Bayesian Sampling Plan for the Exponential Distribution with Generalized Type-I Hybrid Censoring Scheme

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Abstract

This paper discusses Bayesian sampling plan when the lifetime of the experimental units follows an exponential distribution and the data are generalized Type-I hybrid censored. Here we adopt a decision theoretic approach, and the Bayesian decision function is derived under a general loss function. Based on the Bayesian decision function, the Bayesian sampling plan (BSP) has been obtained. Further, for the conjugate prior distribution, the closed-form Bayes decision rule has been provided under the quadratic decision loss. It is noticed that for higher degree polynomial loss functions, closed-form Bayes decision function cannot be obtained analytically. To address this limitation, we discuss a numerical approach to obtain Bayes decision function for other forms of the loss functions. As an illustration, we consider fifth-degree polynomial loss function to obtain the optimum BSP. Optimum BSPs under different scenarios have been reported.

Keywords and Phrases: Bayesian sampling plan, Decision-theoretic approach, Exponential distribution, Generalized hybrid censoring scheme, Maximum Likelihood Estimator.

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1 Introduction

An acceptance sampling plan is a primary statistical technique in the industry by which a decision of acceptance or rejection of a batch is taken. It is one of the oldest methods of quality assurance in the field of statistical quality control and is characterized in two categories viz., attribute sampling plan and variable sampling plan. While designing a sampling plan, there is a wide variety of plans that may be applied using various criteria of optimality. Among them, a decision-theoretic approach is more appealing and widely used because, here the determination of a sampling plan is obtained by making an optimal decision based on some realistic criterion, such as maximizing the return or minimizing the risk; see for example Lam (1994), Lin *et al.* (2002), Huang and Lin (2002), Chen *et al.* (2004), Lin *et al.* (2008, 2010), Liang and Yang (2013), Tsai *et al.* (2014), Yang *et al.* (2017) and the references cited therein.

In practice, most of the life testing experiments are usually censored. The Type-I and Type-II censoring are the most basic censoring schemes. Several other censoring schemes have also been introduced in the literature. Readers are referred to Balakrishnan and Kundu (2013) for more information on different censoring schemes. Due to the complexities of the censoring factors, the decision-theoretic approach for acceptance sampling plans has opened more channels for further research. Hence this approach to obtain optimal sampling plan has been studied by many authors including Lam (1990) and Lam (1994) under Type-I and Type-II censoring schemes, respectively. The decision function based on the estimator of a mean lifetime and the quadratic loss function is generally used to obtain the optimal sampling plan. Chen *et al.* (2004) and Lin *et al.* (2008, 2010) have studied the optimal sampling plans for hybrid censored samples. In a decision-theoretic approach, the Bayesian sampling plan (BSP) is quite appealing as it minimizes the Bayes risks among a class of sampling plans. For Type-I censored sample, Lin *et al.* (2002) have obtained BSP by constructing the Bayes decision function for quadratic loss function. For Type-I hybrid censored sample, Liang and Yang (2013) have obtained the BSP. For interval censored samples, Chen *et al.* (2015) have obtained the BSP. Tsai *et al.* (2014) studied an

efficient BSP for the exponential lifetime distribution with Type-I censored samples. Yang *et al.* (2017) have obtained the BSP for modified Type-II hybrid censoring for linear and quadratic loss functions. Curtailed Bayesian sampling plans for exponential distribution based on Type-II censored samples have been studied by Chen *et al.* (2017). Most of the work for obtaining the optimal BSP is based on the quadratic loss function, because in this case, construction of the Bayes decision function is easy. Recently, Prajapati *et al.* (2020, 2019) have proposed new decision theoretic sampling plan for Type-I censored and Type-I hybrid censored samples, respectively, which are as good as BSP. They have also shown that it can be obtained for a larger class of loss functions quite conveniently as compared to BSP.

The main aim of this paper is to propose BSP for the generalized Type-I hybrid censoring scheme (Type-I GHCS) as proposed by Chandrasekar *et al.* (2004), when the lifetime of the experimental units follow exponential distribution. Now we briefly describe the Type-I GHCS. We use the following notations. Let $(X_1, X_2, \ldots, X_n)$, be the lifetime of sampled units which are assumed to be independent and identically distributed (i.i.d.) random variables. Let the ordered lifetime of these units be denoted by $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$. Here, we fix integers $r, k \in \{1, 2, \ldots, n\}$ such that $k < r < n$, and time $\tau \in (0, \infty)$. If the kth failure occurs before time τ, we terminate the experiment at $\min\{X_{(r)}, \tau\}$, and if the kth failure occurs after time τ, terminate the experiment at $X_{(k)}$. It ensures a minimum k number of failures before the experiment terminates. Allowing $k = 0$, the Type-I GHCS reduces to the Type-I hybrid censoring scheme, and for $r = n$, it reduces to the modified Type-II hybrid censoring scheme. When $\tau \to \infty$, it reduces to the Type-II censoring scheme.

The goal of this paper is to investigate the problem of designing the BSP for Type-I GHCS under a fairly general loss function. The loss function consists of a inspection cost for every unit, cost per unit time of test and cost on the imprecision in the decision taken on the batch. An item which does not fail in the test is reused with a salvage value, and hence this is included as a profit in the loss function. To derive the optimum BSP, we minimize the expected loss based on the Bayes decision function. If the loss function is a

quadratic loss function, then the Bayes decision function can be obtained in explicit form, whereas for any complicated loss function the Bayes decision function cannot be obtained analytically. In that case a numerical algorithm has been proposed to obtain the optimum BSP.

The rest of the paper is organized as follows. In Section 2, we propose the decision function. The Bayes decision function and the Bayes risk of the BSP for quadratic loss are provided in Section 3. In Section 4, we discuss the BSP for higher degree polynomial loss function. To obtain the optimum BSP, we discuss the algorithm in Section 5. Numerical results of the optimum BSP are given in Section 6. Finally, we conclude the paper in Section 7. All proofs and derivations have been provided in the Appendix A and B.

2 Model Construction and Assumptions

Consider a batch of items, which is undergoing an inspection process. The quality of an item is measured by its lifetime. To analyze the quality of this batch of items, a sample of n items is put on life test. It is noted that failed items are not to be replaced. Let $(X_1, X_2, \ldots, X_n)$, denote the lifetimes of these n items which are mutually independent and identically distributed. It is assumed that $X_1, X_2, \ldots, X_n$, follow an exponential distribution with parameter λ with the following probability density function (PDF)

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0, \ \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The mean lifetime of an item is denoted by $\theta = E(X) = \lambda^{-1}$. In Type-I GHCS, the experiment gets terminated at a random time $\tau^* = \min\{X_{(r)}, \tau\} I(X_{(k)} \leq \tau) + X_{(k)} I(X_{(k)} > \tau)$, where $I(A)$ is an indicator function on set A, such that, if A occurs then $I(A) = 1$, otherwise $I(A) = 0$. Let M and M' be the number of failures in the experiment before time τ^* and τ, respectively, and $\mathbf{x} = (x_{(1)}, \ldots, x_{(m)})$ denotes the observed value of the random vector $\mathbf{X} = (X_{(1)}, \ldots, X_{(M)})$ for a given $M = m$. Then the observed data before time τ^* can be categorized in three ways:

(a) If $x_{(k)} < \tau < x_{(r)}$, then $\tau^* = \tau$, $m = m'$, where m' is the observed value of M' and observations are $0 < x_{(1)} < \ldots < x_{(k)} < \ldots < x_{(m')} < \tau$; (b) If $x_{(r)} < \tau$, then $\tau^* = x_{(r)}$, $m = r$ and observations are $0 < x_{(1)} < \ldots < x_{(r)} < \tau$, and (c) If $x_{(k)} > \tau$, then $\tau^* = x_{(k)}$, $m = k$ and observations are $0 < x_{(1)} < \ldots < x_{(m')} < \tau < x_{(m'+1)} < \ldots < x_{(k)}$. For Type-I GHCS, the maximum likelihood estimator (MLE) of the parameter θ is as follows

$$\begin{aligned}\widehat{\theta}_{MLE} &= \begin{cases} \frac{\sum_{i=1}^{k} X_{(i)} + (n-k)X_{(k)}}{k}, & \text{if } X_{(k)} > \tau, \\ \frac{\sum_{i=1}^{M'} X_{(i)} + (n-M')\tau}{M'}, & \text{if } X_{(k)} < \tau < X_{(r)}, \\ \frac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{r}, & \text{if } X_{(r)} < \tau, \end{cases} \\ &= \frac{\sum_{i=1}^{M} X_{(i)} + (n-M)\tau^*}{M} = \frac{Y(n,r,k,\tau,M)}{M}, \quad \forall\, M \geq k.\end{aligned}$$

A decision function δ is a function defined on the random vector $\mathbf{X} = (X_{(1)}, \ldots, X_{(m)})$, such that

$$\delta(\mathbf{X}) = \begin{cases} 1, & \text{if } \widehat{\theta}_{MLE} \in A, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Here A denotes the acceptance region. The above decision function (1) is a very natural decision function to accept or reject a batch of items as it has already been considered by Lam (1990, 1994).

In many life testing situations, it is quite undesirable or even impossible to observe complete sample due to various restrictions on the cost or time of the experiment. Therefore, it is desirable to have the experiment terminated as soon as the observed data is sufficient for a meaningful analysis. It means that when we take a decision, our sample size has to be ideal, i.e., neither too large nor too small. To get a sufficient sample for making a meaningful decision, we consider the efficiency of the estimated parameter as an important factor in the loss function. Hence, we use a loss function incorporating costs corresponding to sample size, duration of the experiment, action taken on batch and also a cost on mean square error (MSE) of the estimator, to deal with the imprecision in the decision. Such a loss function is considered by Bhattacharya *et al.* (2014) to obtain the optimum life testing plans. We consider the loss function used by Liang and Yang (2013)

and Yang *et al.* (2017), and additionally include the cost on MSE of $\widehat{\theta}_{MLE}$. Then the loss function is given by

$$L(n,r,k,\tau,\delta(\mathbf{X}),\lambda) = \begin{cases} nC_s-(n-M)r_s+\tau^*C_\tau+\text{MSE}(\widehat{\theta}_{MLE})C_v+g(\lambda), & \text{if } \delta(\mathbf{X})=1, \\ nC_s-(n-M)r_s+\tau^*C_\tau+\text{MSE}(\widehat{\theta}_{MLE})C_v+C_r, & \text{if } \delta(\mathbf{X})=0, \end{cases} \tag{2}$$

where n is the sample size, $MSE(\widehat{\theta}_{MLE})$ is the MSE of $\widehat{\theta}_{MLE}$, C_s is the inspection cost per item, C_τ is the cost per unit experimental time, C_v is the cost associated with the estimator, $g(\lambda)$ is the cost on accepting the batch, since as the mean lifetime increases the cost on accepting the batch decreases, therefore, $g(\lambda)$ is increasing in λ for $\lambda > 0$, and C_r is the cost of rejecting the batch. Note that we presume that C_r is fixed because when we reject a batch it will be discarded or returned to the supplier, and so the loss is fixed. If an item does not fail, then it can be reused with a salvage value r_s, such that $C_s > r_s \geq 0$. It may be noted that the precision of an estimator is an essential component in making any decision. Therefore, it depends on the practitioner to choose C_v compared to other costs. For example, if a practitioner believes that to him/her, the precision in deciding whether to accept or reject a batch is essential, then C_v should be large; otherwise, it can be small.

Further, the sampling plan is denoted by (n,r,k,τ,δ) when we use decision function (1), where n is the sample size, positive integers r and k are used for r^{th} and k^{th} failures, respectively, and τ is the censoring time. Here we consider a conjugate prior gamma distribution $G(a,b)$, with density

$$\pi(\lambda) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-\lambda b}, \quad \lambda > 0, \;\; a,b > 0 \tag{3}$$

and zero, otherwise. Here a and b are the hyper parameters. Then our aim is to obtain the optimum BSP for Type-I GHCS by determining the optimal decision function under the given loss function which minimizes the Bayes risk among all possible sampling plans of type (n,r,k,τ,δ).

3 Bayes Decision Function and BSP for Type-I GHCS

For a given $M = m$, the joint PDF of the random vector $\mathbf{X} = (X_{(1)}, \ldots, X_{(m)})$ is given by

$$f(\mathbf{x}|\lambda) = \begin{cases} \frac{n!}{(n-k)!}\lambda^k e^{-\lambda[\sum_{i=1}^{k} x_i + (n-k)x_{(k)}]} & m = k, \\ \frac{n!}{(n-m')!}\lambda^{m'} e^{-\lambda[\sum_{i=1}^{m'} x_i + (n-m')\tau]} & m = m', \\ \frac{n!}{(n-r)!}\lambda^r e^{-\lambda[\sum_{i=1}^{r} x_i + (n-r)x_{(r)}]} & m = r. \end{cases}$$

It can be written in a combined form as follows:

$$f(\mathbf{x}|\lambda) = \frac{n!}{(n-m)!}\lambda^m e^{-\lambda y(n,r,k,\tau,m)}, \ \forall\ m \geq k,\ \lambda > 0, \tag{4}$$

where $y(n,r,k,\tau,m) = \sum_{i=1}^{m} x_i + (n-m)\tau^*$. It is assumed that the parameter λ is a positive random variable, having a prior PDF $\pi(\lambda)$, as defined in (3), over the parameter space $(0,\infty)$. Thus, the posterior PDF of λ given $\mathbf{x}$ is

$$\pi(\lambda|\mathbf{x}) = \frac{\pi(\lambda)f(\mathbf{x}|\lambda)}{\int_0^\infty \pi(\lambda)f(\mathbf{x}|\lambda)d\lambda} = \frac{\lambda^m e^{-\lambda y(n,r,k,\tau,m)}\pi(\lambda)}{\int_0^\infty \lambda^m e^{-\lambda y(n,r,k,\tau,m)}\pi(\lambda)d\lambda}. \tag{5}$$

Since the PDF $\pi(\lambda|\mathbf{x})$ depends on $\mathbf{x}$ only through m and $y(n,r,k,\tau,m)$, therefore from (4), $(M,\ Y(n,r,k,\tau,M))$ is jointly sufficient for λ, and hence we may also denote this posterior PDF by $\pi(\lambda|m, y(n,r,k,\tau,m))$. For loss function (2), the Bayes risk of the sampling plan (n,r,k,τ,δ) is

$$R_B^\delta(n,r,k,\tau) \ = \ n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(\mathrm{MSE}(\widehat{\theta}_{MLE}))C_v + R_D(\delta|n,r,k,\tau), \tag{6}$$

where $R_D(\delta|n,r,k,\tau)$ is the risk associated with the decision function δ. Thus,

$$R_D(\delta|n,r,k,\tau) \ = \ R_1(\delta|n,r,k,\tau) + R_2(\delta|n,r,k,\tau), \tag{7}$$

where $R_1(\delta|n,r,k,\tau) = E(g(\lambda))$,

$$R_2(\delta|n,r,k,\tau) \ = \ E_{\mathbf{X}}[(1-\delta(\mathbf{X}))(C_r - \phi_\pi(m, Y(n,r,k,\tau,m)))],$$

and

$$\phi_\pi(m, y(n,r,k,\tau,m)) \ = \ E_{\lambda|\mathbf{x}}[g(\lambda)] = \int_0^\infty g(\lambda)\pi(\lambda|m, y(n,r,k,\tau,m))d\lambda \tag{8}$$

denotes the posterior expectation of $g(\lambda)$ given data $\mathbf{X} = \mathbf{x}$. To obtain the Bayes decision function we need to minimize the posterior risk of the decision function which is equivalent to the minimization of $R_2(\delta|n,r,k,\tau)$ w.r.t. δ. Now for fixed (n,r,k,τ), the Bayes risk $R_B^{\delta}(n,r,k,\tau)$ is minimum when $R_2(\delta|n,r,k,\tau)$ is minimized w.r.t. δ, which can be obtained by considering two cases.

Case 1: $\phi_\pi(m, y(n,r,k,\tau,m)) \leq C_r$:

If $\delta(\mathbf{x}) = 0$, then $R_2(\delta|n,r,k,\tau) \geq 0$, and if $\delta(\mathbf{x}) = 1$, then $R_2(\delta|n,r,k,\tau) = 0$.

Case 2: $\phi_\pi(m, y(n,r,k,\tau,m)) > C_r$:

If $\delta(\mathbf{x}) = 0$, then $R_2(\delta|n,r,k,\tau) < 0$, and if $\delta(\mathbf{x}) = 1$, then $R_2(\delta|n,r,k,\tau) = 0$.

Therefore, for fixed sample size n, positive integers $k < r \leq n$ and censoring time τ, the Bayes decision function, which minimizes Bayes risk $R_B^{\delta}(n,r,k,\tau)$ among all decision functions is given by

$$\delta_B = \delta_B(m, y(n,r,k,\tau,m)|n,r,k,\tau) = \begin{cases} 1, & \text{if } \phi_\pi(m, y(n,r,k,\tau,m)) \leq C_r, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Then our aim is to obtain the optimum BSP $(n_B, r_B, k_B, \tau_B, \delta_B)$, which minimizes the Bayes risk $R_B^{\delta_B}(n,r,k,\tau)$ among all possible BSPs (n,r,k,τ,δ_B). To do so, we need the closed form of Bayes decision function which can be obtained by three Lemmas A.1-A.3 given in Appendix A. These lemmas provide the monotonicity of the Bayes decision function which help to present an alternate form of the Bayes decision function. By Lemma A.2, one can show that for a fixed m, $\phi_\pi(m, y(n,r,k,\tau,m))$ is non-increasing w.r.t. $y(n,r,k,\tau,m)$ for $y(n,r,k,\tau,m) > 0$. Let us define $A(m) = \{y > 0 | \phi_\pi(m,y) \leq C_r\}$ and

$$T_m = \begin{cases} \inf A(m), & \text{if } A(m) \text{ is nonempty,} \\ \infty, & \text{otherwise.} \end{cases} \tag{10}$$

Then from Lemma A.3, the Bayes decision function can be expressed as

$$\delta_B(m, y(n,r,k,\tau,m)|n,r,k,\tau) = \begin{cases} 1, & \text{if } y(n,r,k,\tau,m) \geq T_m, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

which can be written in terms of $\widehat{\theta}_{MLE}$ as follows;

$$\delta_B = \delta_B(m, y(n,r,k,\tau,m)|n,r,k,\tau) = \begin{cases} 1, & \text{if } \widehat{\theta}_{MLE} \geq c_m, \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

where for each $m \geq k$, $c_m = \frac{T_m}{m}$. An alternative Bayes decision function (12), will be used to obtain the Bayes risk of the BSP. In general, Bayes risk of the BSP (n,r,k,τ,δ_B) is given by

$$R_B^{\delta_B}(n,r,k,\tau) = n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(\text{MSE}(\widehat{\theta}_{MLE}))C_v + R_D(\delta_B|n,r,k,\tau).$$

To obtain the Bayes risk, we have to calculate $E(M)$, $E(\tau^*)$, $E(\text{MSE}(\widehat{\theta}_{MLE}))$ and $R_D(\delta_B|n,r,k,\tau)$. Out of which, $E(M)$, $E(\tau^*)$, and $E(\text{MSE}(\widehat{\theta}_{MLE}))$ have been derived in the Appendix B and

$$R_D(\delta_B|n,r,k,\tau) \quad = \quad E(g(\lambda)) + \int_0^\infty [C_r - g(\lambda)] P(\delta_B(M, Y(n,r,k,\tau,M)|n,r,k,\tau) = 0|\lambda)\pi(\lambda)d\lambda$$

Now, from decision function (12)

$$P(\delta_B(M, Y(n,r,k,\tau,M)|n,r,k,\tau) = 0|\lambda) = P\big(\widehat{\theta}_{MLE} < c_M|\lambda\big).$$

Hence to obtain this probability, we need the distribution of $\widehat{\theta}_{MLE}$ which is obtained by Chandrasekar *et al.* (2004) and can be used to obtain Bayes risk. Note that for a fixed m, c_m depends on the number of observations m, and can be evaluated either analytically or numerically, depending on the form of $g(\lambda)$.

3.1 Bayes risk of the BSP for quadratic loss

To obtain the Bayes risk of the BSP, first we consider the quadratic acceptance cost $g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$, with a_0, a_1 and $a_2 \geq 0$. It has been widely used in the literature to obtain the optimal sampling plan, for example, see, Lam (1994); Lin *et al.* (2008, 2010, 2002); Huang and Lin (2002); Chen *et al.* (2004); Liang and Yang (2013); Tsai *et al.* (2014); Yang *et al.* (2017), as it is analytically tractable, and simplifies the problem to a great extent. The

prior distribution of λ is $G(a,b)$, it is well known that the posterior distribution of λ is also gamma, viz.,

$$\lambda|m, y(n,r,k,\tau,m) \sim G(m+a, y(n,r,k,\tau,m)+b).$$

Then the posterior expectation of $g(\lambda)$ based on (8) is

$$\phi_\pi(m, y(n,r,k,\tau,m)) \;=\; a_0 + a_1\frac{m+a}{y(n,r,k,\tau,m)+b} + a_2\frac{(m+a)(m+a+1)}{(y(n,r,k,\tau,m)+b)^2}.$$

We pre-specify the parameter values (n,r,k,τ) and construct the Bayes decision function as follows

$$\delta_B(m, y(n,r,k,\tau,m)|n,r,k,\tau) = \begin{cases} 1, & \text{if } \; a_0 + a_1\frac{m+a}{y(n,r,k,\tau,m)+b} + a_2\frac{(m+a)(m+a+1)}{(y(n,r,k,\tau,m)+b)^2} \le C_r, \\ 0, & \text{otherwise,} \end{cases} \tag{13}$$

and the minimum Bayes risk is obtained as

1. If $a_0 \ge C_r$, then $\phi_\pi(m, y(n,r,k,\tau,m)) > C_r$ for all $(m, y(n,r,k,\tau,m))$ over all possible (n,r,k,τ). Thus,

   $$\delta_B(m, y(n,r,k,\tau,m)|n,r,k,\tau) = 0 \;\forall\; (n,r,k,\tau,m).$$

   In this case, we should take $n_B = 0,\ r_B = 0$, $k_B = 0$, and $\tau_B = 0$. The minimum Bayes risk is $R_B^{\delta_B}(n_B, r_B, k_B, \tau_B) = R_D(\delta_B|n_B, r_B, k_B, \tau_B) = C_r$.

2. Let $0 < a_0 < C_r$, then the Bayes decision function is given by (13). Thus to find the closed form of the Bayes decision function we need to obtain the set

   $$A(m) = \{z > 0 | \phi_\pi\big(m, z\big) \le C_r\},$$

   where $z = y(n,r,k,\tau,m)$ and to construct $A(m)$, we need to obtain the set of $z > 0$, such that

   $$h_1(z) = a_0 + a_1\frac{m+a}{z+b} + a_2\frac{(m+a)(m+a+1)}{(z+b)^2} - C_r \le 0,$$

   which is equivalent to find $z > 0$, such that,

   $$h_2(z) = (C_r - a_0)\big(z+b\big)^2 - a_1(m+a)\big(z+b\big) - a_2(m+a)(m+a+1) \ge 0.$$

It can easily be shown that if $D_n(m)$ is the only real root, or $D_n(m)$ is the maximum real root of $h_2(z) = 0$, i.e.,

$$D_n(m) = \frac{a_1(m+a) + \sqrt{a_1^2(m+a)^2 + 4(C_r - a_0)a_2(m+a)(m+a+1)}}{2(C_r - a_0)},$$

then $T_m = \max\{D_n(m) - b, 0\}$ and for $m \geq k$, $c_m = \frac{T_m}{m}$.

Now using the distribution function of $\widehat{\theta}_{MLE}$ the Bayes risk $R_B^{\delta_B}(n, r, k, \tau)$ of the BSP can be obtained quite easily. Derivation of the Bayes risk of the BSP is given in Appendix B.

4 Bayes risk of the BSP for higher degree polynomial loss function

In many cases form of the acceptance cost is not known. It may be assigned subjectively by experts or it has to be estimated. It is quite likely that the widely used quadratic form of the loss function is only an approximation to the true loss function. The argument given by Lam (1990, 1994) for using the quadratic loss function in fact justifies that higher degree polynomial loss function can also be a good approximation of true loss function. In this section we consider the higher degree polynomial loss function to obtain the Bayes risk of the BSP, by assuming the acceptance cost as a higher degree polynomial, i.e., $g(\lambda) = \sum_{i=0}^{d} a_i \lambda^i$, $a_i \geq 0$ for all $i = 0, \ldots, d$. As $g(\lambda)$ is meaningful only if it is non-decreasing with respect to λ, we assume that this property holds for the proposed acceptance cost. In a bounded region of interest, any smooth cost function can be well approximated by a polynomial. Here, the region of interest means a collection of λ values at which probability of accepting the batch is non negligible. Therefore, it is meaningful to consider the higher degree polynomial loss function. Now when $g(\lambda) = \sum_{i=0}^{d} a_i \lambda^i$ then,

$$\phi_\pi\big(m, y(n, r, k, \tau, m)\big) = a_0 + \sum_{j=1}^{d} a_j \frac{(m+a)\ldots(m+a+j-1)}{(y(n, r, k, \tau, m) + b)^j}$$

and to construct $A(m)$, we need to obtain the set of $z > 0$, such that

$$a_0 + \sum_{j=1}^{d} a_j \frac{(m+a)\ldots(m+a+j-1)}{(z+b)^j} \leq C_r,$$

which is equivalent to finding $z > 0$, such that,

$$h_1(z) = a_0 + \sum_{j=1}^{d} a_j \frac{(m+a)\ldots(m+a+j-1)}{(z+b)^j} - C_r \leq 0.$$

We need a solution of $h_1(z) = 0$, but analytically it cannot be solved for z when $d \geq 5$ (see Herstein (1975)). So, for these cases we consider the following scheme to derive the alternative Bayes decision function.

Scheme A

1. From Lemma A.2, it is clear that for a fixed m, the equation $h_1(z) = \phi_\pi(m,z) - C_r = 0$ has at most one change of sign, i.e., it has only one real root.

2. Obtain the root of the equation $h_1(z) = 0$, by any numerical method and denote it by α_m.

3. Then define $T_m = \max(0, \alpha_m)$ and the Bayes decision function is as given in (12).

The expression for Bayes risk for higher degree polynomial loss function can be obtained exactly in similar manner as in the case of quadratic loss function and it is given in Appendix B.

5 Algorithm

The Bayes risk of the BSP is obtained in explicit form but the expression is very complicated. As a result, obtaining the optimal values of n, r, k, and τ analytically is not straightforward. In the loss function, we have costs on total time of the experiment and on sample size. Therefore, optimal values of n, r, k, and τ are automatically bounded above. For saving the time and computation considerably, the following theorem gives a much smaller upper bound for n, which makes the search procedure simpler and efficient.

Theorem 5.1. *Let n_B, r_B, and k_B be the optimal values of n, r, and k. Then,*

$$n_B \leq \min\left\{\frac{E_\lambda[g(\lambda)]}{C_s - r_s}, \frac{C_r}{C_s - r_s}\right\},$$

and $0 \leq k_B < r_B \leq n_B$.

Proof. See Appendix. ∎

We have used Algorithm A which is similar to Yang *et al.* (2017) and Prajapati *et al.* (2020, 2019) algorithms for deriving the optimum BSP.

Algorithm A

Algorithm for finding optimum BSP $(n_B, r_B, k_B, \tau_B, \delta_B)$

1. For each fixed parameter (n, r, k, τ), derive the closed form of Bayes decision function δ_B to minimize the Bayes risks $R_B^{\delta}(n, r, k, \tau)$ among the class of all decision functions.

2. For each fixed parameter (n, r, k), find the censoring time $\tau_B(n, r, k)$ to minimize the Bayes risks $R_B^{\delta_B}(n, r, k, \tau)$ among all $\tau > 0$. That is, $\tau_B(n, r, k)$ is the time such that

$$R_B^{\delta_B}(n, r, k, \tau_B(n, r, k)) = \inf_{\tau > 0} R_B^{\delta_B}(n, r, k, \tau).$$

3. For each fixed parameter n and r, find an integer $k_B(n, r), 0 \leq k_B(n, r) < r$ such that

$$R_B^{\delta_B}(n, r, k_B(n, r), \tau_B(n, r, k_B(n, r))) = \min_{0 \leq k < r \leq n} R_B^{\delta_B}(n, r, k, \tau_B(n, r, k)).$$

4. For each fixed integer n, find an integer $r_B(n) \leq n$ such that

$$R_B^{\delta_B}(n, r_B(n), k_B(n, r_B(n)), \tau_B(n, r_B(n), k_B(n, r_B(n)))) = \min_{r \leq n} R_B^{\delta_B}(n, r, k_B(n, r), \tau_B(n, r, k_B(n, r))).$$

5. Find an integer $n_B \geq 0$ such that

$$R_B^{\delta_B}(n_B, r_B, k_B, \tau_B) = \min_{n \geq 0} R_B^{\delta_B}(n, r_B(n), k_B(n, r_B(n)), \tau_B(n, r_B(n), k_B(n, r_B(n)))),$$

where $r_B = r_B(n_B)$, $k_B = k_B(n_B, r_B(n_B))$, $\tau_B = \tau_B(n_B, r_B(n_B), k_B(n_B, r_B(n_B))$, and $\delta_B = \delta_B(.|n_B, r_B(n_B), k_B(n_B, r_B(n_B)), \tau_B(n_B, r_B(n_B), k_B(n_B, r_B(n_B))))$. Therefore, $(n_B, r_B, k_B, \tau_B, \delta_B)$ is the optimum BSP.

6 Numerical Results

In this section, numerical study is conducted to provide some optimum BSP for Type-I GHCS. Algorithms A is used to obtain optimum BSP for quadratic and fifth-degree polynomial loss functions. All numerical computations are done using a program in *R* software. The function "optim(.)" provides basic optimization capabilities and is among the most widely used functions in *R* software. Numerical results are presented in Tables 1 - 10 and the Bayes risk $R_B^{\delta_B}(n_B, r_B, k_B, \tau_B)$ of optimum BSP in each table is denoted by the notation $R_B^{\delta_B}$.

6.1 Quadratic Loss Function

In this section, we consider the quadratic loss function for numerical results. Algorithm A as provided in Section 5, is used to obtain the optimum BSP. For illustrative purpose, we take $a = 3.0,\ b = 1.0,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_s = 0.3,\ r_s = 0.1,\ C_v = 3.0,\ C_\tau = 0.8$ and $C_r = 30$. In each table, only the two hyper-parameters or one coefficient are permitted to vary and others are kept fixed. Before obtaining the optimum BSP by optimization function "optim(.)" in *R*, we show graphically that Bayes risk function $R_B^{\delta_B}(n, r, k, \tau)$ has a unique minimum. In Figure 1 we fix the values of n, r and k and draw the plot of the Bayes risk function w.r.t. τ. It appears from Figure 1 that Bayes risk has a unique minimum. Numerical results are presented in Tables 1 - 4 for quadratic loss function. From Table 1, we see that when $a = 3.0, b = 1.0$, the optimum BSP is $(n_B, r_B, k_B, \tau_B, \delta_B) = (7, 6, 4, 0.5633, \delta_B)$, and the associated Bayes risk is $R_B^{\delta_B}(7, 6, 4, 0.5633) = 25.0148$. For an exponential random variable X with scale λ having a gamma prior $G(a, b)$, the expectation $E(X)$ exists when $a > 1$, and is $\frac{b}{a-1}$. Therefore, when $a = 2.2, b = 0.8$, then $E(X) = 0.67$ and minimum Bayes risk is 24.1538 and for $a = 3.5, b = 0.8$, $E(X) = 0.32$ and minimum Bayes risk is 29.2754. Hence, minimum Bayes risk $R_B^{\delta_B}$ is increasing when expected lifetime of item $E(X)$ is decreasing. It is also noticed that the minimum Bayes risk $R_B^{\delta_B}$ is always less than or equal to the values of $E_\lambda[g(\lambda)] = a_0 + a_1\frac{a}{b} + a_2\frac{a(a+1)}{b^2}$ and C_r. For example, when $a = 3.0, b = 1.0$ the minimum Bayes risk $R_B^{\delta_B} = 25.0148$, which is less than equal to $C_r = 30$ and $E_\lambda[g(\lambda)] = 32$.

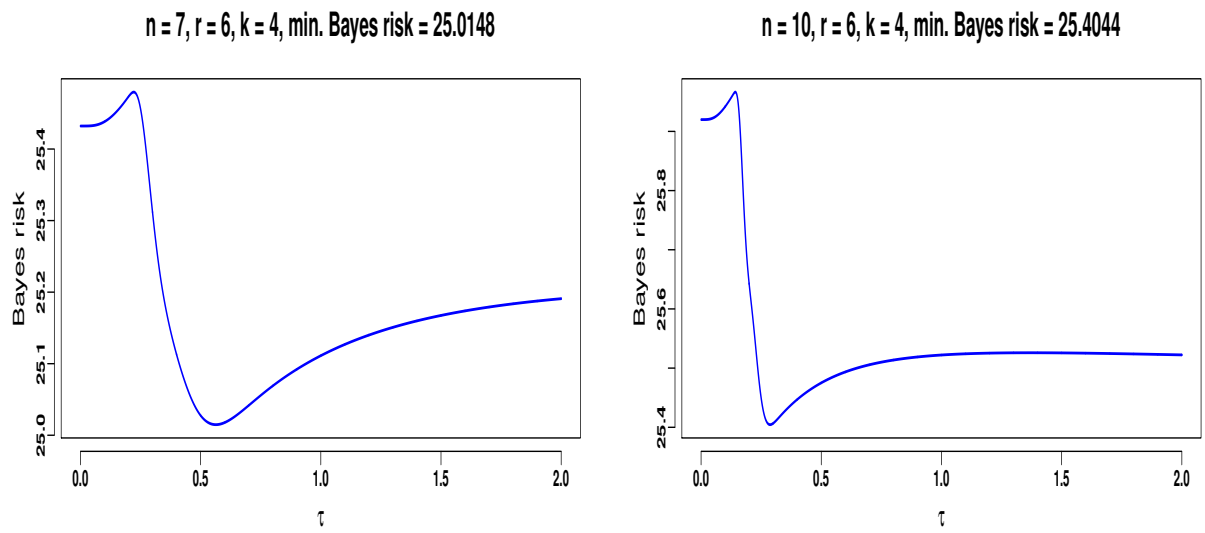

Figure 1: Plot of Bayes risk with set of coefficients and costs $a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_s = 0.3,\ r_s = 0.3,\ C_v = 3.0,\ C_\tau = 0.8,\ C_r = 30$ and parameters $a = 3.0,\ b = 1.0$.

Similarly, when $a = 3.0, b = 1.2$, the minimum Bayes risk $R_B^{\delta_B} = 22.3891$, which is less than $E_\lambda[g(\lambda)] = 23.67$ and $C_r = 30$. However, it has already been discussed in Theorem 5.1 that the minimum Bayes risk $R_B^{\delta_B}$ is always less than $E_\lambda[g(\lambda)]$ and C_r, otherwise we accept or reject the lot without sampling. In Table 1, we observe that for fixed b, as a is increasing, the Bayes risk $R_B^{\delta_B}$ is increasing, and for fixed a, as b is increasing, the Bayes risk is decreasing. It is also observed that as C_s is increasing, the Bayes risk is increasing, and the optimum values of n , r and k are decreasing. It implies that if the inspection cost C_s is high we cannot afford to test a large number of units. It is also observed that the optimal plan seems to be more sensitive to the inspection cost than the choice of the hyperparametrs of the prior distribution. In Table 2, we can see that as costs C_τ and C_r are increasing, the Bayes risk $R_B^{\delta_B}$ is increasing. We can also see that as the rejection cost C_r increases, optimal values of n and r increase, which indicate that it is difficult to reject when the rejection cost is high. In Table 3 we can see that if we increase the cost on MSE, C_v, keeping other costs fixed, the optimal values of n, r, k and τ also increase which indicates that when the cost on MSE is

Table 1: Optimum BSPs and their Bayes risks when a, b and C_s vary with $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, $r_s = 0.1$, $C_v = 3.0$, $C_\tau = 0.8$, $C_r = 30$

a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
2.2	0.8		24.1538	8	7	6	0.6281			0.15	23.7091	12	9	5	0.4029
2.5	0.8		25.1849	7	6	5	0.6234			0.20	24.2459	9	8	5	0.5679
3.0	1.0	0.3	25.0148	7	6	4	0.5633	3.0	1.0	0.30	25.0148	7	6	4	0.5633
3.0	1.2		22.3891	7	6	4	0.4898			0.50	26.0031	4	4	3	0.8469
3.5	0.8		29.2754	4	4	3	1.1500			1.00	27.6727	3	3	2	0.7313

Table 2: Optimum BSPs and their Bayes risks when C_τ and C_r vary with $a = 3.0$, $b = 1.0$, $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, $C_s = 0.3$, $r_s = 0.1$, $C_v = 3.0$

C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.2		24.5389	6	6	5	1.0813		20	19.6657	5	4	3	0.7207
0.4		24.7207	6	6	4	0.9086		30	25.0148	7	6	4	0.5633
0.8	30	25.0148	7	6	4	0.5633	0.8	40	28.4871	7	7	4	0.5805
1.0		25.1209	7	6	4	0.5516		50	30.7622	7	7	4	0.4976
2.0		25.5990	7	5	3	0.3670		55	31.5799	7	7	4	0.4574

large, more units are required to reach the optimal decision. Table 4 presents the optimum BSPs for the coefficients of acceptance cost, and it is clear that when the coefficients of the acceptance costs a_1 and a_2 increase the Bayes risk $R_B^{\delta_B}$ increases, but the optimal plan does not follow any specific pattern.

6.2 Fifth-degree Polynomial Loss Function

To demonstrate the performance of the sampling plans for higher degree polynomial loss function we consider $d = 5$ and for illustrative purpose, we take $a = 3.0$, $b = 1.0$, $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = 2$, $a_5 = 2$, $C_s = 0.4$, $r_s = 0.1$, $C_v = 3.0$, $C_\tau = 0.8$ and $C_r = 75$. Algorithm A is used to obtain the optimum BSPs and their associated Bayes risks. In each

Table 3: Optimum BSPs and their Bayes risks when C_v and a_0 vary with $a = 3.0,\ b = 1.0,\ a_1 = 2,\ a_2 = 2,\ C_s = 0.3,\ r_s = 0.1,\ C_\tau = 0.8,\ C_r = 30$

C_v	a_0	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	C_v	a_0	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.5		24.6611	6	5	2	0.5070		0.5	24.1372	7	6	4	0.5414
1.5		24.8292	6	5	3	0.5234		1.0	24.4338	7	6	4	0.5488
3.0	2.0	25.0148	7	6	4	0.5633	3.0	2.0	25.0148	7	6	4	0.5633
5.0		25.2457	7	6	5	0.6039		3.0	25.5791	7	6	4	0.5781
8.0		25.5379	8	7	6	0.6414		5.0	26.6498	6	5	4	0.6094

Table 4: Optimum BSPs and their Bayes risks when a_1 and a_2 vary with $a = 3.0,\ b = 1.0,\ a_0 = 2,\ C_s = 0.3,\ r_s = 0.1,\ C_v = 3.0,\ C_\tau = 0.8,\ C_r = 30$

a_1	a_2	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a_1	a_2	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.5		23.0513	6	6	4	0.6859		1.5	23.1981	6	6	4	0.6305
1.0		23.7526	6	6	4	0.7156		2.0	25.0148	7	6	4	0.5633
2.0	2.0	25.0148	7	6	4	0.5633	2.0	3.0	27.2454	6	5	4	0.7391
3.0		26.1111	7	6	4	0.6168		4.0	28.5167	6	5	4	0.8961
5.0		27.8323	6	5	4	0.7328		5.0	29.3533	5	5	4	0.0063

table, only the two hyper-parameters or one coefficient are permitted to vary, and others are kept fixed. In Table 5, when $a = 2.2, b = 0.8$, $E(X) = 0.67$ and minimum Bayes risk is 68.0406 and for $a = 3.0, b = 1.0$, $E(X) = 0.50$ and minimum Bayes risk is 72.4593. Thus, again if expected lifetime $E(X)$ decreases, the minimum Bayes risk $R_B^{\delta_B}$ increases. Form Tables 5 - 7, we see that the effect of C_s, C_τ, C_v, or C_r on the optimum BSP follows a similar pattern as it has been observed in the case of the quadratic loss function. Further, it has also been observed that the optimum BSP is not very sensitive to the choice of the hyperparameter a and b. In Tables 8 - 10 we present the optimum BSPs and their Bayes risks for coefficients of acceptance cost. We observed that the effect of the coefficients of acceptance cost is mainly on the optimum Bayes $R_B^{\delta_B}$, but not much on the optimum BSP.

Table 5: Optimum BSPs and their Bayes risks when a, b and C_s vary with $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = 2$, $a_5 = 2$, $r_s = 0.1$, $C_v = 3.0$, $C_\tau = 0.8$, $C_r = 75$

a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
2.2	0.8		68.0406	12	11	9	1.8512			0.20	70.1520	15	13	9	1.5066
2.5	0.8		71.3133	10	9	7	1.8976			0.30	71.4609	11	10	7	1.9086
3.0	1.0	0.4	72.4593	9	8	6	1.9305	3.0	1.0	0.40	72.4593	9	8	6	1.9305
3.0	1.2		68.9658	11	10	7	1.8266			0.50	73.2232	7	7	5	3.1172
3.5	1.5		68.9629	12	11	8	1.8266			0.60	73.8512	6	6	4	3.2484

Table 6: Optimum BSPs and their Bayes risks when C_τ and C_r vary with $a = 3.0$, $b = 1.0$, $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = 2$, $a_5 = 2$, $C_s = 0.4$, $r_s = 0.1$, $C_v = 3.0$

C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.3		71.8677	8	8	6	3.5875		60	59.4836	7	6	5	2.0043
0.4		71.9919	8	8	6	3.4234		75	72.4593	9	8	6	1.9305
0.8	75	72.4593	9	8	6	1.9305	0.8	90	84.9234	11	10	7	1.8594
1.2		72.7890	9	8	6	1.8484		100	92.9909	12	11	8	1.8375
2.0		73.3795	9	7	5	1.3070		125	112.4365	15	14	9	1.7445

Table 8: Optimum BSPs and their Bayes risks when a_0 and a_1 vary with $a = 3.0$, $b = 1.0$, $a_2 = 2$, $a_3 = 2$, $a_4 = 2$, $a_5 = 2$, $C_s = 0.4$, $r_s = 0.1$, $C_v = 3.0$, $C_\tau = 0.8$, $C_r = 75$

a_0	a_1	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a_0	a_1	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.5		72.2291	9	8	6	1.9223		0.5	72.2086	9	8	6	1.9141
1.0		72.3061	9	8	6	1.9223		1.0	72.2928	9	8	6	1.9195
2.0	2.0	72.4593	9	8	6	1.9305	2.0	2.0	72.4593	9	8	6	1.9305
3.0		72.6114	9	8	6	1.9332		3.0	72.6232	9	8	6	1.9469
5.0		72.9122	9	8	6	1.9469		5.0	72.9430	9	8	6	1.9578

Table 7: Optimum BSPs and their Bayes risks when C_v and r_s vary with $a = 3.0,\ b = 1.0,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ a_3 = 2,\ a_4 = 2,\ a_5 = 2,\ C_s = 0.4,\ C_\tau = 0.8,\ C_r = 75$

C_v	r_s	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	C_v	r_s	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.5		72.2611	9	8	4	1.8566		0.05	72.5011	8	8	5	3.0352
1.5		72.3455	9	8	6	1.9086		0.10	72.4593	9	8	6	1.9305
3.0	0.1	72.4593	9	8	6	1.9305	3.0	0.15	72.4030	9	8	6	1.9031
5.0		72.6109	9	8	6	1.9578		0.20	72.3465	9	8	6	1.8813
8.0		72.8046	9	8	7	2.0289		0.30	72.1823	10	8	6	1.3726

Table 9: Optimum BSPs and their Bayes risks when a_2 and a_3 vary with $a = 3.0,\ b = 1.0,\ a_0 = 2,\ a_1 = 2,\ a_4 = 2,\ a_5 = 2,\ C_s = 0.4,\ r_s = 0.1,\ C_v = 3.0,\ C_\tau = 0.8,\ C_r = 75$

a_2	a_3	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a_2	a_3	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.5		72.1326	9	8	6	1.9031		0.5	71.9609	10	9	7	1.8922
1.0		72.2434	9	8	6	1.9141		1.0	72.1356	9	8	6	1.8922
2.0	2.0	72.4593	9	8	6	1.9305	2.0	2.0	72.4593	9	8	6	1.9305
3.0		72.6678	9	8	6	1.9441		3.0	72.7588	9	8	6	1.9578
5.0		73.0571	8	7	6	2.0125		5.0	73.2840	8	7	5	2.0070

Table 10: Optimum BSPs and their Bayes risks when a_4 and a_5 vary with $a = 3.0,\ b = 1.0,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ a_3 = 2,\ C_s = 0.4,\ r_s = 0.1,\ C_v = 3.0,\ C_\tau = 0.8,\ C_r = 75$

a_4	a_5	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B	a_4	a_5	$R_B^{\delta_B}$	n_B	r_B	k_B	τ_B
0.5		71.5585	10	9	7	1.8375		0.5	70.2875	10	9	7	1.6461
1.0		71.8979	10	9	7	1.8703		1.0	71.2586	10	9	7	1.7719
2.0	2.0	72.4593	9	8	6	1.9305	2.0	2.0	72.4593	9	8	6	1.9305
3.0		72.9264	9	8	6	1.9906		3.0	73.2149	9	8	6	2.0563
5.0		73.6403	8	7	5	2.0945		5.0	74.1218	8	7	6	2.2875

7 Concluding remarks

In this paper, we consider the Type-I GHCS censoring scheme and obtain the optimal Bayesian sampling plan under this scheme. A very general loss function has been used, and an explicit expression of the Bayes risk is obtained for a quadratic loss function. Finite algorithm is provided to arrive at the optimum BSP. Numerical algorithm is proposed to obtain the Bayes decision function whenever it cannot be obtained analytically. For illustrative purposes, we have considered a fifth degree polynomial loss function and showed how the Bayes decision function and the optimum BSP can be obtained numerically. In general it is observed that the optimum BSP is more sensitive to the costs involved in the loss function compared to the hyperparameters. Though we have considered the exponential distribution to illustrate the proposed methodology, it will be interesting to extend the results for other lifetime distributions also. It is definitely an important problem to investigate. More work is needed along that direction.

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Appendix A

To present an alternate form of the Bayes decision function, we investigate the monotonicity associated with the Bayes decision function by following three lemmas.

Lemma A.1. *Let* $0 \leq m,\ m^* \leq n$, $y = y(n,r,k,\tau,m)$ *and* $y^* = y(n,r,k,\tau,m^*)$. *Consider the likelihood ratio*

$$L_R(\lambda|(m,y),(m^*,y^*)) = \frac{\pi(\lambda|m^*,y^*)}{\pi(\lambda|m,y)}.$$

When $\pi(\lambda|m,y) \neq 0$*:*

(a) If $m = m^*$ and $y < y^*$, then $L_R(\lambda|(m,y),(m^*,y^*))$ is non-increasing in λ.

(b) If $y = y^*$ and $m < m^*$, then $L_R(\lambda|(m,y),(m^*,y^*))$ is non-decreasing in λ.

Proof. From (5) the likelihood ratio is given by

$$L_R(\lambda|(m,y),(m^*,y^*)) \quad = \quad \frac{w(m,y)}{w(m^*,y^*)}\lambda^{m^*-m}e^{-\lambda(y^*-y)},$$

where $w(m,y) = \int_0^\infty \lambda^m e^{-\lambda y}\pi(\lambda)d\lambda = \frac{b^a\Gamma(m+a)}{\Gamma(a)(y+b)^{m+a}}$.

(a) Clearly, from above , when $m = m^*$ and $y^* - y > 0$, $L_R(\lambda|(m,y),(m^*,y^*))$ is non-increasing in λ.

(b) Similarly, when $y = y^*$ and $m^* - m > 0$, $L_R(\lambda|(m,y),(m^*,y^*))$ is non-decreasing in λ. ∎

Lemma A.2. *Let $g(\lambda)$ be a positive and increasing function of λ for $\lambda > 0$. Then,*

(a) For a fixed integer $m \geq k$, $\phi_\pi(m, y(n,r,k,\tau,m))$ is strictly decreasing in $y(n,r,k,\tau,m)$, and for fixed $y(n,r,k,\tau,m) \geq 0$, $\phi_\pi(m, y(n,r,k,\tau,m))$ is strictly increasing in m.

(b) For fixed integer $m \geq k$, $\delta_B(m, y(n,r,k,\tau,m)|n,r,k,\tau)$ is non-decreasing in $y(n,r,k,\tau,m)$ and for fixed $y(n,r,k,\tau,m) \geq 0$, non-increasing in m.

Proof. Lemma A.1 asserts that the conditional PDF $\pi(\lambda|m, y(n,r,k,\tau,m))$ is a family of densities with MLR in λ considering $y(n,r,k,\tau,m)$ as a fixed quantity. Similarly, the conditional PDF $\pi(\lambda|m, y(n,r,k,\tau,m))$ has MLR property in λ considering m as a fixed quantity.

(a) Proof is similar as given in Lehmann (2005), p.70, Lemma 3.4.2, which is as follows: For a fixed m, let y and y^* are two realization of $y(n,r,k,\tau,m)$ with $y < y^*$. Define sets A and B such that

$$A = \{\lambda : \pi(\lambda|m,y^*) > \pi(\lambda|m,y)\ \forall\ m > 0\} \text{ and } B = \{\lambda : \pi(\lambda|m,y^*) \leq \pi(\lambda|m,y)\ \forall\ m > 0\}.$$

If $g(\lambda)$ is a positive and increasing function of λ, and if $a = \sup_A\{g(\lambda)\}$ and $b = \inf_B\{g(\lambda)\}$, then $b - a > 0$. Now consider

$$\begin{aligned}\phi_\pi(m,y^*) - \phi_\pi(m,y) &= \int g(\lambda)\pi(\lambda|m,y^*)d\lambda - \int g(\lambda)\pi(\lambda|m,y)d\lambda \\ &= \int_A g(\lambda)[\pi(\lambda|m,y^*) - \pi(\lambda|m,y)]d\lambda + \int_B g(\lambda)[\pi(\lambda|m,y^*) - \pi(\lambda|m,y)]d\lambda\end{aligned}$$

$$
\begin{aligned}
&\leq a\int_A(\pi(\lambda|m,y^*)-\pi(\lambda|m,y))d\lambda + b\int_B(\pi(\lambda|m,y^*)-\pi(\lambda|m,y))d\lambda \\
&= -(b-a)\int_A(\pi(\lambda|m,y^*)-\pi(\lambda|m,y))d\lambda < 0.
\end{aligned}
$$

The last step follows from the definition of PDF, i.e.,

$$
\int_A(\pi(\lambda|m,y^*)-\pi(\lambda|m,y))d\lambda = -\int_B(\pi(\lambda|m,y^*)-\pi(\lambda|m,y))d\lambda,
$$

and which implies that $\phi_\pi(m,y^*) < \phi_\pi(m,y)$. Hence, $\phi_\pi(m,y(n,r,k,\tau,m))$ is a strictly decreasing function of $y(n,r,k,\tau,m)$ for a fixed integer m. Similarly, we can prove that $\phi_\pi(m,y(n,r,k,\tau,m))$ is strictly increasing in m for a fixed $y(n,r,k,\tau,m) \geq 0$.

(b) Follows directly from (a) using relation (9), i.e.,

$$
\delta_B(m,y(n,r,k,\tau,m)|n,r,k,\tau) = \begin{cases} 1, & \text{if } \phi_\pi(m,y(n,r,k,\tau,m)) \leq C_r, \\ 0, & \text{otherwise.} \end{cases}
$$

For a fixed m and if $y < y^*$, then from (a) we know that $\phi_\pi(m,y^*) < \phi_\pi(m,y)$. Thus, if $\phi_\pi(m,y) > C_r$, then $\delta_B(m,y|n,r,k,\tau) = 0$, but $\delta_B(m,y^*|n,r,k,\tau)$ can take either of the two values 0 or 1. Now if $\phi_\pi(m,y) < C_r$, then $\phi_\pi(m,y^*) < C_r$, which implies that if $\delta_B(m,y|n,r,k,\tau) = 1$, then $\delta_B(m,y^*|n,r,k,\tau) = 1$.

Therefore, in both the cases $\delta_B(m,y(n,r,k,\tau,m)|n,r,k,\tau)$ is non-decreasing in $y(n,r,k,\tau,m)$ for a fixed m. Similarly, we can prove that for fixed $y(n,r,k,\tau,m) \geq 0$, $\delta_B(m,y(n,r,k,\tau,m)|n,r,k,\tau)$ is non-increasing in m. ∎

By a similar argument of Lemma 2.3 of Liang and Yang (2013), we obtain the following lemma.

Lemma A.3.

(1) (*a*) *When* $T_m = \infty$, *then* $\phi_\pi(m,y) > C_r$ *for all* $y > 0$;

(*b*) *When* $T_m = 0$, *then* $\phi_\pi(m,y) < C_r$ *for all* $y > 0$; *and*

(*c*) *When* $0 < T_m < \infty$, *then*

$\phi_\pi(m,y) > C_r$ *for all* $0 < y < T_m$;

$\phi_\pi(m,y) < C_r$ *for all* $y > T_m$*; and*

$\phi_\pi(m,T_m) = C_r$ *if* $y = T_m$.

(2) *Let* $D = \{m \mid A(m)$ *is nonempty*$\}$. *Then,* T_m *is strictly increasing in* m *for all* $m \in D$.

Proof. (1) Since T_m is defined as

$$T_m = \begin{cases} \inf\ A(m), & \text{if } A(m) \text{ is nonempty,} \\ \infty, & \text{otherwise,} \end{cases}$$

therefore, (a) when $T_m = \infty$, $A(m)$ is an empty set, implying $\forall\ y > 0$, $\phi_\pi(m,y) > C_r$. Similarly, (b) when $T_m = 0$, $A(m)$ is a nonempty set, implying $\forall\ y > 0$, $\phi_\pi(m,y) < C_r$. Now (c) when $0 < T_m < \infty$ and $0 < y < T_m$, then by definition of T_m we have $y \notin A(m)$, which implies that $\phi_\pi(m,y) > C_r$. In this case, if $y > T_m$, then $y \in A(m)$, which implies that $\phi_\pi(m,y) < C_r$ and when $y = T_m$, it is clear that $\phi_\pi(m,T_m) = C_r$.

(2) Let $m < m^*$ and $m, m^* \in D$ then

$$A(m) = \{y > 0 : \phi_\pi(m,y) \le C_r\},\ A(m^*) = \{y > 0 : \phi_\pi(m^*,y) \le C_r\}$$

and $\phi_\pi(m,y) < \phi_\pi(m^*,y)$. Therefore, if $y' \in A(m^*)$, then $\phi_\pi(m^*,y') \le C_r$, which implies that $\phi_\pi(m,y') \le C_r$ and hence $y' \in A(m)$. Thus $A(m^*) \subset A(m)$, which implies that

$$\inf\ A(m) < \inf\ A(m^*) \Rightarrow T_m < T_{m^*}.$$

Thus, T_m is strictly increasing in m for all $m \in D$. ∎

Lemma A.4. *The PDF of* $\widehat{\theta}_{MLE}$ *under Type-I GHCS is given as follows*

$$\begin{aligned} f_{\widehat{\theta}_{MLE}}(y) \;=\; & \sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\lambda(n-i+j)\tau} f_{SG}\left(y|\ k,\ k\lambda, \frac{(n-i+j)\tau}{k}\right) \\ & + \sum_{m=k}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j e^{-\lambda(n-m+j)\tau} f_{SG}\left(y|\ m,\ m\lambda,\ \frac{(n-m+j)\tau}{m}\right) \\ & + \sum_{i=r}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\lambda(n-i+j)\tau} f_{SG}\left(y|\ r,\ r\lambda,\ \frac{(n-i+j)\tau}{r}\right), \end{aligned}$$

where $f_{SG}(.)$ is PDF of shifted gamma distribution given by

$$f_{SG}(y|a,b,c) = \begin{cases} \frac{b^a}{\Gamma(a)}(y-c)^{a-1}e^{-b(y-c)}, & \text{if } y > c, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See Chandrasekar *et al.* (2004). ∎

Appendix B

B.1 Proof of Theorem 5.1

Let $(n_B, r_B, k_B, \tau_B, \delta_B)$ be the optimum BSP, then we have

$$R_B^{\delta_B}(n_B, r_B, k_B, \tau_B) = n_B(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(\text{MSE}(\widehat{\theta}_{MLE}))C_v + R_D(\delta_B|n_B, r_B, k_B, \tau_B).$$

Note that $E(M), E(\tau^*), E(\text{MSE}(\widehat{\theta}_{MLE})), R_D(\delta_B|n_B, r_B, k_B, \tau_B)$ and all costs are positive. Hence

$$R_B^{\delta_B}(n_B, r_B, k_B, \tau_B) \geq n_B(C_s - r_s). \tag{14}$$

Let $(0,0,0,0,1)$ denote the sampling plan when the batch is accepted without sampling with Bayes risk $R_B^{\delta_B=1}(0,0,0,0) = E_\lambda[g(\lambda)]$ and $(0,0,0,0,0)$ denote the sampling plan when the batch is rejected without sampling with Bayes risk $R_B^{\delta_B=0}(0,0,0,0) = C_r$. Then the Bayes risk of optimum BSP is

$$R_B^{\delta_B}(n_B, r_B, k_B, \tau_B) \leq \min\left\{R_B^{\delta_B=1}(0,0,0,0), R_B^{\delta_B=0}(0,0,0,0)\right\} = \min\left\{E_\lambda[g(\lambda)], C_r\right\}. \tag{15}$$

Therefore, from (14) and (15) we have $n_B(C_s - r_s) \leq \min\left\{E_\lambda[g(\lambda)], C_r\right\}$. Then, it follows that $n_B \leq \min\left\{\frac{E_\lambda[g(\lambda)]}{C_s - r_s}, \frac{C_r}{C_s - r_s}\right\}$.

B.2 Derivation of the Bayes risk of the BSP

Bayes risk of the BSP $(n, r, k, \tau, \delta_B)$ is given by

$$R_B^{\delta_B}(n, r, k, \tau) = n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(\text{MSE}(\widehat{\theta}_{MLE}))C_v + a_0 + a_1\mu_1 + a_2\mu_2$$

$$
\begin{aligned}
&+ \quad \int_0^\infty (C_r - a_0 - a_1\lambda - a_2\lambda^2) P(\delta_B(M, Y(n,r,k,\tau,M)|n,r,k,\tau) = 0|\lambda) \frac{b^a \lambda^{a-1} e^{-\lambda b}}{\Gamma(a)} d\lambda \\
&= \quad n(C_s - r_s) + E(M) r_s + E(\tau^*) C_\tau + E(\mathrm{MSE}(\widehat{\theta}_{MLE})) C_v + a_0 + a_1\mu_1 + a_2\mu_2 \\
&\qquad + \sum_{l=0}^{2} C_l \frac{b^a}{\Gamma(a)} \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} P(\widehat{\theta}_{MLE} \le c_M)\, d\lambda.
\end{aligned}
$$

where $\mu_i = E(\lambda^i)$ for $i = 1, 2$,

$$
C_l \quad = \quad \begin{cases} C_r - a_l, & \text{if} \quad l = 0, \\ -a_l, & \text{if} \quad l = 1, 2, \end{cases}
$$

The PDF of $\hat{\theta}_{MLE}$ given in Lemma A.4 of Appendix A and prior density of λ (3) are used to obtain the last integral

$$
\begin{aligned}
&\int_0^\infty \lambda^{a+l-1} e^{-\lambda b} P(\widehat{\theta}_{MLE} \le c_M)\, d\lambda \\
&= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \frac{k^k}{\Gamma(k)} \int_0^\infty \int_0^{c_k} \lambda^{a+l+k-1} \left(x - \frac{(n-i+j)\tau}{k} \right)^{k-1} e^{-\lambda(kx+b)} dx d\lambda \\
&+ \sum_{m=k}^{r-1} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} (-1)^j \frac{m^m}{\Gamma(m)} \int_0^\infty \int_0^{c_m} \lambda^{a+l+m-1} \left(x - \frac{(n-m+j)\tau}{m} \right)^{m-1} e^{-\lambda(mx+b)} dx d\lambda \\
&+ \sum_{i=r}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \frac{r^r}{\Gamma(r)} \int_0^\infty \int_0^{c_r} \lambda^{a+l+r-1} \left(x - \frac{(n-i+j)\tau}{r} \right)^{r-1} e^{-\lambda(rx+b)} dx d\lambda \\
&= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \frac{k^k}{\Gamma(k)} \int_0^{c_k - \frac{(n-i+j)\tau}{k}} \frac{u^{k-1} \Gamma(a+l+k)}{(ku + b + (n-i+j)\tau)^{a+k+l}} du \\
&+ \sum_{m=k}^{r-1} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} (-1)^j \frac{m^m}{\Gamma(m)} \int_0^{c_m - \frac{(n-m+j)\tau}{m}} \frac{u^{m-1} \Gamma(a+l+m)}{(mu + b + (n-m+j)\tau)^{a+m+l}} du \\
&+ \sum_{i=r}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \frac{r^r}{\Gamma(r)} \int_0^{c_r - \frac{(n-i+j)\tau}{r}} \frac{u^{r-1} \Gamma(a+l+r)}{(ru + b + (n-i+j)\tau)^{a+r+l}} du \\
&= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \frac{\Gamma(a+k+l)}{\Gamma(k)(b + (n-i+j)\tau)^{a+l}} \int_0^{\frac{kc_k - (n-i+j)\tau}{b+(n-i+j)\tau}} \frac{z^{k-1}}{(1+z)^{a+k+l}} dz \\
&+ \sum_{m=k}^{r-1} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} (-1)^j \frac{\Gamma(a+m+l)}{\Gamma(m)(b + (n-m+j)\tau)^{a+l}} \int_0^{\frac{mc_m - (n-m+j)\tau}{b+(n-m+j)\tau}} \frac{z^{m-1}}{(1+z)^{a+m+l}} dz \\
&+ \sum_{i=r}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^j \frac{\Gamma(a+r+l)}{\Gamma(r)(b + (n-i+j)\tau)^{a+l}} \int_0^{\frac{rc_r - (n-i+j)\tau}{b+(n-i+j)\tau}} \frac{z^{r-1}}{(1+z)^{a+r+l}} dz.
\end{aligned}
$$

Now taking a transformation $z = u/(1-u)$, the first integral becomes

$$\int_0^{C_{m,j,k}} \frac{z^{m-1}}{(1+z)^{a+l+m}} dz = \int_0^{S_{m,j,k}} u^{m-1}(1-u)^{a+l-1} du = B_{S_{m,j,k}}(m, a+l),$$

where $C_{m,j,k} = \frac{kc_k-(n-m+j)\tau}{b+(n-m+j)\tau}$, $S_{m,j,k} = \frac{C_{m,j,k}}{1+C_{m,j,k}}$, and $B_x(\alpha,\beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1}du, \quad 0 \le x \le 1,$ is the incomplete beta function. We denote the cumulative distribution function of beta as $I_x(\alpha,\beta) = B_x(\alpha,\beta)/B(\alpha,\beta)$. Similarly, other integrals can be transformed and then the Bayes risk is given as

$$\begin{aligned}
R_B^{\delta_B}(n,r,k,\tau) &= n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(\mathrm{MSE}(\widehat{\theta}_{MLE}))C_v + a_0 + a_1\mu_1 + a_2\mu_2 \\
&+ \sum_{l=0}^{2} C_l \frac{b^a}{\Gamma(a)} \left[\sum_{i=0}^{k-1}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \frac{\Gamma(a+l)}{(b+(n-i+j)\tau)^{a+l}} I_{S_{i,j,k}}(k, a+l) \right. \\
&\quad + \sum_{m=k}^{r-1}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j \frac{\Gamma(a+l)}{(b+(n-m+j)\tau)^{a+l}} I_{S_{m,j,m}}(m, a+l) \\
&\quad \left. + \sum_{i=r}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \frac{\Gamma(a+l)}{(b+(n-i+j)\tau)^{a+l}} I_{S_{i,j,r}}(r, a+l) \right],
\end{aligned}$$

where

$$\begin{aligned}
S_{i,j,k} &= \frac{C_{i,j,k}}{1+C_{i,j,k}},\ C_{i,j,k} = \frac{kc_k-(n-i+j)\tau}{b+(n-i+j)\tau},\ S_{m,j,m} = \frac{C_{m,j,m}}{1+C_{m,j,m}},\ C_{m,j,m} = \frac{mc_m-(n-m+j)\tau}{b+(n-m+j)\tau}, \\
S_{i,j,r} &= \frac{C_{i,j,r}}{1+C_{i,j,r}},\ C_{i,j,r} = \frac{rc_r-(n-i+j)\tau}{b+(n-i+j)\tau},
\end{aligned}$$

Similarly, for higher degree polynomial loss function the Bayes risk expression is given by

$$\begin{aligned}
R_B^{\delta_B}(n,r,k,\tau) &= n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(\mathrm{MSE}(\widehat{\theta}_{MLE}))C_v + a_0 + a_1\mu_1 + \ldots + a_d\mu_d \\
&+ \sum_{l=0}^{d} C_l \frac{b^a}{\Gamma(a)} \left[\sum_{i=0}^{k-1}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \frac{\Gamma(a+l)}{(b+(n-i+j)\tau)^{a+l}} I_{S_{i,j,k}}(k, a+l) \right. \\
&\quad + \sum_{m=k}^{r-1}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j \frac{\Gamma(a+l)}{(b+(n-m+j)\tau)^{a+l}} I_{S_{m,j,m}}(m, a+l) \\
&\quad \left. + \sum_{i=r}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}(-1)^j \frac{\Gamma(a+l)}{(b+(n-i+j)\tau)^{a+l}} I_{S_{i,j,r}}(r, a+l) \right],
\end{aligned}$$

where $\mu_i = E(\lambda^i)$ for $i = 1, \ldots d$,

$$C_l = \begin{cases} C_r - a_l, & \text{if} \quad l = 0, \\ -a_l, & \text{if} \quad l = 1, \ldots d, \end{cases}$$

and other notations are as defined in the Bayes risk of the BSP for quadratic loss, which is the special case of $d = 2$.

B.3 Computation of $E(\tau^*)$ and $E(M)$

Total time τ^* is given by

$$\tau^* = \begin{cases} \min\{X_{(r)}, \tau\}, & \text{if } X_{(k)} \leq \tau, \\ X_{(k)}, & \text{if } X_{(k)} > \tau, \end{cases}$$

which can be written as $\tau^* = \min\{X_{(r)}, \tau\} I(X_{(k)} \leq \tau) + X_{(k)} I(X_{(k)} > \tau)$. Now,

$$E(\tau^*) = E_\lambda E(\tau^*|\lambda) = E_\lambda \left[E(\min\{X_{(r)}, \tau\} I(X_{(k)} \leq \tau)|\lambda) + E(X_{(k)} I(X_{(k)} > \tau)|\lambda) \right],$$

$$E(\min\{X_{(r)}, \tau\} I(X_{(k)} \leq \tau)|\lambda) = E\left[X_{(r)} I(X_{(r)} \leq \tau)|\lambda\right] + E\left[\tau I(X_{(k)} \leq \tau < X_{(r)})|\lambda\right],$$

$$\begin{aligned} E\left[X_{(r)} I(X_{(r)} \leq \tau)|\lambda\right] &= \int_0^\tau y f_{X_{(r)}}(y) dy = r\binom{n}{r} \int_0^\tau \lambda y e^{-\lambda y} (1 - e^{-\lambda y})^{r-1} e^{-\lambda(n-r)y} dy \\ &= r\binom{n}{r} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \int_0^\tau \lambda y e^{-\lambda(n-j)y} dy \\ &= r\binom{n}{r} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \left[\frac{1 - e^{-\lambda(n-j)\tau}}{(n-j)^2 \lambda} - \frac{\tau e^{\lambda(n-j)\tau}}{(n-j)} \right], \end{aligned}$$

$$\begin{aligned} E\left[\tau I(X_{(k)} \leq \tau < X_{(r)})|\lambda\right] &= \tau P\big(X_{(k)} \leq \tau < X_{(r)}|\lambda\big) \\ &= \tau P\big(\text{at least } k \ X_i\text{'s} \leq \tau \text{ and at most } (r-1) \ X_i\text{'s} < \tau\big) \\ &= \sum_{i=k}^{r-1} \binom{n}{i} (1 - e^{-\lambda\tau})^i e^{-\lambda(n-i)\tau}, \end{aligned}$$

$$\begin{aligned} E\left[X_{(k)} I(X_{(k)} > \tau)|\lambda\right] &= \int_\tau^\infty \frac{n!}{(k-1)!(n-k)!} \lambda y e^{-\lambda y} (1 - e^{-\lambda y})^{k-1} e^{-\lambda(n-k)y} dy \\ &= k\binom{n}{k} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^{k-1-j} \left[\frac{e^{-\lambda(n-j)\tau}}{(n-j)^2 \lambda} + \frac{\tau e^{\lambda(n-j)\tau}}{(n-j)} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned}
E(\tau^*) &= r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\left[\frac{b}{(a-1)(n-j)^2}-\frac{b^a}{(n-j)^2(a-1)(b+(n-j)\tau)^{a-1}}\right.\\
&\quad\left.-\frac{\tau b^a}{(n-j)(b+(n-j)\tau)^a}\right]+\sum_{i=k}^{r-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\frac{b^a}{(b+(n-i+j)\tau)^a}\\
&+ k\binom{n}{k}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}\left[\frac{b^a}{(n-j)^2(a-1)(b+(n-j)\tau)^{a-1}}+\frac{\tau b^a}{(n-j)(b+(n-j)\tau)^a}\right].
\end{aligned}$$

The distribution function of M is

$$\begin{aligned}
P(M=k|\lambda) &= P(X_{(k)}>\tau|\lambda)=\sum_{m=1}^{k-1}\binom{n}{m}(1-e^{-\lambda\tau})^m e^{-\lambda(n-m)\tau},\\
P(M=m|\lambda) &= P(X_{(m)}<\tau<X_{(m+1)}|\lambda)=\binom{n}{m}(1-e^{-\lambda\tau})^m e^{-\lambda(n-m)\tau}\quad\forall\ m=k,k+1,\ldots,r-1,\\
P(M=r|\lambda) &= P(X_{(r)}<\tau|\lambda)=\sum_{m=r}^{n}\binom{n}{m}(1-e^{-\lambda\tau})^m e^{-\lambda(n-m)\tau}.
\end{aligned}$$

Then expected value of M is given by

$$\begin{aligned}
E(M) &= k\sum_{m=1}^{k-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j\frac{b^a}{(b+(n-m+j)\tau)^a}+\sum_{m=k}^{r-1}\sum_{j=0}^{m}m\binom{n}{m}\binom{m}{j}(-1)^j\frac{b^a}{(b+(n-m+j)\tau)^a}\\
&\quad+r\sum_{m=r}^{n}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j\frac{b^a}{(b+(n-m+j)\tau)^a}.
\end{aligned}$$

B.4 Computation of $E(\mathrm{MSE}(\hat{\theta}_{MLE}))$

The mean square error of the $\widehat{\theta}_{MLE}$ is given by

$$\mathrm{MSE}(\widehat{\theta}_{MLE})=E[(\widehat{\theta}_{MLE}-\theta)^2|\lambda]=E(\widehat{\theta}^2_{MLE}|\lambda)+\theta^2-2\theta E(\widehat{\theta}_{MLE}|\lambda),$$

with

$$\begin{aligned}
E(\widehat{\theta}_{MLE}|\lambda) &= \int_0^\infty y f_{\widehat{\theta}_{MLE}}(y)dy\\
&= \frac{1}{\lambda}+\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\frac{e^{-\lambda(n-i+j)\tau}(n-i+j)\tau}{k}\\
&\quad+\sum_{m=k}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j\frac{e^{-\lambda(n-m+j)\tau}(n-m+j)\tau}{m}
\end{aligned}$$

$$+\sum_{i=r}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\frac{e^{-\lambda(n-i+j)\tau}(n-i+j)\tau}{r}.$$

$$\begin{aligned}
E(\widehat{\theta}^2_{MLE}|\lambda) &= \int_0^\infty y^2 f_{\widehat{\theta}_{MLE}}(y)dy \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\lambda(n-i+j)\tau}\left[\frac{1}{k\lambda^2}+\frac{1}{\lambda^2}+\frac{2(n-i+j)\tau}{k\lambda}+\frac{((n-i+j)\tau)^2}{k^2}\right] \\
&+ \sum_{m=k}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j e^{-\lambda(n-m+j)\tau}\left[\frac{1}{m\lambda^2}+\frac{1}{\lambda^2}+\frac{2(n-m+j)\tau}{m\lambda}+\frac{((n-m+j)\tau)^2}{m^2}\right] \\
&+ \sum_{i=r}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\lambda(n-i+j)\tau}\left[\frac{1}{r\lambda^2}+\frac{1}{\lambda^2}+\frac{2(n-i+j)\tau}{r\lambda}+\frac{((n-i+j)\tau)^2}{r^2}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{MSE}(\widehat{\theta}_{MLE}) &= \sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\lambda(n-i+j)\tau}\left[\frac{1}{k\lambda^2}+\frac{((n-i+j)\tau)^2}{k^2}\right] \\
&\quad+\sum_{m=k}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j e^{-\lambda(n-m+j)\tau}\left[\frac{1}{m\lambda^2}+\frac{((n-m+j)\tau)^2}{m^2}\right] \\
&\quad+\sum_{i=r}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\lambda(n-i+j)\tau}\left[\frac{1}{r\lambda^2}+\frac{((n-i+j)\tau)^2}{r^2}\right]
\end{aligned}$$

and

$$\begin{aligned}
&E(\mathrm{MSE}(\widehat{\theta}_{MLE})) \\
&=\sum_{i=0}^{k-1}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\left[\frac{b^a}{k(a-1)(a-2)(b+(n-i+j)\tau)^{a-2}}+\frac{((n-i+j)\tau)^2 b^a}{k^2(b+(n-i+j)\tau)^a}\right] \\
&+\sum_{m=k}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j\left[\frac{b^a}{m(a-1)(a-2)(b+(n-m+j)\tau)^{a-2}}+\frac{((n-m+j)\tau)^2 b^a}{m^2(b+(n-m+j)\tau)^a}\right] \\
&+\sum_{i=r}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j\left[\frac{b^a}{r(a-1)(a-2)(b+(n-i+j)\tau)^{a-2}}+\frac{((n-i+j)\tau)^2 b^a}{r^2(b+(n-i+j)\tau)^a}\right].
\end{aligned}$$

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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