Optimal Decision-theoretic Sampling Plan For Two Exponential Distributions under Joint Censoring Scheme

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Abstract

In statistical quality control, decision-theoretic approach draws a significant amount of attention due to its economic considerations. In reliability life testing, decision-theoretic approach has been used quite extensively under different censoring schemes. All these implementations are based on single sample of products coming from a particular source. In this work we study decision-theoretic approach on two sample of products coming from two different sources under a joint censoring scheme when the life times are exponentially distributed. The major advantage of such implementation is to take decision on the acceptance or rejection of one or both the batches in a single life testing experiment. Decision making is performed based on minimizing the Bayes risk with respect to a given loss function. It is observed that under certain set-ups, the joint censoring scheme is preferable over the two separate single sample censoring schemes in decision making.

Keywords and Phrases: Bayesian sampling plan, Bayes estimator, Bayes risk, Decision-theoretic approach, Exponential distribution, Joint censoring scheme, Maximum likelihood estimator, Shrinkage estimators.

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1 Introduction

In a reliability life testing experiment, very often, acceptance sampling is implemented to take decision on the acceptance or rejection of a batch of products based on their reliability characteristics. An optimal acceptance sampling plan keeps into account the resources. Among different choices of optimal acceptance sampling plan, decision-theoretic approach draws a significant amount of attention as it is based on making an optimal decision on some economic considerations such as maximizing the return or minimizing the loss. From an economic point of view, this approach is more reasonable and therefore, widely employed in statistical analysis. Some of the relevant works along these lines are by Lam [5], Haung and Lin [3] and the references cited therein for the recent developments in this area.

An important measure of the quality of a product is its reliability. Suppose based on the reliability quality, a decision is made regarding the acceptance or rejection of the batch. The quality of an item in the batch is usually measured by its lifetime. In order to estimate the quality of the items in the batch, a sample of n items is put on a life test and their failure times are observed. In practice, life testing is usually censored in the sense that the testing procedure is terminated before the actual lifetimes of all inspected items are observed. Due to the effectiveness of the censoring factors, the decision-theoretic approach for the acceptance sampling plans has become extremely difficult to handle. To tackle this problem, Lam [6, 7], Lin *et al.* [9, 10], Chen *et al.* [1, 2] adopted a Bayesian approach to study the single variable sampling plan under different censoring schemes, when the lifetime distributions are exponential. The standard Bayesian approach is to determine the decision function so that the Bayes risk is minimized.

Although, the decision-theoretic sampling plans under Bayesian set up have been applied under different censoring schemes, no such attempt has been made on joint censoring schemes. Joint censoring scheme is quite useful to conduct comparative life testing experiment of different products. Mondal and Kundu [12, 13] have introduced a joint progressive censoring scheme applied on two or multiple samples called the balanced joint progressive censoring (BJPC) scheme. Here we briefly discuss the BJPC scheme on two samples. Suppose there are two different kinds of products from product line A and product line B and it is required to study the relative merits of these two kinds of products. A sample of size m is drawn from the product line A and is called sample A. Another sample of size n is drawn from product line B and is called sample B. Let k be the total number of failures to be observed from the life testing experiment and $R_1, \ldots, R_{k-1}$ be pre-specified non-negative integers such that $\sum_{i=1}^{k-1}(R_i+1) < \min(m, n)$. The units of these two samples are simultaneously put on the test. Suppose the first failure comes from sample A and the failure time is denoted by W_1. Then at W_1 time point, R_1 units are removed randomly from the remaining $m-1$ surviving units of sample A and R_1+1 units are chosen randomly from the remaining n surviving units of sample B and they are withdrawn from the experiment. Next, if the second failure comes from sample B at time point W_2, R_2+1 units are withdrawn from the remaining $m - R_1 - 1$ units of sample A and R_2 units are withdrawn from the remaining $n - R_1 - 2$ units of sample B randomly at W_2 time point. The experiment is continued until the k-th failure occurs. At the k-th failure time point W_k, the experiment is terminated with the removal of all the remaining surviving units from both the samples. Along with the ordered failure time W_i, we also record the indicator variable Z_i which is 1, if i-th failure comes from sample A and 0 if it comes from sample B. Let us define, $K_1 = \sum_{i=}^{k} Z_i$ denoting the number of failures from sample A and $K_2 = \sum_{i=}^{k}(1 - Z_i)$ be the number of failures from sample B, in this experiment.

In this work, acceptance sampling plan is implemented under the BJPC scheme when the life time distributions are exponential. The major advantage of such implementation is to take decision on the acceptance or rejection of one batch or both the batches, in a single life testing experiment. Under the BJPC scheme, we do not observe failure of all the items from both the samples, which reduces experiment cost. On the other hand under certain set-up, this joint censoring scheme, provides better estimation than conventional progressive Type-II censoring schemes applied on two samples separately. For detail study see Mondal and Kundu [12, 13, 14]. Due to these factors, application of the acceptance sampling plan under the the BJPC scheme, can be regarded very beneficial and convenient in statistical quality control. An optimal sampling plan is instrumented through decision-theoretic approach using the shrinkage estimators of the mean life times of two different products. These shrinkage estimators will exist even when no failure is observed from any one of the two samples. In the decision-theoretic approach, this kind of shrinkage estimators are studied in Prajapati *et al.* [17, 18]. We consider a very general loss function which contains the sampling cost, the cost per unit time of the experiment, the salvage value, the loss due to decision along with the precision of the shrinkage estimators. Due to analytical tractability of the BJPC scheme, the Bayes risk can be obtained in a simpler form under the assumption that the lifetime distributions are exponential. The optimal decision-theoretic sampling plan (DSP) is determined by minimizing the Bayes risk of that loss function.

The rest of the paper is organized as follows. In Section 2, we propose the decision function, loss function and the prior. The Bayes risk of the DSP is provided in Section 3. Numerical results for the optimal DSP are given in Section 4. In Section 5, we perform a comparative study between the proposed joint censoring scheme and single censoring schemes applied on two samples separately. Finally, we conclude the paper in Section 6. All derivations have been provided in the Appendix.

2 Model Construction and Assumptions

2.1 Life time distribution Assumption

If a random variable X follows exponential distribution with mean $\theta(\theta > 0)$, the probability density function will be

$$f_X(x|\theta) = \frac{1}{\theta} exp(-\frac{x}{\theta});\ x > 0. \tag{1}$$

It is assumed that n items from product line A follow exponential distribution with mean θ_1 and n items from product line B follow exponential distribution with mean θ_2. Applying a BJPC scheme on the two samples simultaneously, we observe the censored data $(\mathbf{w}, \mathbf{z}) = ((w_i, z_i) : \ \forall\ i = 1, \ldots, k)$ and $k_1 = \sum_{i=1}^{k} z_i$, $k_2 = \sum_{i=1}^{k}(1 - z_i)$. Based on the data, the likelihood function can be obtained as

$$L(data|\theta_1, \theta_2) \propto \frac{1}{\theta_1^{k_1}} \frac{1}{\theta_2^{k_2}} e^{(-(\frac{1}{\theta_1} + \frac{1}{\theta_2})U)},$$

where $U = \sum_{i=1}^{k-1}(R_i+1)w_i + (n - \sum_{i=1}^{k-1}(R_i+1))w_k$. From the likelihood function, it is evident that the maximum likelihood estimators (MLEs) of θ_1 and θ_2 exist only when $0 < k_1 < k$ i.e. the MLEs are conditional and they are as follows,

$$\widehat{\theta}_1 = \frac{U}{k_1}, \quad \widehat{\theta}_2 = \frac{U}{k_2}.$$

2.2 Decision function and Loss function

In acceptance sampling plan we draw sample from both the batches from product line A and B. These two samples are put under the BJPC scheme and based on the censored data, we take decision on the acceptance or rejection of the batches. In this section we define a decision function which acts like an instrument to take decision on the acceptance or rejection of the batches. In the literature, though we find the usage of MLEs in decision function, it has some limitations. When the MLEs are conditional, it is not suitable to consider a decision function based on the MLEs. As an alternative approach, one can use shrinkage estimators in the decision function, see Prajapati *et al.* [17]. Here we propose to use shrinkage estimators derived from the MLEs as follows,

$$\widehat{\theta}_{1Sh} = \frac{U}{k_1 + c}, \quad \widehat{\theta}_{2Sh} = \frac{U}{k_2 + c},$$

for some $c > 0$. Eventually we will choose c optimally. Note that it is possible to choose different c values for two different estimators, but it makes the optimization problem more complex. Hence, it has not attempted here. The shrinkage estimators exist even when $k_1 = 0$ or $k_2 = 0$, for any $c > 0$. Based on the shrinkage estimators, the decision function is defined as

$$\delta(\mathbf{W},\mathbf{Z}) = \begin{cases} (d_{10}, d_{20}), \text{ if } \widehat{\theta}_{1Sh} > \xi_1, \ \widehat{\theta}_{2Sh} > \xi_2 \\ (d_{10}, d_{21}), \text{ if } \widehat{\theta}_{1Sh} > \xi_1, \ \widehat{\theta}_{2Sh} < \xi_2 \\ (d_{11}, d_{20}), \text{ if } \widehat{\theta}_{1Sh} < \xi_1, \ \widehat{\theta}_{2Sh} > \xi_2 \\ (d_{11}, d_{21}), \text{ if } \widehat{\theta}_{1Sh} < \xi_1, \ \widehat{\theta}_{2Sh} < \xi_2. \end{cases} \tag{2}$$

Here, $d_{10}(d_{11})$ denotes the action of accepting (rejecting) a batch from product line A and $d_{20}(d_{21})$ denotes the action of accepting (rejecting) a batch from product line B. ξ_1 and ξ_2 denote the minimum acceptable mean survival time from product line A and B, respectively. Naturally, any decision function is always associated with a loss function. In the literature, the quadratic loss function is the most popular one and is widely used to obtain the optimal sampling plan. Some key references are Lam [7], Lin *et al.* [11], Chen *et al.* [1], Lin *et al.* [9], Liang *et al.* [8], Prajapati *et al.* [17, 18]. Most of the authors consider the quadratic loss function to obtain an optimal sampling plan because of computational simplicity. Due to analytical tractability of the BJPC scheme, we are able to consider more general loss function capturing different aspects of the sampling plan. We propose a loss function which takes into account the inspection cost of the experiment items, a salvage value of the remaining surviving items, experiment time, i.e., time till k-th failure, cost of rejecting and accepting a batch and precision of the estimators. In general, acceptance cost will be determined by the inspection requirement or based on some experience before designing a sampling plan. Hence, the form of acceptance cost can vary because it includes costs that are difficult to recognize and accordingly, the form of the loss function also varies. However, as the mean lifetime increases the acceptance cost decreases. So the cost of accepting a batch can be expressed as the polynomial of the hazard rate of the life time distributions, for example see Lam [6, 7]. Therefore, we consider the re-parameterization $\lambda_1 = \frac{1}{\theta_1}$ and $\lambda_2 = \frac{1}{\theta_2}$ and express the acceptance costs as the quadratic function of λ_1 and λ_2. The loss function is defined as,

$$L(\delta,\lambda_1,\lambda_2)=\begin{cases} nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+g_1(\lambda_1)+g_2(\lambda_2), & \text{if } \delta(\mathbf{W},\,\mathbf{Z})=(d_{10},d_{20}) \\ nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+g_1(\lambda_1)+C_{r_2}, & \text{if } \delta(\mathbf{W},\,\mathbf{Z})=(d_{10},d_{21}) \\ nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+C_{r_1}+g_2(\lambda_2), & \text{if } \delta(\mathbf{W},\,\mathbf{Z})=(d_{11},d_{20}) \\ nC_s-(n-K_1)r_{s_1}-(n-K_2)r_{s_2}+W_kC_\tau \\ \quad +\text{MSE}(\widehat{\theta}_{1Sh})C_{e_1}+\text{MSE}(\widehat{\theta}_{2Sh})C_{e_2}+C_r, & \text{if } \delta(\mathbf{W},\,\mathbf{Z})=(d_{11},d_{21}). \end{cases} \tag{3}$$

Here, $C_s=C_{s_1}+C_{s_2}$ where $C_{s_1}(C_{s_2})$ is the inspection cost per item from product line A (line B), C_τ is the cost per unit time for conducting the experiment, $C_{r_1}(C_{r_2})$ is the cost of rejecting the batch from product line A (line B), and C_r is the cost of rejecting both the batches. Further, $g_1(\lambda_1)=l_0'+l_1'\lambda_1+l_2'\lambda_1^2$ is the cost on accepting the batch from product line A, which has to be positive and an increasing function in λ_1 for $\lambda_1>0$, and $g_2(\lambda_2)=l_0''+l_1''\lambda_2+l_2''\lambda_2^2$ is the cost on accepting the batch from product line B, which is also positive and an increasing function in λ_2 for $\lambda_2>0$. If an item from product line A (line B) does not fail, it can be reused with a salvage value $r_{s_1}(r_{s_2})$ such that $C_{s_1}>r_{s_1}\geq 0$ $(C_{s_2}>r_{s_2}\geq 0)$. The precision of the estimators should play an important role in any decision making. If the mean square errors (MSE) of the estimators are high, then from sample to sample the decision may vary significantly even when the production process is under control. Therefore, we consider cost C_{e_1} and C_{e_2} on the mean square errors of the estimators $\widehat{\theta}_{1Sh}$ and $\widehat{\theta}_{2Sh}$, respectively. Note that when we reject a batch, the whole batch is discarded or returned to the supplier, and the loss is fixed, so we presume that C_{r_1}, C_{r_2} and C_r are fixed.

A decision-theoretic sampling plan is defined as any choice of $(n,k,R_1,\ldots,R_{k-1},\xi_1,\xi_2,c)$ and is denoted by DSP. Our aim is to derive the optimal DSP $(n_0,k_0,R_{10},\ldots,R_{k-10},\xi_{10},\xi_{20},c_0)$ which minimizes the Bayes risk of the loss function defined in (3).

2.3 Prior Assumption

Prior knowledge of the model parameters quite governs the Bayes risk of the associated loss function. In this section, we discuss the choice of prior distributions of the model parameters λ_1 and λ_2. It is assumed that $\lambda_1 + \lambda_2 \sim G(a_0, b_0)$, with $a_0 > 0$, $b_0 > 0$ and $\frac{\lambda_1}{\lambda_1+\lambda_2} \sim Beta(a_1, a_2)$, with $a_1 > 0$, $a_2 > 0$, and they are independently distributed. Therefore, $\lambda_1 + \lambda_2$ has the probability density function

$$\pi_1(\lambda_1 + \lambda_2) = \frac{{b_0}^{a_0}}{\Gamma(a_0)}(\lambda_1 + \lambda_2)^{a_0-1}e^{-b_0(\lambda_1+\lambda_2)};\ (\lambda_1 + \lambda_2) > 0,$$

and probability density function of $\frac{\lambda_1}{\lambda_1+\lambda_2}$ is

$$\pi_2\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^{a_1-1}\Big(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^{a_2-1};\ 0 < \frac{\lambda_1}{\lambda_1+\lambda_2} < 1.$$

The joint probability density function of λ_1, λ_2 can be derived as,

$$\pi(\lambda_1, \lambda_2) = \frac{\Gamma(a_1+a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)}{b_0}^{a_0}(\lambda_1+\lambda_2)^{a_0-a_1-a_2}\lambda_1^{a_1-1}\lambda_2^{a_2-1}e^{-b_0(\lambda_1+\lambda_2)}; \qquad \lambda_1, \lambda_2 > 0 \quad (4)$$

The joint prior is known as the Beta-Gamma prior, and it will be denoted by $BG(a_0, b_0, a_1, a_2)$. When $a_0 = a_1 + a_2$ then joint probability density function of λ_1 and λ_2 become as,

$$\pi(\lambda_1, \lambda_2) = \frac{b_0^{a_1}}{\Gamma(a_1)}\lambda_1^{a_1-1}e^{-\lambda_1 b_0}\frac{b_0^{a_2}}{\Gamma(a_2)}\lambda_2^{a_2-1}e^{-\lambda_2 b_0}; \qquad \lambda_1, \lambda_2 > 0$$

which implies that λ_1 and λ_2 are independent and $\lambda_1 \sim G(a_1, b_0)$ and $\lambda_2 \sim G(a_2, b_0)$. In the literature, we find the application of Beta-Gamma prior in Pena and Gupta [16], Kundu and Pradhan [4], Mondal and Kundu [15]. The following result on the Beta-Gamma prior is useful for computation purposes.

Theorem 2.1 $(\lambda_1, \lambda_2) \sim BG(a_0, b_0, a_1, a_2)$ *if and only if* $\lambda_1 + \lambda_2 \sim G(a_0, b_0)$ *and* $\frac{\lambda_1}{\lambda_1+\lambda_2} \sim Beta(a_1, a_2)$ *and they are independently distributed.*

3 Bayes risk of the DSP under the BJPC scheme and the optimal DSP

In this section, we derive the Bayes risk R_B^δ of the DSP $(n, k, R_1, \ldots, R_{k-1}, \xi_1, \xi_2, c)$ for the loss function defined in (3), and eventually find a DSP which minimizes the Bayes risk $R_B^\delta(n, k, R_1, \ldots, R_{k-1}, \xi_1, \xi_2, c)$ among all DSPs based on the proposed prior assumptions. To ensure the Bayes risk to be finite, it is assumed that $\int_0^\infty \int_0^\infty (g_1(\lambda_1)+g_2(\lambda_2))\pi(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 < \infty$. Using the loss function defined in (3), the Bayes risk of the DSP can be obtained as

$$\begin{aligned} & R_B^\delta(n, k, R_1, \ldots, R_{k-1}, \xi_1, \xi_2, c) \\ & \qquad = nC_s - (n - E[K_1])r_{s_1} - (n - E[K_2])r_{s_2} + E[W_k]C_\tau \\ & \qquad\quad + E[\mathrm{MSE}(\widehat{\theta}_{1Sh})]C_{e_1} + E[\mathrm{MSE}(\widehat{\theta}_{2Sh})]C_{e_2} \\ & \qquad\quad + D_1 + D_2 + D_3 + D_4, \end{aligned} \tag{5}$$

where

$$\begin{aligned} D_1 &= E_{(\lambda_1,\lambda_2)}\big[(g_1(\lambda_1) + g_2(\lambda_2))P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2)\big] \\ D_2 &= E_{(\lambda_1,\lambda_2)}\big[(g_1(\lambda_1) + C_{r_2})P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} < \xi_2)\big] \\ D_3 &= E_{(\lambda_1,\lambda_2)}\big[(g_2(\lambda_2) + C_{r_1})P(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} > \xi_2)\big] \\ D_4 &= E_{(\lambda_1,\lambda_2)}\big[C_r P(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} < \xi_2)\big]. \end{aligned} \tag{6}$$

Note that, we define, the expectation $E[\cdot] \equiv E_{(\lambda_1,\lambda_2)} E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)}[\cdot]$. The explicit expression of the Bayes risk is computed in the Appendix.

Under the BJPC scheme, when the life time distributions follow exponential distribution and two sample sizes are equal (i.e. $m = n$), the distributions of the MLEs and consequently the shrinkage estimators and $K_1(K_2)$ are independent of the censoring scheme $R_1, \ldots, R_{k-1}$, for any n and k(for detailed derivation see Mondal and Kundu [12]). Therefore, the expected total salvage costs per sample, the expected mean square errors and the expected cost of acceptance will not involve $R_1, R_2, \ldots, R_{k-1}$, only $E[W_k]$ involves $R_1, R_2, \ldots, R_{k-1}$. Moreover, it is known that for any given n and k, $E[W_k]$ is minimized when $R_1 = \cdots, R_{k-1} = 0$. It may be mentioned that this becomes the famous Self Reallocated Design (SRD) proposed by Srivastava [19]. Therefore, the optimal DSP is a choice of only n, k, ξ_1, ξ_2, c say,$(n_0, k_0, \xi_{10}, \xi_{20}, c_0)$ which minimizes the Bayes risk $R_B^\delta(n, k, \underbrace{0, \ldots, 0}_{\text{(k-1) 0's}}, \xi_1, \xi_2, c)$. From now onwards, we consider,

$$R_B^\delta(n, k, \xi_1, \xi_2, c) = R_B^\delta(n, k, \underbrace{0, \ldots, 0}_{\text{(k-1) 0's}}, \xi_1, \xi_2, c). \tag{7}$$

Therefore, the optimal DSP $(n_0, k_0, \xi_{10}, \xi_{20}, c_0)$ will satisfy

$$R_B^\delta(n_0, k_0, \xi_{10}, \xi_{20}, c_0) = \min_{n,k}\{\min_{\xi_1,\xi_2,c}\{R_B^\delta(n, k, \xi_1, \xi_2, c)\}\}. \tag{8}$$

Since n and k can take only integers values, similar algorithm which is proposed by Lam [7] and Prajapati *et al.* [17, 18] can be adopted to obtain the optimal DSP by minimizing the Bayes risk.

Algorithm:

Step 1: For each fixed parameter (n, k), find $\xi_{10}(n, k)$, $\xi_{20}(n, k)$, and $c_0(n, k)$, which minimize the Bayes risk $R_B^\delta(n, k, \xi_1, \xi_2, c)$ among all $\xi_1, \xi_2, c > 0$. That is, $\xi_{10}(n, k)$, $\xi_{20}(n, k)$, and $c_0(n, k)$, are such that

$$R_B^\delta(n, k, \xi_{10}(n, k), \xi_{20}(n, k), c_0(n, k)) = \underset{\xi_1,\xi_2,c}{inf}\ R_B^\delta(n, k, \xi_1, \xi_2, c).$$

Step 2: For each fixed parameter n, find an integer $k_0(n), 0 \leq k_0(n) \leq n$ such that

$$R_B^\delta(n, k_0(n), \xi_{10}(n, k_0(n)), \xi_{20}(n, k_0(n)), c_0(n, k_0(n)))$$
$$= \underset{k\leq n}{inf} R_B^\delta(n, k, \xi_{10}(n, k), \xi_{20}(n, k), c_0(n, k)).$$

Step 3: Find an integer $n_0 \geq 0$ such that

$$R_B^\delta(n_0, k_0(n_0), \xi_{10}(n_0, k_0(n_0)), \xi_{20}(n_0, k_0(n_0)), c_0(n_0, k_0(n_0)))$$
$$= \underset{n\geq 0}{inf} R_B^\delta(n, k_0(n), \xi_{10}(n, k_0(n)), \xi_{20}(n, k_0(n)), c_0(n, k_0(n))).$$

For simplicity, we denote optimal DSP $(n_0, k_0(n_0), \xi_{10}(n_0, k_0(n_0)), \xi_{20}(n_0, k_0(n_0)), c_0(n_0, k_0(n_0)))$ in short notation by $(n_0, k_0, \xi_{10}, \xi_{20}, c_0)$. The following theorem determines that the algorithm has finite number of steps to reach the optimal solution.

Theorem 3.1 *Assuming* $0 < \xi_1 < \xi_1^*$, $0 < \xi_2 < \xi_2^*$, $0 < c < c^*$, *if* n_0 *and* k_0 *be the optimal values of* n *and* k*, respectively, then,*

$$n_0 \leq \min\left\{\frac{E_{\lambda_1,\lambda_2}[g_1(\lambda_1) + g_2(\lambda_2)]}{C_s - r_{s_1} - r_{s_2}}, \frac{C_r}{C_s - r_{s_1} - r_{s_2}}\right\}$$

and $0 \leq k_0 \leq n_0$.

Proof: See in Appendix.

4 Numerical Results

In this section, we find out the optimal DSP under different set-ups. The *R* software is used for searching the optimal DSP numerically. In Tables 1-2, we obtain the optimal DSPs by varying the prior hyper parameters and the costs one at a time while keeping the other fixed. Since, $E(W_k)$ exists only if $a_0 > 1$ and $E[\text{MSE}(\hat{\theta}_{1Sh})]$ and $E[\text{MSE}(\hat{\theta}_{2Sh})]$ exist only if $a_0 > 2$, so for numerical results we considered $a_0 > 2$, and all the costs should be positive with $C_s > r_{s_1} + r_{s_2}$. Therefore, we considered the specific values of prior hyper parameters $a_0 = 4.0$, $a_1 = 5.0$, $a_2 = 5.0$, $b_0 = 2.0$, costs $C_s = 0.16$, $r_{s_1} = 0.07$, $r_{s_2} = 0.07$, $C_\tau = 0.1$, $C_{r_1} = 8$, $C_{r_2} = 8$, $C_{e_1} = 0.5$, $C_{e_2} = 0.5$, $C_r = 25$ and acceptance cost coefficients $l_0' = 4$, $l_1' = 5$, $l_2' = 4$, $l_0'' = 4$, $l_1'' = 5$, $l_2'' = 4$. In Table 1, we vary hyper-parameters a_0, a_1, a_2 and b_0 one at a time keeping costs and acceptance cost coefficients fixed. In Table 2, we vary the costs and obtain the optimal DSP. In the Tables 1, 2, and 5 the optimal DSP $(n_0, k_0, \underbrace{0, \ldots, 0}_{(\text{k-1})\ 0\text{'s}}, \xi_{10}, \xi_{20}, c_0)$ is mentioned in short notation $(n_0, k_0, \xi_{10}, \xi_{20}, c_0)$.

From Table 1, it is evident that when a_0 increases, the optimal Bayes risk increases while ξ_{10} increases and ξ_{20} decreases. On the other hand, when either a_1 or a_2 increases, the optimal Bayes risk decreases while ξ_{10} decreases and ξ_{20} increases. From Table 2, it is clear that when each of the costs $C_s, C_\tau, C_{e_1}, C_{e_2}$, C_r, C_{r_1}, C_{r_2} increases, the optimal Bayes risk increases. When inspection cost C_s increases, the optimal values of n and k decrease which implies that if inspection cost C_s is high we cannot afford more sample units to test. Further, when the costs C_{e_1} and C_{e_2} increase, the optimal value of n and k increase. This fact indicates that when costs of precession of the shrinkage estimators are large, more sample units are required to reach the optimal decision. From Table 2, it is also evident that when $C_{r_1}(C_{r_2})$ is large, the $\xi_{10}(\xi_{20})$ value is small. These facts indicate that it is difficult to afford the rejection of the batches of items when the rejection costs are high.

Table 1: Minimum Bayes risks and optimal DSPs when a_0, a_1, a_2 and b_0 vary with costs $C_s = 0.16$, $r_{s_1} = 0.07$, $r_{s_2} = 0.07$, $C_\tau = 0.1$, $C_{r_1} = 8$, $C_{r_2} = 8$ $C_{e_1} = 0.5$, $C_{e_2} = 0.5$, $C_r = 25$ and coefficients $l'_0 = 4$, $l'_1 = 5$, $l'_2 = 4$, $l''_0 = 4$, $l''_1 = 5$, $l''_2 = 4$.

a_0	a_1	a_2	b_0	R_B^δ	n_0	k_0	ξ_{10}	ξ_{20}	c_0
3.5				22.7716	13	10	0.4072	3.1507	1.8028
3.8				23.2960	10	8	0.4187	3.4669	1.7645
4.0	5.0	5.0	2.0	23.6713	9	7	0.4304	3.5238	1.7753
4.5				24.5377	8	6	2.9861	0.4696	2.2594
4.6				24.6815	8	6	2.7678	0.4716	2.4799
	3.0			22.9552	14	12	13.5946	0.1477	1.2518
	4.0			22.8511	10	8	6.2124	0.2831	1.5049
4.0	5.0	5.0	2.0	23.6713	9	7	0.4304	3.5238	1.7753
	6.0			22.6493	9	7	0.3048	4.6207	1.6139
	7.0			21.8227	9	7	0.2218	5.6709	1.4927
		3.5		22.5657	11	9	0.2089	8.6699	1.3669
		4.0		22.8511	10	8	0.2831	6.2127	1.5041
4.0	5.0	5.0	2.0	23.6713	9	7	0.4304	3.5238	1.7753
		5.5		23.1332	9	7	4.0768	0.3605	1.6928
		6.0		22.6493	9	7	4.6297	0.3048	1.6138
			1.7	24.7532	8	6	2.6146	0.5124	2.3975
			1.8	24.4310	9	7	2.9361	0.5083	2.2438
4.0	5.0	5.0	2.0	23.6713	9	7	0.4304	3.5238	1.7753
			2.2	22.8554	10	8	0.3612	3.6588	1.6885
			2.5	21.7750	12	9	0.2659	3.2776	1.6753

Table 2: Minimum Bayes risks and optimal DSPs when $C_s, C_\tau, C_{e_1}, C_{e_2}, C_r, C_{r_1}, C_{r_2}$ vary with hyper parameters $a_0 = 4.0,\ a_1 = 5.0,\ a_2 = 5.0,\ b_0 = 2.0$ salvage value $r_{s_1} = 0.07,\ r_{s_2} = 0.07$, and coefficients $l'_0 = 4,\ l'_1 = 5,\ l'_2 = 4,\ l''_0 = 4,\ l''_1 = 5,\ l''_2 = 4$.

C_s	C_τ	C_{e_1}	C_{e_2}	C_r	C_{r_1}	C_{r_2}	R_B^δ	n_0	k_0	ξ_{10}	ξ_{20}	c_0
0.15							23.5660	12	8	0.4507	3.5502	1.8786
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5238	1.7753
0.17							23.7573	8	7	0.4304	3.5239	1.7753
0.18							23.8358	7	6	0.4053	3.5578	1.6680
0.20							23.9729	6	6	0.4053	3.5578	1.6680
	0.05						23.6190	9	8	0.4507	3.5502	1.8786
	0.08						23.6536	9	7	0.4304	3.5239	1.7753
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5239	1.7753
	0.15						23.7123	10	7	0.4304	3.5239	1.7753
	0.30						23.8067	12	7	0.4304	3.5239	1.7753
		0.1					23.2214	8	6	0.3891	2.9018	1.9922
		0.3					23.4579	9	7	0.4246	3.3035	1.8940
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5238	1.7753
		0.7					23.8684	10	8	0.4543	3.7186	1.8054
		1.0					24.1433	11	9	0.4754	3.8873	1.8160
			0.1				23.2214	8	6	0.3891	2.9018	1.9922
			0.3				23.4579	9	7	0.4246	3.3035	1.8940
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5238	1.7753
			0.7				23.8684	10	8	0.4543	3.7186	1.8054
			1.0				24.1433	11	9	0.4753	3.8873	1.8160
				24.0			23.4029	10	8	0.5457	2.6659	1.9405
				24.5			23.5438	9	7	0.4669	3.4633	1.8161
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5238	1.7753
				28.0			24.0970	9	7	0.2653	4.0169	1.6047
				30.0			24.1878	9	7	0.1923	4.1975	1.5513
					5.0		21.1142	9	7	18.8286	0.2656	1.5736
					6.0		22.0243	9	7	9.0346	0.3120	1.6147
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5238	1.7753
					10.0		23.6713	9	7	0.4304	3.5239	1.7753
					12.0		23.6713	9	7	0.4303	3.5536	1.7778
						5.0	21.1142	9	7	0.2655	19.1839	1.5732
						6.0	22.0243	9	7	0.3121	9.0147	1.6152
0.16	0.10	0.5	0.5	25.0	8.0	8.0	23.6713	9	7	0.4304	3.5238	1.7753
						10.0	23.6713	9	7	3.5495	0.4300	1.7787
						12.0	23.6713	9	7	3.5486	0.4303	1.7781

5 Comparison with Lam's sampling plans under Two samples case

In this section, we compare the proposed model based on the BJPC scheme with the model proposed by Lam [6] on Type-II censoring scheme under the exponential distribution. In the optimal DSP based on the BJPC scheme, we obtain $R_1, \ldots, R_{k-1}$ as 0. Therefore, we compare the Bayes risk of the optimal DSP based on the BJPC scheme with the sum of optimal Bayes risks based on two Type-II censoring schemes applied on two samples separately. We perform the following procedure to carry out the comparison study.

We consider the two samples of size n, from product line A and B, respectively and apply conventional Type-II censoring schemes on these two sample separately. According to Lam [6], on a Type-II censoring scheme, the decision function is given by,

$$\delta^L(\mathbf{X}) = \begin{cases} 1, & \text{if} \quad \hat{\theta} \geq \xi, \\ 0, & \text{if} \quad \hat{\theta} < \xi, \end{cases}$$

where $\hat{\theta}$ is the MLE of the mean θ. For comparison study, we consider the following loss functions on the sample A and sample B,

$$L_1(\delta, \lambda_1) = \begin{cases} nC_{s_1} - (n-k_1)r_{s_1} + X_{(k_1)}C_\tau + \text{MSE}(\widehat{\theta}_1)C_{e_1} + g_1(\lambda_1), & \text{if } \delta^L(\mathbf{X}_1) = 1 \\ nC_{s_1} - (n-k_1)r_{s_1} + X_{(k_1)}C_\tau + \text{MSE}(\widehat{\theta}_1)C_{e_1} + C_{r_1}, & \text{if } \delta^L(\mathbf{X}_1) = 0, \end{cases}$$

$$L_2(\delta, \lambda_2) = \begin{cases} nC_{s_2} - (n-k_2)r_{s_2} + X_{(k_2)}C_\tau + \text{MSE}(\widehat{\theta}_2)C_{e_2} + g_2(\lambda_2), & \text{if } \delta^L(\mathbf{X}_2) = 1 \\ nC_{s_2} - (n-k_2)r_{s_2} + X_{(k_2)}C_\tau + \text{MSE}(\widehat{\theta}_2)C_{e_2} + C_{r_2}, & \text{if } \delta^L(\mathbf{X}_2) = 0. \end{cases}$$

In prior Beta-Gamma distribution, we set $a_0 = a_1 + a_2$ so that λ_1 and λ_2 are independent and $\lambda_1 \sim G(a_1, b_0)$ and $\lambda_2 \sim G(a_2, b_0)$. In order to make the comparison, we consider $C_s = C_{s_1} + C_{s_2} = 0.08 + 0.08$, $r_{s_1} = 0.07$, $r_{s_2} = 0.07$, $C_\tau = 0.1$, $C_{r_1} = 9$, $C_{r_2} = 16$ $C_{e_1} = 0.5$, $C_{e_2} = 0.5$, $C_r = C_{r_1} + C_{r_3} = 25$ and coefficients $l'_0 = 4$, $l'_1 = 5$, $l'_2 = 1$, $l''_0 = 4$, $l''_1 = 5$, $l''_2 = 4$. For each sample we obtain the optimal sampling plan and corresponding Bayes risk under Type-II censoring using Lam's approach. The optimal Bayes risks, for

Sample A and B are denoted by R_1^L and R_2^L, respectively. With these specific costs, R_1^L and corresponding Lam's sampling plan for Sample A, R_2^L and corresponding Lam's sampling plan for Sample B are obtained, respectively in Tables 3 and 4. In Table 5, we compute the Bayes risk of the optimal DSP under the BJPC scheme, denoted by R_B^δ and compare it with the sum of the two optimal Bayes risks R_1^L and R_2^L, based on single sample Type-II censoring scheme. From Table 5, it is evident that the Bayes risk R_B^δ of optimal DSP is smaller than the sum of the Bayes risks R_1^L and R_2^L. Therefore, we can conclude that at least for certain sets of hyper parameters, costs and coefficients, our proposed optimal DSP based on the joint censoring scheme, perform better than the Lam's optimal sampling plans based on two separate Type-II censoring schemes in reliability decision making in terms of Bayes risk. It may be mentioned that we have tried with different sets of hyper parameters, costs and coefficients, and it is observed that in all the cases $R_B^\delta < R_1^L + R_2^L$.

Table 3: Lam's sampling plan for Sample A

a_1	b_0	R_1^L	n_0	k_{10}	ξ_{10}^L
5.0	5.0	8.9303	6	6	1.3375
5.0	5.5	8.7875	6	6	1.2543
4.0	5.0	8.4689	8	8	1.1500
3.5	4.0	8.6000	8	8	1.2015
4.7	5.0	8.8173	6	6	1.2789

Table 4: Lam's sampling plan for Sample B

a_2	b_0	R_2^L	n_0	k_{20}	ξ_{20}^L
5.0	5.0	13.1262	8	8	0.7441
5.0	5.5	12.1374	8	8	0.6816
6.0	5.0	14.5789	8	8	0.8471
4.5	4.0	13.8375	8	8	0.8179
4.3	5.0	11.8701	8	8	0.6725

6 Conclusion

In this work we study acceptance sampling plan with decision-theoretic approach under the Bayesian framework on two sample of products coming from two different sources under a joint censoring scheme. The major advantage of such implementation is to take decision on the acceptance or rejection of the batch of two different kind of products in a single life testing experiment. Under flexible prior assumption on the model parameters, decision making

Table 5: Comparison between optimal DSP and Lam's sampling plan in terms of Bayes risks under two samples case

a_0	a_1	a_2	b_0	$R_1^L + R_2^L$	R_B^δ	n_0	k_0	ξ_{10}	ξ_{20}	c_0
10.0	5.0	5.0	5.0	8.9303+13.1262=22.0565	20.3742	5	4	6.9407	0.0036	1.2445
10.0	5.0	5.0	5.5	8.7876+12.1374=20.9250	19.8696	6	4	6.4416	0.0042	1.2451
10.0	4.0	6.0	5.0	8.4689+14.5789=23.0478	19.3091	6	5	10.0423	0.0062	1.1807
8.0	3.5	4.5	4.0	8.6000+13.8375=22.4375	20.0316	7	5	8.8095	0.0013	1.1712
9.0	4.7	4.3	5.0	8.8173+11.8701=20.6874	20.2678	7	5	6.5293	0.0030	1.2919

is performed based on minimizing the Bayes risk of an extensive loss function capturing different aspects of the sampling plan. Under the assumption of equality of two sample and exponentially distributed life times, the Bayes risk can be obtained in simpler form and therefore optimization is quite a feasible task. The sensitivity analysis indicates that the optimal DSP are sensitive to the hyper-parameter values. The costs and acceptance cost coefficients also play a major role to determine the optimal DSP and acceptance or rejection of the batches of items from product line A and B, respectively. Through this study, we have come across a significant finding that based on the assumptions of this work, the optimal decision-theoretic sampling plan under the BJPC scheme potentially superior than optimal sampling plan based on Type II censoring scheme applied on two samples separately, in reliability decision making. In this work, we have restricted our work on two sample problem and a specific prior assumption. In practice, the proposed method can be extended to more than two samples on different prior considerations. More works are needed in those directions.

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Appendix

Notations:

- $\lambda = \lambda_1 + \lambda_2$.
- $G(a, b)$ denotes gamma distribution with shape parameter a and scale parameter b.
- $B(a, b) = \int_0^1 z^{a-1}(1-z)^{b-1}dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.
- $B_x(a, b) = \int_0^x u^{a-1}(1-u)^{b-1}du$, where $0 \leq x \leq 1$, is the incomplete beta function. We denote the cumulative distribution function of beta as, $I_x(a, b) = \frac{B_x(a,b)}{B(a,b)}$.
- $\mathrm{I}(A)$ is indicator function on set A, if A occurs then $\mathrm{I}(A) = 1$, otherwise $\mathrm{I}(A) = 0$

Derivation of Bayes Risk

The Bayes risk given in (5) is

$$
\begin{aligned}
&R_B^\delta(n, k, R_1, \ldots, R_{k-1}, \xi_1, \xi_2, c) \\
&\qquad = E\Big[\big(nC_s - (n-K_1)r_{s_1} - (n-K_2)r_{s_2} + W_kC_\tau + \mathrm{MSE}(\widehat{\theta}_{1Sh})C_{e_1} \\
&\qquad\quad + \mathrm{MSE}(\widehat{\theta}_{2Sh})C_{e_2} + g_1(\lambda_1) + g_2(\lambda_2)\big) I(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2) \\
&\qquad\quad + \big(nC_s - (n-K_1)r_{s_1} - (n-K_2)r_{s_2} + W_kC_\tau + \mathrm{MSE}(\widehat{\theta}_{1Sh})C_{e_1} \\
&\qquad\quad + \mathrm{MSE}(\widehat{\theta}_{2Sh})C_{e_2} + g_1(\lambda_1) + C_{r_2}\big) I(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} < \xi_2) \\
&\qquad\quad + \big(nC_s - (n-K_1)r_{s_1} - (n-K_2)r_{s_2} + W_kC_\tau + \mathrm{MSE}(\widehat{\theta}_{1Sh})C_{e_1} \\
&\qquad\quad + \mathrm{MSE}(\widehat{\theta}_{2Sh})C_{e_2} + g_2(\lambda_2) + C_{r_1}\big) I(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} > \xi_2) \\
&\qquad\quad + \big(nC_s - (n-K_1)r_{s_1} - (n-K_2)r_{s_2} + W_kC_\tau + \mathrm{MSE}(\widehat{\theta}_{1Sh})C_{e_1}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\mathrm{MSE}(\widehat{\theta}_{2Sh})C_{e_2} + C_r\big)I(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} < \xi_2)\big] \\
&= nC_s - (n - E[K_1])r_{s_1} - (n - E[K_2])r_{s_2} + E[W_k]C_\tau + E[\mathrm{MSE}(\widehat{\theta}_{1Sh})]C_{e_1} \\
&\quad +E[\mathrm{MSE}(\widehat{\theta}_{2Sh})]C_{e_2} + E_{(\lambda_1,\lambda_2)}\big[(g_1(\lambda_1) + g_2(\lambda_2))P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2)\big] \\
&\quad +E_{(\lambda_1,\lambda_2)}\big[(g_1(\lambda_1) + C_{r_2})P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} < \xi_2)\big] \\
&\quad +E_{(\lambda_1,\lambda_2)}\big[(C_{r_1} + g_2(\lambda_2))P(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} > \xi_2)\big] \\
&\quad +E_{(\lambda_1,\lambda_2)}\big[C_rP(\hat{\theta}_{1Sh} < \xi_1 \quad \hat{\theta}_{2Sh} < \xi_2)\big] \\
&= nC_s - (n - E[K_1])r_{s_1} - (n - E[K_2])r_{s_2} + E[W_k]C_\tau \\
&\quad +E[\mathrm{MSE}(\widehat{\theta}_{1Sh})]C_{e_1} + E[\mathrm{MSE}(\widehat{\theta}_{2Sh})]C_{e_2} \\
&\quad +D_1 + D_2 + D_3 + D_4, \qquad (9)
\end{aligned}
$$

Computation of $E[K_1]$, $E[K_2]$:

Lemma 1: $K_1(K_2)$ follows a Binomial distribution with parameters k and $\frac{\lambda_1}{\lambda}(\frac{\lambda_2}{\lambda})$.

Proof: See Mondal and Kundu [12].

$$
\begin{aligned}
E[K_1] = E_{(\lambda_1,\lambda_2)}E_{(\mathbf{w},\mathbf{z})|(\lambda_1,\lambda_2)}[K_1] &= E_{(\lambda_1,\lambda_2)}\Big[k\frac{\lambda_1}{\lambda}\Big] \\
&= k\frac{B(a_1+1,a_2)}{B(a_1,a_2)} \\
\text{Similarly, } E(K_2) = E_{(\lambda_1,\lambda_2)}E_{(\mathbf{w},\mathbf{z})|(\lambda_1,\lambda_2)}[K_2] &= k\frac{B(a_1,a_2+1)}{B(a_1,a_2)}
\end{aligned}
$$

Computation of $E[W_k]$:

Lemma 2: For all $i = 1, \ldots, k$, $W_i = \sum_{s=1}^{i} V_s$, where $V_s \sim Exp(\frac{1}{E_s})$ independently and $E_s = \lambda(n - \sum_{j=1}^{s-1}(R_j + 1))$.

Proof: See Mondal and Kundu [12].

Applying Lemma 2, we can obtain

$$
E[W_k] \quad = \quad E_{(\lambda_1,\lambda_2)}E_{(\mathbf{w},\mathbf{z})|(\lambda_1,\lambda_2)}\Big[W_k\Big]
$$

$$
\begin{aligned}
&= E_{(\lambda_1,\lambda_2)} E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)} \Big[\sum_{s=1}^{k} V_s \Big] \\
&= E_{(\lambda_1,\lambda_2)} \Big[\sum_{s=1}^{k} \frac{1}{E_s} \Big] \\
&= E_{(\lambda_1,\lambda_2)} \Big[\frac{1}{\lambda} \Big] \times \sum_{s=1}^{k} \frac{1}{(n - \sum_{j=1}^{s-1}(R_j+1))} \\
&= b_0 \frac{\Gamma(a_0-1)}{\Gamma(a_0)} \times \sum_{s=1}^{k} \frac{1}{(n - \sum_{j=1}^{s-1}(R_j+1))} \qquad \text{(Applying Theorem 2.1)}
\end{aligned}
$$

Therefore, $E[W_k]$ exists only if $a_0 > 1$.

Computation of D_1, D_2, D_3 and D_4 :

Probability distribution of $(\widehat{\theta}_{1Sh}, \widehat{\theta}_{2Sh})$:

$$
\begin{aligned}
U &= \sum_{i=1}^{k-1}(R_i+1)W_i + (n - \sum_{j=1}^{k-1}(R_j+1))W_k, \\
&= \sum_{i=1}^{k-1}(R_i+1)\sum_{s=1}^{i} V_s + (n - \sum_{j=1}^{k-1}(R_j+1))\sum_{s=1}^{k} V_s \\
&= \sum_{s=1}^{k} V_s (n - \sum_{i=1}^{s-1}(R_i+1)) \\
&= \frac{1}{\lambda}\sum_{s=1}^{k} E_s V_s
\end{aligned}
$$

Applying Lemma 2, it is observed that $2\lambda U$ follows a Chi-square distribution with $2k$ degrees of freedom ($2\lambda U \sim \chi^2_{2k}$). Therefore,

$$
\begin{aligned}
P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2) &= \sum_{r=0}^{k} P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2 | K_1 = r) P(K_1 = r) \\
&= \sum_{r=0}^{k} P(U > \xi_1(r+c), U > \xi_2(k-r+c) | K_1 = r) P(K_1 = r)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{r=0}^{k} P(2\lambda U > 2\lambda max(\xi_1(r+c), \xi_2(k-r+c)) \Big| K_1 = r) P(K_1 = r) \\
&= \sum_{r=0}^{k} \binom{k}{r} \left(\frac{\lambda_1}{\lambda}\right)^r \left(\frac{\lambda_2}{\lambda}\right)^{k-r} \int_{2\lambda\eta_{1rc}}^{\infty} \frac{1}{2^k \Gamma(k)} t^{k-1} e^{-\frac{t}{2}} dt \\
&= \sum_{r=0}^{k} \binom{k}{r} \lambda_1^r \lambda_2^{k-r} \int_{\eta_{1rc}}^{\infty} \frac{1}{\Gamma(k)} u^{k-1} e^{-\lambda u} du
\end{aligned}
$$

where $\eta_{1rc} = max(\xi_1(r+c), \xi_2(k-r+c))$.

Similarly we can compute

$$
P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} < \xi_2) = \sum_{r=0}^{k} \binom{k}{r} \lambda_1^r \lambda_2^{k-r} \int_{\eta_{2rc}^L}^{\eta_{2rc}^U} \frac{\mathrm{I}(\xi_2(k-r+c) < \xi_1(r+c))}{\Gamma(k)} u^{k-1} e^{-\lambda u} du,
$$

where $\eta_{2rc}^U = \xi_1(r+c),\ \eta_{2rc}^L = \xi_2(k-r+c)$.

$$
P(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} > \xi_2) = \sum_{r=0}^{k} \binom{k}{r} \lambda_1^r \lambda_2^{k-r} \int_{\eta_{3rc}^L}^{\eta_{3rc}^U} \frac{\mathrm{I}(\xi_2(k-r+c) > \xi_1(r+c))}{\Gamma(k)} u^{k-1} e^{-\lambda u} du,
$$

where $\eta_{3rc}^U = \xi_2(k-r+c),\ \eta_{3rc}^L = \xi_1(r+c)$.

$$
P(\hat{\theta}_{1Sh} < \xi_1, \hat{\theta}_{2Sh} < \xi_2) = \sum_{r=0}^{k} \binom{k}{r} \lambda_1^r \lambda_2^{k-r} \int_{0}^{\eta_{4rc}} \frac{1}{\Gamma(k)} u^{k-1} e^{-\lambda u} du,
$$

where $\eta_{4rc} = min(\xi_1(r+c), \xi_2(k-r+c))$.

In (9), to compute D_1, D_2, D_3, and D_4 we need the joint distribution function of $(\hat{\theta}_{1Sh}, \hat{\theta}_{2Sh})$ which is given above. Applying the joint distribution function of $(\hat{\theta}_{1Sh}, \hat{\theta}_{2Sh})$ and Theorem 2.1, we compute D_1, D_2, D_3 and D_4 as follows.

$$
\begin{aligned}
D_1 &= E_{(\lambda_1,\lambda_2)}\Big[\big(g_1(\lambda_1) + g_2(\lambda_2)\big) P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2)\Big] \\
&= E_{(\lambda_1,\lambda_2)}\Big[\big((l_0' + l_0'') + l_1'\lambda_1 + l_1''\lambda_2 + l_2'\lambda_1^2 + l_2''\lambda_2^2\big) P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2)\Big] \\
&= \sum_{i=1}^{5} C_{1i} E_{(\lambda_1,\lambda_2)}\Big[\lambda_1^{p_i} \lambda_2^{q_i} P(\hat{\theta}_{1Sh} > \xi_1, \hat{\theta}_{2Sh} > \xi_2)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}}{\Gamma(k)}E_{\lambda_1,\lambda_2}\Big[\int_{\eta_{1rc}}^{\infty}\lambda_1^{r+p_i}\lambda_2^{k-r+q_i}u^{k-1}e^{-\lambda u}du\Big]\\
&= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}}{\Gamma(k)}\int_{\eta_{1rc}}^{\infty}E_{\lambda_1,\lambda_2}\Big[\lambda_1^{r+p_i}\lambda_2^{k-r+q_i}e^{-\lambda u}\Big]u^{k-1}du\\
&= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}}{\Gamma(k)}\int_{\eta_{1rc}}^{\infty}E_{\lambda_1,\lambda_2}\Big[\big(\frac{\lambda_1}{\lambda}\big)^{r+p_i}\big(1-\frac{\lambda_1}{\lambda}\big)^{k-r+q_i}\lambda^{k+p_i+q_i}e^{-\lambda u}\Big]u^{k-1}du\\
&= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}}{\Gamma(k)}\int_{\eta_{1rc}}^{\infty}E_{\lambda_1,\lambda_2}\Big[\big(\frac{\lambda_1}{\lambda}\big)^{r+p_i}\big(1-\frac{\lambda_1}{\lambda}\big)^{k-r+q_i}\Big]E_{\lambda_1,\lambda_2}\Big[\lambda^{k+p_i+q_i}e^{-\lambda u}\Big]u^{k-1}du, \quad (10)
\end{aligned}
$$

where the set

$$
(C_{1i},p_i,q_i)=\begin{cases}(l_0'+l_0'',0,0) & \text{if} \quad i=1\\ (l_1',1,0) & \text{if} \quad i=2\\ (l_1'',0,1) & \text{if} \quad i=3\\ (l_2',2,0) & \text{if} \quad i=4\\ (l_2'',0,2) & \text{if} \quad i=5.\end{cases}
$$

Since $\frac{\lambda_1}{\lambda}$ and λ are independent by Theorem 2.1 and $\frac{\lambda_1}{\lambda}\sim Beta(a_1,a_1)$ and $\lambda\sim G(a_0,b_0)$, therefore

$$
\begin{aligned}
E_{\lambda_1,\lambda_2}\Big[\big(\frac{\lambda_1}{\lambda}\big)^{r+p_i}\big(1-\frac{\lambda_1}{\lambda}\big)^{k-r+q_i}\Big] &= \frac{B(r+p_i+a_1,k-r+q_i+a_2)}{B(a_1,a_2)}\\
E_{\lambda_1,\lambda_2}\Big[\lambda^{k+p_i+q_i}e^{-\lambda u}\Big] &= \frac{b_0^{a_0}\Gamma(k+p_i+q_i+a_0)}{\Gamma(a_0)(u+b_0)^{k+p_i+q_i+a_0}}. \quad (11)
\end{aligned}
$$

Now putting (11) into (10) and taking a transformation $u=zb_0$ we will get

$$
\begin{aligned}
D_1 &= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}b_0^{a_0}\Gamma(k+p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(k)\Gamma(a_0)B(a_1,a_2)}\int_{\eta_{1rc}}^{\infty}\frac{u^{k-1}}{(u+b_0)^{k+p_i+q_i+a_0}}du\\
&= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}\Gamma(k+p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(k)\Gamma(a_0)B(a_1,a_2)b_0^{p_i+q_i}}\int_{\frac{\eta_{1rc}}{b_0}}^{\infty}\frac{z^{k-1}}{(1+z)^{k+p_i+q_i+a_0}}dz
\end{aligned}
$$

Now taking a transformation $z=v/(1-v)$, the integral becomes

$$
\int_{\frac{\eta_{1rc}}{b_0}}^{\infty}\frac{z^{k-1}}{(1+z)^{k+p_i+q_i+a_0}}dz=\int_{S_{1rc}}^{1}v^{k-1}(1-v)^{p_i+q_i+a_0-1}dv=B(k,p_i+q_i+a_0)-B_{S_{1rc}}(k,p_i+q_i+a_0),
$$

where $S_{1rc}=\dfrac{\frac{\eta_{1rc}}{b_0}}{1+\frac{\eta_{1rc}}{b_0}}$. Therefore, the final expression of D_1 can be obtained as

$$
D_1 = \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}\Gamma(k+p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)B(k,p_i+q_i+a_0)}{\Gamma(k)\Gamma(a_0)B(a_1,a_2)b_0^{p_i+q_i}}
$$

$$\times[1 - I_{S_{1rc}}(k, p_i + q_i + a_0)]$$

$$= \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}\Gamma(p_i + q_i + a_0)B(r + p_i + a_1, k - r + q_i + a_2)}{\Gamma(a_0)B(a_1, a_2)b_0^{p_i+q_i}}[1 - I_{S_{1rc}}(k, p_i + q_i + a_0)].$$

Similarly, the explicit expressions of D_2, D_3 and D_4 can be obtained as,

$$D_2 = \sum_{i=0}^{2}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{2i}\Gamma(i + a_0)B(r + i + a_1, k - r + a_2)\mathrm{I}(\xi_2(k - r + c) < \xi_1(r + c))}{\Gamma(a_0)B(a_1, a_2)b_0^i}$$
$$\times[I_{S_{2rc}^U}(k, i + a_0) - I_{S_{2rc}^L}(k, i + a_0)]$$

$$D_3 = \sum_{i=0}^{2}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{3i}\Gamma(i + a_0)B(r + a_1, k - r + i + a_2)\mathrm{I}(\xi_2(k - r + c) > \xi_1(r + c))}{\Gamma(a_0)B(a_1, a_2)b_0^i}$$
$$\times[I_{S_{3rc}^U}(k, i + a_0) - I_{S_{3rc}^L}(k, i + a_0)]$$

$$D_4 = \sum_{r=0}^{k}\binom{k}{r}\frac{C_r\Gamma(a_0)B(r + a_1, k - r + a_2)}{\Gamma(a_0)B(a_1, a_2)}I_{S_{4rc}}(k, a_0)$$

where

$$C_{2i} = \begin{cases} l_0' + C_{r_2} & \text{if } i = 0 \\ l_i' & \text{if } i = 1, 2, \end{cases}$$

$$C_{3i} = \begin{cases} l_0'' + C_{r_1} & \text{if } i = 0 \\ l_i'' & \text{if } i = 1, 2, \end{cases}$$

$$S_{2rc}^U = \frac{\frac{\eta_{2rc}^U}{b_0}}{1 + \frac{\eta_{2rc}^U}{b_0}}, S_{2rc}^L = \frac{\frac{\eta_{2rc}^L}{b_0}}{1 + \frac{\eta_{1rc}^L}{b_0}}, S_{3rc}^U = \frac{\frac{\eta_{3rc}^U}{b_0}}{1 + \frac{\eta_{3rc}^U}{b_0}}, S_{3rc}^L = \frac{\frac{\eta_{3rc}^L}{b_0}}{1 + \frac{\eta_{3rc}^L}{b_0}}, \text{ and } S_{4rc} = \frac{\frac{\eta_{4rc}}{b_0}}{1 + \frac{\eta_{4rc}}{b_0}}.$$

Computation of $E[\mathbf{MSE}(\widehat{\theta}_{1Sh})]$, $E[\mathbf{MSE}(\widehat{\theta}_{2Sh})]$:

$$E[\text{MSE}(\widehat{\theta}_{1Sh})] = E_{(\lambda_1,\lambda_2)}[E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)}(\widehat{\theta}_{1Sh} - \theta_1)^2]$$

$$E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)}(\widehat{\theta}_{1Sh} - \theta_1)^2$$
$$= E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)}\left(\frac{A}{K_1 + c} - \theta_1\right)^2$$
$$= \sum_{r=0}^{k} E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)}\left(\frac{A}{r + c} - \theta_1\right)^2 P(K_1 = r)$$

$$
\begin{aligned}
&= \sum_{r=0}^{k} \frac{1}{4\lambda^2(r+c)^2} E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)} \big(2\lambda A - 2\lambda(r+c)\theta_1\big)^2 P(K_1 = r) \\
&= \sum_{r=0}^{k} \frac{1}{4\lambda^2(r+c)^2} \Big[E_{(\mathbf{W},\mathbf{Z})|(\lambda_1,\lambda_2)} \big(2\lambda A - 2k\big)^2 + \big(2k - 2\lambda(r+c)\theta_1\big)^2\Big] P(K_1 = r) \\
&= \sum_{r=0}^{k} \frac{1}{4\lambda^2(r+c)^2} \big[4k + \big(2k - 2\lambda(r+c)\theta_1\big)^2\big] P(K_1 = r) \quad [\text{as } 2\lambda A \sim \chi^2_{2k}\ ] \\
&= \sum_{r=0}^{k} \binom{k}{r} \left(\frac{\lambda_1}{\lambda}\right)^r \left(1 - \frac{\lambda_1}{\lambda}\right)^{k-r} \left[\frac{k(k+1)}{\lambda^2(r+c)^2} - \frac{2k}{\lambda\lambda_1(r+c)} + \frac{1}{\lambda_1^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&E[\mathrm{MSE}(\widehat{\theta}_{1Sh})] \\
&= \sum_{r=0}^{k} \binom{k}{r} \Big[\frac{k(k+1)}{(r+c)^2} \frac{B(a_1 + r, a_2 + k - r)\Gamma(a_0 - 2)}{\Gamma(a_0)B(a_1, a_2)b_0^{-2}} \\
&\quad - \frac{2k}{(r+c)} \frac{B(a_1 + r - 1, a_2 + k - r)\Gamma(a_0 - 2)}{\Gamma(a_0)B(a_1, a_2)b_0^{-2}} + \frac{B(a_1 + r - 2, a_2 + k - r)\Gamma(a_0 - 2)}{\Gamma(a_0)B(a_1, a_2)b_0^{-2}}\Big]
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&E[\mathrm{MSE}(\widehat{\theta}_{2Sh})] \\
&= \sum_{r=0}^{k} \binom{k}{r} \Big[\frac{k(k+1)}{(r+c)^2} \frac{B(a_1 + k - r, a_2 + r)\Gamma(a_0 - 2)}{\Gamma(a_0)B(a_1, a_2)b_0^{-2}} \\
&\quad - \frac{2k}{(r+c)} \frac{B(a_1 + k - r, a_2 + r - 1)\Gamma(a_0 - 2)}{\Gamma(a_0)B(a_1, a_2)b_0^{-2}} + \frac{B(a_1 + k - r, a_2 + r - 2)\Gamma(a_0 - 2)}{\Gamma(a_0)B(a_1, a_2)b_0^{-2}}\Big]
\end{aligned}
$$

Note that $E[\mathrm{MSE}(\widehat{\theta}_{1Sh})]$ and $E[\mathrm{MSE}(\widehat{\theta}_{2Sh})]$ exist only if $a_0 > 2$.

Therefore the exact expression of Bayes Risk is:

$$
\begin{aligned}
R_B^{\delta}(n, k, R_1, \ldots, R_{k-1}, \xi_1, \xi_2, c) &= nC_s - (n - E[K_1])r_{s_1} - (n - E[K_2])r_{s_2} + E[W_k]C_\tau \\
&\quad + E[\mathrm{MSE}(\widehat{\theta}_{1Sh})]C_{e_1} + E[\mathrm{MSE}(\widehat{\theta}_{2Sh})]C_{e_2} + D_1 + D_2 + D_3 + D_4, \\
E[K_1] &= k\frac{B(a_1 + 1, a_2)}{B(a_1, a_2)}, \\
E[K_2] &= k\frac{B(a_1, a_2 + 1)}{B(a_1, a_2)}, \\
E[W_k] &= b_0 \frac{\Gamma(a_0 - 1)}{\Gamma(a_0)} \times \sum_{s=1}^{k} \frac{1}{(n - \sum_{j=1}^{s-1}(R_j + 1))},
\end{aligned}
$$

$$
\begin{aligned}
&E[\mathrm{MSE}(\widehat{\theta}_{1Sh})] \\
&= \sum_{r=0}^{k}\binom{k}{r}\Big[\frac{k(k+1)}{(r+c)^2}\frac{B(a_1+r,a_2+k-r)\Gamma(a_0-2)}{\Gamma(a_0)B(a_1,a_2)b_0^{-2}} \\
&\quad -\frac{2k}{(r+c)}\frac{B(a_1+r-1,a_2+k-r)\Gamma(a_0-2)}{\Gamma(a_0)B(a_1,a_2)b_0^{-2}}+\frac{B(a_1+r-2,a_2+k-r)\Gamma(a_0-2)}{\Gamma(a_0)B(a_1,a_2)b_0^{-2}}\Big], \\
&E[\mathrm{MSE}(\widehat{\theta}_{2Sh})] \\
&= \sum_{r=0}^{k}\binom{k}{r}\Big[\frac{k(k+1)}{(k-r+c)^2}\frac{B(a_1+k-r,a_2+r)\Gamma(a_0-2)}{\Gamma(a_0)B(a_1,a_2)b_0^{-2}} \\
&\quad -\frac{2k}{(k-r+c)}\frac{B(a_1+k-r,a_2+r-1)\Gamma(a_0-2)}{\Gamma(a_0)B(a_1,a_2)b_0^{-2}}+\frac{B(a_1+k-r,a_2+r-2)\Gamma(a_0-2)}{\Gamma(a_0)B(a_1,a_2)b_0^{-2}}\Big], \\
&D_1 = \sum_{i=1}^{5}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{1i}\Gamma(p_i+q_i+a_0)B(r+p_i+a_1,k-r+q_i+a_2)}{\Gamma(a_0)B(a_1,a_2)b_0^{p_i+q_i}}[1-I_{S_{1rc}}(k,p_i+q_i+a_0)], \\
&D_2 = \sum_{i=0}^{2}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{2i}\Gamma(i+a_0)B(r+i+a_1,k-r+a_2)\mathrm{I}(\xi_2(k-r+c)<\xi_1(r+c))}{\Gamma(a_0)B(a_1,a_2)b_0^{i}} \\
&\qquad\qquad \times[I_{S^U_{2rc}}(k,i+a_0)-I_{S^L_{2rc}}(k,i+a_0)], \\
&D_3 = \sum_{i=0}^{2}\sum_{r=0}^{k}\binom{k}{r}\frac{C_{3i}\Gamma(i+a_0)B(r+a_1,k-r+i+a_2)\mathrm{I}(\xi_2(k-r+c)>\xi_1(r+c))}{\Gamma(a_0)B(a_1,a_2)b_0^{i}} \\
&\qquad\qquad \times[I_{S^U_{3rc}}(k,i+a_0)-I_{S^L_{3rc}}(k,i+a_0)], \\
&D_4 = \sum_{r=0}^{k}\binom{k}{r}\frac{C_r\Gamma(a_0)B(r+a_1,k-r+a_2)}{\Gamma(a_0)B(a_1,a_2)}I_{S_{4rc}}(k,a_0),
\end{aligned}
$$

where

$$
(C_{1i},p_i,q_i)=\begin{cases}(l_0'+l_0'',0,0) & \text{if } i=1\\ (l_1',1,0) & \text{if } i=2\\ (l_1'',0,1) & \text{if } i=3\\ (l_2',2,0) & \text{if } i=4\\ (l_2'',0,2) & \text{if } i=5,\end{cases}
$$

$$
C_{2i}=\begin{cases}l_0'+C_{r_2} & \text{if } i=0\\ l_i' & \text{if } i=1,2,\end{cases}\quad C_{3i}=\begin{cases}l_0''+C_{r_1} & \text{if } i=0\\ l_i'' & \text{if } i=1,2,\end{cases}
$$

$S_{1rc}=\frac{\frac{\eta_{1rc}}{b_0}}{1+\frac{\eta_{1rc}}{b_0}}$, $S^U_{2rc}=\frac{\frac{\eta^U_{2rc}}{b_0}}{1+\frac{\eta^U_{2rc}}{b_0}}$, $S^L_{2rc}=\frac{\frac{\eta^L_{2rc}}{b_0}}{1+\frac{\eta^L_{1rc}}{b_0}}$, $S^U_{3rc}=\frac{\frac{\eta^U_{3rc}}{b_0}}{1+\frac{\eta^U_{3rc}}{b_0}}$, $S^L_{3rc}=\frac{\frac{\eta^L_{3rc}}{b_0}}{1+\frac{\eta^L_{3rc}}{b_0}}$, $S_{4rc}=\frac{\frac{\eta_{4rc}}{b_0}}{1+\frac{\eta_{4rc}}{b_0}}$, $\eta_{1rc}=\max(\xi_1(r+c),\xi_2(k-r+c))$, $\eta^U_{2rc}=\xi_1(r+c)$, $\eta^L_{2rc}=\xi_2(k-r+c)$, $\eta^U_{3rc}=\xi_2(k-r+c)$, $\eta^L_{3rc}=\xi_1(r+c)$, and $\eta_{4rc}=\min(\xi_1(r+c),\xi_2(k-r+c))$.

Proof of Theorem 3.1

Let $(n_0, k_0, \xi_{10}, \xi_{20}, c_0)$ be the optimal DSP, as discussed in (7), (8), then minimum Bayes risk is given by $R_B^\delta(n_0, k_0, \xi_{10}, \xi_{20}, c_0)$.

As $E[K_1] \geq 0$, $E[K_2] \geq 0$, $E[W_k] \geq 0$, $E[\text{MSE}(\widehat{\theta}_{1Sh})] \geq 0$, $E[\text{MSE}(\widehat{\theta}_{2Sh})] \geq 0$, decision risk $D_1 + D_2 + D_3 + D_4 \geq 0$, and $C_s > r_{s_1} + r_{s_2}$, therefore it is clear that

$$R_B^\delta(n_0, k_0, \xi_{10}, \xi_{20}, c_0) \geq n_0(C_s - r_{s_1} - r_{s_2}). \tag{12}$$

Let $(0, 0, \infty, \infty, 0)$ denote the sampling plan, when we reject the batch without sampling with Bayes risk $R_B^\delta(0, 0, \infty, \infty, 0) = C_r$ and $(0, 0, 0, 0, 0)$ denote the sampling plan when we accept the batch without sampling with Bayes risk $R_B^\delta(0, 0, 0, 0, 0) = E_{\lambda_1,\lambda_2}[g_1(\lambda_1) + g_2(\lambda_2)]$. Then the Bayes risk of optimal DSP:

$$R_B^\delta(n_0, k_0, \xi_{10}, \xi_{20}, c_0) \leq \min\left\{E_{\lambda_1,\lambda_2}[g_1(\lambda_1) + g_2(\lambda_2)], C_r\right\}. \tag{13}$$

Now from (12) and (13), it follows that

$$n_0(C_s - r_{s_1} - r_{s_2}) \leq \min\left\{E_{\lambda_1,\lambda_2}[g_1(\lambda_1) + g_2(\lambda_2)], C_r\right\},$$

which implies that

$$n_0 \leq \min\left\{\frac{E_{\lambda_1,\lambda_2}[g_1(\lambda_1) + g_2(\lambda_2)]}{C_s - r_{s_1} - r_{s_2}}, \frac{C_r}{C_s - r_{s_1} - r_{s_2}}\right\},$$

and $0 \leq k_0 \leq n_0$.

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