^5) + O_p(N^{5-\delta_2})]$$

$$\tilde{\tilde{\beta}} = \tilde{\beta} + \frac{45}{N^4} \text{Im} \left[ \frac{P_N^\beta}{Q_N} \right]$$

$$\tilde{\tilde{\beta}} = \tilde{\beta} +$$

$$\frac{45}{N^4} \text{Im} \left[ \frac{\left(\frac{A_0 - iB_0}{2}\right) i(\beta_0 - \tilde{\beta}) [O_p(N^5) + O_p(N^{5-\delta_2})] + \sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}}{O_p(N) \left(\frac{A_0 - iB_0}{2}\right)} \right]$$

$$\tilde{\tilde{\beta}} = \tilde{\beta} + (\beta_0 - \tilde{\beta}) + (\beta_0 - \tilde{\beta}) O_p(N^{-\delta_2}) + \frac{45}{N^4} \text{Im} \left[ \frac{\sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}}{O_p(N) \left(\frac{A_0 - iB_0}{2}\right)} \right]$$

Now using Lemma 1 variance of

$$Z_2 = \frac{45}{N^{\frac{3}{2}}} \text{Im} \left[ \frac{\sum_{t=1}^N X(t) \left(t^2 - \frac{N^2}{3}\right) e^{-i(\alpha_0 t + \beta_0 t^2)}}{O_p(N) \left(\frac{A_0 - iB_0}{2}\right)} \right]$$

is asymptotically same as asymptotic variance of LSE of  $\beta_0$ . . So  $\tilde{\tilde{\beta}} - \beta_0 = O_p(N^{(-2-\delta_2)})$

where  $\delta_2 \in (0, \frac{1}{2})$  then  $\tilde{\tilde{\beta}} - \beta_0 = O_p(N^{(-2-2\delta_2)})$  if  $\delta_2 \leq \frac{1}{4}$   $N^{\frac{5}{2}}(\tilde{\tilde{\beta}} - \beta_0) \rightarrow N(0, \sigma_2^2)$  if

$\delta_2 > \frac{1}{4}$  by CLT of stochastic process in Fuller [4] where  $\sigma_2^2 = \frac{360\sigma_c^2 c}{A_0^2 + B_0^2}$  is the asymptotic

variance of the LSE of  $\beta_0$ . ■

PROOF OF THEOREM 3: For covariance of  $Z_1, Z_2$  it is enough to find the quantities  $g_k^1$  or  $g_k^2$  which are given below.

$$\begin{aligned}
g_k^1 &= \lim_{N \rightarrow \infty} \frac{1}{N^4} \sum_{n=1}^{N-|k|} \frac{90 \times 96}{(A^{0^2} + B^{0^2})^2} \left( n^2 - \frac{N^2}{3} \right) \left( n + k - \frac{N}{2} \right) \\
&\quad \left( B^0 \cos(\alpha_0 n + \beta_0 n^2) - A^0 \sin(\alpha_0 n + \beta_0 n^2) \right) \\
&\quad \times \left( B^0 \cos(\alpha_0(n+k) + \beta_0(n+k)^2) - A^0 \sin(\alpha_0(n+k) + \beta_0(n+k)^2) \right) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^5} \sum_{n=1}^{N-|k|} \frac{90^2}{(A^{0^2} + B^{0^2})^2} \left( B^0 \cos(\alpha_0 n + \beta_0 n^2) - A^0 \sin(\alpha_0 n + \beta_0 n^2) \right) \\
&\quad \times \left( B^0 \cos(\alpha_0(n+k) + \beta_0(n+k)^2) - A^0 \sin(\alpha_0(n+k) + \beta_0(n+k)^2) \right) \\
&\quad \left[ n^3 - n^2 \left( \frac{N}{2} - k \right) + n \left( -\frac{N^2}{3} \right) + \left( \frac{N^3}{6} - \frac{N^2 k}{3} \right) \right]
\end{aligned}$$

and

$$\begin{aligned}
g_k^2 &= \lim_{N \rightarrow \infty} \frac{1}{N^4} \sum_{n=1}^{N-|k|} \frac{90 \times 96}{(A^{0^2} + B^{0^2})^2} \left( n - \frac{N}{2} \right) \left( (n+k)^2 - \frac{N^2}{3} \right) \\
&\quad \left( B^0 \cos(\alpha_0 n + \beta_0 n^2) - A^0 \sin(\alpha_0 n + \beta_0 n^2) \right) \\
&\quad \times \left( B^0 \cos(\alpha_0(n+k) + \beta_0(n+k)^2) - A^0 \sin(\alpha_0(n+k) + \beta_0(n+k)^2) \right) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^5} \sum_{n=1}^{N-|k|} \frac{90^2}{(A^{0^2} + B^{0^2})^2} \left( B^0 \cos(\alpha_0 n + \beta_0 n^2) - A^0 \sin(\alpha_0 n + \beta_0 n^2) \right) \\
&\quad \times \left( B^0 \cos(\alpha_0(n+k) + \beta_0(n+k)^2) - A^0 \sin(\alpha_0(n+k) + \beta_0(n+k)^2) \right) \\
&\quad \left[ n^3 - n^2 \left( \frac{N}{2} + k \right) + n \left( k^2 + Nk - \frac{N^2}{3} \right) + \left( \frac{N^3}{6} - \frac{Nk^2}{2} \right) \right]
\end{aligned}$$

and both are  $\frac{360c\sigma^2}{A^{0^2} + B^{0^2}}$  for  $k = 0$  using (20) and 0 otherwise. So, the asymptotic covariance of  $Z_1, Z_2$  has the same order as the asymptotic covariance of the LSEs, see Kundu and Nandi [8]. ■

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