Discriminating Between the Weibull and Log-normal Distributions for Type-II Censored Data

Arabin Kumar Dey[†] & Debasis Kundu[†]

Abstract

Log-normal and Weibull distributions are the two most popular distributions for analyzing lifetime data. In this paper we consider the problem of discriminating between the two distribution functions. It is assumed that the data are coming either from log-normal or Weibull distributions and they are Type-II censored. We use the difference of the maximized log-likelihood functions, in discriminating between the two distribution functions. We obtain the asymptotic distribution of the discrimination statistic. It is used to determine the probability of correct selection in this discrimination process. We perform some simulation studies to observe how the asymptotic results work for different sample sizes and for different censoring proportions. It is observed that the asymptotic results work quite well even for small sizes if the censoring proportions are not very low. We further suggest a modified discrimination procedure. Two real data sets are analyzed for illustrative purpose.

Keywords: Asymptotic distributions; likelihood ratio test; probability of correct selection; log-location-scale family; model selection; Kolmogorov-Smirnov distance.

[†] Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur, Pin 208016, INDIA.

Corresponding Author: Debasis Kundu, Phone no. 91-512-2597141, Fax No. 91-512-2597500, e-mail: kundu@iitk.ac.in.

1 Introduction

Log-normal and Weibull are the two most popular distributions for analyzing skewed lifetime data. The two distributions have several interesting properties and their probability density functions (PDFs) can take different shapes. Although, most of the times the two distribution functions may provide a similar data fit, but still it is desirable to select the correct or more nearly correct model, since the inferences based on the model will often involve the tail probabilities, where the affect of model assumptions are very crucial. The problem becomes more severe if the data are censored. Therefore, it is important to make the best possible choice based on the observed data.

Before progressing further we introduce the following notations. It is assumed that the density function of a log-normal random variable with scale parameter $\eta > 0$ and shape parameter $\sigma > 0$ is

$$f_{LN}(x; \sigma, \eta) = \frac{1}{\sqrt{2\pi}x\sigma} e^{-\frac{1}{2}\left(\frac{\ln x - \ln \eta}{\sigma}\right)^2}; \qquad x > 0,$$

and it will be denoted by $LN(\sigma, \eta)$. Similarly, the density function of a Weibull distribution, with shape parameter $\beta > 0$ and scale parameter $\theta > 0$ is

$$f_{WE}(x; \beta, \theta) = \beta\theta^{\beta} x^{(\beta-1)} e^{-(\theta x)^{\beta}}; \qquad x > 0,$$

and it will be denoted by $WE(\beta, \theta)$.

Now we briefly discuss the necessity of choosing the correct model, if both the models fit the data reasonably well. Cox [8] first discussed the effect of choosing the wrong model. Wiens [22] discussed it nicely by a real data example, and recently Pascual [20] also provided the effect of mis-specification on the maximum likelihood estimates (MLEs) between two distribution functions. The problem due to mis-specification may be observed in other cases also, for example in variable sampling plans, see Schneider [21] or in the construction of confidence bounds, see for example Keats *et al.* [14].

Consider the cumulative distribution function (CDF) of WE(1.67, 0.48) and LN(0.45, 1.50) in Figure 1. From the Figure 1, the closeness of the two distribution functions can be

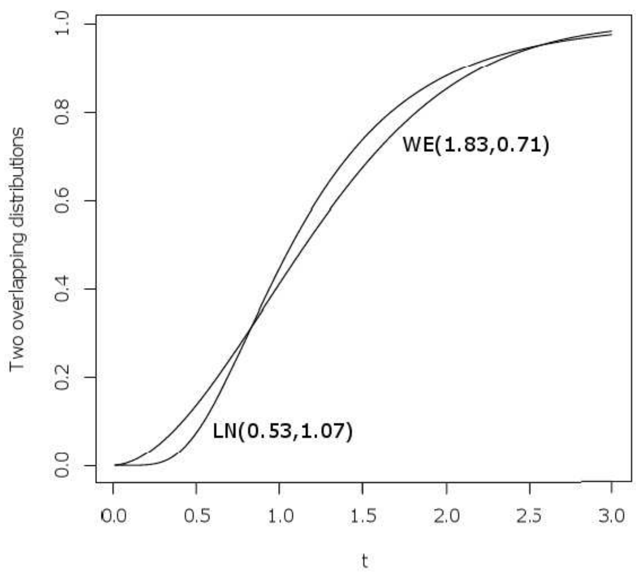

Figure 1: The cumulative distribution functions of WE(1.67, 0.48) and LN(0.45, 1.50)

easily visualized. Therefore, if the data are coming from any one of them, it can be easily modeled by the other one. The problem of choosing the correct distribution becomes more difficult if the sample size is not very large or it is censored. Let us look at the hazard functions and the mean residual life functions of these two distribution functions in Figure 2 and Figure 3 respectively. It is clear that even if the two CDFs are very close to each other, but some of the other characteristics can be quite different. Therefore, even if we have a small or moderate samples, it is still very important to make the best possible decision based on whatever data are available at hand.

In this paper we consider the following problem. Let $X_1, \ldots, X_n$ be a random sample from a log-normal or Weibull distribution and the experimenter observes only the first r of these, namely $X_{(1)} < \ldots < X_{(r)}$. Based on the observed sample the experimenter wants

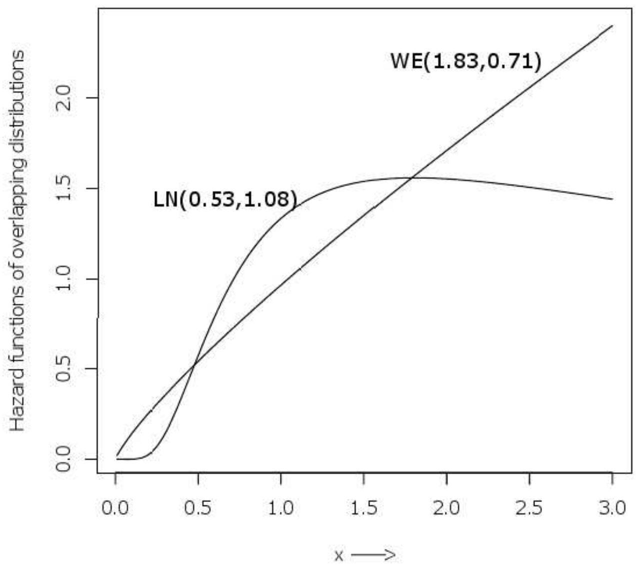

Figure 2: The hazard functions of WE(1.67, 0.48) and LN(0.45, 1.50)

to decide which one is preferable. The problem of testing whether some given observations follow one of the two probability distributions is quite an old problem. See for example the work of Cox [8, 9], Chambers and Cox [6], Atkinson [2, 3], Dyer [11], Bain and Engelhardt [4], Chen [7], Kappeman [13], Fearn and Nebenzahl [12], Kundu, Gupta and Manglick [16], Kundu and Manglick [17], Kim and Yum [15] and the references cited therein.

In this paper we use the difference of the maximized log-likelihood functions in discriminating between the two distribution functions. Suppose, $(\widehat{\beta}, \widehat{\theta})$ and $(\widehat{\sigma}, \widehat{\eta})$ are the MLEs of the Weibull parameters (β, θ) and the log-normal parameters (σ, η) respectively, based on the censored sample $X_{(1)} < \cdots < X_{(r)}$. Then we choose Weibull or log-normal as the preferred model if

$$T_n = L_{WE}(\widehat{\beta}, \widehat{\theta}) - L_{LN}(\widehat{\sigma}, \widehat{\eta}), \tag{1}$$

is greater than zero or less than zero respectively. Here $L_{WE}(\cdot, \cdot)$ and $L_{LN}(\cdot, \cdot)$ denote the log-likelihood functions of the Weibull and log-normal distributions, *i.e.* without the additive

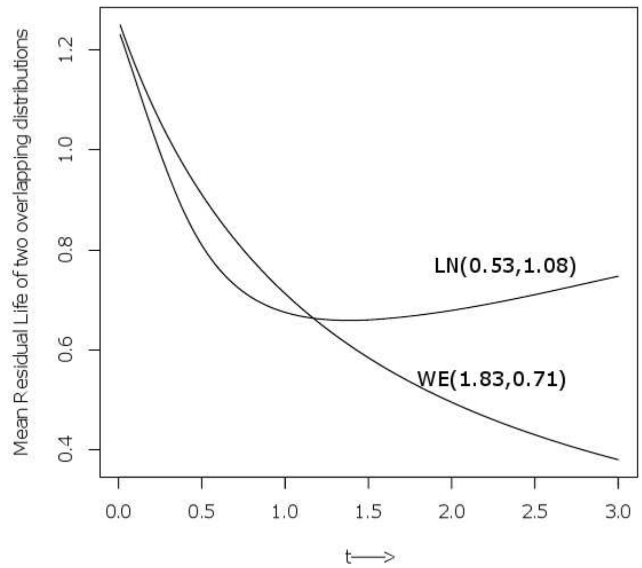

Figure 3: The mean residual life functions of WE(1.67, 0.48) and LN(0.45, 1.50)

constant they can be written as follows;

$$L_{WE}(\beta, \theta) = \sum_{i=1}^{r} \ln f_{WE}(X_{(i)}; \beta, \theta) + (n-r)\ln(1 - F_{WE}(X_{(r)}; \beta, \theta)),$$

and

$$L_{LN}(\sigma, \eta) = \sum_{i=1}^{r} \ln f_{LN}(X_{(i)}; \sigma, \eta) + (n-r)\ln(1 - F_{LN}(X_{(r)}; \sigma, \eta)),$$

respectively.

We obtain the asymptotic distribution of T_n using the approach of Bhattacharyya [5]. It is observed that the asymptotic distributions are normally distributed and they are independent of the unknown parameters. The asymptotic distributions can be used to compute the probability of correct selection (PCS) in selecting the correct model. Moreover, the PCS can be used to discriminate between the two distribution functions for a given PCS. We have performed some simulation experiments to see the behavior of the asymptotic results for different sample sized and also for different censoring proportions and the results are quite satisfactory.

We further propose some modification of the above discrimination procedure. Since the asymptotic distributions are normal and they are independent of the unknown parameters, we suggest to choose Weibull distribution if $T_n > c$ and log-normal otherwise. Here the constant c is chosen such that the total probability of mis-classifications is minimum. Simulation results suggest that for moderate sample sizes the optimal $c \neq 0$.

The rest of the paper is organized as follows. In section 2 we provide the asymptotic distributions of T_n. Sample size determination has been discussed in section 3. The modified method is proposed in section 4. In section 5 and section 6 we present the simulation results and the analysis of two data sets respectively. Finally we conclude the paper in section 7.

2 Asymptotic Results

In this section we derive the asymptotic distribution of the logarithm of RML (ratio of maximized likelihood) when the data are coming from Weibull or from the log-normal distribution. The asymptotic distribution can be used to compute the approximate PCS for different sample sizes. Moreover, for a given user-specified PCS, the minimum sample size needed to discriminate between the Weibull and log-normal distributions also can be obtained. It will be explained in details later.

We use the following notation. Almost sure convergence will be denoted by $a.s.$. For any Borel measurable function, $f_1(\cdot), E_{LN}(f_1(U))$ and $V_{LN}(f_1(U))$ will denote the mean and variance of $f_1(U)$ under the assumption that U follows $LN(\sigma, \eta)$. Similarly, we define, $E_{WE}(f_1(U))$ and $V_{WE}(f_1(U))$ as mean and variance of $f_1(U)$ under the assumption that U follows $WE(\beta, \theta)$. Moreover, if $f_2(\cdot)$ and $f_1(\cdot)$, are two Borel measurable functions, we define

$$Cov_{LN}(f_2(U), f_1(U)) = E_{LN}(f_2(U)f_1(U)) - E_{LN}(f_2(U))E_{LN}(f_1(U))$$

and

$$Cov_{WE}(f_2(U), f_1(U)) = E_{WE}(f_2(U)f_1(U)) - E_{WE}(f_2(U))E_{WE}(f_1(U)),$$

where U follows $LN(\sigma, \eta)$ and $WE(\beta, \theta)$ respectively.

2.1 Case 1: The Data Follow Log-Normal Distribution:

First we present the main result.

Theorem 1 Under the assumption that the data are from $LN(\sigma, \eta)$, the distribution of T_n is approximately normally distributed with mean $E_{LN}(T_n)$ and variance $V_{LN}(T_n)$ for large n and fixed censoring proportion p.

To prove Theorem 1, and for other development we need Lemma 1. The proof of Lemma 1 is provided in the Appendix-A.

Lemma 1 Suppose the data follow $LN(\sigma, \eta)$ and ζ is the p-th percentile point of the $LN(\sigma, \eta)$, *i.e.*, $\lim_{n\to\infty} \frac{r}{n} = p = F_{LN}(\zeta; \sigma, \eta)$; then as $n \to \infty$, we have:

(1) $\widehat{\sigma} \to \sigma \quad a.s., \quad \widehat{\eta} \to \eta \quad a.s.$, where

$$\Pi_1(\sigma, \eta) = \max_{u,v} \Pi_1(u, v)$$

and

$$\Pi_1(u, v) = \int_0^{\zeta} (\ln f_{LN}(x; u, v)) f_{LN}(x; \sigma, \eta) dx + (1-p)\ln(1 - F_{LN}(\zeta, u, v)).$$

(2) $\widehat{\beta} \to \widetilde{\beta} \quad a.s., \quad \widehat{\theta} \to \widetilde{\theta} \quad a.s.$, where

$$\Pi_2(\widetilde{\beta}, \widetilde{\theta}) = \max_{u,v} \Pi_2(u, v), \tag{2}$$

and

$$\Pi_2(u, v) = \int_0^{\zeta} \ln f_{WE}(x; u, v) f_{LN}(x; \sigma, \eta) dx + (1-p)\ln(1 - F_{WE}(\zeta; u, v)).$$

(3) Let us define

$$T_* = L_{WE}(\widetilde{\beta}, \widetilde{\theta}) - L_{LN}(\sigma, \eta).$$

$n^{-1/2}[T_n - E_{LN}(T_n)]$ is asymptotically equivalent to $n^{-1/2}[T_* - E_{LN}(T_*)]$.

Proof of Theorem 1 Using the Theorem 1 of [5] and Lemma 1, the result follows. ∎

It should be mentioned that $\widetilde{\beta}$ and $\widetilde{\theta}$ as obtained in (2) will depend on σ, η and p, but we do not make it explicit for brevity. Moreover, they cannot be obtained explicitly. Both of them can be obtained numerically only. We call $\widetilde{\beta}$ and $\widetilde{\theta}$ as the misspecified Weibull parameters when the data are coming from the log-normal distribution.

Note that $\lim_{n\to\infty} \frac{E_{LN}(T_n)}{n}$ and $\lim_{n\to\infty} \frac{V_{LN}(T_n)}{n}$ exist. We denote $AM_{LN}(p) = \lim_{n\to\infty} \frac{E_{LN}(T_n)}{n}$ and $AV_{LN}(p) = \lim_{n\to\infty} \frac{V_{LN}(T_n)}{n}$. Since the distribution of T_* is independent of σ and η (proof shown in the Appendix-B), without loss of generality we assume $\sigma = \eta = 1$. Now using Theorem 1 of Bhattacharyya [5] we obtain

$$\begin{aligned} AM_{LN}(p) &= \int_0^{\zeta} \left[\ln \frac{f_{WE}(x; \widetilde{\beta}, \widetilde{\theta})}{f_{LN}(\zeta; 1, 1)}\right] f_{LN}(x; 1, 1) dx + (1-p) \ln \frac{(1 - F_{WE}(\zeta; \widetilde{\beta}, \widetilde{\theta}))}{(1 - F_{LN}(\zeta; 1, 1))} \\ &= \left(\frac{1}{2} \ln 2\pi + \ln \widetilde{\beta} + \widetilde{\beta} \ln \widetilde{\theta}\right) p + \widetilde{\beta} E_{LN}((\ln Z) \cdot 1_{Z \le \zeta}) + \frac{1}{2} E_{LN}((\ln Z)^2 \cdot 1_{Z \le \zeta}) \\ &\quad - \widetilde{\theta}^{\widetilde{\beta}} E_{LN}(Z^{\widetilde{\beta}} \cdot 1_{Z \le \zeta}) - (1-p) h_1(\zeta). \end{aligned} \tag{3}$$

Here Z follows $LN(1,1)$, $1_{Z \le \zeta}$ is an indicator random variable. It takes the value 1, if $Z \le \zeta$ and 0 otherwise, Φ is the distribution function of the standard normal random variable and

$$h_1(\zeta) = \left[\ln(1 - \Phi(ln(\zeta))) + (\widetilde{\theta}\zeta)^{\widetilde{\beta}}\right]. \tag{4}$$

Similarly we obtain

$$AV_{LN}(p) = \tau + \frac{p(1-p)}{f_{LN}^2(\zeta; 1, 1)} b^2 \tag{5}$$

where

$$\tau = \int_0^{\zeta} g_{LN}^2(x) f_{LN}(x; 1, 1)\, dx - \frac{1}{p} \left[\int_0^{\zeta} g_{LN}(x) f_{LN}(x; 1, 1) dx\right]^2$$

and

$$b = -f_{LN}(\zeta; 1, 1) g_{LN}(\zeta) + \frac{f_{LN}(\zeta; 1, 1)}{p} \int_0^{\zeta} g_{LN}(x) f_{LN}(x; 1, 1)\, dx - (1-p) h_1^{'}(\zeta).$$

Here

$$g_{LN}(x) = \ln\left[\frac{f_{WE}(x; \widetilde{\beta}, \widetilde{\theta})}{f_{LN}(x; 1, 1)}\right],$$

and $h_1'(\zeta)$ is the derivative of $h_1(\zeta)$ as defined in (4). Note that $\int_0^{\zeta} g_{LN}^2(x) f_{LN}(x; 1, 1)\, dx$ can be written as,

$$\begin{aligned}
\int_0^{\zeta} g_{LN}^2(x) f_{LN}(x; 1, 1)\, dx \;=\; & p\; a_2^2 + (\widetilde{\beta^2} - a_2) E_{LN}((\ln Z)^2 \cdot 1_{Z \le \zeta}) + \frac{1}{4} E_{LN}((\ln Z)^4 \cdot 1_{Z \le \zeta}) \\
& + \widetilde{\theta}^{2\widetilde{\beta}} E_{LN}(Z^{2\tilde{\beta}} \cdot 1_{Z \le \zeta}) + \tilde{\beta} E_{LN}((\ln Z)^3 \cdot 1_{Z \le \zeta}) \\
& - \widetilde{\theta}^{\widetilde{\beta}} E_{LN}(Z^{\widetilde{\beta}} (\ln Z)^2 \cdot 1_{Z \le \zeta}) - 2\widetilde{\beta}\widetilde{\theta}^{\widetilde{\beta}} E_{LN}(Z^{\widetilde{\beta}} \ln Z \cdot 1_{Z \le \zeta}) \\
& - 2\tilde{\beta} a_2 E_{LN}(\ln Z \cdot 1_{Z \le \zeta}) + 2\tilde{\theta}^{\tilde{\beta}} a_2 E_{LN}(Z^{\beta} \cdot 1_{Z \le \zeta}).
\end{aligned}$$

Here $a_2 = \frac{1}{2} \ln 2\pi + \ln \tilde{\beta} + \tilde{\beta}$, Z and $1_{Z \le \zeta}$ are same as defined above.

2.2 Case 2: The Data Follow Weibull Distribution:

In this case also first we present the main result.

Theorem 2 Under the assumption that the data follow $WE(\beta, \theta)$, the distribution of T_n is approximately normally distributed with mean $E_{WE}(T_n)$ and variance $V_{WE}(T_n)$ for large n and fixed censoring proportion p.

Similarly as Lemma 1, we need Lemma 2 to prove Theorem 2 and also for some other developments.

Lemma 2 Suppose the data follow $WE(\beta, \theta)$ and ζ denotes the p-th percentile point of the $WE(\beta, \theta)$ distribution, *i.e.* $\lim_{n \to \infty} \frac{r}{n} = p = F_{WE}(\zeta, \beta, \theta)$, then as $n \to \infty$, we have

(1) $\widehat{\beta} \to \beta \quad a.s., \quad \widehat{\theta} \to \theta \quad a.s.$, where

$$\Pi_3(\beta, \theta) = \max_{u,v} \Pi_3(u, v)$$

and

$$\Pi_3(u, v) = \int_0^{\zeta} (\ln f_{WE}(x; u, v)) f_{WE}(x; \beta, \theta)\, dx + (1-p) \ln \bar{F}_{WE}(\zeta, u, v).$$

(2) $\widehat{\sigma} \to \tilde{\sigma}$ $a.s.$, $\widehat{\eta} \to \tilde{\eta}$ $a.s.$, where

$$\Pi_4(\tilde{\sigma}, \tilde{\eta}) = \max_{u,v} \Pi_4(u, v).$$

and

$$\Pi_4(u, v) = \int_0^{\zeta} (\ln f_{LN}(x; u, v)) f_{WE}(x; u, v)\, dx + (1-p) \ln \bar{F}_{LN}(\zeta, u, v).$$

(3) If we define

$$T_* = L_{WE}(\beta, \theta) - L_{LN}(\tilde{\sigma}, \tilde{\eta}),$$

$n^{-1/2}[T_n - E_{WE}(T_n)]$ is asymptotically equivalent to $n^{-1/2}[T_* - E_{WE}(T_*)]$

Proof of Lemma 2 It follows along the same line as Lemma 1. ∎

Proof of Theorem 2 It follows along the same line as Theorem 1. ∎

In this case also it should be mentioned that $\tilde{\sigma}$ and $\tilde{\eta}$ depend on β, θ and p. Moreover $\tilde{\sigma}$ and $\tilde{\eta}$ have to be obtained numerically only. We call $\tilde{\sigma}$ and $\tilde{\eta}$ as the misspecified log-normal parameters when the data are coming from the Weibull distribution.

Now $\lim_{n\to\infty} \frac{1}{n} E_{WE}(T_n)$ and $\lim_{n\to\infty} \frac{1}{n} V_{WE}(T_n)$ exist and we denote them as $AM_{WE}(p)$ and $AV_{WE}(p)$ respectively. Without loss of generality we assume $\beta = \theta = 1$. Similarly as AM_{LN} and AV_{LN}, AM_{WE} and AV_{WE} can be obtained using Theorem 2 of Bhattacharyya [5] and they are as follows;

$$AM_{WE}(p) \quad = \quad -\int_0^{\zeta} \left[\ln \frac{f_{LN}(x; \tilde{\sigma}, \tilde{\eta})}{f_{WE}(x; 1, 1)} \right] f_{WE}(x; 1, 1) dx - (1-p) \ln \frac{\bar{F}_{LN}(\zeta; \tilde{\sigma}, \tilde{\eta})}{\bar{F}_{WE}(\zeta; 1, 1)}$$

$$
\begin{aligned}
&= -\left(\frac{1}{2}\ln 2\pi + \ln\tilde{\sigma} + \frac{(\ln\tilde{\eta})^2}{2\tilde{\sigma}^2}\right)p - \left(1 - \frac{\ln\tilde{\eta}}{\tilde{\sigma}^2}\right)E_{WE}(\ln(W)\cdot 1_{W\le\zeta}) \\
&\quad + E_{WE}(W\cdot 1_{W\le\zeta}) - \frac{E_{WE}((\ln W)^2\cdot 1_{Z\le\zeta})}{2\tilde{\sigma}^2} - (1-p)h_2(\zeta).
\end{aligned}
$$

Here $h_2(\zeta) = \left[-\zeta - \ln(1 - \Phi(\frac{ln(\zeta) - \ln\tilde{\eta}}{\tilde{\sigma}}))\right]$ and W follows $WE(1,1)$. Similarly,

$$
AV_{WE}(p) = \tau + \frac{pq}{f_{WE}^2(x;\beta,\theta)}b^2, \tag{6}
$$

where

$$
\tau = \int_0^\zeta g_{WE}^2(x)f_{WE}(x;1,1)\,dx - \frac{1}{p}\left[\int_0^\zeta g_{WE}(x)f_{WE}(x;1,1)\,dx\right]^2 \tag{7}
$$

and

$$
b = f_{WE}(\zeta;1,1)g_{WE}(\zeta) - \frac{f_{WE}(\zeta;1,1)}{p}\int_0^\zeta g_{WE}(x)f_{WE}(x;1,1)\,dx + (1-p){h_2}'(\zeta). \tag{8}
$$

Here $h_2'(\zeta)$ is the derivative of $h_2(\zeta)$ and

$$
g_{WE}(x) = \ln\left[\frac{f_{LN}(x;\tilde{\sigma},\tilde{\eta})}{f_{WE}(x;1,1)}\right].
$$

Note that $\int_0^\zeta g_{WE}^2(x)f_{WE}(x;1,1)\,dx$ can be written as

$$
\begin{aligned}
\int_0^\zeta g_{WE}^2(x)f_{WE}(x;1,1)\,dx &= pa_1^2 + 2\beta^* a_1 E_{WE}(\ln W\cdot 1_{W\le\zeta}) - 2a_1E_{WE}(W\cdot 1_{W\le\zeta}) \\
&\quad + \frac{a_1}{\tilde{\sigma}^2}E_{WE}((\ln W)^2\cdot 1_{W\le\zeta}) + (\beta^*)^2E_{WE}((\ln W)^2\cdot 1_{W\le\zeta}) \\
&\quad + E_{WE}(W^2\cdot 1_{W\le\zeta}) + \frac{1}{4\tilde{\sigma}^4}E_{WE}((\ln W)^4\cdot 1_{W\le\zeta}) \\
&\quad - 2\beta^*E_{WE}(W\ln W\cdot 1_{W\le\zeta}) + \frac{\beta^*}{\tilde{\sigma}^2}E_{WE}((\ln W)^3\cdot 1_{W\le\zeta}) \\
&\quad - \frac{1}{\tilde{\sigma}^2}E_{WE}(W(\ln W)^2\cdot 1_{W\le\zeta}),
\end{aligned}
$$

where $a_1 = \left(\frac{1}{2}\ln 2\pi + \ln\tilde{\sigma} + \frac{(\ln\tilde{\eta})^2}{2\tilde{\sigma}^2}\right)$, $\beta^* = \left(1 - \frac{(\ln\tilde{\eta})^2}{2\tilde{\sigma}^2}\right)$ and W is same as before.

3 Determination of Sample Size

In the previous section we obtain the asymptotic distributions of the ratio of the maximized likelihood functions. It can be used to compute PCS for different sample sizes and for different censoring proportions. In this section we propose a method to determine the minimum sample size needed to discriminate between the two distribution functions for a given user specified probability of correct selection and when the censoring proportion is known. Moreover the asymptotic distributions can be used for testing purposes also.

Suppose it is assumed that the data are coming from $LN(\sigma, \eta)$ and the censoring proportion is p. Since T_n is asymptotically normally distributed with mean $E_{LN}(T_n)$ and variance $V_{LN}(T_n)$, therefore, $PCS_{LN} = P(T_n \leq 0)$. Similarly, if it is assumed that the data are coming from $WE(\beta, \theta)$, then for the censoring proportion p, the PCS can be written as $PCS_{WE} = P(T_n > 0)$. Therefore, for a given p, to determine the minimum sample size required to achieve at least α^* protection level, we equate

$$\Phi\left(-\frac{n \times AM_{LN}(p)}{\sqrt{n \times AV_{LN}(p)}}\right) = \alpha^*, \quad \text{and} \quad \Phi\left(\frac{n \times AM_{WE}(p)}{\sqrt{n \times AV_{WE}(p)}}\right) = \alpha^*, \tag{9}$$

and obtain n_1 and n_2 from the above two equations as

$$n_1 = \frac{z^2_{\alpha^*} AV_{LN}(p)}{(AM_{LN}(p))^2} \quad \text{and} \quad n_2 = \frac{z^2_{\alpha^*} AV_{WE}(p)}{(AM_{WE}(p))^2},$$

here z_α is the α-th percentile point of a standard normal distribution. Therefore, we can take $n = \max\{n_1, n_2\}$ the minimum sample size required to achieve at least α^* protection level for a given p.

We are providing $AM_{LN}(p)$, $AV_{LN}(p)$, $AM_{WE}(p)$ and $AV_{WE}(p)$ for different values of p for practical use in Table 1 and we also provide the minimum sample size required to discriminate between the two distribution functions for different α^* and for different p values in Table 2.

Suppose one wants a 0.99 protection level and it is known that approximately 10% observations will be censored at the end (Type-II), in that case one needs at least $n = 294$, to meet that criterion. On the other hand if it is known that only first 20% observations will be

Table 1: The values of $AM_{LN}(p)$, $AV_{LN}(p)$, $AM_{WE}(p)$ and $AV_{WE}(p)$ for different p.

$p \rightarrow$	0.9	0.8	0.7	0.6	0.5	0.4	0.3
$AM_{LN}(p)$	-0.0448	-0.0319	-0.0233	-0.0169	-0.0120	-0.0081	-0.0050
$AV_{LN}(p)$	0.0737	0.0498	0.0356	0.0256	0.0181	0.0123	0.0077
$AM_{WE}(p)$	0.0617	0.0454	0.0336	0.0244	0.0172	0.0115	0.0071
$AV_{WE}(p)$	0.2066	0.1533	0.1128	0.0812	0.0563	0.0368	0.0219

Table 2: Minimum Sample Size to achieve a level of significance level α^* for different p.

$p \rightarrow$ $\alpha^* \downarrow$	0.9	0.8	0.7	0.6	0.5	0.4	0.3
0.99	294	403	543	736	1026	1500	2385
0.95	147	202	272	368	513	750	1192
0.9	90	123	164	224	312	456	724

available, *i.e.* approximately 80% observations will be censored towards the right tail, then to meet 0.99 protection level one needs at least $n = 4450$. Interestingly, in the first case the effective sample size is only 265, where as in the second case the effective sample size is 890 for the same protection level. It may not be very surprising, because the effective discrimination between log-normal and Weibull is possible if we have more observations toward the right tail. Hence if p is small we need larger n. Therefore, if the experimenter has the option to choose p, the best possible choice of p is 1, *i.e.* the complete sample.

4 Modified Method

So far we have discussed about the discrimination procedure based on $T_n > 0$ or $T_n < 0$, and most of the work in the literature based on this method. But it is observed that the asymptotic distribution of T_n is normal, whether the null distribution is log-normal or Weibull and moreover the asymptotic distributions are independent of the unknown parameters. They only depend on the censoring proportion.

The mean and variances of the asymptotic distributions for both the cases are provided in Table 3. Based on that we propose the following discrimination procedure. Choose Weibull distribution if $T_n > c$ and log-normal otherwise. The constant c is chosen such that the total probability of mis-classification is minimum, *i.e.*

$$1 - \Phi\left(\frac{c - \mu_2}{\sigma_2}\right) + \Phi\left(\frac{c - \mu_1}{\sigma_1}\right), \tag{10}$$

is minimum. Here μ_1 and σ_1^2 denote the mean and variance correspond to Weibull distribution and similarly, μ_2 and σ_2^2 denote the mean and variance correspond to log-normal distribution as provided in Table 1. The constant c can be easily obtained by solving a quadratic equation involving μ_1, μ_2, σ_1 and σ_2. For practical users, we provide the optimum c for different censoring proportion.

Table 3: The optimum values of c for different p.

$p \rightarrow$	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2
c	0.2537	0.2262	0.1998	0.1746	0.1499	0.1253	0.1004	0.0739

5 Numerical Results

In this section we perform some simulation experiments mainly to observe how the PCSs based on the asymptotic distributions derived in Section 3, work for different sample sizes and for different censoring proportions. All the computations are performed at the Indian Institute of Technology Kanpur using S-PLUS or R in a Pentium IV machine. The programs can be obtained from the authors on request.

We consider different sample sizes, namely n = 20, 40, 60, 80, 100, 200, and different censoring proportions, for example p = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3. We compute the PCS based on simulation and also based on the asymptotic results obtained in Section 3. Since the distribution of T_n is independent of the shape and scale parameters, we consider the shape and scale parameters to be one in all cases. For a given n and p, we generate a sample of size n either from a $WE(1,1)$ or from a $LN(1,1)$ and compute T_n based on the Type-II censored sample of size $r = [np]$, here $[x]$, means the largest integer less than or equal to x. We check whether $T_n \leq 0$ or $T_n > 0$. We replicate the process 10,000 times and obtain the PCS. We also compute the PCS based on the asymptotic results obtained in section 3. The results are reported in Table 4 and Table 5.

We also computed the PCS based on the modified method suggested in section 4 and using the c values reported in Table 3. It is also based on 10,000 replications and the results are reported in Tables 6 and 7.

It is quite clear from the tables that for fixed n as p increases the PCS increases and similarly for fixed p as n increases the PCS increases, as expected. In all the cases if p is large and n is also large then the asymptotic results match quite well with the simulated results. Comparing Tables 4 and Table 5 with Tables 6 and 7, it is clear that the modified method performs slightly better than the classical one. For the modified method the total PCS (PCS under WE + PCS under LN) is slightly larger than the classical method.

6 Data Analysis

In this section we analyze two real data sets for illustrative purpose. In both the cases the sample size is not very large. It is observed that for one case the choice is quite clear but for the other case it is not very easy to make a decision about the model choice.

Data Set 1: The first data set is obtained from Linhardt and Zucchini [19] (page 69) and it represents the failure times of the air conditioning system of an airplane. They are as follows: 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 90,

Table 4: The probability of correct selection(PCS) based on Monte Carlo simulations(MC) and also based on asymptotic results(AS) when the data are from Weibull distribution for different p

n		20	40	60	80	100	200
p=0.9	MC	0.708	0.789	0.866	0.898	0.926	0.986
	AS	0.728	0.805	0.853	0.888	0.913	0.973
p=0.8	MC	0.662	0.746	0.826	0.846	0.888	0.970
	AS	0.698	0.768	0.815	0.850	0.877	0.949
p=0.7	MC	0.631	0.718	0.739	0.821	0.851	0.940
	AS	0.673	0.737	0.781	0.816	0.841	0.921
p=0.6	MC	0.642	0.659	0.751	0.767	0.812	0.901
	AS	0.649	0.706	0.747	0.778	0.804	0.887
p=0.5	MC	0.579	0.609	0.709	0.733	0.761	0.856
	AS	0.627	0.677	0.713	0.742	0.766	0.848
p=0.4	MC	0.593	0.608	0.686	0.707	0.736	0.805
	AS	0.606	0.648	0.679	0.704	0.726	0.802
p=0.3	MC	0.513	0.559	0.595	0.619	0.648	0.739
	AS	0.585	0.619	0.645	0.666	0.684	0.751

95, 120, 120, 225, 246, 261.

For the complete data set the MLEs of the unknown parameters are $\widehat{\sigma} = 1.3192$, $\widehat{\eta} = 28.7343$, $\widehat{\beta} = 0.8554$, $\widehat{\theta} = 0.0183$. The log-likelihood values correspond to log-normal and Weibull distributions are -151.706 and -152.007. It indicates that for complete sample the preferred model is the log-normal one. Interestingly, although not very surprisingly the Kolmogorov Smirnov (K-S) distances between the empirical distribution function and the fitted log-normal (0.1047) and fitted Weibull (0.1540) also suggest the log-normal as the preferred model.

Note that the log-normal has always unimodal hazard function where as the Weibull distribution can have both increasing ($\beta > 1$) as well as decreasing ($\beta \leq 1$) hazard functions. In this case the fitted Weibull has a decreasing hazard function whereas the fitted log-normal has the unimodal hazard function. From the observed data we try to obtain an estimate of the shape of the hazard function. A device called scaled TTT transform and

Table 5: The probability of correct selection(PCS) based on Monte Carlo simulations(MC) and also based on asymptotic results(AS) when the data are from log-normal distribution for different values of p

n		20	40	60	80	100	200
p=0.9	MC	0.731	0.814	0.869	0.913	0.931	0.987
	AS	0.770	0.852	0.900	0.930	0.951	0.990
p=0.8	MC	0.693	0.782	0.846	0.877	0.923	0.974
	AS	0.739	0.817	0.866	0.899	0.923	0.978
p=0.7	MC	0.689	0.761	0.810	0.848	0.878	0.948
	AS	0.710	0.783	0.831	0.865	0.892	0.960
p=0.6	MC	0.662	0.728	0.749	0.784	0.826	0.911
	AS	0.682	0.748	0.793	0.828	0.829	0.932
p=0.5	MC	0.617	0.665	0.686	0.749	0.750	0.842
	AS	0.655	0.713	0.755	0.787	0.813	0.896
p=0.4	MC	0.568	0.666	0.663	0.690	0.715	0.786
	AS	0.628	0.678	0.714	0.743	0.768	0.849
p=0.3	MC	0.598	0.647	0.676	0.683	0.717	0.785
	AS	0.601	0.641	0.671	0.695	0.716	0.789

its empirical version are relevant in this context. For a family with the survival function $S(y) = 1 - F(y)$, the scaled TTT transform, with $H_F^{-1}(u) = \int_0^{F^1(u)} S(y)dy$ defined for $0 < u < 1$ is $\phi_F(u) = H_F^{-1}(u)/H_F^{-1}(1)$. The empirical version of the scaled TTT transform is given by

$$\phi_n(j/n) = H_n^{-1}(j/n)/H_n^{-1}(1) = \left(\sum_{i=1}^{j} x_{(i)} + (n-j)x_{(j)}\right) / (\sum_{i=1}^{n} x_{(i)}),$$

here $j = 1, \ldots, n$ and $x_{(i)}$ for $i = 1, \ldots, n$ represent the order statistics of the sample. Aarset [1] showed that the scaled TTT transform is convex (concave) if the hazard rate is decreasing (increasing), and for bathtub (unimodal) hazard rates, the scaled TTT transform is first convex (concave) and then concave (convex). We have plotted the empirical version of the scaled TTT transform of the data set 1 in Figure 4.

From the Figure 4 it is at least quite clear that the scaled TTT transform is not concave or convex in the entire range. Therefore the empirical hazard function is not a monotone function. Therefore, between the two models based on the scaled TTT transform clearly,

Table 6: The probability of correct selection(PCS) based on Monte Carlo simulations(MC) and also based on asymptotic results(AS) when the data are from Weibull distribution for different p

n		20	40	60	80	100	200
p=0.9	MC	0.643	0.776	0.844	0.894	0.922	0.983
	AS	0.685	0.779	0.836	0.875	0.903	0.970
p=0.8	MC	0.591	0.717	0.790	0.844	0.878	0.960
	AS	0.652	0.740	0.795	0.834	0.865	0.945
p=0.7	MC	0.547	0.668	0.739	0.794	0.831	0.932
	AS	0.623	0.705	0.757	0.796	0.827	0.915
p=0.6	MC	0.511	0.627	0.694	0.749	0.787	0.898
	AS	0.597	0.672	0.720	0.757	0.787	0.879
p=0.5	MC	0.479	0.590	0.653	0.701	0.744	0.852
	AS	0.573	0.640	0.684	0.718	0.746	0.837
p=0.4	MC	0.445	0.538	0.595	0.637	0.679	0.785
	AS	0.549	0.609	0.648	0.678	0.703	0.789
p=0.3	MC	0.405	0.492	0.542	0.574	0.609	0.717
	AS	0.525	0.578	0.612	0.638	0.660	0.736

log-normal is preferable.

Now instead of the complete sample suppose we had observed only the first $r < n$ observations, then what would have been the preferred model? We have reported the results for different values of r, namely 27, 24, 21, 18, 15, 12, 9 and 6 in Table 8.

Interestingly, for all r, the preferred model is log-normal. It is also observed that in all cases the plot never shows the concavity. Therefore, from the scaled TTT transform also it is clear that log-normal is preferable for all r.

DATA SET 2: The second data set represents the number of revolution before failure of the 23 ball bearings in the life-test. It was originally reported in Lawless [18] (page 228). It is as follows: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88. 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

For Data Set 2, the MLEs of the unknown parameters are $\widehat{\sigma} = 0.5215$, $\widehat{\eta} = 63.4890$, $\widehat{\beta} =$

Table 7: The probability of correct selection(PCS) based on Monte Carlo simulations(MC) and also based on asymptotic results(AS) when the data are from Log-normal distribution for different p

n		20	40	60	80	100	200
p=0.9	MC	0.799	0.861	0.905	0.928	0.951	0.988
	AS	0.828	0.883	0.919	0.943	0.959	0.992
p=0.8	MC	0.788	0.832	0.873	0.900	0.927	0.974
	AS	0.807	0.856	0.892	0.918	0.937	0.982
p=0.7	MC	0.768	0.803	0.845	0.871	0.897	0.954
	AS	0.799	0.845	0.880	0.906	0.926	0.976
p=0.6	MC	0.747	0.782	0.813	0.841	0.861	0.926
	AS	0.763	0.780	0.831	0.857	0.878	0.942
p=0.5	MC	0.726	0.754	0.780	0.799	0.819	0.884
	AS	0.742	0.770	0.798	0.822	0.842	0.910
p=0.4	MC	0.719	0.730	0.760	0.774	0.793	0.858
	AS	0.735	0.750	0.770	0.789	0.807	0.870
p=0.3	MC	0.701	0.713	0.726	0.738	0.752	0.809
	AS	0.695	0.706	0.722	0.738	0.753	0.812

2.1050, $\widehat{\theta} = 0.0122$. The log-likelihood values for log-normal and Weibull distributions are -113.1017 and -113.6887 respectively. Therefore, for complete sample the log-normal is the preferred model. The K-S distances for log-normal and for Weibull are 0.0901 and 0.1521 respectively. We have reported the results for different values of $r = 20, 17, 14, 11, 8$ in Table 9 and the scaled TTT transform is plotted in Figure 5.

In this case also from the scaled TTT transform of the complete sample, it is clear that the hazard function is not monotone. Therefore it is not surprising that for complete sample the log-normal is preferred than Weibull. But as r decreases, for example when $r \leq 11$, the scaled TTT transform is convex and therefore the based on the censored observations for ($r \leq 11$) clearly Weibull is the preferred model and that is the choice using log-likelihood functions also.

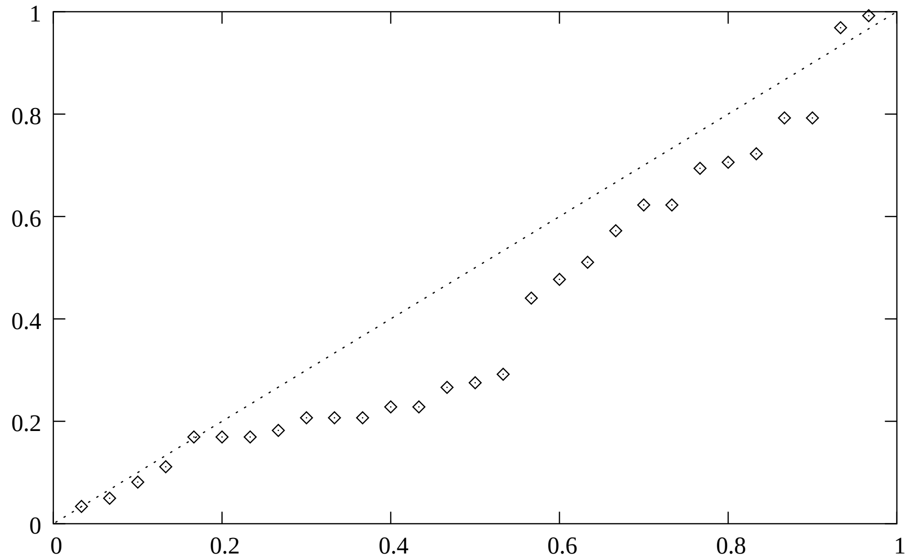

Figure 4: The empirical scaled TTT transform of the data set 1

Table 8: The performance of likelihoods for different choices of r.

p	r	Weibull log-likelihood	Log-normal log-likelihood	Preferred model
0.9	27	-144.82	-143.31	Log-normal
0.8	24	-136.79	-134.69	Log-normal
0.76	21	-133.79	-131.94	Log-normal
0.6	18	-114.86	-113.79	Log-normal
0.5	15	-100.48	-99.79	Log-normal
0.4	12	-85.32	-85.09	Log-normal
0.266	9	-59.72	-59.36	Log-normal
0.166	6	-40.70	-40.59	Log-normal

7 Conclusions

In this paper we consider the problem of discriminating between the log-normal and Weibull families when the data are Type-II censored. We consider the ratio of maximized likelihoods and make the decision based on that ratio. We also obtain the asymptotic distributions of the RML statistics. It is observed the asymptotic distributions are asymptotically normal and they are independent of the parameters of the corresponding parent distribution. We

Table 9: The performance of likelihoods for different choices of r.

p	r	Weibull log-likelihood	Log-normal log-likelihood	Preferred Model
0.9	20	-99.43	-99.23	Log-normal
0.74	17	-87.20	-86.50	Log-normal
0.61	14	-70.34	-70.70	Weibull
0.48	11	-55.00	-56.28	Weibull
0.35	8	-43.14	-43.64	Weibull

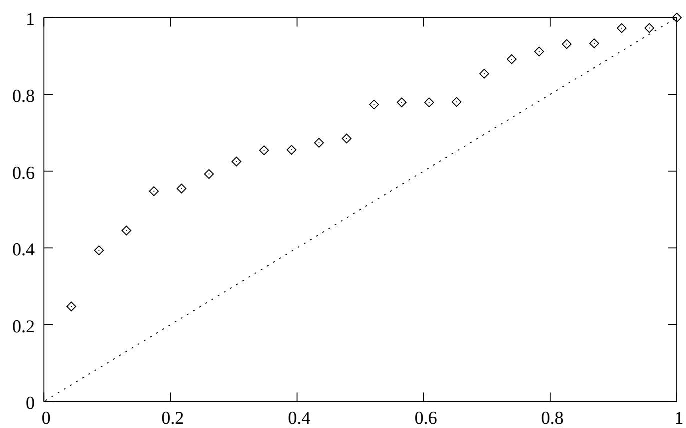

Figure 5: The empirical scaled TTT transform of the data set 2

compare the probability of correct selection using Monte Carlo simulations and using the proposed asymptotic results for different sample sizes and for different censoring proportions. It is observed that the asymptotic results work quite well even for small sample sizes when Weibull is parent distribution, for log-normal as the parent distribution asymptotic results match with simulated result for moderate or large sample sizes. We have also used these asymptotic results to calculate the minimum sample size required to discriminate between the two probability distributions for given a PCS. Although we have considered discriminating between the Weibull and log-normal distributions, but the results can be extended for any two members of the location and scale families or for more than two families also. Work is in progress it will be reported elsewhere.

Acknowledgements:

The authors would like to thank two referees and the associate for their valuable comments.

Appendix - A

Proof of Lemma 1: The proof of (1) and (2) mainly follow using Theorem 1 and Theorem 2 of Bhattacharyya [5]. It can be easily shown that the log-normal distribution satisfies all the regularity conditions mentioned in Theorem 1 and Theorem 2 of Bhattacharyya [5]. The proof of (3) is given below. Note that

$$\frac{T_n}{n} = \frac{1}{n}\left[L_{WE}(\widehat{\beta},\widehat{\theta}) - L_{LN}(\widehat{\sigma},\widehat{\eta})\right] = \frac{1}{n}\left[\sum_{i=1}^{r} g(X_{(i)},\widehat{\gamma}) + (n-r)h(X_{(r)},\widehat{\gamma})\right]$$

and

$$\frac{T_*}{n} = \frac{1}{n}\left[\sum_{i=1}^{r} g(X_{(i)},\widetilde{\gamma}) + (n-r)h(X_{(r)},\widetilde{\gamma})\right]$$

where, $\gamma = (\sigma,\eta,\beta,\theta)^T$, $\widehat{\gamma} = \left(\widehat{\sigma},\widehat{\eta},\widehat{\beta},\widehat{\theta}\right)^T$, $g(X_{(i)};\gamma) = \ln\left(\frac{f_{WE}(X_{(i)},\beta,\theta)}{f_{LN}(X_{(i)},\sigma,\eta)}\right)$, $h(X_{(i)};\gamma) = \ln\left(\frac{\bar{F}_{WE}(X_{(i)};\beta,\theta)}{\bar{F}_{LN}(X_{(i)};\sigma,\eta)}\right)$, and $\widetilde{\gamma} = \left(\sigma,\eta,\widetilde{\beta},\widetilde{\theta}\right)^T$. We need to show

$$\frac{1}{\sqrt{n}}\left[T_n - E_{LN}(T_n)\right] - \frac{1}{\sqrt{n}}\left[T_* - E_{LN}(T_*)\right] \xrightarrow{P} 0, \tag{11}$$

here '$\xrightarrow{P}$' means converges in probability. To prove (11) it is enough to prove

$$\frac{1}{\sqrt{n}}\left[T_n - T_*\right] \xrightarrow{P} 0 \tag{12}$$

and $E_{LN}\left[\frac{1}{\sqrt{n}}\left[T_n - T_*\right]\right] =$

$$E_{LN}\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{r}\left[(g(X_{(i)};\widehat{\gamma})) - (g(X_{(i)};\widetilde{\gamma}))\right] + \sqrt{n}\left(\frac{n-r}{n}\right)\left[h(X_{(r)};\widehat{\gamma}) - h(X_{(r)};\widetilde{\gamma})\right]\right\} \longrightarrow 0. \tag{13}$$

Note that $\frac{1}{\sqrt{n}}[T_n - T_*]$ is

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{r}\left[g(X_{(i)};\widehat{\gamma}) - g(X_{(i)};\widetilde{\gamma})\right] - \left(\frac{n-r}{n}\right)\left[h(X_{(r)};\widehat{\gamma}) - h(X_{(r)};\widetilde{\gamma})\right]\right] =$$

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}[g(X_i;\widehat{\gamma}) - g(X_i;\widetilde{\gamma})]\cdot 1_{X_i\le X_{(r)}} - \left(\frac{n-r}{n}\right)\left[h(X_{(r)};\widehat{\gamma}) - h(X_{(r)};\widetilde{\gamma})\right]\right] \overset{a.e.}{=}$$

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}[g(X_i;\widehat{\gamma}) - g(X_i;\widetilde{\gamma})]\cdot 1_{X_i\le \zeta} - \left(\frac{n-r}{n}\right)\left[h(X_{(r)};\widehat{\gamma}) - h(X_{(r)};\widetilde{\gamma})\right]\right] =$$

$$\sum_{j=1}^{4}\left[\frac{1}{n}\sum_{i=1}^{n} g_j^{'}(X_i;\widetilde{\gamma}_j)1_{X_i\le\zeta} + \left(\frac{n-r}{n}\right)h_j^{'}(X_{(r)};\widetilde{\gamma}_j)\right]\sqrt{n}\,(\widehat{\gamma}_j - \widetilde{\gamma_j}) + o_p(1), \tag{14}$$

here $\overset{a.e.}{=}$ means asymptotically equivalent, $o_p(1)$ means converges to zero in probability, where $g_j^{'} = \frac{\partial}{\partial\gamma_j}g(\cdot)$, $h_j^{'} = \frac{\partial}{\partial\gamma_j}h(\cdot)$, $\widehat{\gamma}_j$ is the j-th component of the $\widehat{\gamma}$ and $\widetilde{\gamma_j}$ is the j-th component of the $\widetilde{\gamma}$ for $j = 1,\ldots,4$. Note that the second equality follows from Theorem-2 of [5], as $\sum_{i=1}^{n} g(X_i;\cdot)\cdot 1_{X_i\le\zeta}$ and $\sum_{i=1}^{n} g(X_i;\cdot)\cdot 1_{X_i\le X_{(r)}}$ are asymptotically equivalent. Now, using weak law of large number and the fact that $h^{'}(\cdot)$ is a continuous function we can say

$$\left[\frac{1}{n}\sum_{i=1}^{n} g_j^{'}(X_i;\widetilde{\gamma}_j)\cdot 1_{X_i\le\zeta} + \frac{n-r}{n}h_j^{'}(X_{(r)};\widetilde{\gamma}_j)\right] \xrightarrow{P} \left[E_{LN}(g_j^{'}(X_i;\gamma_j)\cdot 1_{X_i\le\zeta}) + (1-p)h_j^{'}(\zeta;\gamma_j)\right]\Big|_{\gamma=\widetilde{\gamma}}$$

Again using a similar argument as in Fearn and Nebenzahl [12], it follows that

$$\left|\frac{1}{n}\sum_{i=1}^{n} g^{'}(X_i;\widetilde{\gamma_j})\cdot 1_{X_i\le\zeta} + \frac{n-r}{n}h^{'}(X_{(r)};\widetilde{\gamma_j})\right| \text{ is asymptotically bounded.} \tag{15}$$

Therefore using dominated convergence theorem we can express

$$\begin{aligned}
&\left[E_{LN}(g_j^{'}(X_i;\gamma_j)\cdot 1_{X_i\le\zeta}) + (1-p)h_j^{'}(\zeta;\gamma_j)\right]\Big|_{\gamma=\widetilde{\gamma}} \\
&= \frac{\partial}{\partial\gamma_j}\left[\int_0^{\zeta}\ln\left(\frac{f_{WE}(x;\beta,\theta)}{f_{LN}(x;\sigma,\eta)}\right)f_{LN}(x;\sigma,\eta)dx + (1-p)\ln\left(\frac{\bar{F}_{WE}(\zeta,\beta,\theta)}{\bar{F}_{LN}(\zeta;\sigma,\eta)}\right)\right]\Big|_{\gamma=\widetilde{\gamma}} \\
&= 0,
\end{aligned} \tag{16}$$

Now, using (16) and $\sqrt{n}\,(\widehat{\gamma}_j - \widetilde{\gamma_j})$ converges to a normal distribution with mean zero and finite variance, (12) follows.

Now to prove (13), observe that

$$E_{LN}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{r}\left[g(X_{(i)};\widehat{\gamma})-g(X_{(i)};\widetilde{\gamma})\right]+\sqrt{n}\left(\frac{n-r}{n}\right)\left[h(X_{(r)};\widehat{\gamma})-h(X_{(r)};\widetilde{\gamma})\right]\right|\leq$$
$$\sum_{j=1}^{4}E_{LN}\left|\left[\frac{1}{n}\sum_{i=1}^{n}g_j'(X_i;\widetilde{\gamma}_j)\cdot 1_{X_i\leq\zeta}+\frac{n-r}{n}h^{'}(X_{(r)};\widetilde{\gamma}_j)\right]\sqrt{n}(\widehat{\gamma}_j-\widetilde{\gamma}_j)\right|+o(1)\leq$$
$$\sum_{j=1}^{4}\left[E_{LN}\left[\sqrt{n}\left(\widehat{\gamma}_j-\widetilde{\gamma_j}\right)\right]^2 E_{LN}\left[\frac{1}{n}\sum_{i=1}^{n}g^{'}(X_i;\widetilde{\gamma_j})\cdot 1_{X_i\leq\zeta}+\frac{n-r}{n}h^{'}(X_{(r)};\widetilde{\gamma_j})\right]^2\right]^{\frac{1}{2}}+o(1).$$

Therefore, using (15) and (16), it is observed that

$$E_{LN}\left|\frac{1}{n}\sum_{i=1}^{n}g^{'}(X_i;\widetilde{\gamma_j})\cdot 1_{X_i\leq\zeta}+\frac{n-r}{n}h^{'}(X_{(r)};\widetilde{\gamma_j})\right|^2\longrightarrow 0.$$

Since $\lim_{n\to\infty}E_{LN}\left[\sqrt{n}\left(\widehat{\gamma}_j-\widetilde{\gamma_j}\right)\right]^2<\infty$ for all $j=1,\ldots,4$, (13) follows.

Appendix - B

T_* is independent of parameters of the parent distribution

We can maximize $\Pi_2(\beta,\theta)$ to get $\widetilde{\beta}$ and $\widetilde{\theta}$. It may be observed that equivalently we can maximize

$$\begin{aligned}\Lambda(\beta,\theta) &= \int_0^{\zeta}\ln f_{WE}(x;\beta,\theta)f_{LN}(x;\sigma,\eta)dx+q\ln(1-F_{WE}(\zeta;\beta,\theta))\\ &- \int_0^{\zeta}\ln f_{LN}(x;\sigma,\eta)f_{LN}(x;\sigma,\eta)dx-q\ln(1-F_{LN}(\zeta;\sigma,\eta)),\end{aligned}\tag{17}$$

$q=1-p$. From definition of ζ, we have

$$\begin{aligned}&\Phi\left(\frac{\ln\zeta-\ln\eta}{\sigma}\right)=p\Rightarrow\ln\zeta=\ln\eta+\sigma\Phi^{-1}(p)\\ \Rightarrow\ &\ln\zeta^{\widetilde{\beta}}=\ln\eta^{\widetilde{\beta}}+\sigma\widetilde{\beta}\Phi^{-1}(p)\Rightarrow\ln\zeta^{\widetilde{\beta}}=\widetilde{\tau}_1+\widetilde{\tau}_2\Phi^{-1}(p).\end{aligned}\tag{18}$$

where, $\widetilde{\tau}_1=\ln\eta^{\widetilde{\beta}}$ and $\widetilde{\tau}_2=\sigma\widetilde{\beta}$. Now,

$$\frac{\delta\Lambda(\beta,\theta)}{\delta\theta} = \beta p-\beta\theta^{\beta}\left[\int_0^{\zeta}x^{\beta}f_{LN}(x;\sigma,\eta)dx+q\zeta^{\beta}\right]=0$$

$$
\begin{aligned}
\Rightarrow \quad \widetilde{\theta}^{\widetilde{\beta}} &= \frac{p}{q\zeta^{\widetilde{\beta}} + \int_0^{\zeta} x^{\widetilde{\beta}} f_{LN}(x,\sigma,\eta)dx} \\
&= \frac{p}{\left(qe^{\widetilde{\tau}_1+\widetilde{\tau}_2\Phi^{-1}(p)} + e^{\widetilde{\tau}_1+\frac{1}{2}\widetilde{\tau}_2^2}(\Phi(\Phi^{-1}(p)-\widetilde{\tau}_2))\right)} \qquad (19)
\end{aligned}
$$

Now, substituting $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ in the expression of T_* we get,

$$
\begin{aligned}
T_* &= r\ln\widetilde{\tau}_2 + r\ln p - r\ln\left[qe^{\widetilde{\tau}_1+\widetilde{\tau}_2\Phi^{-1}(p)} + e^{\widetilde{\tau}_1+\frac{1}{2}\widetilde{\tau}_2^2}\left(\Phi(\Phi^{-1}(p)-\widetilde{\tau}_2)\right)\right] \\
&+ \widetilde{\tau}_2\sum_{i=1}^{r} z_i + r\widetilde{\tau}_1 - \frac{p\sum_{i=1}^{r} e^{\widetilde{\tau}_1+z_i\widetilde{\tau}_2}}{q\left[e^{\widetilde{\tau}_1+\widetilde{\tau}_2\Phi^{-1}(p)}\right] + e^{\widetilde{\tau}_1+\frac{1}{2}\widetilde{\tau}_2^2}\left[\Phi(\Phi^{-1}(p)-\widetilde{\tau}_2)\right]} \\
&+ (n-r)\frac{pe^{z_r\widetilde{\tau}_2+\widetilde{\tau}_1}}{qe^{\widetilde{\tau}_1+\Phi^{-1}(p)\widetilde{\tau}_2} + e^{\widetilde{\tau}_1+\frac{1}{2}\widetilde{\tau}_2^2}\left[\Phi(\Phi^{-1}(p)-\widetilde{\tau}_2)\right]} + \frac{1}{2}\sum_{i=1}^{r} z_i^2 - (n-r)\ln\Phi(z_r)
\end{aligned}
$$

$$
\begin{aligned}
&= r\ln\widetilde{\tau}_2 + r\ln p - r\ln\left[qe^{\widetilde{\tau}_2\Phi^{-1}(p)} + e^{\frac{1}{2}\widetilde{\tau}_2^2}\left[\Phi(\Phi^{-1}(p)-\widetilde{\tau}_2)\right]\right] \\
&+ \widetilde{\tau}_2\sum_{i=1}^{r} z_i - \frac{p\sum_{i=1}^{r} e^{\widetilde{\tau}_2 z_i}}{qe^{\widetilde{\tau}_2\Phi^{-1}(p)} + e^{\frac{1}{2}\widetilde{\tau}_2^2}\left[\Phi(\Phi^{-1}(p)-\widetilde{\tau}_2)\right]} + \frac{1}{2}\sum_{i=1}^{r} z_i^2 - (n-r)\ln\Phi(z_r). \qquad (20)
\end{aligned}
$$

Where $z_i = (\frac{\ln x_i - \ln\eta}{\sigma})$ follows $N(0,1)$. Clearly from (20), T_* is independent of $\widetilde{\tau}_1$ i.e. it is independent of η. Therefore without loss of generality we can choose $\eta = 1$ or $\widetilde{\tau}_1 = 0$. Now we have to show T_* is independent of σ i.e. τ_2 is a function of p only.

For given β, maximizing $\Lambda(\beta,\theta)$ w.r.t. θ, we can get

$$
\begin{aligned}
\widetilde{\theta}(\beta) &= \left[\frac{p}{q\zeta^{\beta} + \int_0^{\zeta} x^{\beta} f_{LN}(x;\sigma,\eta)dx}\right]^{\frac{1}{\beta}} \\
&= \frac{p}{qe^{\tau_2\Phi^{-1}(p)} + e^{\frac{1}{2}\tau_2^2}\left(\Phi\left(\Phi^{-1}(p)-\tau_2\right)\right)}
\end{aligned}
$$

where $\tau_2 = \sigma\beta$. Now we get $\widetilde{\beta}$ by maximizing $\Lambda(\beta,\widetilde{\theta}(\beta))$ w.r.t. β. Therefore for given β, we can express

$$
\begin{aligned}
\Lambda(\beta,\widetilde{\theta}(\beta)) &= p\ln\tau_2 + p\ln\frac{p}{q\tau_2\Phi^{-1}(p) + e^{\frac{1}{2}\tau_2^2}(\Phi(\Phi^{-1}(p)-\tau_2))} + \tau_2\int_{-\infty}^{\Phi^{-1}(p)}\frac{1}{\sqrt{2\pi}}ze^{-\frac{z^2}{2}}dz \\
&- \frac{p}{qe^{\tau_2\Phi^{-1}(p)} + e^{\frac{\tau_2^2}{2}}[\Phi(\Phi^{-1}(p)-\tau_2)]}\int_{-\infty}^{\Phi^{-1}(p)} e^{z\tau_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + (\frac{1}{\sqrt{2\pi}}\ln 2\pi)p
\end{aligned}
$$

$$+ \quad \frac{1}{2}\int_{-\infty}^{\Phi^{-1}(p)} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - q \left[\frac{p}{q e^{\tau_2 \Phi^{-1}(p)} + e^{\frac{\tau_2^2}{2}} [\Phi(\Phi^{-1}(p) - \tau_2)]} e^{\tau_2 \Phi^{-1}(p)} \right] - q \ln q$$

Therefore maximizing w.r.t. β is equivalent to maximizing w.r.t. τ_2. Clearly the function has maximum at some τ_2 (say $\widetilde{\tau}_2$) depending only on p. Therefore, $\widetilde{\tau}_2$ is a constant for fixed p. Now, T_* is independent of parameters of the parent distribution.

Similarly, we can show T_* is independent of parameters of the parent distribution when Weibull is the parent distribution.

References

[1] Aarset, M. V. (1987), "How to identify a bathtub hazard rate", *IEEE Transactions on Reliability*, vol. 36, 106 - 108.

[2] Atkinson, A. (1969), "A test of discriminating between models", *Biometrika*, vol. 56, 337-341.

[3] Atkinson, A. (1970), "A method for discriminating between models" (with discussions), *Journal of the Royal Statistical Society*, Ser. B, vol. 32, 323-353.

[4] Bain, L.J. and Engelhardt, M. (1980), "Probability of correct selection of Weibull versus gamma based on likelihood ratio", *Communications in Statistics*, Ser. A., vol. 9, 375-381.

[5] Bhattacharyya, G.K. (1985), "The Asymptotic of Maximum Likelihood and Related Estimators Based on Type II Censored Data",*Journal of the American Statistical Association* vol. 80, 398 - 404.

[6] Chambers, E.A. and Cox, D.R. (1967), "Discriminating between alternative binary response models", *Biometrika*, 54, 573-578.

[7] Chen, W.W. (1980), "On the tests of separate families of hypotheses with small sample size", *Journal of Statistical Computations and Simulations*, vol. 2, 183-187.

[8] Cox, D.R. (1961), "Tests of separate families of hypotheses", *Proceedings of the Fourth Berkeley Symposium in Mathematical Statistics and Probability*, Berkeley, University of California Press, 105-123.

[9] Cox, D.R. (1962), "Further results on tests of separate families of hypotheses", *Journal of the Royal Statistical Society*, Ser. B, vol. 24, 406-424.

[10] Dumonceaux, R. and Antle, C.E. (1973), "Discriminating between the log-normal and Weibull distribution", *Technometrics*, vol. 15, 923-926.

[11] Dyer, A.R. (1973), "Discrimination procedure for separate families of hypotheses", *Journal of the American Statistical Association*, vol. 68, 970-974.

[12] Fearn, D.H. and Nebenzahl, E. (1991), "On the maximum likelihood ratio method of deciding between the Weibull and Gamma distributions", *Communications in Statistics - Theory and Methods*, vol. 20, 579-593.

[13] Kappeman, R.F. (1982), "On a method for selecting a distributional model", *Communications in Statistics - Theory and Methods*, vol.11, 663-672.

[14] Keats, J. B., Nahar, P.C. and Korbell, K.M. (2000), "A study of the effect of mis-specification of the Weibull shape parameter on confidence bounds based on the Weibull-exponential transformation", *Quality and Reliability Engineering International*, vol. 16, 27 - 31.

[15] Kim, J.S. and Yum, B-J (2008), "Selection between Weibull and log-normal distributions: A comparative simulation study", *Computational Statistics and Data Analysis*, vol. 53, 477 - 485.

[16] Kundu, D., Gupta, R.D. and Manglick, A. (2005), "Discriminating between the log-normal and generalized exponential distributions", *Journal of Statistical Planning and Inference*, vol. 127, 213 - 227.

[17] Kundu, D. and Manglick, A. (2004), "Discriminating between the Weibull and Log-Normal distributions", *Naval Research Logistics*, vol. 51, 893-905, 2004.

[18] Lawless, J.F. (1982), *Statistical Models and Methods for Lifetime Data*, John Wiley and Sons, New York.

[19] Linhardt, H. and Zucchini, W. (1986), *Model Selection*, Wiley, New York.

[20] Pascual, F.G. (2005), "Maximum likelihood estimation under misspecified log-normal and Weibull distributions", *Communications in Statistics - Simulation and Computations* vol. 34, 503 - 524, 2005.

[21] Schneider, H. (1989), "Failure-censored variables-sampling plans for log-normal and Weibull distributions", *Technometrics*, vol. 31, no. 2, 199 - 206.

[22] Wiens, B.L. (1999), "When log-normal and gamma models give different results: a case study", *American Statistician*, vol. 53, 89-93.

[23] Yu, H.F. (2007), " Mis-specification analysis between normal and extreme value distributions for a linear regression model", *Communications in Statistics - Theory and Methods*, vol. 36, 499 - 521.