Estimating the Parameters of Multiple Chirp Signals

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Abstract

Chirp signals occur naturally in different areas of signal processing. Recently, Kundu and Nandi [6] considered the least squares estimators of the unknown parameters of a chirp signal model and established their consistency and asymptotic normality properties. It is observed that the dispersion matrix of the asymptotic distribution of the least squares estimators is quite complicated. The aim of this paper is twofold. First, using a number theoretic result of Vinogradov [12], we present a simplified form of the above mentioned dispersion matrix. Secondly, using the orthogonal structure of the different chirp components, we propose a step by step sequential estimation procedure of the unknown parameters of the model. Under the proposed sequential procedure, the problem of estimation of the parameters of a multiple chirp signal model reduces to solving only a two dimensional optimization problem at each step. It is observed that the estimators obtained by the proposed method are strongly consistent. Due to the complicated nature of the model, we could not establish the asymptotic distribution of the proposed sequential estimators. We perform some simulation experiments to compare the performance of the proposed and least squares estimators for small sample sizes, and for different parameter values. It is observed that the mean squared errors of the proposed estimators are very close to the corresponding mean squared errors of the least squares estimators. Two real data sets have been analyzed for illustrative purposes.

Key Words and Phrases: Chirp signals; least squares estimators; strong consistency, asymptotic distribution; linear process.

AMS Subject Classifications: 60F05, 62P30, 62F10

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1 Introduction

In this paper we consider the following multiple chirp model;

$$y(n) = \sum_{k=1}^{p} \left\{ A_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) \right\} + X(n), \quad n = 1, \cdots, N. \quad (1)$$

Here $y(n)$ is the real valued signal observed at $n = 1, \cdots, N$; A_k^0, B_k^0 are amplitudes, and α_k^0 and β_k^0 are frequency and frequency rate, respectively for $k = 1, \cdots, p$. The additive error $\{X(n)\}$ is a sequence of stationary random variables with mean zero and finite fourth moment. The explicit assumptions on $X(n)$ will be provided later.

The model (1) is popularly known as the 'chirp signal model' in the signal processing literature, and occurs naturally in various areas of science and engineering, particularly in physics, sonar, radar and communications. Chirp signal models have been used extensively to measure the distance of a moving object, emitting chirp signal, from a fixed receiver. This phenomena is used by different animal species, like bats, whales, to detect their prey. Shrill voices of birds are also examples of chirp signal. In the class of periodic models wherein the frequency varies with time, chirp signal model is the simplest possible model. In other words, chirp signal has time dependent frequency or can incorporate frequency modulation, in electrical engineering terminology. Extensive work on chirp signal model, mainly when $p = 1$, has been carried out by several authors, see for example, Abatzoglou [1], Kumaresan and Verma [5], Djuric and Kay [2], Gini *et al.* [4], Nandi and Kundu [8], Kundu and Nandi [6] and the references cited therein.

Saha and Kay [11] first introduced the multiple chirp signal model (1), and provided the maximum likelihood estimators of the unknown parameters using importance sampling procedure under the assumptions that $X(n)$s are independent and identically distributed (*i.i.d.*) normal random variables. Kundu and Nandi [6] studied the properties of the least squares estimators (LSEs) of the model (1), and proved the strong consistency and asymptotic normality when $X(n)$s are obtained from a linear stationary process. However, the structure of the dispersion matrix of the asymptotic distribution of the LSEs in Kundu and Nandi [6], is quite complicated.

It is interesting to observe that the LSE of α_k^0 has the convergence rate $O_p(N^{-3/2})$, whereas the LSE of β_k^0 has the convergence rate $O_p(N^{-5/2})$. But it is also observed that, if $p \geq 2$, finding the LSEs is a numerically challenging problem. For the model (1), it involves solving a $2p$-dimensional optimization problem. Therefore, for large p, it becomes a highly computationally intensive procedure.

The aim of this paper is twofold. First, using a number theoretic result of Vinogradov [12] we provide a simplified structure of the dispersion matrix of the asymptotic distribution of the LSEs. The second aim of this paper is to provide an estimation procedure which is computationally less demanding and produces estimators of the unknown parameters, which behave in a manner very similar to the LSEs. If p is known, using the fact that the regressor vectors are orthogonal, we provide a step by step sequential estimation procedure for estimation of the amplitudes, frequencies and frequency rates. It is observed that $2p$-dimensional optimization procedure can be reduced to p sequential 2-dimensional (2-D) optimization problems. Therefore, if p is large, the proposed sequential procedure can be very effective.

It is observed that if p is not known, and we fit a lower order model, *i.e.* when the assumed number of components is less than the actual number of components, then the proposed estimators converge almost surely to the corresponding true parameter values. If we fit a higher order model, *i.e.* assumed number of components is more than the actual number of components, then the amplitude estimates obtained after the p-th step converge to zero almost surely.

Due to the complicated nature of the model, we could not establish the asymptotic distribution of the proposed sequential estimators. However, based on a conjecture in number theory, it can be shown that the asymptotic distributions of the LSEs and the proposed sequential estimators are the same. We perform some simulation experiments to study the behavior of the proposed estimators, and compare their performances with the LSEs. It is observed that the mean squared errors (MSEs) of the LSEs and sequential estimators are very close to each other. Finally, we provide the analysis of two real data sets for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we mainly state the model assumptions and preliminary results. Simplified form of the dispersion matrix of the asymptotic distribution of the LSEs is presented in Section 3. The sequential estimators are provided in Section 4. Numerical results and the analysis of two data sets are presented in Section 5, and finally we conclude the paper in Section 6. All the proofs are supplied in the Appendices.

2 Model Assumptions and Preliminary Results

2.1 Model Assumptions

We make the following assumptions on the error random variables.

Assumption 1: The error random variable $X(n)$ satisfies the following condition;

$$X(n) = \sum_{j=-\infty}^{\infty} a(j)e(n-j), \tag{2}$$

where $\{e(n)\}$ is a sequence of *i.i.d.* random variables with mean zero,variance σ^2, finite fourth moment, and

$$\sum_{j=-\infty}^{\infty} |a(j)| < \infty. \tag{3}$$

It may be mentioned that Assumption 1 is a standard assumption for stationary linear process, and any finite dimensional stationary AR, MA or ARMA process can be represented as (2), when $a(j)$s satisfy (3).

We use the following notations. The parameter vector of the k-th component is represented as $\theta_k = (A_k, B_k, \alpha_k, \beta_k)$, the true parameter vector as $\theta_k^0 = (A_k^0, B_k^0, \alpha_k^0, \beta_k^0)$, for $k = 1, \cdots, p$, and the parameter space as $\Theta = [-M, M] \times [-M, M] \times [0, \pi] \times [0, \pi]$; where $M > 0$ is a real number.

Assumption 2: It is assumed that θ_k^0 is an interior point of Θ, α_k^0s are distinct, and β_k^0s are also distinct for $k = 1, \cdots, p$.

Assumption 3: A_k^0s and B_k^0s satisfy the following relation;

$$M^2 > A_1^{0^2} + B_1^{0^2} > \cdots > A_p^{0^2} + B_p^{0^2} > 0.$$

2.2 Preliminary Results

We will be requiring the following results for further development of the asymptotic theory.

Lemma 1: If $(\theta_1, \theta_2) \in (0, \pi) \times (0, \pi)$, and θ_2 is an irrational number, then the following results hold.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \cos(\theta_1 n + \theta_2 n^2) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sin(\theta_1 n + \theta_2 n^2) = 0. \tag{4}$$

$$\lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N} n^t \cos^2(\theta_1 n + \theta_2 n^2) = \frac{1}{2(t+1)}, \tag{5}$$

$$\lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N} n^t \sin^2(\theta_1 n + \theta_2 n^2) = \frac{1}{2(t+1)}, \tag{6}$$

$$\lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N} n^t \sin(\theta_1 n + \theta_2 n^2) \cos(\theta_1 n + \theta_2 n^2) = 0, \tag{7}$$

where $t = 0, 1, 2, 3, 4, \cdots$.

It may be noted that (4) is one of the crucial results which has been used in this paper and it can be obtained from Vinogradov's [12] result. We also note that although the results are not true for rationals but given any rational there exists infinitely many N's for which the difference between the left hand sides (without $\lim_{N\to\infty}$) and right hand sides of (4), (5), (6), (7) can be made arbitrarily small.

Proof: See in the Appendix A.

Lemma 2: If $X(n)$ satisfies Assumptions 1-3 then as $N \to \infty$ and for $s \geq 0$.

$$\sup_{\alpha,\beta} \left| \frac{1}{N^{s+1}} \sum_{n=1}^{N} n^s X(n) e^{i(\alpha n + \beta n^2)} \right| \to 0 \;\; a.s. \tag{8}$$

Proof: The proof can be obtained along the same line as the proof of Lemma 1 of Kundu and Nandi [6].

3 Asymptotic Distribution of the LSEs

In this section, first we provide the asymptotic distribution of the LSEs, as obtained in Kundu and Nandi [6].

Theorem 1: Suppose, $\widetilde{\theta}_k = (\widetilde{A}_k, \widetilde{B}_k, \widetilde{\alpha}_k, \widetilde{\beta}_k)$ is the LSE of $\theta_k^0 = (A_k^0, B_k^0, \alpha_k^0, \beta_k^0)$, for $k = 1, \cdots, p$. Then under the Assumptions 1-3,

$$((\tilde{\theta}_1 - \theta_1^0)D^{-1}, \cdots, (\tilde{\theta}_p - \theta_p^0)D^{-1}) \xrightarrow{d} N_{4p}(0, 2\sigma^2 \Lambda^{-1}(\theta^0) H(\theta^0) \Lambda^{-1}(\theta^0)), \tag{9}$$

where, $D = diag(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{1}{N\sqrt{N}}, \frac{1}{N^2\sqrt{N}})$ is a diagonal matrix of order 4×4, $\xrightarrow{d}$ means converges in distribution, $\theta^0 = (\theta_1^0, \cdots, \theta_p^0)$, $N_{4p}(0, \Sigma)$ means a $4p$ variate normal distribution with mean vector 0 and the dispersion matrix Σ. The matrices $\Lambda(\theta^0)$ and $H(\theta^0)$ have the following structures;

$$\Lambda(\theta^0) = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1p} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \Lambda_{p1} & \Lambda_{p2} & \cdots & \Lambda_{pp} \end{bmatrix} \quad \text{and} \quad H(\theta^0) = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1p} \\ H_{21} & H_{22} & \cdots & H_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ H_{p1} & H_{p2} & \cdots & H_{pp} \end{bmatrix},$$

where $\Lambda_{jk} \equiv \Lambda_{jk}(\theta_j^0, \theta_k^0) = ((\lambda_{rs}))$ and $H_{jk} \equiv H_{jk}(\theta_j^0, \theta_k^0) = ((h_{rs}))$ are 4×4 square matrices. For the explicit expression of the different entries of $((\lambda_{rs}))$, $((h_{rs}))$, the readers are referred to the original article of Kundu and Nandi [6]. Using Lemma 1 and Lemma 2, these terms get simplified and the following version of the asymptotic distribution of the LSEs can be obtained.

THEOREM 2: If the Assumptions 1-3 are satisfied, then the LSEs of the unknown parameters have the following asymptotic distribution

$$((\tilde{\theta}_1 - \theta_1^0)D^{-1}, \cdots, (\tilde{\theta}_p - \theta_p^0)D^{-1}) \xrightarrow{d} N_{4p}(0, 2c\sigma^2\Sigma(\theta^0)). \tag{10}$$

Here, $\Sigma(\theta^0)$ is a $4p \times 4p$ matrix with the following block-diagonal structure

$$\Sigma(\theta^0) = \begin{bmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Sigma_p \end{bmatrix},$$

for $k = 1, \cdots, p$,

$$\Sigma_k = \frac{1}{A_k^{0^2} + B_k^{0^2}} \begin{bmatrix} \frac{1}{2}\left(A_k^{0^2} + 9B_k^{0^2}\right) & -4A_k^0B_k^0 & 18B_k^0 & -15B_k^0 \\ -4A_k^0B_k^0 & \frac{1}{2}\left(9A_k^{0^2} + B_k^{0^2}\right) & -18A_k^0 & 15A_k^0 \\ 18B_k^0 & -18A_k^0 & 96 & -90 \\ -15B_k^0 & 15A_k^0 & -90 & 90 \end{bmatrix}, \tag{11}$$

and $c = \sum_{j=-\infty}^{\infty} a(j)^2$.

PROOF: See in the Appendix B.

4 Sequential Estimation Procedure

In this section, we propose a sequential procedure to estimate the unknown parameters of the model (1), and prove that the estimators are strongly consistent. Let us use the following notations. The $N \times 2$ matrix, $\mathbf{W}(\alpha, \beta)$ is defined as follows;

$$\mathbf{W}(\alpha, \beta) = \begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ \cos(2\alpha + 4\beta) & \sin(2\alpha + 4\beta) \\ \vdots & \vdots \\ \cos(N\alpha + N^2\beta) & \sin(N\alpha + N^2\beta) \end{bmatrix}. \tag{12}$$

We first minimize $Q_1(A, B, \alpha, \beta)$ with respect to A, B, α, β, where

$$Q_1(A, B, \alpha, \beta) = \left[\mathbf{Y} - \mathbf{W}(\alpha, \beta)\begin{bmatrix} A \\ B \end{bmatrix}\right]^T \left[\mathbf{Y} - \mathbf{W}(\alpha, \beta)\begin{bmatrix} A \\ B \end{bmatrix}\right], \tag{13}$$

and $\mathbf{Y} = [y(1), \cdots, y(N)]^T$ is the data vector. For a fixed α and β,

$$(\widehat{A}(\alpha,\beta), \widehat{B}(\alpha,\beta)) = [\mathbf{W}(\alpha,\beta)^T\mathbf{W}(\alpha,\beta)]^{-1}\mathbf{W}(\alpha,\beta)^T\mathbf{Y} \tag{14}$$

minimizes (13) with respect to A and B. Therefore, using the separable regression technique of Richards [10], the minimization of $Q_1(A, B, \alpha, \beta)$ can be obtained by first minimizing

$$R_1(\alpha,\beta) = Q_1(\widehat{A}(\alpha,\beta), \widehat{B}(\alpha,\beta), \alpha, \beta) = \mathbf{Y}^T(\mathbf{I} - \mathbf{P_1}(\alpha,\beta))\mathbf{Y} \tag{15}$$

with respect to (α, β), where $\mathbf{P_1}(\alpha,\beta) = \mathbf{W}(\alpha,\beta)\left[\mathbf{W}(\alpha,\beta)^T\mathbf{W}(\alpha,\beta)\right]^{-1}\mathbf{W}(\alpha,\beta)^T$ is the projection matrix on the column space of the matrix $\mathbf{W}(\alpha,\beta)$. If $(\widehat{\alpha}_1, \widehat{\beta}_1)$ minimizes $R_1(\alpha,\beta)$, then the estimators of $(A_1^0, B_1^0, \alpha_1^0, \beta_1^0)$, which are unique minimizer of (13), become $\widehat{A}_1 = \widehat{A}(\widehat{\alpha}_1, \widehat{\beta}_1), \widehat{B}_1 = \widehat{B}(\widehat{\alpha}_1, \widehat{\beta}_1), \widehat{\alpha}_1, \widehat{\beta}_1$, respectively.

To compute the estimators of $(A_2^0, B_2^0, \alpha_2^0, \beta_2^0)$, we take out the effect of the first component from the signal, *i.e.*, we consider a new data vector

$$\mathbf{Y}^1 = \mathbf{Y} - \mathbf{W}(\hat{\alpha}_1, \hat{\beta}_1)\begin{bmatrix}\widehat{A}_1\\ \\ \widehat{B}_1\end{bmatrix}. \tag{16}$$

Using the new data vector $\mathbf{Y}^1$, following the same procedure as before we get $\widehat{A}_2, \widehat{B}_2, \widehat{\alpha}_2, \widehat{\beta}_2$, the estimators of $A_2^0, B_2^0, \alpha_2^0, \beta_2^0$, respectively. Continuing in this manner, at the k-th stage we can obtain estimators of $A_k^0, B_k^0, \alpha_k^0, \beta_k^0$, say $\widehat{A}_k, \widehat{B}_k, \widehat{\alpha}_k, \widehat{\beta}_k$, respectively. The practical implementation aspects of the sequential procedure, like the choice of optimization algorithm and finding initial values, are discussed in the simulation section.

Finally, we will provide the consistency results of the proposed estimators. We consider two cases separately; (i) the number of components of the fitted model (1) is less than or equal to the actual number of components and (ii) it is more than the actual number. We have the following results:

THEOREM 3: If the Assumptions 1-3 are satisfied then $(\widehat{A}_1, \widehat{B}_1, \widehat{\alpha}_1, \widehat{\beta}_1)$ is a strongly consistent estimator of $(A_1^0, B_1^0, \alpha_1^0, \beta_1^0)$.

Proof: See in the Appendix C.

We next show that the estimators obtained at the second step also are strongly consistent.

Theorem 4: If the Assumptions 1-3 are satisfied and $p \geq 2$, then $\widehat{\theta}_2$, the estimator obtained by minimizing $Q_2(A, B, \alpha, \beta)$, where $Q_2(A, B, \alpha, \beta)$ is obtained by replacing $\mathbf{Y}$ with $\mathbf{Y}^1$ in (13), is a strongly consistent estimator of θ_2^0.

Proof: See in the Appendix D.

Theorem 5: If the Assumptions 1-3 are satisfied and $p \geq k$, then the estimators obtained at the k-th step are strongly consistent.

Proof: See in the Appendix D.

It is interesting to investigate the properties of the estimators for the case when the sequential process is continued beyond the p-th step. We have the following result.

Theorem 6: If the Assumptions 1-3 are satisfied, and if $\widehat{A}_k, \widehat{B}_k, \widehat{\alpha}_k, \widehat{\beta}_k$ are the estimators obtained at the k-th step for $k > p$, then $\widehat{A}_k \rightarrow 0 \ a.s.$ and $\widehat{B}_k \rightarrow 0 \ a.s.$

Proof: See in the Appendix E.

Comments: If the following conjecture, see Montgomery [7], holds, then it can be shown that the proposed sequential estimators have the same asymptotic distribution as that of the ordinary LSEs.

Conjecture: If $\theta_1, \theta_2, \theta_1', \theta_2' \in (0, \pi)$, then except for countable number of points

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N} N^t} \sum_{n=1}^{N} n^t \cos(\theta_1 n + \theta_2 n^2) \sin(\theta_1' n + \theta_2' n^2) = 0; \quad t = 0, 1, 2. \tag{17}$$

In addition if $\theta_2 \neq \theta_2'$, then

$$\lim_{N \rightarrow \infty} \frac{1}{\sqrt{N} N^t} \sum_{n=1}^{N} n^t \cos(\theta_1 n + \theta_2 n^2) \cos(\theta_1' n + \theta_2' n^2) = 0; \quad t = 0, 1, 2. \tag{18}$$

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}N^t}\sum_{n=1}^{N} n^t \sin(\theta_1 n+\theta_2 n^2)\sin(\theta_1' n+\theta_2' n^2)=0; \quad t=0,1,2. \qquad (19)$$

5 Numerical Results and Data Analysis

5.1 Numerical Results

It has already been observed in Kundu and Nandi [6] that the performances of the LSEs are quite satisfactory. In this section, we perform some simulation experiments to see how the proposed sequential estimators behave as compared to the LSEs in terms of biases and MSEs. We consider the following model: $p=2$, $A_1=2.0, B_1=2.0, \alpha_1=1.75, \beta_1=1.05, A_2=1.0, B_2=1.0, \alpha_2=0.8, \beta_2=1.2$. $X(n)$s are assumed to be *i.i.d.* Gaussian random variables with mean 0 and variance σ^2. We have taken three different sample sizes, 150, 180, 210, and two different σ^2, 0.5 and 1.5 for our simulations.

First, we consider the LSEs, and compute the average estimates and the MSEs over 1000 independent replications. For comparison purposes we report the corresponding asymptotic variances also. Next, we consider the proposed sequential method over the same simulated data sets. In both the cases we have used the two dimensional periodogram like function

$$\frac{1}{N}\left|\sum_{n=1}^{N} y(n)e^{-i(\alpha n+\beta n^2)}\right|^2,$$

as suggested by Kundu and Nandi [6], for initial guesses. In case of LSEs, we need four initial guesses, whereas in case of sequential estimators we need two initial guesses in each step. In both the cases we have used Downhill Simplex Algorithm (see for example Press *et al.* [9]) for minimization purposes.

Note that although theoretically, only irrationals are allowed as frequency rates to achieve convergence, but in practice if true frequency rate is rational then there are infinitely many N's to satisfy the limits in (4), (5), (6), (7), so there is no contradic-

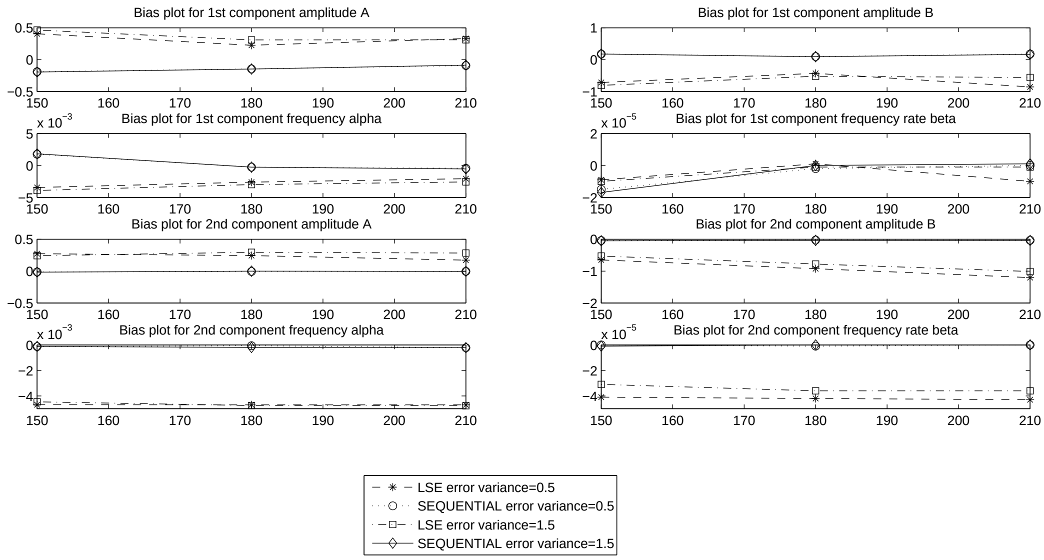

Figure 1: Bias plot against sample size

tion between theory and practical simulation implementation. In case of sequential method, at the first step we obtain the estimates of all the four parameters of the dominating component. At the second step, we remove the effect of the first component, and obtain the estimates of the parameters of the second component. We replicate the process 1000 times, and obtain the average estimates and MSEs. We have provided the bias and MSE plots of eight different parameters present in the two component model for different sample sizes, different error variances and for two different methods in Figure 1 and Figure 2. We have not presented here all the numerical values for paucity of space, but they can be obtained on request from the corresponding author.

It is observed that in case of small sample simulations, sequential estimators perform better than the LSEs. The performance of the LSEs improve for higher sample sizes, see Kundu and Nandi [6], but it is tremendously time consuming due to the requirement of solving a $2p$-dimensional optimization problem. In the present case of $p = 2$, we need to solve a four dimensional optimization problem for finding the LSEs and only a two dimensional optimization problem for finding the sequential estima-

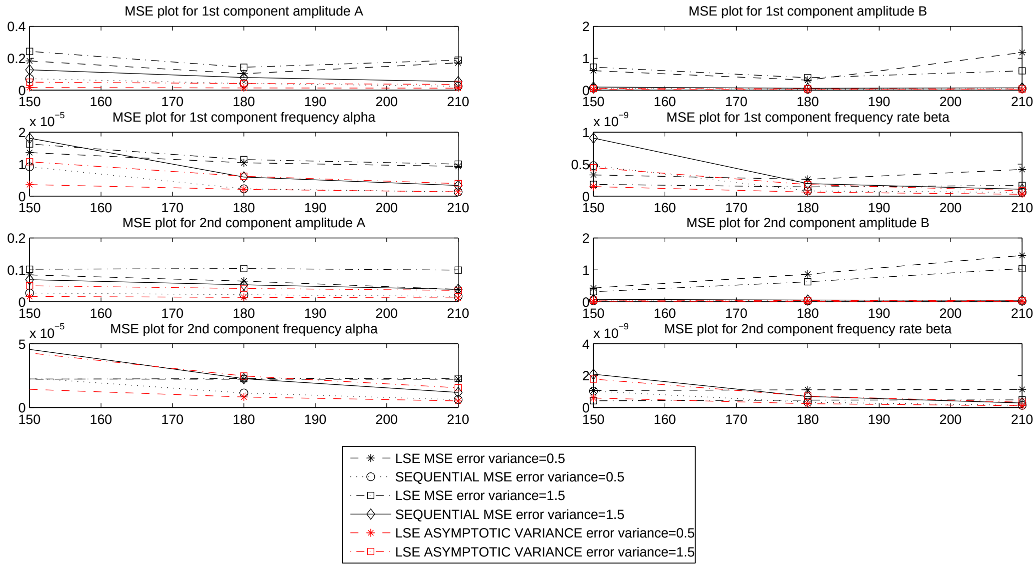

Figure 2: MSE plot against sample size

tors. It is observed from the simulation results that the MSEs of the LSEs are quite comparable with their asymptotic variances. It verifies the accuracy of the proposed simplified asymptotic variances of the LSEs. Moreover, the biases are also small, which verifies the consistency properties of the LSEs. Along the same line, we compare the MSEs of sequential estimators with asymptotic variance of LSEs. Although we do not have explicit expressions of the asymptotic variances of the sequential estimators but MSEs of sequential estimators are comparable with the asymptotic variances of LSEs. From the graphs we observe that, sequential method is getting stabilized earlier (with respect to smaller sample sizes) than the LSEs. We also observe that the MSEs of the sequential estimators not only decrease as sample size increases, but are also having parity with the corresponding asymptotic variances of LSEs, whereas LSEs are not performing that well. This observation highlights the potential advantage of using sequential estimators. The performances of both methods improve (in terms of MSEs) as error variance decreases. The fact that, the MSEs of the sequential estimators for the first two components are very close to the corresponding asymptotic variances of the LSEs, indicates that the sequential estimators might have the same rate of convergence as that of LSEs.

5.2 Real Data Analysis

In this section, we perform the analysis of two speech signal data sets; “AHH” and “AWW” vowel sound, mainly for illustrative purposes. Both these data sets are obtained from a sound instrument at the Speech Signal Processing laboratory of the Indian Institute of Technology Kanpur. There are 469 data points of “AHH” signal and 512 data points of “AWW” signal, both sampled at 10 kHz frequency.

From the plot of the sound signals we have observed that the signals are quasi periodic. Hence, we expect that a variable frequency model would provide a better fit of the data than a fixed frequency model. Further, a single component chirp model seems an unlikely model for the data sets. This is because the change in frequency over time is not monotone, i.e. increasing or decreasing over time, and hence we have tried to fit the sound data using multi component chirp model.

We have fitted the chirp signal model to these data sets and use the proposed sequential method to compute the unknown parameters. Since the number of components of the model is not known in these cases, we use the Bayesian Information Criterion (BIC) to estimate the number of components. The BIC takes the following form

$$BIC(k) = N\ln(SSE) + \frac{1}{2}(4k + ar_k + 1)\ln(N)$$

in this case, where k is the number of components of the fitted model, ar_k is the number of parameters of the fitted stationary process $X(n)$ and N is the sample size. We choose that model order for which the BIC is minimum. For “AHH” data set, the estimate of p becomes 5, and for “AWW” it is 7. After fitting the AHH and AWW signals we have analyzed the residuals. Using SAS proc ARIMA, we have performed the augmented Dickey-Fuller unit root test on both the residual series. The null hypothesis is that there is a unit root, and hence the residuals are non stationary, whereas the alternate hypothesis is absence of unit root, and hence stationary residuals. In both the cases, the null hypotheses have been rejected, and hence we conclude that the residuals are stationary in nature. We have provided the autocorrelation function (ACF) and partial auto correlation function (PACF) plots of the residuals for both the cases. We also provide the plots for the fitted and predicted signals and they match quite well in both the cases.

6 Conclusion

In this paper, we have considered the problem of estimating the parameters of multiple chirp signal model, originally proposed by Saha and Kay [11]. First, we have considered the properties of the LSEs of the unknown parameters of this model. Using a number theoretic result of Vinogradov [12], we have obtained a simplified form of the asymptotic dispersion matrix of the LSEs. It is observed that the LSEs of the different chirp components are asymptotically independent. Then we have provided a sequential estimation procedure for estimation of the unknown parameters and it is proved that these estimators are strongly consistent. Due to the sequential nature, the proposed method can be very useful in fitting multiple chirp signal model, particularly when the number of chirp components is large. Although, we could not prove the rate of convergence of the proposed estimators, but our simulation results suggest that the proposed estimators might have the same rate of convergence as the LSEs. More work is needed in that direction.

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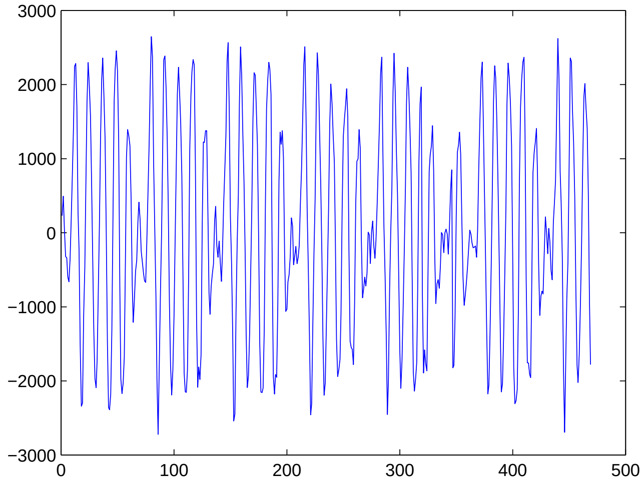

Figure 3: AHH; Original signal

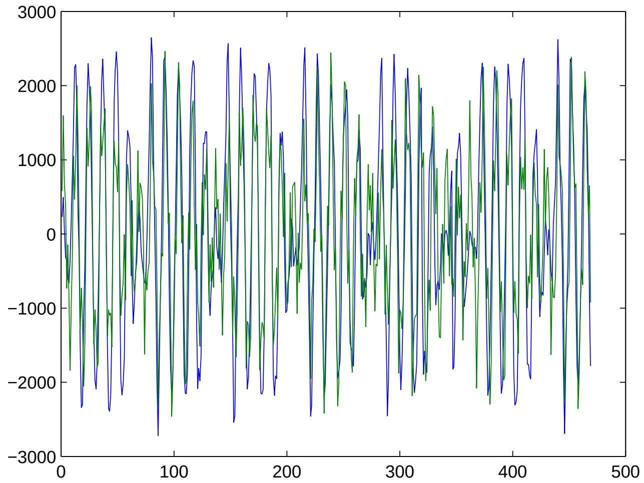

Figure 4: AHH; Fitted signal vs Original signal

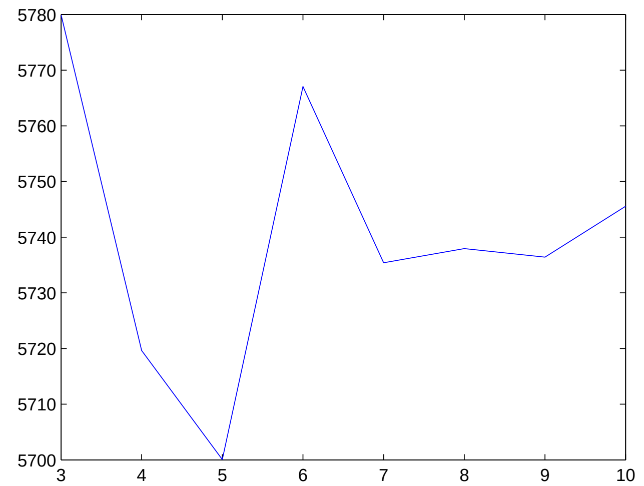

Figure 5: AHH; BIC plot

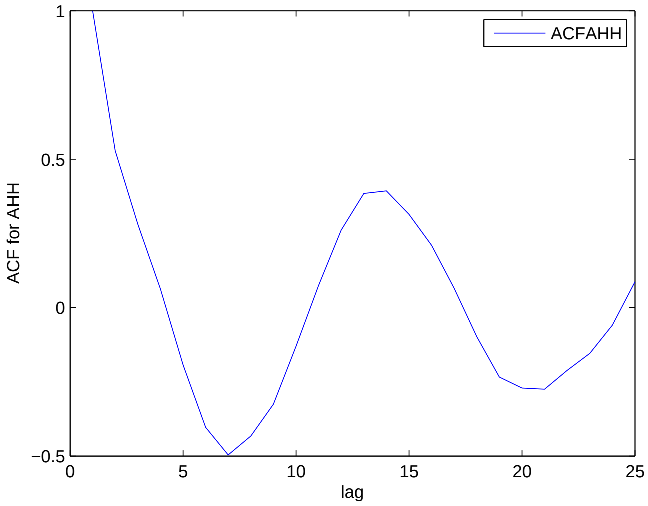

Figure 6: AHH; ACF plot of the residuals

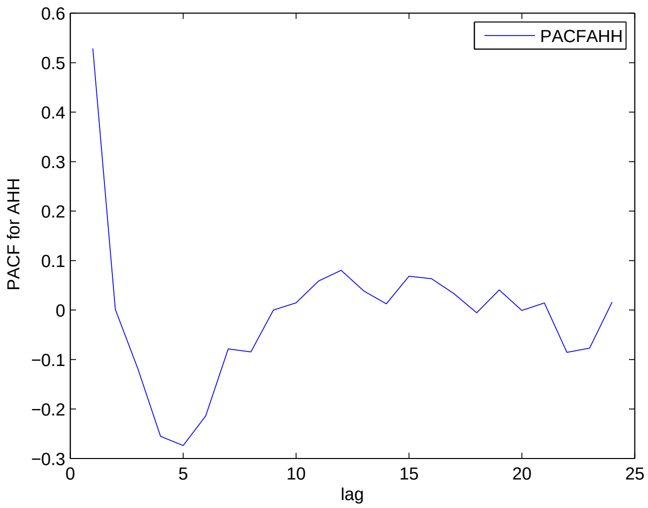

Figure 7: AHH; PACF plot of the residuals

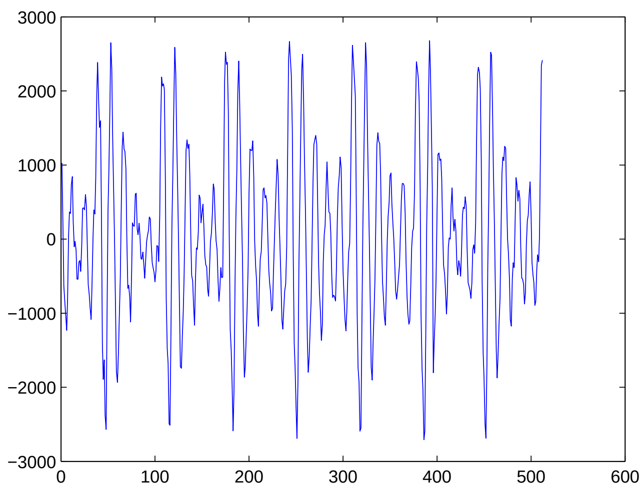

Figure 8: AWW; Original signal

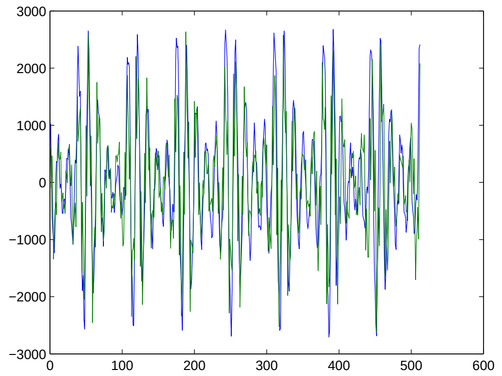

Figure 9: AWW; Fitted signal vs Original signal

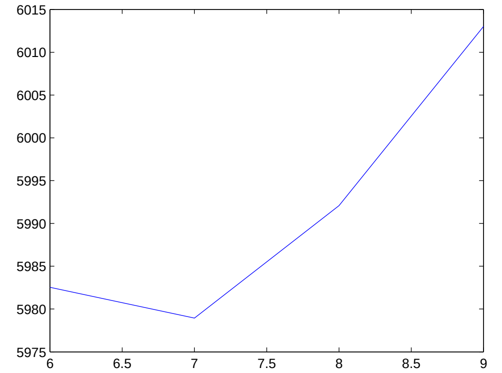

Figure 10: AWW; BIC plot

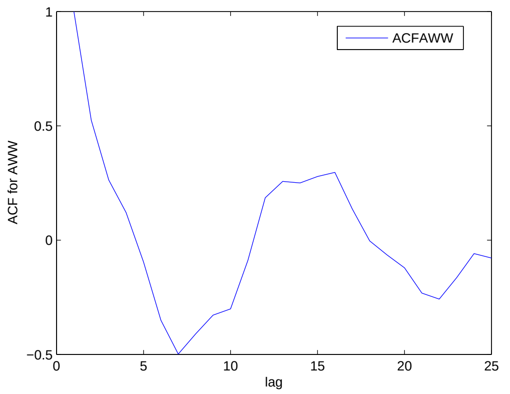

Figure 11: AWW; ACF plot of the residuals

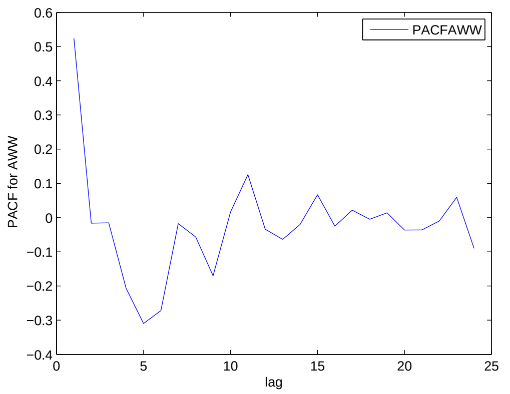

Figure 12: AWW; PACF plot of the residuals

Appendix A

Proof of Lemma 1:

First we look at an existing result by Vinogradov [12] for estimating Weyl's [14] sum. The result can be stated as follows;

Let $f(n) = \alpha_1 n + \alpha_2 n^2$ for $n = 1, 2, \cdots$, where α_1 and α_2 are real numbers, and for a positive integer N, $S = \sum_{n=1}^{N} e^{2\pi i f(n)}$. Suppose $\alpha_2 = \frac{a_N}{q_N} + \frac{b_N}{q_N^2}$, where $q_N > 0$, a_N, q_N are integers and a_N and q_N are relatively prime to each other, and $-1 < b_N < 1$. Then $S = O(N^{1-\rho})$, for some $\rho > 0$, when $c_1 N \leq q_N \leq c_2 N$ for some selected positive constants c_1 and c_2. It may be noted that if α_2 is an irrational number, then the corresponding q_N satisfies the above property. Note that N here is not the sample size but just a index.

To establish (4), it is enough to show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e^{2\pi i(\theta_1 t + \theta_2 t^2)} = 0. \tag{20}$$

Note that (20) follows immediately if θ_2 is an irrational number.

Let us define

$$S_1 = \sum_{n=1}^{N} n e^{2\pi i(\alpha n + \beta n^2)} \quad \text{and} \quad S_t = \sum_{n=1}^{N} n^t e^{2\pi i(\alpha n + \beta n^2)},\ t > 1. \tag{21}$$

To establish (5), (6), (7), it is enough to show

$$\lim_{N\to\infty} \frac{1}{N^2} S_1 = 0 \quad \text{and} \lim_{N\to\infty} \frac{1}{N^{t+1}} S_t = 0. \tag{22}$$

Now for some $\epsilon > 0$,

$$\begin{aligned}
|S_1| &= \left| N \sum_{n=1}^{N} e^{2\pi i(\alpha n + \beta n^2)} - \sum_{n=1}^{N} (N-n) e^{2\pi i(\alpha n + \beta n^2)} \right| \\
&\leq O(N.N^{1-\epsilon}) + \left| \sum_{n=1}^{1} e^{2\pi i(\alpha n + \beta n^2)} + \sum_{n=1}^{2} e^{2\pi i(\alpha n + \beta n^2)} + \cdots + \sum_{n=1}^{N-1} e^{2\pi i(\alpha n + \beta n^2)} \right| \\
&\leq O(N^{2-\epsilon}) + O(N^{2-\epsilon}) = O(N^{2-\epsilon}).
\end{aligned}$$

The result for $S_t,\ t>1$, can be obtained in a similar manner and is given below.

$$\begin{aligned}
|S_t| &= \left| N^t \sum_{n=1}^{N} e^{2\pi i(\alpha n+\beta n^2)} - \sum_{n=1}^{N}(N^t - n^t) e^{2\pi i(\alpha n+\beta n^2)} \right| \\
&\leq O(N^t . N^{1-\epsilon}) + \left| (N^t - (N-1)^t) \sum_{n=1}^{1} e^{2\pi i(\alpha n+\beta n^2)} \right. \\
&\qquad \left. + ((N-1)^t - (N-2)^t) \sum_{n=1}^{2} e^{2\pi i(\alpha n+\beta n^2)} + \cdots + (2^t - 1) \sum_{n=1}^{N-1} e^{2\pi i(\alpha n+\beta n^2)} \right| \\
&\leq O(N^{t+1-\epsilon}) + O(N^{t+1-\epsilon}) = O(N^{t+1-\epsilon}).
\end{aligned}$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N} n^t \cos(2\pi(\alpha n+\beta n^2)) = 0 \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N} n^t \sin(2\pi(\alpha n+\beta n^2)) = 0.$$

Now using

$$\cos^2\alpha = \frac{1}{2} + \frac{\cos 2\alpha}{2}, \quad \sin^2\alpha = \frac{1}{2} - \frac{\cos 2\alpha}{2}, \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N^{t+1}} \sum_{n=1}^{N} n^t = \frac{1}{t+1},$$

(5), (6) and (7) can be easily obtained.

Appendix B

Proof of Theorem 2: Using similar steps as in Kundu and Nandi [6], we obtain

$$\left((\tilde{\theta}_1 - \theta_1^0), \cdots, (\tilde{\theta}_p - \theta_p^0) \right) \Delta^{-1} = \left[-Q'(\theta_1^0, \cdots, \theta_p^0)\Delta \right] \left[\Delta Q''(\bar{\theta})\Delta \right]^{-1}. \qquad (23)$$

Here $\Delta = diag\left(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{1}{N\sqrt{N}}, \frac{1}{N^2\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \frac{1}{N\sqrt{N}}, \frac{1}{N^2\sqrt{N}} \right)$ is a $4p \times 4p$ diagonal matrix and $Q'(\theta_1^0, \cdots, \theta_p^0)$ is the derivative of $Q(\theta_1, \cdots, \theta_p)$ evaluated at θ^0, where

$$Q(\theta_1, \cdots, \theta_p) = \sum_{n=1}^{N} \left(y(n) - \sum_{k=1}^{p} \left(A_k \cos(\alpha_k n + \beta_k n^2) + B_k \sin(\alpha_k n + \beta_k n^2) \right) \right)^2$$

and $\bar{\theta}$ is on the line joining the estimator and the true parameter. Furthermore,

$$[Q'(\theta_1^0, \cdots, \theta_p^0)\Delta]^T = \begin{bmatrix} -\frac{2}{\sqrt{N}} \sum_{n=1}^{N} \cos(\alpha_1^0 n + \beta_1^0 n^2) X(n) \\ -\frac{2}{\sqrt{N}} \sum_{n=1}^{N} \sin(\alpha_1^0 n + \beta_1^0 n^2) X(n) \\ \frac{2}{N\sqrt{N}} \sum_{n=1}^{N} n[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)] X(n) \\ \frac{2}{N^2\sqrt{N}} \sum_{n=1}^{N} n^2[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)] X(n) \\ \vdots \\ -\frac{2}{\sqrt{N}} \sum_{n=1}^{N} \cos(\alpha_p^0 n + \beta_p^0 n^2) X(n) \\ -\frac{2}{\sqrt{N}} \sum_{n=1}^{N} \sin(\alpha_p^0 n + \beta_p^0 n^2) X(n) \\ \frac{2}{N\sqrt{N}} \sum_{n=1}^{N} n[A_p^0 \sin(\alpha_p^0 n + \beta_p^0 n^2) - B_p^0 \cos(\alpha_p^0 n + \beta_p^0 n^2)] X(n) \\ \frac{2}{N^2\sqrt{N}} \sum_{n=1}^{N} n^2[A_p^0 \sin(\alpha_p^0 n + \beta_p^0 n^2) - B_p^0 \cos(\alpha_p^0 n + \beta_p^0 n^2)] X(n) \end{bmatrix}.$$

Using similar arguments as in Kundu and Nandi [6], it follows that $\left[Q'(\theta_1^0, \cdots, \theta_p^0)\Delta\right]^T$ converges to a multivariate normal distribution. We now present the steps for calculating the dispersion matrix.

To find the asymptotic variance of $\left[\frac{2}{\sqrt{N}} \sum_{n=1}^{N} \cos(\alpha_1^0 n + \beta_1^0 n^2) X(n)\right]$, we need to calculate for $h = 0, \pm 1, \pm 2, \cdots$, the term

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-|h|} \cos(\alpha_1^0 n + \beta_1^0 n^2) \cos(\alpha_1^0 (n+h) + \beta_1^0 (n+h)^2).$$

Using Lemma 1, and after some calculations it can be shown that for $h = 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-|h|} \cos(\alpha_1^0 n + \beta_1^0 n^2) \cos(\alpha_1^0 (n+h) + \beta_1^0 (n+h)^2) = \frac{1}{2}$$

and it is 0 otherwise. To find the variance of $\left[\frac{2}{\sqrt{N}} \sum_{n=1}^{N} \sin(\alpha_1^0 n + \beta_1^0 n^2) X(n)\right]$, we need to calculate the above limits where both the cos terms are replaced by sin terms, and we will get similar result using Lemma 1.

Now for all h, using the following results;

$$\lim_{N\to\infty}\sum_{n=1}^{N-|h|}\cos(\alpha_j^0 n+\beta_j^0 n^2)\sin(\alpha_j^0(n+h)+\beta_j^0(n+h)^2)=0,$$

$$\lim_{N\to\infty}\sum_{n=1}^{N-|h|}\cos(\alpha_j^0 n+\beta_j^0 n^2)\cos(\alpha_l^0(n+h)+\beta_l^0(n+h)^2)=0,$$

$$\lim_{N\to\infty}\sum_{n=1}^{N-|h|}\cos(\alpha_j^0 n+\beta_j^0 n^2)\sin(\alpha_l^0(n+h)+\beta_l^0(n+h)^2)=0,$$

$$\lim_{N\to\infty}\sum_{n=1}^{N-|h|}\sin(\alpha_j^0 n+\beta_j^0 n^2)\sin(\alpha_l^0(n+h)+\beta_l^0(n+h)^2)=0,$$

where $j,l=1,\cdots,p$, and $j\neq l$, the variances of the other terms and covariance between any two terms can be obtained along the same line. Therefore, $\left[-Q'(\theta_1^0,\cdots,\theta_p^0)\Delta\right]$ converges in distribution to

$$N_{4p}(0,2c\sigma^2\Sigma(\theta^0)^{-1}). \tag{24}$$

Next we calculate $\lim_{N\to\infty}\left[\Delta Q''(\bar\theta)\Delta\right]^{-1}$, for which we need the following;

$$\frac{\partial^2 Q(\theta)}{\partial A_k^2}\Big|_{\theta^0}=2\sum_{n=1}^{N}\cos^2(\alpha_k^0 n+\beta_k^0 n^2),$$

$$\frac{\partial^2 Q(\theta)}{\partial A_k\partial B_k}\Big|_{\theta^0}=2\sum_{n=1}^{N}\sin(\alpha_k^0 n+\beta_k^0 n^2)\cos(\alpha_k^0 n+\beta_k^0 n^2),$$

$$\frac{\partial^2 Q(\theta)}{\partial B_k^2}\Big|_{\theta^0}=2\sum_{n=1}^{N}\sin^2(\alpha_k^0 n+\beta_k^0 n^2),$$

$$\frac{\partial^2 Q(\theta)}{\partial A_k\partial \alpha_k}\Big|_{\theta^0}=2\sum_{n=1}^{N}n\sin(\alpha_k^0 n+\beta_k^0 n^2)\times X(n)$$
$$-2\sum_{n=1}^{N}n\cos(\alpha_k^0 n+\beta_k^0 n^2)\times[A_k^0\sin(\alpha_k^0 n+\beta_k^0 n^2)-B_k^0\cos(\alpha_k^0 n+\beta_k^0 n^2)],$$

$$\frac{\partial^2 Q(\theta)}{\partial B_k\partial \alpha_k}\Big|_{\theta^0}=-2\sum_{n=1}^{N}n\cos(\alpha_k^0 n+\beta_k^0 n^2)\times X(n)$$
$$-2\sum_{n=1}^{N}n\sin(\alpha_k^0 n+\beta_k^0 n^2)\times[A_k^0\sin(\alpha_k^0 n+\beta_k^0 n^2)-B_k^0\cos(\alpha_k^0 n+\beta_k^0 n^2)],$$

$$\frac{\partial^2 Q(\theta)}{\partial A_k \partial \beta_k}\Big|_{\theta^0} = 2\sum_{n=1}^{N} n^2 \sin(\alpha_k^0 n + \beta_k^0 n^2) \times X(n)$$

$$- 2\sum_{n=1}^{N} n^2 \cos(\alpha_k^0 n + \beta_k^0 n^2) \times [A_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) - B_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2)],$$

$$\frac{\partial^2 Q(\theta)}{\partial B_k \partial \beta_k}\Big|_{\theta^0} = -2\sum_{n=1}^{N} n^2 \cos(\alpha_k^0 n + \beta_k^0 n^2) \times X(n)$$

$$- 2\sum_{n=1}^{N} n^2 \sin(\alpha_k^0 n + \beta_k^0 n^2) \times [A_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) - B_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2)],$$

$$\frac{\partial^2 Q(\theta)}{\partial \alpha_k^2}\Big|_{\theta^0} = 2\sum_{n=1}^{N} n^2 [A_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2)] \times X(n)$$

$$+ 2\sum_{n=1}^{N} n^2 [A_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) - B_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2)]^2,$$

$$\frac{\partial^2 Q(\theta)}{\partial \alpha_k \partial \beta_k}\Big|_{\theta^0} = 2\sum_{n=1}^{N} n^3 [A_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2)] \times X(n)$$

$$+ 2\sum_{n=1}^{N} n^3 [A_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) - B_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2)]^2,$$

$$\frac{\partial^2 Q(\theta)}{\partial \beta_k^2}\Big|_{\theta^0} = 2\sum_{n=1}^{N} n^4 [A_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2)] \times X(n)$$

$$+ 2\sum_{n=1}^{N} n^4 [A_k^0 \sin(\alpha_k^0 n + \beta_k^0 n^2) - B_k^0 \cos(\alpha_k^0 n + \beta_k^0 n^2)]^2.$$

Using Lemma 1, Lemma 2 and the above calculations, it can be shown that

$$[\Delta Q''(\bar{\theta})\Delta] \to \Sigma(\theta^0)^{-1} \quad a.s. \tag{25}$$

Finally, using (24) and (25), we get the desired result.

Appendix C

To prove Theorem 3, we need the following lemma.

Lemma 3: Consider the set $S_c = \{\theta : \ \theta \in \Theta, \ |\theta - \theta_1^0| \geq 3c\}$; where $\theta = (A, B, \alpha, \beta)$, $\theta_1^0 = (A_1^0, B_1^0, \alpha_1^0, \beta_1^0)$ and $\Theta = [-M, M] \times [-M, M] \times [0, \pi] \times [0, \pi]$. If for any $c > 0$,

$\liminf \inf_{\theta \in S_c} \frac{1}{N}(Q_1(\theta) - Q_1(\theta_1^0)) > \ 0 \ a.s.$ then $\widehat{\theta}_1$, which minimizes $Q_1(\theta)$, is a strongly consistent estimator of θ_1^0.

Proof : The proof can be obtained by contradiction, along the same lines as the proof of Lemma 1 of Wu [15].

Proof of Theorem 3: Consider the following expression of $\frac{1}{N}\left[Q_1(\theta) - Q_1(\theta_1^0)\right]$.

$$\frac{1}{N}\left[Q_1(\theta) - Q_1(\theta_1^0)\right] = f(\theta) + g(\theta),$$

where

$$\begin{aligned} f(\theta) &= \frac{1}{N}\sum_{n=1}^{N}\left[A\cos(\alpha n + \beta n^2) + B\sin(\alpha n + \beta n^2)\right. \\ &\left. -A_1^0\cos(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0\sin(\alpha_1^0 n + \beta_1^0 n^2)\right]^2 \\ &+\frac{2}{N}\sum_{n=1}^{N}\left(A_1^0\cos(\alpha_1^0 n + \beta_1^0 n^2) + B_1^0\sin(\alpha_1^0 n + \beta_1^0 n^2)\right. \\ &\left. - A\cos(\alpha n + \beta n^2) - B\sin(\alpha n + \beta n^2)\right) \times \\ &\sum_{k=2}^{p}\left[A_k^0\cos(\alpha_k^0 n + \beta_k^0 n^2) + B_k^0\sin(\alpha_k^0 n + \beta_k^0 n^2)\right] \end{aligned}$$

and

$$\begin{aligned} g(\theta) &= \frac{2}{N}\sum_{n=1}^{N}X(n)\left[A_1^0\cos(\alpha_1^0 n + \beta_1^0 n^2) + B_1^0\sin(\alpha_1^0 n + \beta_1^0 n^2)\right. \\ &\left. -A\cos(\alpha n + \beta n^2) - B\sin(\alpha n + \beta n^2)\right]. \end{aligned}$$

Now using Lemma 2, it immediately follows that $\sup_{\theta \in S_c}|g(\theta)| \to \ 0 \ a.s.$ Note that the proof will be complete if we can show $\liminf \inf_{\theta \in S_c} f(\theta) > 0 \ a.s.$

For notational simplicity, we assume $p = 2$. Since

$$S_c = \{\theta : |\theta - \theta_1^0| \geq 3c\} \subseteq S_c^A \cup S_c^B \cup S_c^{(\alpha,\beta)}, \tag{26}$$

where

$$\begin{aligned} S_c^A &= \{\theta : |A - A_1^0| \geq c\} \subseteq \{\theta : |A - A_1^0| \geq c, (\alpha, \beta) = (\alpha_1^0, \beta_1^0)\} \\ &\cup\{\theta : |A - A_1^0| \geq c, (\alpha, \beta) = (\alpha_2^0, \beta_2^0) \text{ and } (A, B) = (A_2^0, B_2^0)\} \\ &\cup\{\theta : |A - A_1^0| \geq c, (\alpha, \beta) = (\alpha_2^0, \beta_2^0) \text{ and } (A, B) \neq (A_2^0, B_2^0)\} \\ &\cup\{\theta : |A - A_1^0| \geq c, (\alpha, \beta) \neq (\alpha_k^0, \beta_k^0); k = 1, 2\}, \end{aligned}$$

$$\begin{aligned} S_c^B &= \{\theta : |B - B_1^0| \geq c\} \subseteq \{\theta : |B - B_1^0| \geq c, (\alpha,\beta) = (\alpha_1^0, \beta_1^0)\} \\ &\cup \{\theta : |B - B_1^0| \geq c, (\alpha,\beta) = (\alpha_2^0, \beta_2^0) \text{ and} (A, B) = (A_2^0, B_2^0)\} \\ &\cup \{\theta : |B - B_1^0| \geq c, (\alpha,\beta) = (\alpha_2^0, \beta_2^0) \text{ and } (A, B) \neq (A_2^0, B_2^0)\} \\ &\cup \{\theta : |B - B_1^0| \geq c, (\alpha,\beta) \neq (\alpha_k^0, \beta_k^0); k = 1, 2\}, \end{aligned}$$

$$\begin{aligned} S_c^{(\alpha,\beta)} &= \{\theta : |(\alpha,\beta) - (\alpha_1^0, \beta_1^0)| \geq c\} \\ &\subseteq \{\theta : |(\alpha,\beta) - (\alpha_1^0, \beta_1^0)| \geq c, \alpha = \alpha_1^0\} \cup \{\theta : |(\alpha,\beta) - (\alpha_1^0, \beta_1^0)| \geq c, \beta = \beta_1^0\} \\ &\cup \{\theta : |(\alpha,\beta) - (\alpha_1^0, \beta_1^0)| \geq c, (\alpha,\beta) = (\alpha_2^0, \beta_2^0) \text{ and } (A, B) = (A_2^0, B_2^0)\} \\ &\cup \{\theta : |(\alpha,\beta) - (\alpha_1^0, \beta_1^0)| \geq c, (\alpha,\beta) = (\alpha_2^0, \beta_2^0) \text{ and} (A, B) \neq (A_2^0, B_2^0)\} \\ &\cup \{\theta : |(\alpha,\beta) - (\alpha_1^0, \beta_1^0)| \geq c, (\alpha,\beta) \neq (\alpha_k^0, \beta_k^0); k = 1, 2\}, \end{aligned}$$

and for each of the above sets, $\liminf \inf_{\theta \in S} f(\theta) > 0$ *a.s.*, where S can be S_c^A, S_c^B or $S_c^{(\alpha,\beta)}$, the result follows.

Appendix D

To prove Theorem 4, we need the following lemma.

Lemma 4: If the Assumptions 1-3 are satisfied, then $N(\hat{\alpha}_1 - \alpha_1^0) \to 0$ *a.s.* and $N^2(\hat{\beta}_1 - \beta_1^0) \to 0$ *a.s.*

Proof of Lemma 4:

We have,

$$Q_1(\theta) = \sum_{n=1}^{N} \left[y(n) - A\cos(\alpha n + \beta n^2) - B\sin(\alpha n + \beta n^2) \right]^2. \tag{27}$$

Let us denote $Q_1'(\theta)$ as the 4×1 first derivative vector and $Q_1''(\theta)$ as the 4×4 second derivative matrix. Now using multivariate Taylor series expansion we have

$$Q_1'(\widehat{\theta}_1) - Q_1'(\theta_1^0) = (\widehat{\theta}_1 - \theta_1^0) Q_1''(\bar{\theta}), \tag{28}$$

where $\bar{\theta}$ is a point on line joining $\widehat{\theta}_1$ and θ_1^0. Since, $Q_1'(\widehat{\theta}_1) = 0$, (28) can be written as

$$-Q_1'(\theta_1^0)D = (\widehat{\theta}_1 - \theta_1^0)D^{-1}[DQ_1''(\bar{\theta})D]. \tag{29}$$

In order to calculate $Q_1''(\bar{\theta})$, we evaluate the following;

$$\frac{\partial Q_1(\theta)}{\partial A}\Big|_{\theta_1^0} = -2\sum_{n=1}^{N} \cos(\alpha_1^0 n + \beta_1^0 n^2)$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n)),$$

$$\frac{\partial Q_1(\theta)}{\partial B}\Big|_{\theta_1^0} = -2\sum_{n=1}^{N} \sin(\alpha_1^0 n + \beta_1^0 n^2)$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n)),$$

$$\frac{\partial Q_1(\theta)}{\partial \alpha}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} n[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)]$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n)),$$

$$\frac{\partial Q_1(\theta)}{\partial \beta}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} n^2[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)]$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n)),$$

$$\frac{\partial^2 Q_1(\theta)}{\partial A^2}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} \cos^2(\alpha_1^0 n + \beta_1^0 n^2),$$

$$\frac{\partial^2 Q_1(\theta)}{\partial A \partial B}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} \sin(\alpha_1^0 n + \beta_1^0 n^2)\cos(\alpha_1^0 n + \beta_1^0 n^2),$$

$$\frac{\partial^2 Q_1(\theta)}{\partial B^2}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} \sin^2(\alpha_1^0 n + \beta_1^0 n^2),$$

$$\frac{\partial^2 Q_1(\theta)}{\partial A \partial \alpha}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} n \sin(\alpha_1^0 n + \beta_1^0 n^2)$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$- 2\sum_{n=1}^{N} n \cos(\alpha_1^0 n + \beta_1^0 n^2) \times [A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)],$$

$$\frac{\partial^2 Q_1(\theta)}{\partial B \partial \alpha}\Big|_{\theta_1^0} = -2\sum_{n=1}^{N} n\cos(\alpha_1^0 n + \beta_1^0 n^2)$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$-2\sum_{n=1}^{N} n\sin(\alpha_1^0 n + \beta_1^0 n^2) \times [A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)],$$

$$\frac{\partial^2 Q_1(\theta)}{\partial A \partial \beta}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} n^2\sin(\alpha_1^0 n + \beta_1^0 n^2)$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$-2\sum_{n=1}^{N} n^2\cos(\alpha_1^0 n + \beta_1^0 n^2) \times [A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)],$$

$$\frac{\partial^2 Q_1(\theta)}{\partial B \partial \beta}\Big|_{\theta_1^0} = -2\sum_{n=1}^{N} n^2\cos(\alpha_1^0 n + \beta_1^0 n^2)$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$-2\sum_{n=1}^{N} n^2\sin(\alpha_1^0 n + \beta_1^0 n^2) \times [A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)],$$

$$\frac{\partial^2 Q_1(\theta)}{\partial \alpha^2}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} n^2[A_1^0\cos(\alpha_1^0 n + \beta_1^0 n^2) + B_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2)]$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$+2\sum_{n=1}^{N} n^2[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)]^2,$$

$$\frac{\partial^2 Q_1(\theta)}{\partial \alpha \partial \beta}\Big|_{\theta_1^0} = 2\sum_{n=1}^{N} n^3[A_1^0\cos(\alpha_1^0 n + \beta_1^0 n^2) + B_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2)]$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$+2\sum_{n=1}^{N} n^3[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)]^2,$$

$$\left.\frac{\partial^2 Q_1(\theta)}{\partial \beta^2}\right|_{\theta_1^0} = 2\sum_{n=1}^{N} n^4[A_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2) + B_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2)]$$
$$\times (\sum_{k=2}^{p}[A_k^o \cos(\alpha_k^o n + \beta_k^o n^2) + B_k^o \sin(\alpha_k^o n + \beta_k^o n^2)] + X(n))$$
$$+ 2\sum_{n=1}^{N} n^4[A_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) - B_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2)]^2.$$

Using Lemma 1 and Lemma 2

$$DQ_1''(\bar{\theta})D \to \lim_{N\to\infty} DQ_1''(\theta_1^0)D = \Sigma_1^{-1}, \tag{30}$$

which is a positive definite matrix. (29) gives

$$(\widehat{\theta}_1 - \theta_1^0)D^{-1} = [-Q_1'(\theta_1^0)D][DQ_1''(\bar{\theta})D]^{-1}. \tag{31}$$

Dividing by $\sqrt{N}$, (31) becomes

$$(\widehat{\theta}_1 - \theta_1^0)(\sqrt{N}D)^{-1} = [-\frac{1}{\sqrt{N}}Q_1'(\theta_1^0)D][DQ_1''(\bar{\theta})D]^{-1}. \tag{32}$$

Using Lemma 1 and Lemma 2,

$$\frac{1}{\sqrt{N}}Q_1'(\theta_1^0)D \to 0 \ a.s. \tag{33}$$

Hence the result follows.

Proof of Theorem 4: Using Theorem 1 and Lemma 4, we obtain

$$\widehat{A}_1 = A_1^0 + o(1), \quad \widehat{B}_1 = B_1^0 + o(1), \quad \widehat{\alpha}_1 = \alpha_1^0 + o(N^{-1}), \quad \widehat{\beta}_1 = \beta_1^0 + o(N^{-2}).$$

Here the random variable $U = o(1)$ means, $U \to 0 \ \ a.s.$, and $U = o(N^{-k})$ means $UN^k \to 0 \ \ a.s.$ Therefore, the result can be obtained by following the same argument as in Theorem 1 and by using

$$\widehat{A}_1 \cos(\widehat{\alpha}_1 n + \widehat{\beta}_1 n^2) + \widehat{B}_1 \sin(\widehat{\alpha}_1 n + \widehat{\beta}_1 n^2) = A_1^0 \cos(\alpha_1^0 n + \beta_1^0 n^2) + B_1^0 \sin(\alpha_1^0 n + \beta_1^0 n^2) + o(1).$$

Proof of Theorem 5: In the proof of Theorem 4, replace the suffix 1 by $k-1$, and suffix 2 by k, for $3 < k \le p$ and the result follows.

Appendix E

To prove Theorem 6, we need the following Lemma.

Lemma 5: If $X(n)$ satisfies Assumption 1 and $\widehat{A}, \widehat{B}, \widehat{\alpha}, \widehat{\beta}$ are obtained by minimizing

$$\frac{1}{N} Q_{(p+1)}(\theta) = \frac{1}{N} \sum_{n=1}^{N} \left[X(n) - A\cos(\alpha n + \beta n^2) - B\sin(\alpha n + \beta n^2)\right]^2,$$

then $\widehat{A} \to \ 0 \ \ a.s.; \widehat{B} \to \ 0 \ \ a.s.$

Proof: Note that $\frac{1}{N} Q_{(p+1)}(\theta) = \frac{1}{N} R(\theta) + o(1)$ where,

$$R(\theta) = \sum_{n=1}^{N} X(n)^2 - 2\sum_{n=1}^{N} X(n)\left[A\cos(\alpha n + \beta n^2) + B\sin(\alpha n + \beta n^2)\right] + N\frac{A^2 + B^2}{2}.$$

Now we will use similar argument as in Walker [13]. We note that the minimizer of $R(\theta)$, with respect to A and B is $\tilde{A} = \frac{2}{N}\sum_{n=1}^{N} X(n)\cos(\tilde{\alpha} n + \tilde{\beta} n^2)$ and $\tilde{B} = \frac{2}{N}\sum_{n=1}^{N} X(n)\sin(\tilde{\alpha} n + \tilde{\beta} n^2)$, respectively. While the minimizer of $\frac{1}{N} Q_{(p+1)}(\theta)$ is $\widehat{A}$ and $\widehat{B}$, minimizer of $\frac{1}{N} R(\theta)$ is $\tilde{A}$ and $\tilde{B}$. Let us introduce the notation, $\tilde{\theta} = (\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{\beta})$. $Q(\widehat{\theta}) = Q(\tilde{\theta}) + (\widehat{\theta} - \tilde{\theta})Q''(\theta^*)(\widehat{\theta} - \tilde{\theta})^T$, for some θ^*, $Q'(\widehat{\theta}) = 0$, $R(\widehat{\theta}) = R(\tilde{\theta}) - (\widehat{\theta} - \tilde{\theta})R''(\theta^{**})(\widehat{\theta} - \tilde{\theta})^T$, for some θ^{**}, $R'(\tilde{\theta}) = 0$. Then,

$$Q(\widehat{\theta}) - R(\widehat{\theta}) = Q(\tilde{\theta}) - R(\tilde{\theta}) + (\widehat{\theta} - \tilde{\theta})[Q''(\theta^*) + R''(\theta^{**})](\widehat{\theta} - \tilde{\theta})^T$$

and on dividing by N, $\widehat{A} = \tilde{A} + o(1) = \frac{2}{N}\sum_{n=1}^{N} X(n)\sin(\tilde{\alpha} n + \tilde{\beta} n^2) + o(1)$ and similarly $\widehat{B} = \tilde{B} + o(1) = \frac{2}{N}\sum_{n=1}^{N} X(n)\sin(\tilde{\alpha} n + \tilde{\beta} n^2) + o(1)$. Now using Lemma 2 we get $\tilde{A} \to 0 \ a.s., \quad \tilde{B} \to 0 \ a.s.$ Hence the result follows.

References

[1] Abatzoglou, T. (1986), "Fast maximum likelihood joint estimation of frequency and frequency rate", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 22, 708 - 714.

[2] Djuric, P.M. and Kay, S.M. (1990), "Parameter estimation of chirp signals", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 38, 2118 - 2126.

[3] Fuller, W. A. (1996), *Introduction to Statistical Time Series*, 2nd. edition, John Wiley and Sons, New York.

[4] Gini, F., Montanari, M. and Verrazzani, L. (2000), "Estimation of chirp signals in compound Gaussian clutter: A cyclostationary approach", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 48, 1029 - 1039.

[5] Kumaresan, R. and Verma, S. (1987), "On estimation the parameters of chirp signals using rank reduction techniques", *Proceedings of 21 st Asilomar Conference*, 555 - 558, Pacific Grove, California.

[6] Kundu, D. and Nandi, S. (2008), "Parameter estimation of chirp signals in presence of stationary noise", *Statistica Sinica*, vol. 18, 187 - 202.

[7] Montgomery H.L. (1990), *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, American Mathematical Society, 196.

[8] Nandi, S. and Kundu, D. (2004), "Asymptotic properties of the least squares estimators of the parameters of the chirp signals", *Annals of the Institute of Statistical Mathematics*, vol. 56, 529 - 544.

[9] Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P. (1996), *Numerical Recipes in Fortran 90*, Cambridge University Press. Page 402.

[10] Richards, F. S. G. (1961) "A method of maximum likelihood estimation", *Jour. Royal Stat. Soc.* B, 23, 469-475.

[11] Saha, S. and Kay, S. (2002), "Maximum likelihood parameter estimation of superimposed chirps using Monte Carlo importance sampling", *IEEE Transactions on Signal Processing*, vol. 50, 224 - 230.

[12] Vinogradov, I.M. (1954), "The method of trigonometrical sums in the theory of numbers" , *Interscience*, Translated from Russian. Revised and annotated by K. F. Roth and Anne Davenport. Reprint of the 1954 translation. Dover Publications, Inc., Mineola, NY, 2004.

[13] Walker, A.M. (1971), "On the estimation of a Harmonic components in a time series with stationary residuals", *Biometrika*, vol. 58, 21-36.

[14] Weyl, H. (1916), "Ueber die Gleichverteilung von Zahlen mod. Eins" , *Math. Ann.*, vol. 77, 313 - 352.

[15] Wu, C. F. (1981), "Asymptotic theory of non-linear least-squares estimation" , *Annals of Statistics*, vol. 9, 501 - 513.