SUPER EFFICIENT FREQUENCY ESTIMATION

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Abstract. In this paper we propose a modified Newton-Raphson method to obtain super efficient estimators of the frequencies of a sinusoidal signal in presence of stationary noise. It is observed that if we start from an initial estimator with convergence rate $O_p(n^{-1})$ and use Newton-Raphson algorithm with proper step factor modification, then it produces super efficient frequency estimator in the sense its asymptotic variance is lower than the asymptotic variance of the corresponding least squares estimator. The proposed frequency estimator is consistent and it has the same rate of convergence, namely $O_p(n^{-\frac{3}{2}})$, as the least squares estimator. Monte Carlo simulations are performed to observe the performance of the proposed estimator for different sample sizes and for different models. The results are quite satisfactory. One real data set has been analyzed for illustrative purpose.

1. Introduction

In this paper, we consider the problem of estimating the unknown parameters of the following sinusoidal signal model, observed in presence of stationary noise. The model can be described as follows;

$$y(t) = \sum_{j=1}^{p} [A_j \cos(\omega_j t) + B_j \sin(\omega_j t)] + Z(t); \qquad t = 1, \ldots, n. \tag{1}$$

The unknown amplitudes A_j, B_j, $j = 1, \ldots, p$ are real numbers and $\omega_1, \ldots, \omega_p \in (0, \pi)$ are known as frequencies. The additive noise $\{Z(t)\}$ is a sequence of stationary random variables with mean zero and finite variance and it satisfies the following Assumption.

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Assumption 1. *The random variables* $Z(t)$ *have the following linear structure*

$$Z(t) = \sum_{k=-\infty}^{\infty} a(k)e(t-k), \tag{2}$$

where $\{e(t)\}$ *is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and finite variance* σ^2*. The arbitrary real valued sequence* $\{a(t)\}$ *satisfies the following condition;*

$$\sum_{k=-\infty}^{\infty} |a(k)| < \infty. \tag{3}$$

We are interested to estimate the unknown parameters under Assumption 1. Several authors considered this model under different assumptions on the error, see Walker [13], Hannan [6] etc. Many real life phenomena can be described quite effectively using model 1. Baldwin and Thomson [2] used model (1) to describe the visual observation of S. Carinae, a variable star in the Southern Hemisphere sky, under the restriction $\lambda_j = j\lambda$ for all j. Greenhouse, Kass and Tsay [5] proposed the use of higher-order harmonic terms of one or more fundamentals and ARMA processes for the errors (which is again a particular case of model (1)) for fitting biological rhythms (human core body temperature data). The harmonic regression model has also been used to assess the static properties of human circadian systems; see Brown and Czeisler [4]. Model (1) is also useful for describing musical sound waves produced by musical instruments, see Rodet [11]. Irizarry [7] studied a segment of sound generated from pipe organ using a similar model. Nandi and Kundu [8] proposed the generalized multiple fundamental frequency model and several short duration speech data were studied. We shall use model (1) to analyze a real electrocardiograph (ECG) signal later in this paper.

The main problem is to estimate the unknown amplitudes and the frequencies, given a sample of size n, namely $\{y(1), \ldots, y(n)\}$ assuming p, the number of components, to be known. It is known, see Walker [13], that if the frequency (non-linear parameter) is estimated with the convergence rate $O_p(n^{-\frac{3}{2}})$, here $O_p(n^{-\delta})$ means $n^{\delta}O_p(n^{-\delta})$ is bounded in probability, then the amplitudes (linear parameters) can be estimated efficiently with the convergence rate $O_p(n^{-\frac{1}{2}})$. In this paper, we mainly address the estimation of the frequencies efficiently and propose an algorithm by modifying Newton-Raphson iterative method. We observe that the estimator has the asymptotic variance smaller than that of the least squares estimator (LSE). We call it as a super efficient frequency estimator.

The rest of the paper is organized as follows. In section 2, we briefly describe the least squares and approximate least squares procedures for estimating the frequencies. In Section 3, we propose the modified Newton-Raphson algorithm in case of one component model. In Section 4, we consider the case when more than one frequency is present. Simulation results are presented in Section 5. For illustrative purpose, one data set has been analyzed in Section 6, and finally we conclude the paper in Section 7. All the proofs are presented in the Appendix.

2. Estimating the Unknown Parameters

In the literature there are two well known methods available for estimating the unknown frequency of model (1). One is the standard least squares method and the other is the periodogram maximizer, known as the approximate least squares method. We briefly describe them for one component model, given below, for further development.

$$y(t) = A\cos(\omega t) + B\sin(\omega t) + Z(t), \quad t = 1, \ldots, n. \tag{4}$$

2.1. Least Squares Method.

The LSEs of A, B and ω of model (4) can be obtained by minimizing

$$Q(A, B, \omega) = \sum_{t=1}^{n} \left(y(t) - A\cos(\omega t) - B\sin(\omega t)\right)^2, \tag{5}$$

with respect to A, B and ω. It can be done in two stages, first minimize $Q(A,B,\omega)$ with respect to ($w.r.t.$) A and B for fixed ω, say $\widehat{A}(\omega)$ and $\widehat{B}(\omega)$ and then minimize $Q(\widehat{A}(\omega),\widehat{B}(\omega),\omega)$ $w.r.t.$ ω. We use the following notation;

$$\mathbf{X} = \begin{bmatrix} \cos(\omega) & \sin(\omega) \\ \vdots & \vdots \\ \cos(n\omega) & \sin(n\omega) \end{bmatrix}, \quad \mathbf{Y} = (y(1),\ldots,y(n))^T. \tag{6}$$

Minimizing $Q(A,B,\omega)$ $w.r.t.$ A, B and ω boils down to maximizing $R(\omega)$ $w.r.t.$ ω, where

$$R(\omega) = \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}. \tag{7}$$

The matrix $\mathbf{X}$ depends on ω, it is not made explicit unless necessary. Suppose $\widehat{\omega}$ maximizes $R(\omega)$, then the estimators of A and B are obtained using the separable regression technique as

$$(\widehat{A} \ \ \widehat{B})^T = (\mathbf{X}(\widehat{\omega})^T\mathbf{X}(\widehat{\omega}))^{-1}\mathbf{X}(\widehat{\omega})^T\mathbf{Y}. \tag{8}$$

Most of the numerically efficient algorithms, for example, the methods proposed by Bresler and Macovski [3] and Smyth [12] attempt to maximize $R(\omega)$ in (7). This naturally saves computational time as the optimization takes place in one dimension.

2.2. Approximate Least Squares Method. The approximate LSE (ALSE) of the frequency is obtained by maximizing the periodogram function $I(\omega)$,

$$I(\omega) = \left| \frac{1}{n}\sum_{t=1}^{n} y(t)e^{-i\omega t} \right|^2, \tag{9}$$

$w.r.t.$ $\omega \in (0,\pi)$. Let $\widehat{\widehat{\omega}}$ be the ALSE of ω, then the amplitudes are obtained as before. Both $\widehat{\omega}$ and $\widehat{\widehat{\omega}}$ are consistent estimators of ω, see Hannan [6], Walker [13] and are asymptotically equivalent with the following asymptotic distributions;

$$n^{\frac{3}{2}}(\widehat{\omega}-\omega) \xrightarrow{d} \mathcal{N}(0, \frac{24\sigma^2 c}{A^2+B^2}), \quad n^{\frac{3}{2}}(\widehat{\widehat{\omega}}-\omega) \xrightarrow{d} \mathcal{N}(0, \frac{24\sigma^2 c}{A^2+B^2}), \tag{10}$$

where $c = \left| \sum_{k=-\infty}^{\infty} a(k) e^{-ik\omega} \right|^2$. In equation (10), '$\xrightarrow{d}$' means convergence in distribution and $\mathcal{N}(a, b)$, normal distribution with mean a and variance b. In the following section, we modify Newton-Raphson Algorithm using the set-up of this section.

3. Modified Newton-Raphson Algorithm

We modify Newton-Raphson algorithm in this section, in the aim of finding a consistent estimator of the frequency ω of model (4). Before proceeding further, we briefly describe Newton-Raphson algorithm in case of $R(\omega)$. Let $\widehat{\omega}_1$ be the initial estimate and $\widehat{\omega}_k$ be the estimate at the k^{th} iteration. Then at $(k+1)^{th}$ iteration, the estimate $\widehat{\omega}_{k+1}$ is updated as

$$\widehat{\omega}_{k+1} = \widehat{\omega}_k - \frac{R'(\widehat{\omega}_k)}{R''(\widehat{\omega}_k)}, \tag{11}$$

where $R'(\widehat{\omega}_k)$ and $R''(\widehat{\omega}_k)$ are first and second derivatives of $R(\omega)$, evaluated at $\widehat{\omega}_k$.

The standard Newton-Raphson algorithm (11) is modified with a smaller correction factor, and the iteration uses

$$\widehat{\omega}_{k+1} = \widehat{\omega}_k - \frac{1}{4} \frac{R'(\widehat{\omega}_k)}{R''(\widehat{\omega}_k)}. \tag{12}$$

A smaller step factor $\frac{1}{4} \frac{R'(\widehat{\omega}_k)}{R''(\widehat{\omega}_k)}$ prevents the procedure to diverge or converge to a local minima. The following theorem motivates us to modify the algorithm in this fashion.

Theorem 1. Let $\widehat{\omega}_0$ be an estimate of ω and $\widehat{\omega}_0 - \omega = O_p(n^{-1-\delta})$, $\delta \in \left(0, \frac{1}{2}\right]$. Suppose $\widehat{\omega}_0$ is updated as $\widehat{\omega}$, using $\widehat{\omega} = \widehat{\omega}_0 - \frac{1}{4} \times \frac{R'(\widehat{\omega}_0)}{R''(\widehat{\omega}_0)}$, then

(a) $\widehat{\omega} - \omega = O_p(n^{-1-3\delta}) \qquad \text{if } \delta \leq \frac{1}{6}$,

(b) $n^{\frac{3}{2}}(\widehat{\omega} - \omega) \xrightarrow{d} \mathcal{N}\left(0, \dfrac{6\sigma^2 c}{A^2 + B^2}\right) \qquad \text{if } \delta > \frac{1}{6}$,

where c is same as defined in last section.

We apply this theorem in modifying Newton-Raphson algorithm. We start with an initial estimator having convergence rate $O_p(n^{-1})$ and a fraction (subset) of the available data points to implement the above result. The subset is selected such that the dependence structure in the data remains the same. The argument maximum of $I(\omega)$, defined in (9), or $R(\omega)$, defined in (7), over Fourier frequencies $\{\frac{2\pi k}{n}, k = 0, \ldots, [\frac{n}{2}]\}$, provides an estimator with convergence rate $O_p(n^{-1})$. We use ideas similar to Bai *et al.* [1] or Nandi and Kundu [9] for implementation of Theorem 1 as an algorithm.

Algorithm:

(1) Find the argument maximum of $R(\omega)$ over Fourier frequencies. Suppose it is $\widehat{\omega}_0$. Then $\widehat{\omega}_0 = O_p(n^{-1})$.

(2) Take $n_1 = n^{\frac{6}{7}}$, and calculate

$$\widehat{\omega}_1 = \widehat{\omega}_0 - \frac{1}{4} \times \frac{R'_{n_1}(\widehat{\omega}_0)}{R''_{n_1}(\widehat{\omega}_0)},$$

where $R'_{n_1}(\widehat{\omega}_0)$ and $R''_{n_1}(\widehat{\omega}_0)$ are same as $R'(\widehat{\omega}_0)$ and $R''(\widehat{\omega}_0)$ respectively, computed using a sub-sample of size n_1. Since $\widehat{\omega}_0 - \omega = O_p(n^{-1})$ and $n^{-1} = n_1^{-\frac{7}{6}} = n_1^{-1-\frac{1}{6}}$, therefore, $\widehat{\omega}_0 - \omega = O_p(n_1^{-1-\frac{1}{6}})$. Now using Theorem 1(a),

$$\widehat{\omega}_1 - \omega = O_p(n_1^{-1-\frac{1}{2}}) = O_p(n^{-\frac{3}{2}\times\frac{6}{7}}) = O_p(n^{-\frac{9}{7}}) = O_p(n^{-1-\frac{2}{7}}), \quad \delta = \frac{2}{7}.$$

(3) As $\delta = \frac{2}{7} > \frac{1}{6}$, apply Theorem 1(b). With $n_j = n$, repeat

$$\widehat{\omega}_{k+1} = \widehat{\omega}_k - \frac{1}{4} \times \frac{R'_{n_{k+1}}(\widehat{\omega}_k)}{R''_{n_{k+1}}(\widehat{\omega}_k)}, \quad k = 1, 2, \ldots, \tag{13}$$

until a suitable stopping criterion is satisfied.

The algorithm suggests to begin with a subset of size $n^{6/7}$ where n is the given sample size, but which subset to begin with. It has been observed in simulation study that one can start from any sub-sample (so that the dependence/correlation structure in the error process remains the same) and the choice of the initial sub-sample does not have any visible effect on the final estimator. To keep the dependence structure intact, we choose any n_1 consecutive points to start. Note that the factor 6/7 in the exponent is not so important and is not unique. There are several other ways the algorithm can be initiated. For example, one can start with $n_1 = n^{\frac{7}{8}}$ and apply Theorem 1, similarly as above. Basically we need to ensure that δ is greater than 1/6 to use part (b) of Theorem 1.

Remark 1. Write $(\widehat{A}, \widehat{B})^T = (A(\widehat{\omega}), B(\widehat{\omega}))^T$, in equation (8). Expand $A(\widehat{\omega})$ at ω^0 by Taylor series,

$$A(\widehat{\omega}) - A(\omega^0) = (\widehat{\omega} - \omega^0)A'(\bar{\omega}) + o(n^2),$$

where $A'(\bar{\omega})$ is the first derivative of $A(\omega)$ at $\bar{\omega}$,; $\bar{\omega}$ is a point between $\widehat{\omega}$ and ω^0; $\widehat{\omega}$ can be either the LSE or the estimator obtained by modified Newton Raphson method. Comparing the variances (asymptotic) of the two estimators of ω, we note that the asymptotic variance of the corresponding estimator of A is four times less than that of the LSE. The same is true for the estimator of B.

$$\begin{aligned} \mathrm{Var}(A(\widehat{\omega}) - A(\omega^0)) &\approx \mathrm{Var}(\widehat{\omega} - \omega^0)[A'(\bar{\omega})]^2 \\ \mathrm{Var}(A(\widehat{\omega}) - A^0) &= \frac{\mathrm{Var}(\widehat{\omega}_{LSE})}{4}[A'(\bar{\omega})]^2 \\ &= \frac{\mathrm{Var}(\widehat{A}_{LSE})}{4} \end{aligned}$$

where $\widehat{\omega}_{LSE}$ and $\widehat{A}_{LSE}$ are the LSEs of ω and A, respectively. Similarly, $\mathrm{Var}(B(\widehat{\omega}) - B(\omega^0)) = \frac{\mathrm{Var}(\widehat{B}_{LSE})}{4}$ and different notation have same meaning replacing A by B.

Remark 2. Newton-Raphson algorithm in its original form does not work well for sinusoidal model. But with the step factor modification, it provides estimator with the same rate of convergence as the LSE with smaller asymptotic variance than that of the LSE.

4. Multiple Sinusoid

In this section, the problem of estimating the unknown parameters of model (1). We assume that $\omega_j \neq \omega_k$, $j \neq k$ and the number of components, p is known in advance. The sequence of $\{Z(t)\}$ satisfies Assumption 1, as before. The following restriction has been imposed for identifiability;

$$A_1^2 + B_1^2 \geq \cdots \geq A_p^2 + B_p^2. \tag{14}$$

and if for any $1 \leq k \leq p-1$, $A_k^2 + B_k^2 = A_{k+1}^2 + B_{k+1}^2$, then $\omega_k > \omega_{k+1}$.

We concentrate on the problem of estimation of the unknown frequencies $\omega_1, \ldots, \omega_p$ from the given sample $\{y(1), \ldots, y(n)\}$. Amplitudes are linear parameters and can be estimated using the separable regression technique. The sequential estimation procedure is used to estimate the unknown parameters if more than one frequency is present. The procedure can be described as follows. Starting from the maximizer of $R(\omega)$, estimate ω_1 by $\widehat{\omega}_1$, the estimator obtained using the algorithm described in the previous section. Then

$$n^{\frac{3}{2}}(\widehat{\omega}_1 - \omega_1) \xrightarrow{d} \mathcal{N}\left(0, \frac{6\sigma^2 c}{A_1^2 + B_1^2}\right).$$

Calculate $\widehat{A}_1$ and $\widehat{B}_1$ replacing $\widehat{\omega}$ by $\widehat{\omega}_1$ in (8). Define the modified data by taking out the effect of ω_1 as follows;

$$\tilde{y}(t) = y(t) - \widehat{A}_1 \cos(\widehat{\omega}_1 t) - \widehat{B}_1 \sin(\widehat{\omega}_1 t), \qquad t = 1, \ldots, n. \tag{15}$$

Now apply the same procedure on the modified data $\{\tilde{y}(1), \ldots, \tilde{y}(n)\}$ and estimate ω_2. Repeat the procedure p times.

The method automatically gives an estimate of the number of components, p. If the above procedure is repeated, say q times and we are left with a pure random sequence, then q can be treated as an estimate of p.

5. Numerical Experiment

In this section, we present some results based on numerical experiments to observe small sample performances for different models. We use the random deviate generator RAN2 of Press *et al.* [10]. All the programs are written in FORTRAN and they can be obtained from the corresponding author on request. The following two models are considered for the study;

- Model 1: $y(t) = 2\cos(0.5t + \frac{\pi}{4}) + Z(t)$.
- Model 2: $y(t) = 2\cos(1.5t + \frac{\pi}{3}) + 2\cos(0.5t + \frac{\pi}{4}) + Z(t)$.

The sequence $\{Z(t)\}$ is from an autoregressive process of order one, of the form

$$Z(t) = .75\ Z(t-1) + e(t),$$

where $\{e(t)\}$ is a sequence of $i.i.d.$ Gaussian random variables with mean zero and variance σ^2. We consider different sample sizes and different error variances, namely $n = 100$, 200, 300, 400, 500, 1000 and $\sigma = 0.25$, 0.50, 0.75, 1.00, 1.25.

In each case, we generate a sample from a selected model and find the initial estimate from $R(\omega)$ as discussed in section 3. We compute the final estimate $\widehat{\omega}$ for Model 1, using the iterative process proposed in (13). The iterative process is terminated if the absolute difference between two consecutive iterates is less than $\frac{\sigma^2}{n^3}$. In most of the cases, the iteration converges in 8-10 iterations. In case of Model 2, we repeat the process after removing the effect of the first frequency to estimate the parameters of the second component. The process is replicated 1000 times and we have computed the average estimates (AEs) and their mean squared errors (MSEs). The asymptotic variances of the least squares estimators (ASV) as given in (10) as well as the proposed asymptotic variance (ASVP) as provided in part (b) of Theorem 1 are reported for comparison. Results for the frequencies of both the models are reported. The results for Model 1 are reported in Tables 1 and for Model 2 in Tables 2 and 3. According to Remark 1, we expect that MSEs of $\widehat{A}$ and $\widehat{B}$ may be smaller than the corresponding Cramer-Rao Lower Bounds. To study the effect we have reported results of A in case of Model 1 in Table 4. Similar trend has been observed in case of other amplitude estimators and are not reported here.

Some of the points are quite clear from Tables 1-4. As the sample size increases or the variance decreases the MSEs and the biases decrease for different parameter estimators of both the models. It verifies the consistency of the proposed estimators. The MSEs of the proposed estimates are usually smaller than the asymptotic variances of the LSEs. This is observed in case of the frequencies as well as the amplitudes (Remark 1). Therefore, the improvement is achievable in practice.

We now analyze a synthesized data which were generated using Model 2 with error variance $\sigma^2 = 0.5$ and sample size $n = 100$ and presented in Fig 1. The $R(\omega)$ function of the generated data is presented in Fig 2. There are two sharp distinct peaks and it looks like that two frequency components are present. We take a note that the plot of $R(\omega)$ may not reveal the actual information of number of components present in the signal depending on the error variance and the magnitudes of the amplitudes associated with different frequencies. Now we maximize $R(\omega)$ over Fourier frequencies in $(0, \pi)$ and obtain the initial estimate of ω_1, say $\widehat{\omega}_{10}$. Using $\widehat{\omega}_{10}$ as the starting estimate and a subset of size $n^{6/7}$ (it is 51 when $n = 100$), we update it to $\widehat{\omega}_{11}$ by implementing Theorem 1(a). From second step onward, we use all the available data points and apply Theorem 1(b) iteratively till they converge according to the stopping criterion given above. Let $\widehat{\omega}_1$ be the final estimate of ω_1 and plugging it, we estimate A_1 and B_1, assuming that the data contain only one frequency. Suppose $\widehat{A}_1$ and $\widehat{B}_1$ are the estimates. Then we define the modified data $\tilde{y}(t)$ (similarly as (15)) to remove the effect of ω_1. To estimate the second frequency ω_2, and the corresponding linear parameters, we apply the same procedure to $\tilde{y}(t)$ instead of $y(t)$. The plot of the estimated signal along

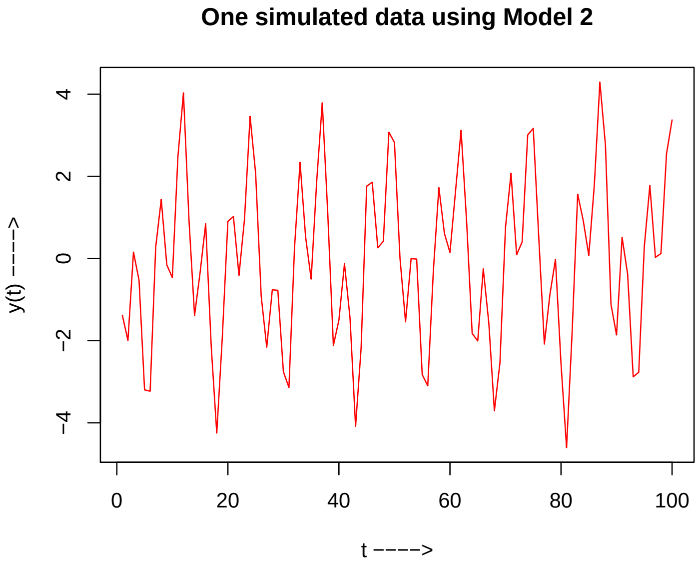

FIGURE 1. One simulated data using Model 2.

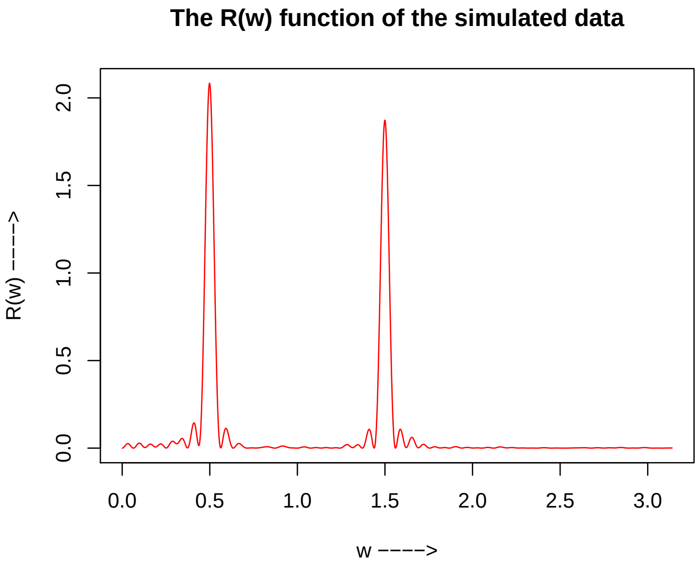

FIGURE 2. The $R(\omega)$ Function of the Data in Fig. 1.

with the generated data is presented in Fig 3 to understand how well the fit is. The estimated signal match quite well with the generated one.

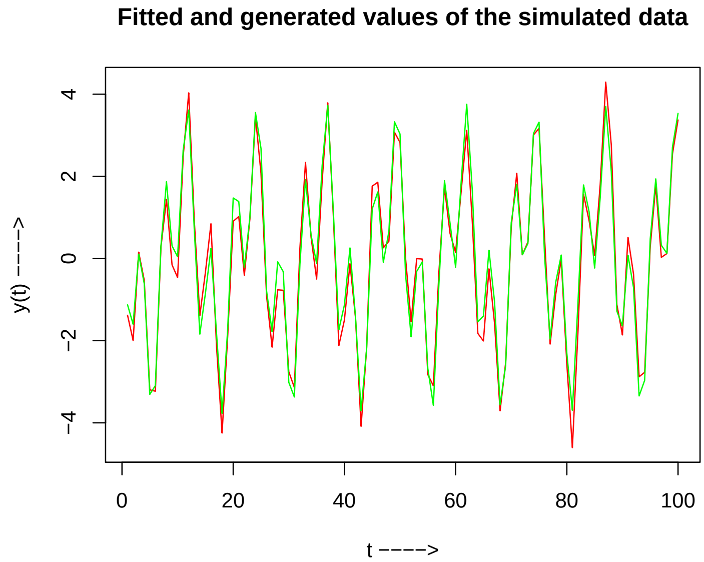

FIGURE 3. The Observed (red) & Fitted Values (green) of the Simulated Data.

6. DATA ANALYSIS

In this section, we apply the multiple sinusoidal model to analyze a real data set. The data represent a segment of ECG signal and contains 512 data points, see Fig 4. The $R(\omega)$ function over a fine grid of $(0, \pi)$ is plotted in Fig 5. This gives an idea of the number of sinusoidal components, p, present in the model. Estimating p from Fig 5 is not easy as all the frequencies present in the model for this data set, may not be visible. We observe that $\widehat{p}$ is much larger than what Fig 5 reveals. We use the following procedure. Estimate the unknown parameters of model (1) sequentially for $k = 1, \ldots, 100$ with an autoregressive (AR) model to the residual sequence for each k. Here k represents the number of components. Let ar_k be the number of the AR parameters in the AR model fitted to the residuals when k sinusoidal components are estimated and $\widehat{\sigma}^2$ be the estimated innovation variance. Then minimize the Bayes Information Criterion (BIC) for estimating the number of components, p. The BIC takes the following form in this case;

$$BIC(k) = n \log \widehat{\sigma}_k^2 + \frac{1}{2}(3k + ar_k + 1) \log n.$$

The BIC values are presented in Fig. 6 for $k = 75(1)85$ and at $k = 78$, the BIC takes its minimum value, therefore, we estimate p as $\widehat{p} = 78$. The initial estimates are obtained by maximizing the function $R(\omega)$ over the Fourier frequencies and have the same rate of convergence as the periodogram maximizer, i.e., $O_p(n^{-1})$. The parameters are estimated using the proposed method applied sequentially. We stop the iterative process if the absolute difference between two iterates is less than n^{-3} which is approximately 7.5×10^{-9} for the given sample size. With the estimated $\widehat{p}$, we plug-in the other estimates of the linear parameters and frequencies and finally obtained the fitted values, $\widehat{y}(t)$. They are plotted in Fig. 7 along with their observed values. The data were mean-corrected and scaled by the square root of the estimated variance of $y(t)$. Then the above stopping criterion has been used. In Figs. 4 and 7, the mean-corrected and scaled version are shown. The fitted values match reasonably well with the observed one. The final residual vector satisfy the assumption of stationarity. Note that it is possible to fit such a large order model as the method proposed here is a sequential procedure for $p \geq 2$ and every step requires an one-dimensional optimization. Otherwise in simultaneous estimation, if global optimization is required, it would be difficult to get a satisfactory outcome.

7. Conclusions

In this paper, we have considered the problem of estimation of the unknown parameters of a classical model, namely the sum of sinusoidal model. We have mainly discussed the estimation of the unknown frequencies efficiently. It has been criticized heavily in the literature against the use of Newton-Raphson algorithm for computing the LSEs of the unknown

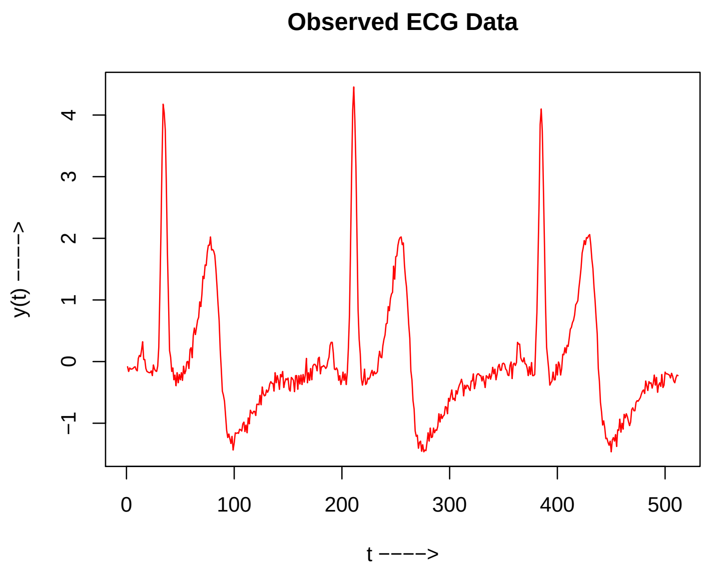

FIGURE 4. 512 Observation of a Real ECG Signal.

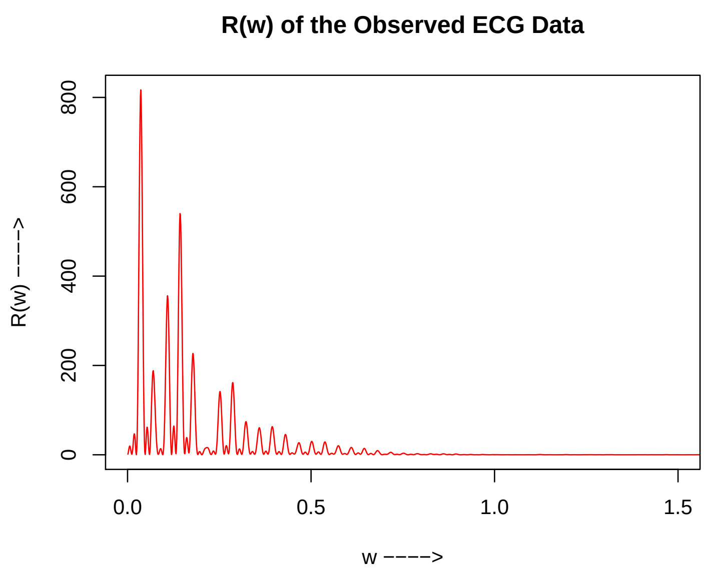

FIGURE 5. $R(\omega)$ Function of the ECG data.

frequencies of this model, but Newton-Raphson algorithm with proper step factor modification, works very well. The asymptotic variance of the proposed frequency estimator is smaller than that of the LSE. The method, combined with sequential procedure, can easily

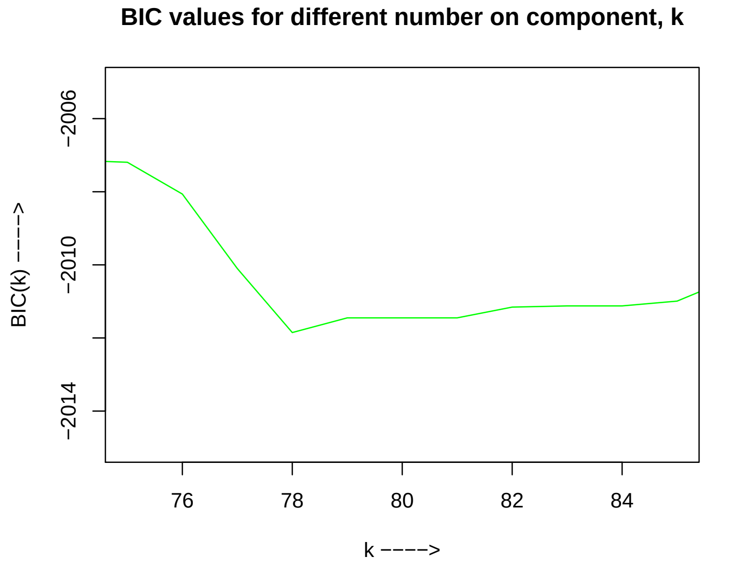

FIGURE 6. BIC values for different number of components.

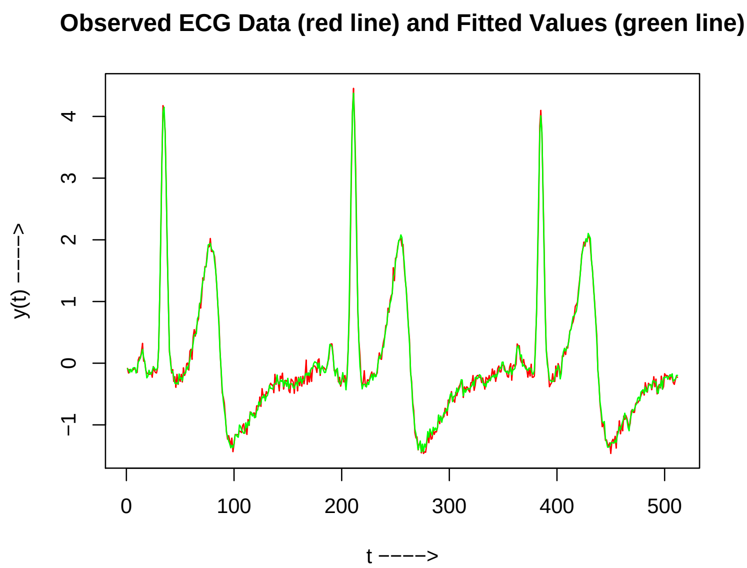

FIGURE 7. The Observed (red) and Fitted Values (green) of the ECG data.

be used in case of multiple sinusoid and it does not require any higher dimensional optimization. The MSEs of the proposed estimators are often less than the asymptotic variance of the LSEs. The iteration converges very quickly, and so the proposed method can be used quite effectively for on-line implementation purposes.

Appendix

We use the following notation to prove Theorem 1;

$$\mathbf{D} = \text{diag}\{1, 2, \ldots, n\}, \quad \mathbf{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \dot{\mathbf{X}} = \frac{d}{d\omega}\mathbf{X} = \mathbf{DXE}, \quad \ddot{\mathbf{X}} = \frac{d^2}{d\omega^2}\mathbf{X} = -\mathbf{D}^2\mathbf{X}.$$

Note that $\mathbf{EE} = -\mathbf{I}$, $\mathbf{EE}^T = \mathbf{I} = \mathbf{E}^T\mathbf{E}$ and

$$\frac{d}{d\omega}(\mathbf{X}^T\mathbf{X})^{-1} = -(\mathbf{X}^T\mathbf{X})^{-1}[\dot{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\dot{\mathbf{X}}](\mathbf{X}^T\mathbf{X})^{-1}.$$

Now we compute $R'(\omega)$ and $R''(\omega)$;

$$\frac{1}{2}\, R'(\omega) = \mathbf{Y}^T\dot{\mathbf{X}}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} - \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y},$$

and

$$\begin{aligned}
\frac{1}{2}\, R''(\omega) \;&=\; \mathbf{Y}^T\ddot{\mathbf{X}}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} - \mathbf{Y}^T\dot{\mathbf{X}}(\mathbf{X}^T\mathbf{X})^{-1}(\dot{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\dot{\mathbf{X}})(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&+\; \mathbf{Y}^T\dot{\mathbf{X}}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{Y} - \mathbf{Y}^T\dot{\mathbf{X}}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&+\; \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}(\dot{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\dot{\mathbf{X}})(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&-\; \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}(\ddot{\mathbf{X}}^T\mathbf{X})(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} - \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}(\dot{\mathbf{X}}^T\dot{\mathbf{X}})(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&+\; \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}(\dot{\mathbf{X}}^T\mathbf{X} + \mathbf{X}^T\dot{\mathbf{X}})(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&-\; \mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{Y}.
\end{aligned}$$

In order to approximate $\frac{1}{n^3}R''(\tilde{\omega})$ for large n, assume $\tilde{\omega}-\omega = O_p(n^{-1-\delta})$. So, for large n, $(\frac{1}{n}\mathbf{X}(\tilde{\omega})^T\mathbf{X}(\tilde{\omega}))^{-1} = 2\,\mathbf{I} + O_p(\frac{1}{n})$ and

$$\begin{aligned}
\frac{1}{2n^3}\ R''(\tilde{\omega}) &= \frac{2}{n^4}\mathbf{Y}^T\ddot{\mathbf{X}}\mathbf{X}^T\mathbf{Y} - \frac{4}{n^5}\mathbf{Y}^T\dot{\mathbf{X}}(\dot{\mathbf{X}}^T\mathbf{X}+\mathbf{X}^T\dot{\mathbf{X}})\mathbf{X}^T\mathbf{Y} + \frac{2}{n^4}\mathbf{Y}^T\dot{\mathbf{X}}\dot{\mathbf{X}}^T\mathbf{Y}\\
&- \frac{4}{n^5}\mathbf{Y}^T\dot{\mathbf{X}}\dot{\mathbf{X}}^T\mathbf{X}\mathbf{X}^T\mathbf{Y} + \frac{8}{n^6}\mathbf{Y}^T\mathbf{X}(\dot{\mathbf{X}}^T\mathbf{X}+\mathbf{X}^T\dot{\mathbf{X}})\dot{\mathbf{X}}^T\mathbf{X}\mathbf{X}^T\mathbf{Y}\\
&- \frac{4}{n^5}\mathbf{Y}^T\mathbf{X}\ddot{\mathbf{X}}^T\mathbf{X}\mathbf{X}^T\mathbf{Y} - \frac{4}{n^5}\mathbf{Y}^T\mathbf{X}\dot{\mathbf{X}}^T\dot{\mathbf{X}}\mathbf{X}^T\mathbf{Y} + \frac{8}{n^6}\mathbf{Y}^T\mathbf{X}\dot{\mathbf{X}}^T\mathbf{X}(\dot{\mathbf{X}}^T\mathbf{X}+\mathbf{X}^T\dot{\mathbf{X}})\mathbf{X}^T\mathbf{Y}\\
&- \frac{4}{n^5}\mathbf{Y}^T\mathbf{X}\dot{\mathbf{X}}^T\mathbf{X}\dot{\mathbf{X}}^T\mathbf{Y} + O_p(\frac{1}{n}).
\end{aligned}$$

Substituting $\dot{\mathbf{X}}$ and $\ddot{\mathbf{X}}$ in terms of $\mathbf{D}$ and $\mathbf{X}$, we obtain

$$\begin{aligned}
\frac{1}{2n^3}\ R''(\tilde{\omega}) &= -\frac{2}{n^4}\mathbf{Y}^T\mathbf{D}^2\mathbf{X}\mathbf{X}^T\mathbf{Y} - \frac{4}{n^5}\mathbf{Y}^T\mathbf{DXE}(\mathbf{E}^T\mathbf{X}^T\mathbf{DX}+\mathbf{X}^T\mathbf{DXE})\mathbf{X}^T\mathbf{Y}\\
&+ \frac{2}{n^4}\mathbf{Y}^T\mathbf{DXEE}^T\mathbf{X}^T\mathbf{DY} - \frac{4}{n^5}\mathbf{Y}^T\mathbf{DXEE}^T\mathbf{X}^T\mathbf{DXX}^T\mathbf{Y}\\
&+ \frac{8}{n^6}\mathbf{Y}^T\mathbf{X}(\mathbf{E}^T\mathbf{X}^T\mathbf{DX}+\mathbf{X}^T\mathbf{DXE})\mathbf{E}^T\mathbf{X}^T\mathbf{DXX}^T\mathbf{Y} + \frac{4}{n^5}\mathbf{Y}^T\mathbf{XX}^T\mathbf{D}^2\mathbf{XX}^T\mathbf{Y}\\
&- \frac{4}{n^5}\mathbf{Y}^T\mathbf{XE}^T\mathbf{X}^T\mathbf{D}^2\mathbf{XEX}^T\mathbf{Y} + \frac{8}{n^6}\mathbf{Y}^T\mathbf{XE}^T\mathbf{X}^T\mathbf{DX}(\mathbf{E}^T\mathbf{X}^T\mathbf{DX}+\mathbf{X}^T\mathbf{DXE})\mathbf{X}^T\mathbf{Y}\\
&- \frac{4}{n^5}\mathbf{Y}^T\mathbf{XE}^T\mathbf{X}^T\mathbf{DXE}^T\mathbf{X}^T\mathbf{DY} + O_p(\frac{1}{n}).
\end{aligned}$$

Use the following results; for $0<\omega<\pi$,

$$\sum_{t=1}^{n} t\ \cos^2(\omega t) = \frac{n^2}{4}+O(n), \qquad \sum_{t=1}^{n} t\ \sin^2(\omega t) = \frac{n^2}{4}+O(n),$$

$$\sum_{t=1}^{n} \cos^2(\omega t) = \frac{n}{2}+o(n), \qquad \sum_{t=1}^{n} \sin^2(\omega t) = \frac{n}{2}+o(n),$$

$$\sum_{t=1}^{n} t^2\ \cos^2(\omega t) = \frac{n^3}{6}+O(n^2), \qquad \sum_{t=1}^{n} t^2\ \sin^2(\omega t) = \frac{n^3}{6}+O(n^2),$$

and

$$\frac{1}{n^2}\mathbf{Y}^T\mathbf{DX} = \frac{1}{4}(A\ \ B)+O_p(\frac{1}{n}), \quad \frac{1}{n^3}\mathbf{Y}^T\mathbf{D}^2\mathbf{X} = \frac{1}{6}(A\ \ B)+O_p(\frac{1}{n}), \quad \frac{1}{n^3}\mathbf{X}^T\mathbf{D}^2\mathbf{X} = \frac{1}{6}\mathbf{I}+O_p(\frac{1}{n}),$$

$$\frac{1}{n}\mathbf{X}^T\mathbf{Y} = \frac{1}{2}(A\ \ B)^T+O_p(\frac{1}{n}), \quad \frac{1}{n^2}\mathbf{X}^T\mathbf{DX} = \frac{1}{4}\mathbf{I}+O_p(\frac{1}{n}).$$

Therefore,

$$\frac{1}{2n^3}\ R''(\tilde{\omega}) = (A^2+B^2)\left[-\frac{1}{6}-0+\frac{1}{8}-\frac{1}{8}+0+\frac{1}{6}-\frac{1}{6}+0+\frac{1}{8}\right]+O_p(\frac{1}{n}) = -\frac{1}{24}(A^2+B^2)+O_p(\frac{1}{n}).$$

Consider the two terms of the numerator separately. The first one

$$\begin{aligned}\frac{1}{n^3}\mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\dot{\mathbf{X}}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} &= \frac{1}{n^3}\mathbf{Y}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{E}^T\mathbf{X}^T\mathbf{D}\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\ &= \frac{1}{n^3}\mathbf{Y}^T\mathbf{X}(2\mathbf{I}+O_p(\frac{1}{n}))\mathbf{E}^T(\frac{1}{4}\mathbf{I}+O_p(\frac{1}{n}))(2\mathbf{I}+O_p(\frac{1}{n}))\mathbf{X}^T\mathbf{Y} \\ &= \frac{1}{n^3}\mathbf{Y}^T\mathbf{X}\mathbf{E}^T\mathbf{X}^T\mathbf{Y}+O_p(\frac{1}{n}) = O_p(\frac{1}{n}),\end{aligned}$$

and the second one $\frac{1}{n^3}\mathbf{Y}^T\dot{\mathbf{X}}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \frac{2}{n^4}\mathbf{Y}^T\mathbf{D}\mathbf{X}\mathbf{E}\mathbf{X}^T\mathbf{Y}$. We consider $\mathbf{Y}^T\mathbf{D}\mathbf{X}\mathbf{E}\mathbf{X}^T\mathbf{Y}$.

$$\begin{aligned}\mathbf{Y}^T\mathbf{D}\mathbf{X}(\tilde{\omega})\mathbf{E}\mathbf{X}^T(\tilde{\omega})\mathbf{Y} &= \left(\sum_{t=1}^n y(t)t\cos(\tilde{\omega}t)\right)\left(\sum_{t=1}^n y(t)\sin(\tilde{\omega}t)\right) \\ &- \left(\sum_{t=1}^n y(t)t\sin(\tilde{\omega}t)\right)\left(\sum_{t=1}^n y(t)\cos(\tilde{\omega}t)\right).\end{aligned}$$

If $\delta \in \big(0,\ \frac{1}{2}\big]$, along the same line as Bai *et al.* [1] (see also Nandi and Kundu [9]), it can be shown that,

$$\sum_{t=1}^n y(t)\cos(\tilde{\omega}t) = \frac{n}{2}\left(A+O_p(n^{-\delta})\right), \quad \sum_{t=1}^n y(t)\sin(\tilde{\omega}t) = \frac{n}{2}\left(B+O_p(n^{-\delta})\right).$$

We observe that

$$\begin{aligned}\sum_{t=1}^n y(t)te^{-i\tilde{\omega}t} &= \sum_{t=1}^n \left(A\cos(\omega t)+B\sin(\omega t)+X(t)\right)te^{-i\tilde{\omega}t} \\ &= \frac{1}{2}(A-iB)\sum_{t=1}^n t\ e^{i(\omega-\tilde{\omega})t} + \frac{1}{2}(A+iB)\sum_{t=1}^n t\ e^{-i(\omega+\tilde{\omega})t} + \sum_{t=1}^n X(t)te^{-i\tilde{\omega}t}.\end{aligned}$$

Following Bai *et al.* [1],

$$\begin{aligned}\sum_{t=1}^n t\ e^{-i(\omega+\tilde{\omega})t} &= O_p(n), \\ \sum_{t=1}^n t\ e^{i(\omega-\tilde{\omega})t} &= \sum_{t=1}^n t + i(\omega-\tilde{\omega})\sum_{t=1}^n t^2 - \frac{1}{2}(\omega-\tilde{\omega})^2\sum_{t=1}^n t^3 \\ &\quad -\frac{1}{6}i(\omega-\tilde{\omega})^3\sum_{t=1}^n t^4 + \frac{1}{24}(\omega-\tilde{\omega})^4\sum_{t=1}^n t^5e^{i(\omega-\omega^*)t}.\end{aligned}$$

Note $\frac{1}{24}(\omega-\tilde{\omega})^4\sum_{t=1}^{n} t^5 e^{i(\omega-\omega^*)t} = O_p(n^{2-4\delta})$. Choose L large enough such that $L\delta > 1$ and using Taylor series expansion of $e^{-i\tilde{\omega}t}$ we have,

$$\begin{aligned}
\sum_{t=1}^{n} X(t)te^{-i\tilde{\omega}t} &= \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{n} e(t-k)te^{-i\tilde{\omega}t} \\
&= \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{n} e(t-k)te^{-i\omega t} + \sum_{k=-\infty}^{\infty} a(k)\sum_{l=1}^{L-1}\frac{(-i(\tilde{\omega}-\omega))^l}{l!}\sum_{t=1}^{n} e(t-k)t^{l+1}e^{-i\omega t} \\
&\quad + \sum_{k=-\infty}^{\infty} a(k)\frac{\theta(n(\tilde{\omega}-\omega))^L}{L!}\sum_{t=1}^{n} t|e(t-k)| \ (\text{here } |\theta|<1) \\
&= \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{n} e(t-k)te^{-i\omega t} + \sum_{l=1}^{L-1} O_p(n^{-(1+\delta)l})O_p(n^{l+\frac{3}{2}}) + \sum_{k=-\infty}^{\infty} a(k)O_p(n^{\frac{5}{2}-L\delta}) \\
&= \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{n} e(t-k)te^{-i\omega t} + O_p(n^{\frac{5}{2}-L\delta}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{t=1}^{n} y(t)t\cos(\tilde{\omega}t) &= \frac{1}{2}\left[A\left(\sum_{t=1}^{n} t - \frac{1}{2}(\omega-\tilde{\omega})^2\sum_{t=1}^{n} t^3\right) + B\left(\sum_{t=1}^{n}(\omega-\tilde{\omega})t^2 - \frac{1}{6}(\omega-\tilde{\omega})^3\sum_{t=1}^{n} t^4\right)\right] \\
&\quad + \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{n} e(t-k)t\cos(\omega t) + O_p(n^{\frac{5}{2}-L\delta}) + O_p(n) + O_p(n^{2-4\delta}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{t=1}^{n} y(t)t\sin(\tilde{\omega}t) &= \frac{1}{2}\left[B\left(\sum_{t=1}^{n} t - \frac{1}{2}(\omega-\tilde{\omega})^2\sum_{t=1}^{n} t^3\right) - A\left(\sum_{t=1}^{n}(\omega-\tilde{\omega})t^2 - \frac{1}{6}(\omega-\tilde{\omega})^3\sum_{t=1}^{n} t^4\right)\right] \\
&\quad + \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{n} e(t-k)t\sin(\omega t) + O_p(n^{\frac{5}{2}-L\delta}) + O_p(n) + O_p(n^{2-4\delta}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\widehat{\omega} &= \tilde{\omega} - \frac{1}{4}\,\frac{R'(\tilde{\omega})}{R''(\tilde{\omega})} \\
&= \tilde{\omega} - \frac{1}{4}\,\frac{\frac{1}{2n^3}R'(\tilde{\omega})}{-\frac{1}{24}(A^2+B^2)+O_p(\frac{1}{n})} \\
&= \tilde{\omega} - \frac{1}{4}\,\frac{\frac{2}{n^4}\mathbf{Y}^T\mathbf{DXEX}^T\mathbf{Y}}{-\frac{1}{24}(A^2+B^2)+O_p(\frac{1}{n})} \\
&= \tilde{\omega} + 12\frac{\frac{1}{n^4}\mathbf{Y}^T\mathbf{DXEX}^T\mathbf{Y}}{(A^2+B^2)+O_p(\frac{1}{n})}
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\omega} + 12\frac{\frac{1}{4n^3}(A^2+B^2)\left\{(\omega-\tilde{\omega})\sum_{t=1}^n t^2 - \frac{1}{6}(\omega-\tilde{\omega})^3\sum_{t=1}^n t^4\right\}}{(A^2+B^2)+O_p(\frac{1}{n})} \\
&+ \frac{6}{(A^2+B^2)n^3+O_p(\frac{1}{n})}\left[B\sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^n e(t-k)t\cos(\omega t) + A\sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^n e(t-k)t\sin(\omega t)\right] \\
&+ O_p(n^{-\frac{1}{2}-L\delta}) + O_p(n^{-2}) + O_p(n^{-1-4\delta}) \\
&= \omega + (\omega-\tilde{\omega})O_p(n^{-2\delta}) \\
&+ \frac{6}{(A^2+B^2)n^3+O_p(\frac{1}{n})}\left[B\sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^n e(t-k)t\cos(\omega t) + A\sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^n e(t-k)t\sin(\omega t)\right] \\
&+ O_p(n^{-\frac{1}{2}-L\delta}) + O_p(n^{-2}) + O_p(n^{-1-4\delta}).
\end{aligned}
$$

If $\delta \le \dfrac{1}{6}$, clearly $\widehat{\omega}-\omega = O_p(n^{-1-3\delta})$, and if $\delta > \frac{1}{6}$, then

$$
\begin{aligned}
n^{\frac{3}{2}}(\widehat{\omega}-\omega) &\stackrel{d}{=} \frac{6n^{-\frac{3}{2}}}{(A^2+B^2)}\left[B\sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^n e(t-k)t\cos(\omega t) + A\sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^n e(t-k)t\sin(\omega t)\right] \\
&= \frac{6n^{-\frac{3}{2}}}{(A^2+B^2)}\left[B\sum_{t=1}^n X(t)e(t-k)t\cos(\omega t) + A\sum_{t=1}^n X(t)e(t-k)t\sin(\omega t)\right] \\
&\stackrel{d}{\longrightarrow} \mathcal{N}\left(0, \frac{6\sigma^2 c}{A^2+B^2}\right).
\end{aligned}
$$

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Table 1. Model 1: The average estimates (AEs), the mean squared errors (MSEs), the asymptotic variances of the LSE (ASVs) and the asymptotic variances of the modified Newton-Raphson method (ASVPs) of $\widehat{\omega}$, are reported.

		$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$	$N = 1000$
$\sigma = .25$	AE	.4999851	.4999956	.4999999	.4999996	.4999990	.5000000
	MSE	6.16253e-7	8.21868e-8	2.40528e-8	1.08736e-8	5.13852e-9	4.65843e-10
	ASV	1.07958e-6	1.34947e-7	3.99844e-8	1.68684e-8	8.63662e-9	1.07958e-9
	ASVP	2.69894e-7	3.9.04893e-8	2.61876e-8	1.13173e-8	5.75471e-9	5.27773e-10
$\sigma = .50$	AE	.4999621	.4999898	.4999966	.4999988	.4999977	.4999997
	MSE	2.47168e-6	3.28498e-7	9.63056e-8	4.34961e-8	2.05522e-8	1.86335e-9
	ASV	4.31831e-6	5.39789e-7	1.59937e-7	6.74736e-8	3.45465e-8	4.31831e-9
	ASVP	1.07958e-6	1.34947e-7	3.99844e-8	1.68684e-8	8.63662e-9	1.07958e-9
$\sigma = .75$	AE	.4999306	.4999826	.4999942	.4999974	.4999962	.4999995
	MSE	5.60401e-6	7.39757e-7	2.17069e-7	9.79201e-8	4.62503e-8	4.19441e-9
	ASV	9.71620e-6	1.21452e-6	3.59859e-7	1.51816e-7	7.77296e-8	9.71620e-9
	ASVP	2.42905e-6	3.03631e-7	8.99648e-8	3.79539e-8	1.94324e-8	2.42905e-9
$\sigma = 1.0$	AE	.4998900	.4999740	.4999912	.4999956	.4999944	.4999992
	MSE	1.00936e-5	1.31844e-6	3.86862e-7	1.74267e-7	8.22594e-8	7.46342e-9
	ASV	1.72732e-5	2.15916e-6	6.39750e-7	2.69894e-7	1.38186e-7	1.72732e-8
	ASVP	4.31831e-6	5.39789e-7	1.59937e-7	6.74736e-8	3.45465e-8	4.31831e-9
$\sigma = 1.25$	AE	.4998397	.4999638	.4999877	.4999933	.4999924	.4999989
	MSE	1.607244-5	2.06877e-6	6.06408e-7	2.72734e-7	1.28627e-7	1.16773e-8
	ASV	2.69894e-5	3.37368e-6	9.99609e-7	4.21710e-7	2.15916e-7	2.69894e-8
	ASVP	6.74736e-6	8.43420e-7	2.49902e-7	1.05428e-7	5.39789e-8	6.74736e-9

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Table 2. Model 2: The average estimates (AEs), the mean squared errors (MSEs), the asymptotic variances of the LSE (ASVs) and the asymptotic variances of the modified Newton-Raphson method (ASVPs) of $\widehat{\omega}_1$, are reported.

		$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$	$N = 1000$
$\sigma = .25$	AE	1.5006574	1.500252	1.500099	1.5000190	1.4999851	1.5000002
	MSE	5.60926e-7	8.07241e-8	1.40301e-8	2.18556e-9	1.20245e-9	1.16658e-10
	ASV	6.25727e-7	7.82159-8	2.31751e-8	9.77699e-9	5.00582e-9	6.25727e-10
	ASVP	1.56432e-7	1.95540e-8	5.79377e-9	2.44425e-9	1.25145e-9	1.56432e-10
$\sigma = .50$	AE	1.5006512	1.5002527	1.5000990	1.5000189	1.4999849	1.5000001
	MSE	9.39911e-7	1.33550e-7	2.68708e-8	7.65196e-9	4.14277e-9	4.66536e-10
	ASV	2.50291e-6	3.12864e-7	9.27003e-8	3.91079e-8	2.00232e-8	2.50291e-9
	ASVP	6.25727e-7	7.82159e-8	2.31751e-8	9.77699e-9	5.00582e-9	6.25727e-10
$\sigma = .75$	AE	1.5006435	1.5002537	1.5000992	1.5000187	1.4999848	1.4999999
	MSE	1.57729e-6	2.21140e-7	4.82457e-8	1.67635e-8	9.04066e-9	1.04995e-9
	ASV	5.63154e-6	7.03943e-7	2.08576e-7	8.79929e-8	4.50524e-8	5.63154e-9
	ASVP	1.40789e-6	1.75986e-7	5.21439e-8	2.19982e-8	1.12631e-8	1.40789e-9
$\sigma = 1.0$	AE	1.5006345	1.5002545	1.5000994	1.5000186	1.4999846	1.4999998
	MSE	2.47635e-6	3.43549e-7	7.81688e-8	2.95226e-8	1.58971e-8	1.86720e-9
	ASV	1.00116e-5	1.25145e-6	3.70801e-7	1.56432e-7	8.00931e-8	1.00116e-8
	ASVP	2.50291e-6	3.12864e-7	9.27003e-8	3.91079e-8	2.00233e-8	2.50291e-9
$\sigma = 1.25$	AE	1.5006241	1.5002550	1.5000996	1.5000184	1.4999845	1.4999996
	MSE	3.64155e-6	5.00934e-7	1.16665e-7	4.59359e-8	2.47146e-8	2.91868e-9
	ASV	1.564312e-5	1.95540e-6	5.79377e-7	2.44425e-7	1.25145e-7	1.56432e-8
	ASVP	3.91079e-6	4.88849e-7	1.44844e-7	6.11062e-8	3.12864e-8	3.91079e-9

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TABLE 3. Model 2: The average estimates (AEs), the mean squared errors (MSEs), the asymptotic variances of the LSE (ASVs) and the asymptotic variances of the modified Newton-Raphson method (ASVPs) of $\widehat{\omega}_2$, are reported.

		$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$	$N = 1000$
$\sigma = .25$	AE	.4999985	.4999991	.4999990	.4999994	.4999987	.4999999
	MSE	6.16658e-7	8.22836e-8	2.40645e-8	1.08731e-8	5.14001e-9	4.65847e-10
	ASV	1.07958e-6	1.34947e-7	3.99844e-8	1.68684e-8	8.63662e-9	1.07958e-9
	ASVP	2.69894e-7	3.37368e-8	9.99609e-9	4.21710e-9	2.15916e-9	2.69894e-10
$\sigma = .50$	AE	.4999745	.4999932	.4999971	.4999985	.4999974	.4999997
	MSE	2.47331e-6	3.28900e-7	9.63532e-8	4.34934e-8	2.05570e-8	1.86337e-9
	ASV	4.31831e-6	5.39789e-7	1.59937e-7	6.74736e-8	3.45465e-8	4.31831e-9
	ASVP	1.07958e-6	1.34947e-7	3.99844e-8	1.68684e-8	8.63662e-9	1.07958e-9
$\sigma = .75$	AE	.4999416	.4999860	.4999946	.4999972	.4999959	.4999995
	MSE	5.60775e-6	7.40676e-7	2.17176e-7	9.79131e-8	4.62600e-8	4.19446e-9
	ASV	9.71620e-6	1.21452e-6	3.59859e-7	1.51816e-7	7.77296e-8	9.71620e-9
	ASVP	2.42905e-6	3.03631e-7	8.99648e-8	3.79539e-8	1.94324e-8	2.42905e-9
$\sigma = 1.0$	AE	.4998992	.4999772	.4999916	.4999954	.4999941	.4999992
	MSE	1.01000e-5	1.32008e-6	3.87051e-7	1.74253e-7	8.22754e-8	7.46353e-9
	ASV	1.72732e-5	2.15916e-6	6.39750e-7	2.69894e-7	1.38186e-7	1.72732e-8
	ASVP	4.31831e-6	5.39789e-7	1.59937e-7	6.74736e-8	3.45465e-8	4.31831e-9
$\sigma = 1.25$	AE	.4998465	.4999670	.4999881	.4999931	.4999921	.4999989
	MSE	1.60816e-5	2.07131e-6	6.06699e-7	2.72710e-7	2.47146e-8	1.16775e-8
	ASV	2.69894e-5	3.37368e-6	9.99609e-7	4.21710e-7	2.15916e-7	2.69894e-8
	ASVP	6.74736e-6	8.43420e-7	2.49902e-07	1.05428e-7	5.39789e-8	6.74736e-9

TABLE 4. Model 1: The average estimates (AEs), the mean squared errors (MSEs), the asymptotic variances of the LSE (ASVs) and the asymptotic variances of the modified Newton-Raphson method (ASVPs) of $\widehat{A}$ are reported.

		$N = 100$	$N = 200$	$N = 300$	$N = 400$	$N = 500$	$N = 1000$
$\sigma = .25$	AE	1.412627	1.413344	1.413683	1.413859	1.413772	1.414180
	MSE	5.60946e-3	2.62114e-3	1.54835e-3	1.13729e-3	7.93309e-4	3.35889e-4
	ASV	8.99648e-3	4.49824e-3	2.99883e-3	2.24912e-3	1.79930e-3	8.99648e-4
	ASVP	2.24912e-3	1.12456e-3	7.49707e-4	5.62280e-4	4.49824e-4	2.24912e-4
$\sigma = .50$	AE	1.409218	1.411501	1.412534	1.412915	1.412942	1.413996
	MSE	2.24744e-2	1.05072e-2	6.19872e-3	4.55484e-3	3.17618e-3	1.34412e-3
	ASV	3.59859e-2	1.79930e-2	1.19953e-2	8.99648e-3	7.19718e-3	3.59859e-3
	ASVP	8.99648e-3	4.49824e-3	2.99883e-3	2.24912e-3	1.79930e-3	8.99648e-4
$\sigma = .75$	AE	1.403949	1.408677	1.410761	1.411385	1.411726	1.413661
	MSE	5.07373e-2	2.37244e-2	1.39716e-2	1.02714e-2	7.15734e-3	3.02664e-3
	ASV	8.09683e-2	4.04842e-2	2.69894e-2	2.02421e-2	1.61937e-2	8.09683e-3
	ASVP	2.02421e-2	1.01210e-2	6.74736e-3	5.06052e-3	4.04842e-3	2.02421e-3
$\sigma = 1.0$	AE	1.396750	1.404860	1.408361	1.409268	1.410121	1.413174
	MSE	9.06768e-2	4.23841e-2	2.49047e-2	1.83191e-2	1.27515e-2	5.38682e-3
	ASV	.143944	7.19718e-2	4.79812e-2	3.59859e-2	2.87887e-2	1.43944e-2
	ASVP	3.59859e-2	1.79930e-2	1.19953e-2	8.99648e-3	7.19718e-3	3.59859e-3
$\sigma = 1.25$	AE	1.387489	1.400025	1.405323	1.406566	1.408126	1.412536
	MSE	.1427504	6.66468e-2	3.90543e-2	2.87439e-2	1.99796e-2	8.42952e-3
	ASV	.2249120	.1124560	7.49708e-2	5.62280e-2	4.49824e-2	2.24912e-2
	ASVP	5.62280e-2	2.81140e-2	1.87427e-2	1.40570e-2	1.12456e-2	5.62280e-3