Misspecification of Copula for One-Shot Devices under Constant Stress Accelerated Life-Tests

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Abstract

Copula models have attracted significant attention in the recent literature for modelling multivariate observations. An essential feature of copulas is that they enable us to specify the univariate marginal distributions and their joint behaviours separately. This paper provides asymptotic results for misspecification of copula models, and examines the consequences and detection of misspecification in copula models under constant-stress accelerated life-tests for one-shot devices. The one-shot device is considered as a two-component system. The reliability is an important factor in lifetime data applications, and we focus on the effect of misspecification on the estimation of the reliability of one-shot devices. Moreover, the Akaike information criterion is used as a specification test for copula model validation. A simulation study is carried out to evaluate the effect of misspecification under Gumbel-Hougaard, Frank and Clayton copulas incorporated with Weibull and gamma distributions as marginal distributions, in terms of asymptotic bias, asymptotic relative bias, and root mean square error.

KEYWORDS AND PHRASES: copula, constant-stress accelerated life testing, likelihood function, one-shot devices, quasi-maximum likelihood estimator, series and parallel systems.

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1 INTRODUCTION

The reliability of a product can be defined as the ability of the product to perform as planned. Ideally, it is obtained by continuously testing the product again and again and observing its failure rate. Certain products called "one-shot" devices, make this approach challenging. One-shot devices can be used only once, after used; either they get destroyed or must be rebuilt. Statistical inference for one-shot devices, namely electro-explosive devices, space shuttles, military weapons, and automobile airbags, has also been developed recently. Sometimes, the structure of a device is complicated in the sense that the device is composed of multiple components. Thus, the reliability of the device depends on the reliability of each component. In such complex device (or system), the components are connected in series or/and in parallel.

For mathematical ease, the assumption of independence on the components is usually made. However, this assumption may be unreliable in many situations and may have severe consequences in evaluating the reliability of the systems, wherein different components are connected; therefore, the component may influence each other, causing the system failure earlier than expected. It happened due to the correlation among components. For example, the lifetime of the motherboard of a computer typically depends on the condition of its components, such as the central processing unit (CPU), power, hard disk drive (HDD), and so on. Such dependencies can be modelled by considering a joint distribution of the component lifetimes that are associated with various marginal distributions.

A copula approach is popular to construct the joint distribution and to relax the independence assumption by providing dependence between lifetime variables. It was first proposed by Sklar [14], and it becomes quite popular in reliability studies. Readers may consider Nelsen [7] and Chiyoshi [15] for a detailed review of copula models. Jia et al. [16] considered copula models for measuring the reliability of systems with dependent components. Ling et al. [18] studied the copula models for one-shot device testing data, which are collected from constant-stress accelerated life-tests. However, misspecification of copula models for one-shot devices with components under constant stress accelerated life-tests has not been studied in the literature.

In real data applications, the problem is deciding a suitable distribution to model a given data set when the two or more distributions fit the data set well. Even when the model fits the data set well, the estimation and inference may have significant differences due to misspecification of distributions, dependence structures, link functions, and other structural components. Therefore, the model misspecification is always an important concern when statistical inferences are based on a parametric likelihood. White [1] examined the effect of model misspecification using maximum likelihood estimation method and associated inference. It has been observed that point estimators of some parameters of interest can be inconsistent due to the model misspecification. Later, Chow [2] indicated that the properties of misspecified models are corrected if and only if data are independent and identically distributed. Some of the relevant work along these lines have been studied by Bai et al. [3], Yu [4], and Pascual [5].

An accelerated life-test is utilized to evaluate the quality of data under different stress levels for quick failures within a shorter time. In accelerated life-tests, products are put on higher levels of stress (e.g., higher temperature, voltage, and pressure) to produce failures more quickly, which reduces the cost and length of tests. For more details on accelerated life-tests, readers may refer to Nelsen [8]. Pascual [6] derived expressions for the asymptotic distribution of maximum likelihood estimators of model parameters in accelerated life-tests when the model distribution is misspecified under the lognormal and Weibull models. Recently, Khakifirooz [9] discussed the behaviour of the relative bias and relative variability of the p^{th} quantile of the accelerated lifetime experiment when the generalized Gamma distribution is incorrectly specified as Log-normal or Weibull distribution.

In one-shot device test data, the lifetimes of one-shot devices are always either left or right-censored due to the destructive nature of the devices, which implies that the exact failure times of the devices cannot be observed. One can only observe the condition of the device at a specific inspection time, so only binary data (successes or failures) are found. To obtain sufficient lifetime information within a limited time from the devices for predicting the reliability at a specific mission time at normal operating conditions becomes quite tricky. To overcome this issue, accelerated life-tests are often used by applying higher levels of stress to the devices for quick failure. For more details on a one-shot device, readers may refer to Balakrishnan and Ling[10, 11, 12]. Ling and Balakrishnan [13] examined the effect of model misspecification between Weibull and gamma models, and the inference on the reliability at some mission times based on one-shot device test data. For an overview of such inferential results for one-shot devices, one may refer to the recent book [17].

In this paper, we are interested in copula model misspecification analysis when the one-shot device with two components is put on constant-stress accelerated life-tests. The main aim is to evaluate the effect of the misspecification on reliability estimation. Assuming the marginal failure times belong to unknown distributions, we model the dependence among components via the copula. To study the effect on the estimation of reliability under a misspecified copula model, we consider three popular copula models used in the reliability literature, including Gumbel-Hoggard, Frank, and Clayton copulas. Moreover, the Akaike information criterion (AIC) can also be used as a specification test, and it performs well in identifying the correct copula model. So, we compare the misspecification results for the reliability of one-shot devices and evaluate the detection power of the AIC to identify the correct copula model through a simulation study.

The rest of the paper is organized as follows. In Section 2, we first provide the log-likelihood for one-shot devices treated as two-component systems under constant-stress accelerated life-tests using a copula model. Besides, three popular copulas in the reliability literature are presented, namely Gumbel-Hougaard, Frank, and Clayton copulas. In Section 3, we investigate the effect of misspecification between copula models and obtain reliability estimates under the influence of misspecified copula. These formulas can be used to estimate empirical and asymptotic bias in the reliability estimation. In Section 4, an comprehensive simulation study is conducted to examine the asymptotic results and the effect of copula misspecification on the reliability estimation for two commonly used lifetime distributions, such as Weibull and gamma distributions. Finally, some concluding remarks are given in Section 5.

2 Assumptions and Likelihood function

Let us assume that one-shot device is a system with two components. Suppose that the devices are tested at I higher-than-normal stress conditions, each of which is subject to an accelerating factor, and the tests are inspected J times. Let K_{ij} devices are placed on test at stress level s_i and only inspected at inspection time τ_j, where $i = 1,2,\ldots,I$ and $j = 1,2,\ldots,J$. The numbers of devices without failures, with only failure of component 1, with only failure of component 2, and the number of devices with both components failures are denoted by $n_{ij,0}$, $n_{ij,1}$, $n_{ij,2}$, and $n_{ij,12}$ respectively. Then the observed data is $\{s_i, \tau_j, K_{ij}, n_{ij,0}, n_{ij,1}, n_{ij,2}, n_{ij,12}, i = 1,2,\ldots,I, j = 1,2,\ldots,J\}$ which we denote by $\boldsymbol{z}$ and $K_{ij} = n_{ij,0} + n_{ij,1} + n_{ij,2} + n_{ij,12}$. Moreover, the total number of devices with failure of component m is denoted by $N_{ij,m} = n_{ij,m} + n_{ij,12}$, for $m = 1,2$.

Let $T_{i,m}$ denote the time to failure of component m at stress level s_i, where $i = 1,2,\ldots,I$, $m = 1,2$ and $p_{ij,0}, p_{ij,1}, p_{ij,2}$, and $p_{ij,12}$ denote the probability that no component failed in device, the probability that only component 1 failed in device, the probability that only component 2 failed in device, and the probability that both components 1 and 2 failed, respectively, at inspection time τ_j. Then

$$
\begin{aligned}
p_{ij,0} &= P(T_{i,1} > \tau_j, T_{i,2} > \tau_j) \\
p_{ij,1} &= P(T_{i,1} \leq \tau_j, T_{i,2} > \tau_j) \\
p_{ij,2} &= P(T_{i,1} > \tau_j, T_{i,2} \leq \tau_j) \\
p_{ij,12} &= P(T_{i,1} \leq \tau_j, T_{i,2} \leq \tau_j)
\end{aligned}
$$

A copula function is used to find this probability by modelling joint distribution and it is done by using the marginal distribution. Sklar's Theorem in Sklar [14] stated that a unique copula function exists such that any joint distribution functions can be written in terms of marginal distribution functions when marginal distributions are continuous

2.1 COPULA

A copula function can be used to model multivariate distribution and it is done by using the marginal distributions, where marginals are uniformly distributed on closed interval of 0 and 1. In the bivariate case, from the Sklar's theorem, if C is copula and F_{X_1} and F_{X_2} are marginal CDFs, then the joint distribution G with marginals F_{X_1} and F_{X_2} is given by:

$$G(x_1,x_2) = P(X_1 \leq x_1, X_2 \leq x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2); \alpha),$$

where X_1 and X_2 are continuous random variables and α is a dependence parameter. The bivariate copula density is given by $c(u,v;\alpha) = \partial C(u,v;\alpha)/\partial u \partial v$, where $u, v \in [0,1]$. Then joint PDF of X_1 and X_2 can be derived as:

$$g(x_1,x_2) = c(F_{X_1}(x_1), F_{X_2}(x_2); \alpha) f_{X_1}(x_1) f_{X_2}(x_2)$$

where $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are marginal PDFs of the random variables X_1 and X_2, respectively. For more information on copula functions, like, the applications of copula, and procedure to generating random variables based on a specific copula, one may refer to the books written on copula by Nelsen [7] and Chiyoshi [15]. There are several class of copulas for example, Archimedean copulas, extreme value copula and Gaussian copula. Among them the Archimedean copulas are very much popular and has many applications due to flexibility, ease in construction, and capability of modelling multivariate joint distributions with one or few parameters. In our numerical studies, we have considered three Archimedean copulas families: Gumbel-Hougaard, Frank, and Clayton copulas, the forms and properties of these copulas are summarized in Table 1.

Table 1: Forms, ranges and generators of Gumbel-Hougaard, Frank, and Clayton copulas.

Copula	Form $C(u,v;\alpha)$	Range	Generator
Gumbel-Hougaard	$\exp\left(-[(-\log u)^{\alpha} + (-\log v)^{\alpha}]^{1/\alpha}\right)$	$\alpha \in [1,\infty)$	$\phi(t) = (-\ln(t))^{\alpha}$
Frank	$-\frac{1}{\alpha}\ln\left(1 + \frac{(\exp(-\alpha u)-1)(\exp(-\alpha v)-1)}{\exp(-\alpha)-1}\right)$	$\alpha \in \mathbb{R} - \{0\}$	$\phi(t) = -\ln\left(\frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}\right)$
Clayton	$\left(\max\left\{(u^{-\alpha} + v^{-\alpha} - 1), 0\right\}\right)^{-1/\alpha}$	$\alpha \in [-1,\infty) - \{0\}$	$\phi(t) = \frac{t^{\alpha}-1}{\alpha}$

Gumbel-Hougaard copula is utilized to model asymmetric dependence in the data. This copula is well-known for its strength to capture strong upper tail dependence and weak lower tail dependence. If outcomes are anticipated to be strongly correlated at high values but less correlated at low values, then Gumbel-Hougaard copula is a suitable selection. When α approaches 1, the marginals become independent, and when it goes to infinity, Gumbel-Hougaard copula approaches the Fréchet-Hoeffding upper bound.

Frank copula provides the maximum range of dependence. This means that the dependence parameter of the copula allows the approximation of the upper and the lower Fréchet-Hoeffding bounds, and thus, it permits modelling positive and negative dependence in the data. When α approaches $+\infty$ and $-\infty$, the Fréchet-Hoeffding upper and lower bound will be achieved. Likewise, the independence case will be achieved when α approaches zero. But, Frank copula has neither lower nor upper tail dependence, this copula is thus suitable for modelling data described by weak tail dependence.

Clayton copula is mainly used to analyze correlated risks because of their strength to capture lower tail dependence. If $\alpha = 0$, then the marginal distributions become independent, and when $\alpha \to \infty$, Clayton copula approximates the Fréchet-Hoeffding upper bound. However, due to the restriction on the dependence parameter α, the Fréchet-Hoeffding lower bound cannot be reached by Clayton copula. It hints that Clayton copula cannot account for negative dependence.

Suppose $T_{i,m}$ has a marginal distribution $F_{i,m}(.)$ and it is further assumed that the dependence parameter, $\alpha_i(\boldsymbol{\theta})$, is related to stress levels, s_i, in a way that $\alpha_i(\boldsymbol{\theta}) = h(s_i, \boldsymbol{\theta})$. Then, the joint cumulative distribution function of $T_{i,1}$ and $T_{i,2}$ under the copula can be written as

$$P(T_{i,1} \leq t_1, T_{i,2} \leq t_2) = C(F_{i,1}(t_1), F_{i,2}(t_2); \alpha_i(\boldsymbol{\theta})).$$

The observed likelihood function is then given by

$$L(\boldsymbol{\theta}|\boldsymbol{z}) = \prod_{i=1}^{I}\prod_{j=1}^{J} p_{ij,0}^{n_{ij,0}} p_{ij,1}^{n_{ij,1}} p_{ij,2}^{n_{ij,2}} p_{ij,12}^{n_{ij,12}},$$

where

$$p_{ij,0} = 1 - F_{i,1}(\tau_j) - F_{i,2}(\tau_j) + C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))$$

$$p_{ij,1} = F_{i,1}(\tau_j) - C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))$$
$$p_{ij,2} = F_{i,2}(\tau_j) - C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))$$
$$p_{ij,12} = C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))$$

Therefore the log-likelihood is given by

$$\begin{aligned} l(\boldsymbol{\theta}) = \ln(L(\boldsymbol{\theta}|\boldsymbol{z})) &= \sum_{i=1}^{I}\sum_{j=1}^{J} \left[n_{ij,0}\ln(p_{ij,0}) + n_{ij,1}\ln(p_{ij,1}) + n_{ij,2}\ln(p_{ij,2}) + n_{ij,12}\ln(p_{ij,12})\right] \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J} \Bigg[n_{ij,0}\ln\left(1 - F_{i,1}(\tau_j) - F_{i,2}(\tau_j) + C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))\right) \\ &\quad + n_{ij,1}\ln\left(F_{i,1}(\tau_j) - C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))\right) \\ &\quad + n_{ij,2}\ln\left(F_{i,2}(\tau_j) - C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))\right) + n_{ij,12}\ln\left(C(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta}))\right)\Bigg] \end{aligned}$$

For estimation and inference purposes, there are many bivariate copulas available for two-component systems for testing data under accelerated life-tests. Hence, there is a possibility of mis-specification of copula models under accelerated life-tests. Therefore, misspecification analysis of copula models will provide useful insights into the choice of the most appropriate copula model among a set of candidate copula models.

2.2 Reliability representations

A system consists of two dependent components connected in the series or/and in parallel. A copula model can be used to construct the bivariate distribution of two components and then obtain the reliability of the system.

2.2.1 Series system

Let us assume that one-shot device is a two-component system wherein the components are place in series. If one of the components fails, the device will fail. The occurrence of any component failures causes the failure of the device to work. Then the lifetime of the device at stress level s_i is denoted by $T_i^S = \min\{T_{i,1}, T_{i,2}\}$. Then, in the series system, the reliability of the device at stress level s_i and at mission time t is given by

$$R_i(t;\boldsymbol{\theta}) = P(T_i^S > t) = 1 - F_{i,1}(t) - F_{i,2}(t) + C(F_{i,1}(t), F_{i,2}(t); \alpha_i(\boldsymbol{\theta})).$$

2.2.2 Parallel system

A parallel system is another fundamental structure of a complex system where components are connected in parallel. Let us assume that one-shot device is a two-component parallel system. The failure time of the device is the maximum failure time of the components. Then the lifetime of the device at stress level s_i is denoted by $T_i^P = \max\{T_{i,1}, T_{i,2}\}$. The reliability of the device at stress level s_i and at mission time t is given by

$$R_i(t;\boldsymbol{\theta}) = P(T_i^P > t) = 1 - C(F_{i,1}(t), F_{i,2}(t); \alpha_i(\boldsymbol{\theta})).$$

3 Copula Model Misspecification Analysis

In this section, the results of White [1] are used to derive MLEs when the assumed copula is incorrect. We shall refer to these incorrect MLEs as quasi-MLEs (QMLEs), as done in White [1], and use MLEs to denote true estimators under no misspecification. Let us model the data with copula $C_1(.;\alpha_i(\boldsymbol{\theta_1}))$, denoted by CM_1, and with copula $C_2(.;\alpha_i(\boldsymbol{\theta_2}))$, denoted by CM_2, with dependence parameters $\alpha_i(\boldsymbol{\theta_1})$ and $\alpha_i(\boldsymbol{\theta_2})$, respectively, for $i = 1, 2, \ldots, I$.

3.1 CM_1 is Correct, CM_2 is Incorrect

If the copula model is correctly specified, then the MLE of the model parameters is the value of the parameter vector that maximizes the likelihood for given observed data. Suppose that the lifetime of the device comes from the correct copula model CM_1, but it is wrongly fitted with another copula model CM_2 then the log-likelihood will be

$$\ln(L(\boldsymbol{\theta_2}|z)) = \sum_{i=1}^{I}\sum_{j=1}^{J}\Big[n_{ij,0}\ln\big(1 - F_{i,1}(\tau_j) - F_{i,2}(\tau_j) + C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\big)$$
$$+ n_{ij,1}\ln\big(F_{i,1}(\tau_j) - C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\big)$$

$$+ n_{ij,2} \ln\left(F_{i,2}(\tau_j) - C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\right)$$
$$\left. + n_{ij,12} \ln\left(C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\right)\right].$$

Here, we propose a 2-step procedure for the parameter estimation, where the marginal distribution are first estimated by the proportion of failures, i.e., $\hat{F}_{i,m}(\tau_j) = N_{ij,m}/K_{ij}$, and then plugged into the likelihood to obtain a quasi-likelihood. Then quasi log-likelihood is given by

$$\ln(\widehat{L}(\boldsymbol{\theta_2}|\boldsymbol{z})) = \sum_{i=1}^{I}\sum_{j=1}^{J}\left[n_{ij,0} \ln\left(1 - \hat{F}_{i,1}(\tau_j) - \hat{F}_{i,2}(\tau_j) + C_2(\hat{F}_{i,1}(\tau_j), \hat{F}_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\right)\right.$$
$$+ n_{ij,1} \ln\left(\hat{F}_{i,1}(\tau_j) - C_2(\hat{F}_{i,1}(\tau_j), \hat{F}_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\right)$$
$$+ n_{ij,2} \ln\left(\hat{F}_{i,2}(\tau_j) - C_2(\hat{F}_{i,1}(\tau_j), \hat{F}_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\right)$$
$$\left. + n_{ij,12} \ln\left(C_2(\hat{F}_{i,1}(\tau_j), \hat{F}_{i,2}(\tau_j); \alpha_i(\boldsymbol{\theta_2}))\right)\right]. \tag{1}$$

Let $\hat{\boldsymbol{\theta}}_{\mathbf{2}}$ is the QMLE which maximizes the quasi log-likelihood (1), i.e.,

$$\hat{\boldsymbol{\theta}}_{\mathbf{2}} = arg \max_{\boldsymbol{\theta_2}} \ \ln(\hat{L}(\boldsymbol{\theta_2}|\boldsymbol{z})).$$

Suppose that the objective is to estimate some lifetime characteristics of the device ϕ (e.g., reliability, mean, distribution quantiles), which are functions of the model parameters. To evaluate the effect of mis-specification of copula on the estimation of the function ϕ, let $B_1(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}}; s_0))$ denote the bias of the estimate of the parameter of interest $\phi(\boldsymbol{\theta_2}; s_0)$ under the CM_2 model at use level s_0 which is equivalent to the parameter of interest under the correct copula model CM_1. Then, the bias can be expressed as

$$B_1(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}}; s_0)) = E(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}}; s_0)) - \phi(\boldsymbol{\theta_1}; s_0), \tag{2}$$

and the relative bias is given by

$$\kappa_1(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}}; s_0)) = \frac{B(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}}; s_0))}{\phi(\boldsymbol{\theta_1}; s_0)}. \tag{3}$$

For example when $\phi(\boldsymbol{\theta_1}; s_0) = R_0(\tau_j; \boldsymbol{\theta_1})$ at inspection time τ_j, then

$$B_1(R_0(\tau_j; \hat{\boldsymbol{\theta}}_{\mathbf{2}})) = E[R_0(\tau_j; \hat{\boldsymbol{\theta}}_{\mathbf{2}})] - R_0(\tau_j; \boldsymbol{\theta_1}),$$

and the relative bias of reliability at inspection time τ_j is given by

$$\kappa_1(R_0(\tau_j;\hat{\boldsymbol{\theta}}_{\mathbf{2}})) = \frac{E[R_0(\tau_j;\hat{\boldsymbol{\theta}}_{\mathbf{2}})]}{R_0(\tau_j;\boldsymbol{\theta}_{\mathbf{1}})} - 1.$$

Now the expected value of the negative CM_2 log-likelihood with respect to CM_1 is

$$\begin{aligned}
&E[-\ln(L(\boldsymbol{\theta}_{\mathbf{2}}|\boldsymbol{z}))] \\
&= -\sum_{i=1}^{I}\sum_{j=1}^{J}\Big[E[n_{ij,0}]\ln\big(1 - F_{i,1}(\tau_j) - F_{i,2}(\tau_j) + C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big) \\
&\qquad + E[n_{ij,1}]\ln\big(F_{i,1}(\tau_j) - C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big) \\
&\qquad + E[n_{ij,2}]\ln\big(F_{i,2}(\tau_j) - C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big) \\
&\qquad + E[n_{ij,12}]\ln\big(C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big)\Big],
\end{aligned}$$

Since $E[n_{ij,m}] = K_{ij}p_{ij,m}$ and $E[n_{ij,12}] = K_{ij}p_{ij,12}$, the expected negative log-likelihood function under the model CM_1 can be expressed as

$$\begin{aligned}
E[-\ln(L(\boldsymbol{\theta}_{\mathbf{2}}|\boldsymbol{z}))] = -\sum_{i=1}^{I}\sum_{j=1}^{J}\Big[&K_{ij}\big(1 - F_{i,1}(\tau_j) - F_{i,2}(\tau_j) + C_1(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{1}}))\big) \\
&\times \ln\big(1 - F_{i,1}(\tau_j) - F_{i,2}(\tau_j) + C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big) \\
&+ K_{ij}\big(F_{i,1}(\tau_j) - C_1(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{1}}))\big)\ln\big(F_{i,1}(\tau_j) - C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big) \\
&+ K_{ij}\big(F_{i,2}(\tau_j) - C_1(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{1}}))\big)\ln\big(F_{i,2}(\tau_j) - C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big) \\
&+ K_{ij}C_1(\tau_j, \tau_j;\alpha_i(\boldsymbol{\theta}_{\mathbf{1}}))\ln\big(C_2(F_{i,1}(\tau_j), F_{i,2}(\tau_j);\alpha_i(\boldsymbol{\theta}_{\mathbf{2}}))\big)\Big], \qquad (4)
\end{aligned}$$

Since it is the negative of expected log-likelihood so for the solution of $\boldsymbol{\theta}_{\mathbf{2}}$ we have to minimize it. Let $\boldsymbol{\theta}_{\mathbf{2}}^{*}$ is the value of $\boldsymbol{\theta}_{\mathbf{2}}$ which minimizes (4), i.e.,

$$\boldsymbol{\theta}_{\mathbf{2}}^{*} = arg\ \underset{\boldsymbol{\theta}_{\mathbf{2}}}{min}\ E[-\ln(L(\boldsymbol{\theta}_{\mathbf{2}}|\boldsymbol{z}))].$$

Note that $\boldsymbol{\theta}_{\mathbf{2}}^{*}$ is a function of $\boldsymbol{\theta}_{\mathbf{1}}$. Let $AB_1(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}};s_0))$ denote the asymptotic bias of the estimate of the parameter of interest $\phi(\boldsymbol{\theta}_{\mathbf{2}};s_0)$ under the copula model CM_2 at used level s_0, which is equivalent to the parameter of interest under the correct copula model CM_1. Then, the asymptotic bias can be expressed as

$$AB_1(\phi(\hat{\boldsymbol{\theta}}_{\mathbf{2}};s_0)) = \phi(\boldsymbol{\theta}_{\mathbf{2}}^{*};s_0) - \phi(\boldsymbol{\theta}_{\mathbf{1}};s_0), \qquad (5)$$

and the asymptotic relative bias is given by

$$A\kappa_1(\phi(\hat{\boldsymbol{\theta}}_2;s_0)) = \frac{AB(\phi(\hat{\boldsymbol{\theta}}_2;s_0))}{\phi(\boldsymbol{\theta}_1;s_0)} = \frac{\phi(\boldsymbol{\theta}_2^*;s_0)}{\phi(\boldsymbol{\theta}_1;s_0)} - 1. \tag{6}$$

When $\phi(\boldsymbol{\theta}_1;s_0) = R_0(\tau_j;\boldsymbol{\theta}_1)$ at inspection time τ_j, then the asymptotic bias of reliability is

$$AB_1(R_0(\tau_j;\hat{\boldsymbol{\theta}}_2)) = R_0(\tau_j;\boldsymbol{\theta}_2^*) - R_0(\tau_j;\boldsymbol{\theta}_1)$$

and the asymptotic relative bias of reliability at inspection time τ_j is given by

$$A\kappa_1(R_0(\tau_j;\hat{\boldsymbol{\theta}}_2)) = \frac{R_0(\tau_j;\boldsymbol{\theta}_2^*)}{R_0(\tau_j;\boldsymbol{\theta}_1)} - 1.$$

3.2 Tests for Detecting a Misspecification of Copula Model

In this section, we first introduce the Akaike information criteria (AIC) as a specification test for the selection of copula models. White [1] stated that a quasi-maximum likelihood estimator converges to a well-defined limit, but maybe inconsistent for a particular parameter of interest. Consequently, it results in invalid inference and prediction. The AIC is commonly used to measure the relative quality of fits achieved by probability models for a given set of data. If a copula model has p model parameters to be estimated, then The AIC value is given by

$$\text{AIC} = -2\ln(L(\boldsymbol{\theta}|z)) + 2p.$$

If the assumed model fits the data well, then the log-likelihood value is large, and the AIC value becomes small. The smaller the AIC, the closer the expected model is to the correct model. When there are several competing copula models, we want to know which copula model best fits the data. We can calculate the AIC value for each copula model with the same data, then the “ best ” model is the one with the least AIC value. Also, the AIC value provides a versatile procedure for statistical model identification and is free from the ambiguities inherent in the application of the conventional hypothesis testing procedures.

4 SIMULATION STUDY

In this section, we consider three different copulas to study the mis-specification analysis. The first one is Gumbel-Hougaard copula with

$$C_1(u,v;\alpha_i) = \exp\Big(-[(-\log u)^{\alpha_i} + (-\log v)^{\alpha_i}]^{1/\alpha_i}\Big),\ \ \alpha_i \geq 1, \forall\, i = 1,2,\ldots,I,$$

where U and V are standard uniform distributed random variables. Since we have further assumed that the correlation between times to two failure components changes over the stress levels, and so

$$\ln(\alpha_i - 1) = a_0 + a_1 s_i, \forall\, i = 1,2,\ldots,I.$$

The second is Frank copula with

$$C_2(u,v;\alpha_i) = \begin{cases} -\frac{1}{\alpha_i}\ln\Big(1 + \frac{(\exp(-\alpha_i u)-1)(\exp(-\alpha_i v)-1)}{\exp(-\alpha_i)-1}\Big), & \alpha_i \in (-\infty,\infty), \\ uv, & \alpha_i = 0, \end{cases}$$

$\forall\ i = 1,2,\ldots,I$, where U and V are standard uniform distributed random variables. Again, the dependence parameter is assumed to relate to stress level s_i in a linear form, namely,

$$\alpha_i = b_0 + b_1 s_i, \forall\, i = 1,2,\ldots,I.$$

The third is Clayton copula with

$$C_3(u,v;\alpha_i) = \Big(\max\big\{(u^{-\alpha_i} + v^{-\alpha_i} - 1), 0\big\}\Big)^{-1/\alpha_i},\ \ \alpha_i \in [-1,\infty]/\{0\}\ \forall\, i = 1,2,\ldots,I,$$

where U and V are standard uniform distributed random variables and the dependence parameter is assumed to relate to stress level s_i as following,

$$\ln(\alpha_i + 1) = c_0 + c_1 s_i, \forall\, i = 1,2,\ldots,I.$$

The parameters settings for the copulas are given in Table 2. Gumbel-Hougaard (Gumbel-H), Frank, and Clayton copulas used to investigate the effect of the misspecification of copula models are summarized as follows:

Table 2: The parameters settings for Gumbel-Hougaard, Frank and Clayton copulas used in the simulation study.

True model parameters	
Gumbel-Hougaard Copula	$(a_0, a_1) = (1, 1)$
Frank Copula	$(b_0, b_1) = (5, 1)$
Clayton Copula	$(c_0, c_1) = (3, 1)$

1. Gumbel-H copula model is CM_1 with $\boldsymbol{\theta_1} = (a_0, a_1)$, misspecified by Frank copula model CM_2 with $\boldsymbol{\theta_2} = (b_0, b_1)$.

2. Frank copula model is CM_1 with $\boldsymbol{\theta_1} = (b_0, b_1)$, misspecified by Gumbel-H copula CM_2 with $\boldsymbol{\theta_2} = (a_0, a_1)$.

3. Gumbel-H copula model is CM_1 with $\boldsymbol{\theta_1} = (a_0, a_1)$, misspecified by Clayton copula model CM_2 with $\boldsymbol{\theta_2} = (c_0, c_1)$.

4. Clayton copula model is CM_1 with $\boldsymbol{\theta_1} = (c_0, c_1)$, misspecified by Gumbel-H copula model CM_2 with $\boldsymbol{\theta_2} = (a_0, a_1)$.

5. Frank copula model is CM_1 with $\boldsymbol{\theta_1} = (b_0, b_1)$, misspecified by Clayton copula model CM_2 with $\boldsymbol{\theta_2} = (c_0, c_1)$.

6. Clayton copula model is CM_1 with $\boldsymbol{\theta_1} = (c_0, c_1)$, misspecified by Frank copula model CM_2 with $\boldsymbol{\theta_2} = (b_0, b_1)$.

For the simulation study, we generate one-shot device test data from bivariate copula with marginal following Weibull and gamma distributions.

If the component lifetime T_{im} follows Weibull distribution with scale parameter λ_{im} and shape parameter η_{im}, the corresponding cumulative distribution function is given by

$$F_{T_{im}}(t) = 1 - \exp\left(- \left(\frac{t}{\lambda_{im}} \right)^{\eta_{im}} \right), \qquad t > 0.$$

The relationship between λ_{im} and s_i is assumed to be a log-link function of the form

$$\lambda_{im} = \exp(u_{m1} + u_{m2}s_i), \;\; \forall \;\; i = 1, 2, \ldots, I,$$

where $\mathbf{u}_m = (u_{m1}, u_{m2})$ are known constants and the relationship between η_{im} and s_i is assumed to be a log-link function of the form

$$\eta_{im} = \exp(v_{m1} + v_{m2}s_i), \;\; \forall \;\; i = 1, 2, \ldots, I,$$

where $\mathbf{v}_m = (v_{m1}, v_{m2})$ are known constants.

If the component lifetime T_{im} follows gamma distribution with scale parameter β_{im} and shape parameter γ_{im}, the corresponding probability density function is

$$f_{T_{im}}(t) = \frac{1}{\Gamma\gamma_{im}\beta_{im}^{\gamma_{im}}} t^{\gamma_{im}-1} \exp\big(-\frac{t}{\beta_{im}}\big) \qquad t > 0$$

The relationship between β_{im} and s_i is assumed to be a log-link function of the form

$$\beta_{im} = \exp(u_{m1} + u_{m2}s_i), \;\; \forall \;\; i = 1, 2, \ldots, I,$$

where $\mathbf{u}_m = (u_{m1}, u_{m2})$ are known constants and the relationship between η_{im} and s_i is assumed to be a log-link function of the form

$$\gamma_{im} = \exp(v_{m1} + v_{m2}s_i), \;\; \forall \;\; i = 1, 2, \ldots, I,$$

where $\mathbf{v}_m = (v_{m1}, v_{m2})$ are known constants.

The settings of the parameters for the Weibull and gamma distributions are summarized in Tables 3 and 4, respectively.

4.1 Detection Power of the Correct Copula Model

First, we assume that one of the three copula models is the true copula model CM_1, which is misspecified by another copula model CM_2. Since the quasi log-likelihood function in the series and the parallel systems are same for one-shot device data. So, we obtain the QMLEs by maximizing

Table 3: The settings of parameters for Weibull distribution.

Parameters	Settings
Sample Sizes K_{ij}	$(30, 100)$
Stress levels s_i	$(-2.5, -1.0, -0.1)$
Inspection Time τ_j	$(0.15, 1.3, 2)$
$\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2$	$(0.7, 0.9),\ (1.2, 1),\ (0.6, 0.8),\ (1.3, 1.1)$

Table 4: The settings of parameters for gamma distribution.

Parameters	Settings
Sample Sizes K_{ij}	$(30, 100)$
Stress levels s_i	$(-2.5, -1.0, -0.1)$
Inspection Time τ_j	$(0.15, 1.3, 2)$
$\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2$	$(0.7, -0.9),\ (1.2, 1),\ (0.6, -0.8),\ (1.3, 1.1)$

the quasi log-likelihood with the misspecified copula. In one-shot device test data, there is no analytic expression for the QMLEs of the parameters of interest. So we conduct the simulation study based on 1000 samples of $(n_{ij,0}, n_{ij,1}, n_{ij,2}, n_{ij,12})$ to obtain the QMLEs. In Tables 5 and 6, the QMLEs are obtained for various sample sizes $K_{ij} \in \{30, 100\}$ when the true copula models incorporated with the Weibull and gamma distributions are misspecified. As the sample size K_{ij} increases, the QMLEs are close to the asymptotic results (a_0^*, a_1^*), (b_0^*, b_1^*), and (c_0^*, c_1^*). A simulation study is also carried out for evaluating the proportion of choosing the correct copula model based on the AIC values for different sample sizes $K_{i,j}$ and different marginal distributions. The results in Tables 7 and 8 reveal that the AIC is more likely to select the true copula model as sample size increases. Besides, we observe that the AIC does not generally perform very well when the marginal distributions are gamma distributions, compared to the cases of the Weibull distributions. It implies that the marginal distributions also have an impact on the effect of misspecification. In case of small sample sizes, when the marginals are Weibull distributed, the AIC performs better in selecting the correct copula model, in comparison of when marginals are gamma distributed.

Table 5: QMLEs when the true copula model with Weibull distributions is fitted with misspecified copula models.

True Model : Gumbel-Hougaard Copula											
Fitted model	b_0^*	b_1^*	K_{ij}	$\hat{b}_0$	$\hat{b}_1$	Fitted model	c_0^*	c_1^*	K_{ij}	$\hat{c}_0$	$\hat{c}_1$
Frank	12.46475	4.13879	30	9.54703	3.29515	Clayton	1.64714	0.42137	30	1.84930	0.49860
			100	13.14028	4.38593				100	1.68389	0.43638
True Model : Frank Copula											
Fitted model	a_0^*	a_1^*	K_{ij}	$\hat{a}_0$	$\hat{a}_1$	Fitted model	c_0^*	c_1^*	K_{ij}	$\hat{c}_0$	$\hat{c}_1$
Gumbel-H	-0.32316	0.44717	30	-0.24396	0.78169	Clayton	0.83451	0.06299	30	0.91739	0.09188
			100	-0.30896	0.45552				100	0.85321	0.06920
True Model : Clayton Copula											
Fitted model	a_0^*	a_1^*	K_{ij}	$\hat{a}_0$	$\hat{a}_1$	Fitted model	b_0^*	b_1^*	K_{ij}	$\hat{b}_0$	$\hat{b}_1$
Gumbel-H	2.19891	1.68336	30	2.97569	2.29788	Frank	26.37671	9.93157	30	33.12386	12.54165
			100	2.49104	1.91393				100	28.73118	10.87054

Table 7: Proportion of choosing the true copula model with Weibull distributions based on AIC values

True model	Fitted model	Proportion	
		$K_{ij} = 30$	$K_{ij} = 100$
Gumbel-H	Frank	0.772	0.855
Frank	Gumbel-H	0.744	0.866
Gumbel-H	Clayton	0.880	0.998
Clayton	Gumbel-H	0.890	0.993
Frank	Clayton	0.770	0.926
Clayton	Frank	0.866	0.963

Table 6: QMLEs when the true copula model with gamma distributions is fitted with misspecified copula models.

True Model : Gumbel-Hougaard Copula											
Fitted model	b_0^*	b_1^*	K_{ij}	$\hat{b}_0$	$\hat{b}_1$	Fitted model	c_0^*	c_1^*	K_{ij}	$\hat{c}_0$	$\hat{c}_1$
Frank	8.47751	2.78486	30	10.09768	2.47590	Clayton	0.72858	0.14290	30	0.96071	0.23836
			100	9.22167	2.21706				100	0.78603	0.16893
True Model : Frank Copula											
Fitted model	a_0^*	a_1^*	K_{ij}	$\hat{a}_0$	$\hat{a}_1$	Fitted model	c_0^*	c_1^*	K_{ij}	$\hat{c}_0$	$\hat{c}_1$
Gumbel-H	-0.06201	0.33909	30	-0.15287	0.41642	Clayton	0.31240	-0.11797	30	0.44673	-0.06689
			100	-0.06315	0.34438				100	0.33569	-0.11168
True Model : Clayton Copula											
Fitted model	a_0^*	a_1^*	K_{ij}	$\hat{a}_0$	$\hat{a}_1$	Fitted model	b_0^*	b_1^*	K_{ij}	$\hat{b}_0$	$\hat{b}_1$
Gumbel-H	4.49601	2.28013	30	5.18854	2.57581	Frank	78.35772	30.53095	30	88.94450	34.74594
			100	4.95676	2.48176				100	82.67208	32.23651

Table 8: Proportion of choosing the true copula model with gamma distributions based on AIC values

True model	Fitted model	Proportion	
		$K_{ij} = 30$	$K_{ij} = 100$
Gumbel-H	Frank	0.842	0.975
Frank	Gumbel-H	0.617	0.681
Gumbel-H	Clayton	0.504	0.737
Clayton	Gumbel-H	0.571	0.758
Frank	Clayton	0.623	0.788
Clayton	Frank	0.868	0.975

4.2 SERIES SYSTEMS

Furthermore, we evaluate the effect of copula misspecification on the reliability of the one-shot device with two components treated as a series system. The asymptotic bias (AB_1), asymtotic relative bias ($A\kappa_1$), empirical bias (B_1), empirical relative bias (κ_1), and root mean square error (RMSE) are obtained for the reliability estimation. In this regard, we carry out 1000 simulations of one-shot device test data $(n_{ij,0}, n_{ij,1}, n_{ij,2}, n_{ij,12})$ for different sample size K_{ij} at different used stress levels. A Monte Carlo simulation based on 1000 samples is investigated to see how effectively the asymptotic behaviour matches the sample size behaviour of the observed bias and root mean square error. The simulated results of the observed bias B_1 and relative bias κ_1 is compared with the asymptotic bias AB_1 and relative bias $A\kappa_1$ in Tables 9-11 (Weibull distributions) and 12-14 (gamma distributions). The results reveal that the empirical bias is close to the asymptotic bias and the RMSE is decreasing as the sample size K_{ij} increases. In all the cases of Weibull distributions, we observed that asymptotic bias are very small which implies that there is no significant effect of copula misspecification on the reliability of the device. Moreover, we observe that asymptotic bias are very small when Gumbel-Hougaard copula wrongly fits to the data that actually come from either Frank or Clayton copula, which clearly indicate that there is no significant effect of misspecification of copula on the estimation of reliability of one-shot device. Therefore, in the series system the Gumbel-Hougaard copula is recommended to fit data when the true model is unknown. For the cases of gamma distributions, it is clear that Clayton copula is not preferable to fit the data that actually come from either Frank or Gumbel-Hougaard copula in terms of the bias, relative bias and RMSE. So, Gumbel-Hougaard or Frank copula model is recommended to fit the data when the true model is unknown because small bias, small relative bias and small RMSE are observed.

Table 9: $AB_1, A\kappa_1, B_1, \kappa_1$ and RMSE of reliability when Gumbel-Hougaard copula with Weibull distributions is misspecified by Frank or Clayton copula in series systems.

True Model : Gumbel-Hougaard Copula									
Fitted model	s_0	τ	$R_{GH}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Frank	1.0	3.0	0.96414	-0.00579	-0.00601	30	-0.00539	-0.00559	0.03795
						100	-0.00547	-0.00568	0.02127
		4.0	0.42325	-0.00003	-0.00008	30	0.00139	0.00329	0.08837
						100	0.00022	0.00053	0.04939
	1.5	5.0	0.97411	-0.00088	-0.00090	30	-0.00069	-0.00071	0.03028
						100	-0.00196	-0.00202	0.01644
		6.0	0.42557	-0.00001	-0.00002	30	-0.00204	-0.00479	0.09160
						100	0.00112	0.00265	0.04903
Clayton	1.0	3.0	0.96414	0.00043	0.00044	30	0.00055	0.00057	0.03389
						100	0.00060	0.00062	0.01819
		4.0	0.42325	0.00000	0.00000	30	-0.00109	-0.00258	0.09206
						100	0.00031	0.00074	0.04882
	1.5	4.0	0.97411	0.00000	0.00000	30	0.00017	0.00018	0.02966
						100	-0.00018	-0.00019	0.01581
		6.0	0.42557	0.00000	0.00000	30	0.00479	0.01127	0.09200
						100	-0.00018	-0.00042	0.05082

Table 10: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Frank copula with Weibull distributions is misspecified by Gumbel-Hougaard or Clayton copula in series systems.

True Model : Frank Copula									
Fitted model	s_0	τ	$R_{FR}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	1.0	3.0	0.95581	0.00200	0.00209	30	0.00279	0.00292	0.03355
						100	0.00141	0.00148	0.01966
		4.0	0.41731	-0.00179	-0.00430	30	0.00444	0.01064	0.09123
						100	0.00266	0.00637	0.04893
	1.5	5.0	0.97291	0.00054	0.00056	30	0.00111	0.00114	0.02901
						100	0.00045	0.00046	0.01609
		6.0	0.42506	0.00019	0.00047	30	0.00223	0.00524	0.09132
						100	0.00099	0.00232	0.04785
Clayton	1.0	3.0	0.95581	0.00762	0.00798	30	0.00335	0.00350	0.03360
						100	0.00615	0.00643	0.01831
		4.0	0.41731	0.00005	0.00012	30	-0.00338	-0.00810	0.08993
						100	0.00107	0.00256	0.04724
	1.5	5.0	0.97291	0.00119	0.00122	30	-0.00012	-0.00012	0.02919
						100	0.00140	0.00144	0.01501
		6.0	0.42506	0.00046	0.00108	30	0.00114	0.00269	0.09139
						100	0.00088	0.00207	0.04770

Table 11: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Clayton copula with Weibull distributions is misspecified by Gumbel-Hougaard or Frank copula in series systems.

True Model : Clayton Copula									
Fitted model	s_0	τ	$R_{CL}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	1.0	3.0	0.96457	0.00000	0.00000	30	0.00204	0.00212	0.03215
						100	0.00001	0.00001	0.01809
		4.0	0.42325	0.00000	0.00000	30	-0.00245	-0.00580	0.08932
						100	0.00147	0.00348	0.05039
	1.5	5.0	0.97411	0.00000	0.00000	30	-0.00018	-0.00018	0.02982
						100	-0.00001	-0.00001	0.01625
		6.0	0.42557	0.00000	0.00000	30	-0.00357	-0.00839	0.09158
						100	-0.00002	-0.00005	0.04885
Frank	1.0	3.0	0.96457	-0.00342	-0.00354	30	-0.00159	-0.00165	0.03338
						100	-0.00316	-0.00328	0.01869
		4.0	0.42325	0.00000	0.00000	30	-0.00039	-0.00092	0.08850
						100	0.00078	0.00185	0.05221
	1.5	5.0	0.97411	-0.00049	-0.00051	30	-0.00059	-0.00061	0.02898
						100	-0.00050	-0.00051	0.01492
		6.0	0.42557	0.00000	0.00000	30	-0.00210	-0.00494	0.08879
						100	-0.00009	-0.00021	0.05019

Table 12: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Gumbel-Hougaard copula with gamma distributions is misspecified by Frank or Clayton copula in series systems.

True Model : Gumbel-Hougaard Copula									
Fitted model	s_0	τ	$R_{GH}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Frank	-0.5	3.0	0.65774	0.00011	0.00017	30	0.00994	0.01511	0.08201
						100	0.00724	0.01101	0.04379
		6.0	0.32265	-0.00649	-0.02013	30	-0.00011	-0.00035	0.07919
						100	-0.00243	-0.00713	0.04445
	0.5	3.0	0.94028	-0.01206	-0.01283	30	-0.00867	-0.00922	0.05151
						100	-0.01102	-0.01172	0.02951
		6.0	0.57248	-0.01719	-0.03004	30	-0.01477	-0.02580	0.08356
						100	-0.01567	-0.02738	0.04971
Clayton	-0.5	3.0	0.65773	-0.02705	-0.04113	30	-0.01196	-0.01818	0.09005
						100	-0.02510	-0.03816	0.05597
		6.0	0.32265	-0.08745	-0.27104	30	-0.06619	-0.20516	0.10660
						100	-0.08085	-0.25059	0.09208
	0.5	3.0	0.94028	-0.00214	-0.00227	30	-0.00151	-0.00160	0.04248
						100	-0.00111	-0.00118	0.02492
		6.0	0.57247	-0.07085	-0.12377	30	-0.04596	-0.08028	0.10879
						100	-0.06373	-0.11132	0.08358

Table 13: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Frank copula with gamma distributions is misspecified by Gumbel-Hougaard or Clayton copula in series systems.

True Model : Frank Copula									
Fitted model	s_0	τ	$R_{FR}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	-0.5	3.0	0.63002	-0.00773	-0.01227	30	-0.01055	-0.01675	0.08331
						100	-0.00795	-0.01262	0.04369
		6.0	0.28022	-0.00054	-0.00192	30	-0.00974	-0.03475	0.06755
						100	-0.00141	-0.00503	0.03613
	0.5	3.0	0.92448	0.00385	0.00417	30	0.00015	0.00016	0.04552
						100	0.00318	0.00344	0.02516
		6.0	0.51778	-0.00629	-0.01216	30	-0.01296	-0.02504	0.08538
						100	-0.00656	-0.01266	0.04529
Clayton	-0.5	3.0	0.63001	-0.03149	-0.04999	30	-0.01929	-0.03063	0.09449
						100	-0.02994	-0.04753	0.06001
		6.0	0.28022	-0.07434	-0.26530	30	-0.06227	-0.22223	0.09733
						100	-0.07294	-0.26028	0.08228
	0.5	3.0	0.92448	0.00054	0.00058	30	0.00097	0.00104	0.04541
						100	0.00081	0.00088	0.02581
		6.0	0.51778	-0.08186	-0.15811	30	-0.05791	-0.11185	0.10999
						100	-0.07763	-0.14993	0.09232

Table 14: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Clayton copula with gamma distributions is misspecified by Gumbel-Hougaard or Frank copula in series systems.

True Model : Clayton Copula									
Fitted model	s_0	τ	$R_{CL}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	-0.5	3.0	0.72476	0.00456	0.00629	30	0.00018	0.00024	0.07932
						100	0.00639	0.00883	0.04382
		6.0	0.36986	0.01539	0.04161	30	0.01489	0.04026	0.09133
						100	0.01724	0.04662	0.05108
	0.5	3.0	0.94453	0.00000	0.00000	30	-0.00066	-0.00070	0.04148
						100	-0.00008	-0.00009	0.02364
		6.0	0.58546	0.00000	0.00000	30	0.00224	0.00382	0.09129
						100	0.00078	0.00114	0.04665
Frank	-0.5	3.0	0.72476	0.00305	0.00421	30	0.00136	0.00188	0.08008
						100	0.00264	0.00364	0.04347
		6.0	0.3698641	0.01546	0.04181	30	0.01402	0.03791	0.08701
						100	0.01463	0.03957	0.05026
	0.5	3.0	0.94453	-0.00064	-0.00068	30	0.00164	0.00173	0.04103
						100	0.00024	0.00025	0.02132
		6.0	0.58546	0.00000	0.00000	30	-0.00066	-0.00113	0.08761
						100	0.00076	0.00131	0.04907

4.3 Parallel Systems

We also evaluate the effect of copula misspecification on the reliability of the one-shot device with two components treated as a parallel system. From Tables 15-20, we observe that the bias B_1 and relative bias κ_1 converge to the asymptotic bias AB_1 and the asymptotic relative bias $A\kappa_1$, respectively, as sample size increases. In addition, From Tables 15-17, we can say that there is no significant effect of copula misspecification on the reliability estimation for the one-shot device with two components placed in parallel. The Gumbel-Hougaard, Frank, and Clayton copulas are equally well in modelling the joint distribution function when the interest is reliability estimation. However, from Tables 18 and 19, it is clear that when Gumbel-Hougaard and Frank copula models are wrongly fitted by Clayton copula, the bias, relative bias and RMSE are large in most of the cases. The effect of misspecification of copula model is more significant for gamma distributions compare to the Weibull distributions. Therefore, in the cases of gamma distributions, Gumbel-Hougaard or Frank copula model is recommended to fit the data when the true model is unknown.

Table 15: $AB_1, A\kappa_1, B_1$, κ_1 and RMSE of reliability when Gumbel-Hougaard copula with Weibull distributions is misspecified by Frank or Clayton copula in parallel systems.

True Model : Gumbel-Hougaard Copula										
Fitted model	s_0	τ	$R_{GH}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Frank	1.0	3.0	0.98965	0.00579	0.00585	30	0.00350	0.00353	0.01376
						100	0.00521	0.00526	0.00802
		4.0	0.86473	0.00003	0.00004	30	0.00275	0.00318	0.05952
						100	0.00122	0.00142	0.03350
	1.5	7.0	0.80856	0.00000	0.00000	30	0.00060	0.00075	0.07220
						100	-0.00165	-0.00204	0.04019
		8.0	0.21230	0.00000	0.00000	30	-0.00074	-0.00347	0.07232
						100	0.00054	0.00256	0.04012
Clayton	1.0	3.0	0.98965	-0.00043	-0.00043	30	0.00053	0.00054	0.01758
						100	-0.00032	-0.00032	0.01034
		4.0	0.86473	0.00000	0.00000	30	0.00234	0.00270	0.05920
						100	-0.00126	-0.00145	0.03474
	1.5	7.0	0.80856	0.00000	0.00000	30	-0.00359	-0.00445	0.07517
						100	0.00035	0.00044	0.03831
		8.0	0.21230	0.00000	0.00000	30	-0.00110	-0.00520	0.07529
						100	-0.00005	-0.00026	0.03941

Table 16: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Frank copula with Weibull distributions is misspecified by Gumbel-Hougaard or Clayton copula in parallel systems.

True Model : Frank Copula									
Fitted model	s_0	τ	$R_{FR}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	1.0	4.0	0.87067	0.00179	0.00206	30	-0.00062	-0.00071	0.06181
						100	0.00157	0.00180	0.03190
		5.0	0.33659	0.00000	0.00000	30	-0.00315	-0.00937	0.08732
						100	-0.00072	-0.00216	0.0443
	1.5	7.0	0.808563	0.00000	0.00000	30	0.00017	0.00021	0.07323
						100	0.00045	0.00056	0.03839
		8.0	0.21230	0.00000	0.00000	30	0.00133	0.00625	0.07270
						100	0.00100	0.00473	0.04190
Clayton	1.0	4.0	0.87067	-0.00004	-0.00005	30	-0.00417	-0.00479	0.05879
						100	-0.00004	-0.00005	0.03111
		5.0	0.33658	0.00001	0.00003	30	0.00220	0.00655	0.08439
						100	0.00109	0.00324	0.04552
	1.5	7.0	0.80856	0.00000	0.00000	30	0.00114	0.00141	0.06955
						100	-0.00191	-0.00236	0.03892
		8.0	0.21230	0.00000	0.00000	30	0.00009	0.00045	0.07583
						100	0.00202	0.00953	0.04003

Table 17: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Clayton copula with Weibull distributions is misspecified by Gumbel-Hougaard or Frank copula in parallel systems.

True Model : Clayton Copula									
Fitted model	s_0	τ	$R_{CL}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	1.0	4.0	0.86473	0.00000	0.00000	30	-0.00099	-0.00115	0.06330
						100	-0.00040	-0.00046	0.03455
		5.0	0.33658	0.00000	0.00000	30	0.00131	0.00391	0.08703
						100	-0.00169	-0.00503	0.04789
	1.5	7.0	0.80856	0.00000	0.00000	30	0.00013	0.00016	0.07317
						100	-0.00003	-0.00004	0.03770
		8.0	0.21230	0.00000	0.00000	30	-0.00007	-0.00336	0.07659
						100	0.00017	0.00082	0.04053
Frank	1.0	4.0	0.86473	0.00000	0.00000	30	-0.00156	-0.00180	0.06184
						100	-0.00078	-0.00090	0.03483
		4.5	0.65655	0.00000	0.00000	30	0.00378	0.00576	0.08484
						100	0.00201	0.00307	0.04816
	1.5	7.0	0.80856	0.00000	0.00000	30	0.00107	0.00132	0.07368
						100	-0.00179	-0.00221	0.04029
		7.4	0.61521	0.00000	0.00000	30	-0.00247	-0.00401	0.08996
						100	0.00136	0.00221	0.04806

Table 18: $AB_1, A\kappa_1, B_1, \kappa_1$ and RMSE of reliability when Gumbel-Hougaard copula with gamma distributions is misspecified by Frank or Clayton copula in parallel systems.

True Model : Gumbel-Hougaard Copula									
Fitted model	s_0	τ	$R_{GH}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Frank	-0.5	3.0	0.83049	-0.00011	-0.00013	30	-0.01105	-0.01330	0.07113
						100	-0.00762	-0.00918	0.03836
		6.0	0.50061	0.00649	0.01297	30	0.00032	0.00065	0.08142
						100	0.00221	0.00442	0.04589
	0.5	3.0	0.97751	0.01206	0.01230	30	0.00887	0.00907	0.02054
						100	0.01088	0.01113	0.01426
		6.0	0.70060	0.01719	0.02455	30	0.01256	0.01793	0.08026
						100	0.01629	0.02326	0.04728
Clayton	-0.5	3.0	0.83049	0.02705	0.03257	30	0.01212	0.01459	0.05012
						100	0.02182	0.02628	0.03312
		6.0	0.50061	0.08745	0.17469	30	0.06783	0.13550	0.10606
						100	0.08371	0.16722	0.09489
	0.5	3.0	0.97751	0.00214	0.00219	30	0.00234	0.00239	0.02207
						100	0.00264	0.00270	0.01165
		6.0	0.70060	0.07086	0.10113	30	0.04465	0.06373	0.08039
						100	0.06363	0.09083	0.07214

Table 19: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Frank copula with gamma distributions is misspecified by Gumbel-Hougaard or Clayton copula in parallel systems.

True Model : Frank Copula									
Fitted model	s_0	τ	$R_{FR}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	-0.5	3.0	0.85822	0.00773	0.00900	30	0.01705	0.01986	0.05299
						100	0.00858	0.00999	0.02832
		6.0	0.54304	0.00054	0.00099	30	0.01061	0.01954	0.08993
						100	0.00203	0.00373	0.04766
	0.5	3.0	0.99331	-0.00385	-0.00388	30	-0.00152	-0.00153	0.01112
						100	-0.00316	-0.00318	0.00681
		6.0	0.75529	0.00629	0.00833	30	0.00829	0.01097	0.07357
						100	0.00754	0.00998	0.03924
Clayton	-0.5	3.0	0.85822	0.03149	0.03670	30	-0.09103	-0.10608	0.09103
						100	0.05261	0.06131	0.05262
		6.0	0.54304	0.07434	0.13690	30	0.02759	0.05082	0.02759
						100	0.09289	0.17105	0.09289
	0.5	3.0	0.99331	-0.00054	-0.00054	30	0.00668	0.00673	0.00668
						100	-0.00307	-0.00310	0.00308
		6.0	0.75529	0.08186	0.10839	30	0.10331	0.13679	0.10331
						100	0.07963	0.10542	0.07963

Table 20: AB_1, $A\kappa_1$, B_1, κ_1 and RMSE of reliability when Clayton copula with gamma distributions is misspecified by Gumbel-Hougaard or Frank copula in parallel systems.

True Model : Clayton Copula									
Fitted model	s_0	τ	$R_{CL}(\tau;s_0)$	AB_1	$A\kappa_1$	K_{ij}	B_1	κ_1	RMSE
Gumbel-H	-0.5	3.0	0.76347	-0.00456	-0.00598	30	-0.00292	-0.00382	0.07744
						100	-0.00229	-0.00301	0.04322
		6.0	0.45339	-0.01539	-0.03394	30	-0.00825	-0.01820	0.09486
						100	-0.01322	-0.02917	0.05103
	0.5	3.0	0.97327	0.00000	0.00000	30	0.00069	0.00071	0.02980
						100	0.00009	0.00010	0.01648
		6.0	0.68761	0.00000	0.00000	30	0.00235	0.00342	0.08577
						100	0.00003	0.00006	0.04950
Frank	-0.5	3.0	0.76347	-0.00304	-0.00399	30	0.00266	0.00349	0.07801
						100	-0.00093	-0.00122	0.04298
		6.0	0.45339	-0.01546	-0.03410	30	-0.00222	-0.00488	0.08712
						100	-0.01455	-0.03208	0.05331
	0.5	3.0	0.97327	0.00064	0.00066	30	0.00065	0.00071	0.02811
						100	0.00155	0.00159	0.01591
		6.0	0.68761	0.00000	0.00000	30	0.00002	0.00003	0.08437
						100	0.00031	0.00046	0.04593

5 CONCLUSION

In this paper, we present a model misspecification analysis by modelling the dependence among the components lifetimes by using Gumbel-Hougaard, Frank, and Clayton copulas for one-shot device test data. Assuming that the marginal distributions are unknown, the reliability estimate is obtained when the correct copula model is misspecified by an incorrect model. In simulation study, the Weibull and gamma distributions are considered in the simulation study. A specification test of AIC is used for model selection, and the AIC generally performs very well in detecting the correct model when the sample size is sufficiently large. In case of small sample size, when the marginals are Weibull distributed, the AIC performs better in detecting the correct copula, in comparison of when marginals are gamma distributed. For the reliability estimation, we observe that the bias and relative bias of estimated reliability are small when marginals are Weibull distributed, while they are relatively serious in the gamma cases. In addition, Gumbel-Hougaard, Frank, and Clayton copulas are equally well in modelling the joint distribution function for the reliability estimation in the Weibull cases. However, Clayton copula is not highly recommended for the gamma cases.

Our analysis is essential because industrial engineers often need to check the performance of the one-shot device in terms of reliability. For one-shot devices connected in a series or parallel structure, they sometimes have a strong dependence among the components. The simulation study shows that the system reliability can be estimated by using these copulas quite accurately regardless of correlation among components of the one-shot device. There is still scope for improvement in our study. In this study, we only focus on the effect of misspecification on the estimate of reliability. Confidence intervals of reliability are also crucial in engineering. The misspecification of copula can also influence the confidence interval of reliability. Hence it is exciting and meaningful to study the effect of misspecification of copula on the confidence interval of reliability.

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