Decision Theoretic Sampling Plan for One-Parameter Exponential Distribution Under Type-I and Type-I Hybrid Censoring Schemes

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Abstract In this paper, we design a decision theoretic sampling plan (DSP) based on Type-I and Type-I hybrid censored lifetime data from a one-parameter exponential distribution. The Bayes estimator of the mean lifetime is used to define a decision function. A suitable loss function is considered to derive the Bayes risk of this DSP. A finite algorithm is provided to obtain the optimum DSP and the corresponding Bayes risk. It has been observed numerically that the optimum DSP is better than the sampling plan proposed by Lam [12] and Lin et al. [16, 17] and it is as good as the Bayesian sampling plan (BSP) of Lin et al. [18] and Liang and Yang [14]. It is observed that the Bayes risk of the optimum DSP is approximately equal to the Bayes risk of the BSP. In case of higher degree polynomials and for a non-polynomial loss function the DSP can be obtained without any additional effort as compared to BSP.

1 Introduction

In reliability life testing experiments, manufacturers usually choose a suitable acceptance sampling plan and do inspection to make decision on the reliability of the experimental items in the batch. If we can find the optimal acceptance sampling plan then we make a better decision on batch of items. So acceptance sampling plan plays

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an important role in reliability life testing or quality control. In the vast literature of acceptance sampling plans, various approaches have been discussed. The approach based on decision theory has become more popular because here the sampling plan is obtained by making an optimal decision on some economic considerations such as maximizing the return or minimizing the loss. Many researchers have widely used this approach to obtain the optimal sampling plan. See, for example, Hald [6], Fertig and Mann [5], Lam [10, 11, 12], Lam and Choy [13], Lin et al. [18], Huang and Lin [8, 9], Chen et al. [3], Lin et al. [16, 17], Liang and Yang [14] and the references cited therein.

In life testing experiments, usually we collect a censored or incomplete data. The experimenter does not observe the failure of all items because inspection cost increases with time, items may be expensive, etc. So, to save time or resources we try to get the optimal result based on censored or incomplete data. If the inspection cost increases with time, then we put n items on test and terminate the test at a preassigned time τ. This type of censoring is called Type-I censoring. In Type-I hybrid censoring, the experiment is terminated at time of r^{th} failure or at time τ, which ever occurs first. It is useful when items are expensive and the inspection cost also increases with time. In this paper it is assumed that the lifetimes of the experimental units follow one parameter exponential distribution. If M is the number of failures in the experiment, then in both the censoring schemes for $M = 0$, the maximum likelihood (ML) estimator of the mean lifetime does not exist. Lam [12] has obtained the Bayesian sampling plan for Type-I censoring where decision function is based on an estimator which is equal to the ML estimator of the mean lifetime when $M \geq 1$ and at $M = 0$, the estimator is equal to $n\tau$. Lin et al. [18] has observed that the loss function used by Lam [12] does not involve any cost on duration of the experiment. So if we extend the duration of the experiment, we can observe complete sample to get a better decision on the batch of items. Therefore, the sampling plan of Lam [12] is " neither optimal, nor Bayes ". Lin et al.[16, 17] proposed optimal sampling plans for Type-I and Type-I hybrid censoring schemes using a decision function based on the ML estimator of the mean lifetime conditioning on the event that $M \geq 1$.

In this study, we consider the case where life tests are Type-I and Type-I hybrid censored. To avoid the non-existence of the ML estimator of the mean lifetime, we use its Bayes estimator which exists for all values of M. We find the decision theoretic sampling plan (DSP) using the Bayes estimator of the mean lifetime in the decision function. A loss function which includes the sampling cost, the cost on per unit duration of the experiment, the salvage value and the decision loss are used to determine the optimal sampling plan by minimizing the Bayes risk. Numerical results for quadratic loss function show that the optimum DSP is a better plan than the sampling plans proposed by Lam [12] and Lin et al. [16, 17], and as good as the Bayesian sampling plan (BSP) given by Lin et al. [18] and Liang and Yang [14]. Theoretically, for fifth or higher degree polynomial and for a non-polynomial loss function, the optimum DSP is quite easy to derive but BSP cannot be obtained very conveniently.

The rest of the paper is organized as follows. In Section 1.1 we introduce the decision function based on the Bayes estimator of the mean lifetime. All necessary theoretical results and algorithm to obtain the optimum DSP are provided in Section 2. In Section 3 we compare the proposed DSP and the BSP for higher degree polynomial and for a non-polynomial loss function. Numerical comparisons and results on DSP are given in Section 4. Finally we conclude the paper in Section 5. All necessary derivations are provided in the Appendix.

1.1 Model Formulation, Assumptions and DSP

Let $X_1, X_2, \ldots, X_n$ denote the lifetime of n items put to test in an experiment. It is assumed that lifetimes of these components are mutually independent and follow the exponential distribution with probability density function (PDF)

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0, \ \lambda > 0 \\ 0, & otherwise. \end{cases} \tag{1}$$

Let θ denote the mean lifetime of the item and for the above distribution, the mean lifetime of the item is $\theta = \frac{1}{\lambda}$. So use of λ is equivalent to the use of θ. Henceforth, we will use whichever is more convenient. Let $X_{(1)} < \ldots < X_{(n)}$ be the ordered observations from the given sample of n lifetimes $X_1, \ldots, X_n$. For Type-I and Type-I hybrid censoring schemes, if τ^* denotes the duration of the experiment, then $\tau^* = \tau$ in a Type-I censoring and $\tau^* = min\{X_{(r)}, \tau\}$ in a Type-I hybrid censoring schemes. Here τ denotes the pre-fixed time and r is a pre-fixed integer, $1 \leq r \leq n$. We observe that τ^* is fixed in Type-I censoring and random for Type-I hybrid censoring. Define $M = max\{i : X_{(i)} \leq \tau^*\}$, i.e., M is the number of failures among the n items put on the life test before time τ^*. For Type-I censoring the joint PDF of $(X_{(1)}, X_{(2)}, \ldots, X_{(M)})$ is as follows:

$$f_{X_{(1)},\ldots,X_{(M)}}(x_1, \ldots, x_m, M = m|\lambda) = \frac{n!}{(n-m)!} \ \lambda^m e^{-\lambda \left[\sum_{i=1}^{m} x_i + (n-m)\tau\right]}; \quad 0 < x_1 < \ldots < x_m < \infty, \tag{2}$$

and for Type-I hybrid censoring the joint PDF is given as follows:

$$f_{X_{(1)},\ldots,X_{(M)}}(x_1, \ldots, x_m, M = m|\lambda) = \begin{cases} \frac{n!}{(n-m)!} \lambda^m e^{-\lambda \left[\sum_{i=1}^{m} x_i + (n-m)\tau\right]}, & \text{if } x_r \geq \tau \\ \frac{n!}{(n-r)!} \lambda^r e^{-\lambda \left[\sum_{i=1}^{r} x_i + (n-r)x_r\right]}, & \text{if } x_r < \tau. \end{cases} \tag{3}$$

Lam [12] proposed the Bayesian sampling plan for Type-I censoring using a decision function

$$\delta(\mathbf{x}) = \begin{cases} 1, & \text{if } \widehat{\theta} \geq \xi \\ 0, & \text{if } \widehat{\theta} < \xi, \end{cases} \tag{4}$$

where $\xi(>0)$ denotes the minimum acceptable surviving time to take a decision on a batch, i.e., whether to accept it with an action 1 or reject it with an action 0. Estimator $\widehat{\theta}$ is given by

$$\widehat{\theta} = \begin{cases} \widehat{\theta}_M, & \text{if } M \geq 1 \\ n\tau, & \text{if } M = 0, \end{cases}$$

where $\widehat{\theta}_M$ is the ML estimator of the mean lifetime θ and is defined as:

$$\widehat{\theta}_M = \begin{cases} \frac{\sum_{i=1}^{M} X_{(i)} + (n-M)\tau}{M}, & \text{if } M \geq 1 \\ \text{does not exist}, & \text{if } M = 0. \end{cases}$$

Similarly, for Type-I hybrid censoring also the ML estimator of the mean lifetime does not exist for $M = 0$. Therefore, in place of ML estimator, we propose to use the Bayes estimator of θ in decision function because it exists for all values of M. The Bayes estimator with respect to the squared error loss function (see Berger[2]) when λ has a prior distribution $G(a,b)$, with PDF

$$\pi(\lambda; a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-\lambda b}; \ \ \lambda > 0, \ \ a, b > 0, \tag{5}$$

is as follows:

1. *Type-I Censoring* :

$$\widehat{\theta}_B = \frac{b + \sum_{i=1}^{M} X_{(i)} + (n-M)\tau}{M + a},$$

2. *Type-I Hybrid Censoring* :

$$\widehat{\theta}_B = \begin{cases} \frac{b + \sum_{i=1}^{M} X_{(i)} + (n-M)\tau}{M+a}, & \text{if } X_{(r)} \geq \tau \\ \frac{b + \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{r+a}, & \text{if } X_{(r)} < \tau. \end{cases}$$

Based on the observed sample, we define our decision function as:

$$\delta(\mathbf{x}) = \begin{cases} 1, & \text{if } \hat{\theta}_B \geq \xi \\ 0, & \text{if } \hat{\theta}_B < \xi, \end{cases} \tag{6}$$

where $\hat{\theta}_B$ is the Bayes estimator of θ, $\xi(>0)$ is the minimum acceptable surviving time to take a decision on a batch , i.e., whether to accept it with an action 1 or reject it with an action 0. We consider the loss function:

$$L(\delta(\mathbf{x}), \lambda) = \begin{cases} nC_s - (n-M)r_s + \tau^* C_\tau + g(\lambda), & \text{if } \delta(\mathbf{x}) = 1 \\ nC_s - (n-M)r_s + \tau^* C_\tau + C_r, & \text{if } \delta(\mathbf{x}) = 0, \end{cases} \tag{7}$$

which depends on various costs. C_r is the cost due to rejecting the batch, C_s is the cost due to inspection of each item and C_τ is the cost per unit duration of the experiment. Cost due to accepting the batch is denoted by $g(\lambda)$, so that $g(\lambda)$ has to be positive and increasing in λ. In general, $g(\lambda)$ will be determined by the inspection requirement or experience before designing a sampling plan. Hence, the form of $g(\lambda)$ can vary and accordingly, form of the loss function also varies. If an item does not fail, then this item can be reused with a salvage value r_s such that $C_s > r_s \geq 0$, see, Liang and Yang [14], Liang et al. [15] etc.

Any sampling plan consists of the sampling parameters and the decision parameter ξ. Therefore, (n, τ, ξ) and (n, r, τ, ξ) are the sampling plans for Type-I censoring and Type-I hybrid censoring, respectively, which we denote by DSP. To determine the optimum DSP namely, (n_0, τ_0, ξ_0) and $(n_0, r_0, \tau_0, \xi_0)$ for Type-I censoring and Type-I hybrid censoring, respectively, we determine that decision function in (6) which minimizes the Bayes risk of the DSP under the given loss function in (7) among all possible DSPs.

2 Computation of Bayes risk and Optimum DSP

In this section we compute the Bayes risk of the DSP and an algorithm is presented for obtaining the optimum DSP. First, we derive the general expression of the Bayes risk for any given sampling plan and the distribution of $\hat{\theta}_B$. In Section 2.1 and 2.3 we derive the exact expression of the Bayes risks for Type-I and Type-I hybrid censoring schemes.

2.1 Computation of Bayes risk and Distribution of $\hat{\theta}_B$

A number of authors used a quadratic loss function to obtain the Bayesian sampling plans (for example see, Lam [11, 12], Lam and Choy [13], Lin et al. [18], Huang and Lin [8, 9], Lin et al. [16, 17], Liang and Yang [14], Liang et al. [15], etc). They use this functional form because computation becomes easier and $g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$ is a reasonable approximation of the true acceptance cost. However, a higher degree polynomial may be a better approximation of the true acceptance cost. So we consider the functional form of the loss function defined as

$$L(\delta(\mathbf{x}), \lambda) = \begin{cases} nC_s - (n-M)r_s + \tau^* C_\tau + a_0 + a_1\lambda + \ldots + a_k\lambda^k, & \text{if } \delta(\mathbf{x}) = 1 \\ nC_s - (n-M)r_s + \tau^* C_\tau + C_r, & \text{if } \delta(\mathbf{x}) = 0, \end{cases} \quad (8)$$

where $a_0, a_1, \ldots$ and a_k are fixed positive constants. Based on the decision rule in (6) and the loss function in (8), the Bayes risk takes the form

$$\begin{aligned} R_B(\mathbf{S}) &= E\{L(\delta(\mathbf{x}), \lambda)\} = E_\lambda E_{X/\lambda}\{L(\delta(\mathbf{x}), \lambda)\} \\ &= E_\lambda E_{X/\lambda}\{nC_s - (n-M)r_s + \tau^* C_\tau + a_0 + a_1\lambda + \ldots + a_k\lambda^k \\ &\quad +(1-\delta(\mathbf{x}))(C_r - a_1\lambda + \ldots + a_k\lambda^k)\} \end{aligned}$$

$$= n(C_s - r_s) + r_s E(M) + C_\tau E(\tau^*) + a_0 + a_1\mu_1 + \ldots + a_k\mu_k$$
$$+ E_\lambda\left\{(C_r - a_0 - a_1\lambda - \ldots - a_k\lambda^k)P(\hat{\theta}_B < \xi\ |\lambda)\right\}, \tag{9}$$

where **S** is the sampling plan under Type-I or Type-I hybrid censoring and $\mu_i = E(\lambda^i)$ for $i = 1, 2, \ldots, k$. Therefore, in order to derive the Bayes risk of the DSP from (9), it is clear that we need a distribution of $\hat{\theta}_B$ given λ for Type-I and Type-I hybrid censoring schemes.
So, first we compute the distribution of $\hat{\theta}_B$ for a given λ, which is

$$\begin{aligned} P(\widehat{\theta}_B \le x|\lambda) &= P(M=0|\lambda)P(\widehat{\theta}_B \le x|M=0,\lambda) \\ &\quad + P(M \ge 1|\lambda)P(\widehat{\theta}_B \le x|M \ge 1,\lambda) \\ &= pS_\lambda(x) + (1-p)H_\lambda(x), \end{aligned} \tag{10}$$

where $p = P(M = 0|\lambda) = e^{-n\lambda\tau}$ and

$$S_\lambda(x) = P(\widehat{\theta}_B \le x|M=0,\lambda) = \begin{cases} 1, & \text{if } x \ge \frac{b+n\tau}{a} \\ 0, & otherwise. \end{cases}$$
$$H_\lambda(x) = P(\widehat{\theta}_B \le x|M \ge 1,\lambda) = \begin{cases} \int_0^x h(u|\lambda)du, & \text{if } 0 < x < \frac{b+n\tau}{a} \\ 0, & otherwise, \end{cases}$$

where $h(u|\lambda)$ is the PDF of the absolutely continuous part of $\widehat{\theta}_B$ given λ.

2.2 Type- I censoring

When the samples are Type-I censored then as mentioned above, evaluation of the Bayes risk of DSP requires the PDF $h(.|\lambda)$. The following lemma provides the PDF of the absolutely continuous part of the distribution.

Lemma 1. *The PDF $h(.|\lambda)$ of $\widehat{\theta}_B$ given λ under a Type-I censoring scheme is given by*

$$h(x|\lambda) = \frac{1}{1-p}\sum_{m=1}^{n}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j e^{-\lambda(n-m+j)\tau}\pi\big(x-\tau_{j,m,a,b};m,(m+a)\lambda\big),$$

for $\tau_{j,m,a,b} < x < \frac{b+n\tau}{a}$, where $\tau_{j,m,a,b} = \frac{b+(n-m+j)\tau}{m+a}$ and $\pi(\cdot)$ is as defined in (5).

Proof. The PDF can be easily obtained using the result of Bartholomew [1]. □

Hence using the distribution function of $\hat{\theta}_B$ and the prior density in (5) of λ, the following theorem provides the Bayes risk of the DSP (n,τ,ξ) for the loss function in (8).

Theorem 1. *The Bayes risk of the DSP (n,τ,ξ) w.r.t the loss function in (8) is given as*

$$R_B(n,\tau,\xi) = n(C_s - r_s) + r_s E(M) + \tau C_\tau + a_0 + a_1\mu_1 + \ldots + a_k\mu_k + \sum_{l=0}^{k} \frac{b^a}{\Gamma(a)}$$
$$\times \left[\frac{\Gamma(a+l) I_{(\frac{b+n\tau}{a}<\xi)}}{(b+n\tau)^{(a+l)}} + \sum_{m=1}^{n}\sum_{j=0}^{m} C_l (-1)^j \binom{n}{m}\binom{m}{j} \frac{\Gamma(a+l) I_{S^*_{j,m,a,b}}(m,a+l)}{((m+a)\tau_{j,m,a,b})^{a+l}} \right],$$

where $I_{(\frac{b+n\tau}{a}<\xi)}$ *is an indicator function and the exact expressions for* $E(M)$, C_l *and* $I_{S^*_{j,m,a,b}}(m,a+l)$ *are provided in the proof of the theorem.*

Proof. See Appendix. □

Based on the explicit expression of the Bayes risk $R_B(n,\tau,\xi)$, an optimum DSP (n_0,τ_0,ξ_0) can be determined by

$$R_B(n_0,\tau_0,\xi_0) = \min_n \{\min_{\tau,\xi} [R_B(n,\tau,\xi)]\}. \tag{11}$$

The explicit expression of the Bayes risk is very complicated and hence, it is not possible to obtain optimal values of n, τ and ξ analytically. Lam [12] has given a grid search algorithm for obtaining an approximate optimal sampling plan. Following that approach, we present a similar algorithm to find the optimum DSP:
Algorithm A:

1. Fix n and τ, minimize $R_B(n,\tau,\xi)$ with respect to ξ using grid search method and denote the minimum Bayes risk by $R_B(n,\tau,\xi_0(n,\tau))$.
2. For fixed n, minimize $R_B(n,\tau,\xi_0(n,\tau))$ with respect to τ, using grid search method and denote the minimum Bayes risk by $R_B(n,\tau_0(n),\xi_0(n,\tau_0(n)))$.
3. Choose sample size n_0 such that

$$R_B(n_0,\tau_0(n_0),\xi_0(n_0,\tau_0(n_0))) \le R_B(n,\tau_0(n),\xi_0(n,\tau_0(n))) \;\; \forall \;\; n \ge 0.$$

The choice of the number of grid points for τ and ξ depends on the Bayes risk function $R_B(n,\tau,\xi)$. If $R_B(n,\tau,\xi)$ has a unique minimum then the algorithm works very well to find the optimum DSP. In relation to Algorithm A, the following theorem can significantly reduce the effort in the search for the optimum DSP and also ensures that it will be obtained in a finite number of search steps.

Theorem 2. *Assuming* $0 < \xi \le \xi^*$, *let us denote* $R_B(n,\tau,\xi') = min_{0<\xi\le\xi^*} R_B(n,\tau,\xi)$ *for some fixed* n (≥ 1) *and* τ. *Further, let* n_0 *and* τ_0 *be the optimal sample size and censoring time . Then,*

$$n_0 \le \min\left\{ \frac{C_r}{C_s - r_s}, \frac{a_0 + a_1\mu_1 + \ldots + a_k\mu_k}{C_s - r_s}, \frac{R_B(n,\tau,\xi')}{C_s - r_s} \right\},$$
$$\tau_0 \le \min\left\{ \frac{C_r}{C_\tau}, \frac{a_0 + a_1\mu_1 + \ldots + a_k\mu_k}{C_\tau}, \frac{R_B(n,\tau,\xi')}{C_\tau} \right\}.$$

Proof. See Appendix. □

2.3 Type-I hybrid censoring

When the random sample is coming from a Type-I hybrid censoring scheme, the PDF of the absolutely continuous part of the Bayes estimator $\hat{\theta}_B$ given λ is $h(.|\lambda)$, which is given in the following lemma.

Lemma 2. *The PDF $h(.|\lambda)$ of $\widehat{\theta}_B$ given λ under a Type-I hybrid censoring scheme is given by*

$$h(x|\lambda) = \frac{1}{1-p}\Bigg[\sum_{m=1}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j e^{-\lambda\tau(n-m+j)}\pi\big(x-\tau_{j,m,a,b};m,(m+a)\lambda\big) + \pi\Big(x-\frac{b}{r+a};r,(r+a)\lambda\Big) + r\binom{n}{r}\sum_{j=1}^{r}\binom{r-1}{j-1}\frac{(-1)^j e^{-\lambda\tau(n-r+j)}}{n-r+j} \times \pi\big(x-\tau_{j,r,a,b};r,(r+a)\lambda\big)\Bigg],$$

for $0 < x < \frac{b}{a} + \frac{n\tau}{a}$, where $\tau_{j,m,a,b} = \frac{b}{m+a} + \frac{(n-m+j)\tau}{m+a}$ and $\pi(\cdot)$ is as defined in (5).

Proof. It can be easily obtained using the result of Childs et al. [4]. □

Then in the expression (9), the distribution of $\hat{\theta}_B$ given λ and prior density (5) of λ is used to obtain the Bayes risk of DSP (n,r,τ,ξ). The following theorem provides the Bayes risk of DSP.

Theorem 3. *The Bayes risk of DSP (n,r,τ,ξ) w.r.t the loss function (8) is given as follows*

$$R_B(n,r,\tau,\xi) = n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + a_0 + a_1\mu_1 + \ldots + a_k\mu_k + \sum_{l=0}^{k} C_l \frac{b^a}{\Gamma(a)}\Bigg\{\frac{\Gamma(a+l)}{(b+n\tau)^{(a+l)}} I_{(\frac{b+n\tau}{a}<\xi)} + \sum_{m=1}^{r-1}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j}(-1)^j R_{l,j,m} + R_{l,r-n,r} + \sum_{j=1}^{r}\binom{n}{r}\binom{r-1}{j-1}(-1)^j \frac{r}{(n-r+j)} R_{l,j,r}\Bigg\},$$

where $I_{(\frac{b+n\tau}{a}<\xi)}$ is an indicator function as defined before, and the expressions for $E(M)$, $E(\tau^*)$, C_l and $R_{l,j,m}$ are provided in the proof of the theorem.

Proof. See Appendix. □

Based on the explicit expression of the Bayes risk, an optimum DSP (n_0,r_0,τ_0,ξ_0) can be determined by

$$R_B(n_0,r_0,\tau_0,\xi_0) = \min_{n}\{\min_{r\le n}\{\min_{\tau,\xi}[R_B(n,r,\tau,\xi)]\}\}. \tag{12}$$

In this case also the Bayes risk expression is very complicated. So we present a similar algorithm as given in Section 2.2 to obtain the optimum DSP (n_0,r_0,τ_0,ξ_0).

Algorithm B:

1. Fix n, r and τ, minimize $R_B(n,r,\tau,\xi)$ with respect to ξ using a grid search method. Denote the minimum Bayes risk by $R_B(n,r,\tau,\xi_0(n,r,\tau))$.
2. For fixed n and r, minimize $R_B(n,r,\tau,\xi_0(n,r,\tau))$ with respect to τ using the grid search method and denote the minimum Bayes risk by
$$R_B(n,r,\tau_0(n,r),\xi_0(n,r,\tau_0(n,r))).$$
3. For fixed n, choose $r \leq n$ for which $R_B(n,r,\tau_0(n,r),\xi_0(n,r,\tau_0(n,r)))$ is minimum and denote it by
$$R_B(n,r_0(n),\tau_0(n,r_0(n)),\xi_0(n,r_0(n),\tau_0(n,r_0(n)))).$$
4. Choose sample size n_0 such that
$$\begin{aligned} &R_B(n_0,r_0(n_0),\tau_0(n_0,r_0(n_0)),\xi_0(n_0,r_0(n_0),\tau_0(n_0,r_0(n_0)))) \\ &\quad \leq R_B(n,r_0(n),\tau_0(n,r_0(n)),\xi_0(n,r_0(n),\tau_0(n,r_0(n)))) \;\; \forall \;\; n \geq 0. \end{aligned}$$

For each sample size n and for given values of r and ξ, Bayes risk $R_B(n,r,\tau,\xi)$ is a function of τ. If we have to find an optimum DSP, we need an upper bound of τ. Tsai et al. [19] suggested to choose suitable range of τ, say $[0,\tau_\alpha]$, where τ_α is such that

$$\begin{aligned} P(0 < X < \tau_\alpha) &= 1-\alpha \\ \int_0^\infty \int_0^{\tau_\alpha} \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-\lambda b} \lambda e^{-\lambda x} dx d\lambda &= 1-\alpha \\ 1 - \left(1+\frac{\tau_\alpha}{b}\right)^{-a} &= 1-\alpha, \end{aligned}$$

hence

$$\tau_\alpha = b\left(\alpha^{-\frac{1}{a}} - 1\right), \tag{13}$$

where α is a preassigned number satisfying $0 < \alpha < 1$. The choice of α depends on the prescribed precision. Higher the precision required, smaller the value of α should be chosen. In the range $[0,\tau_\alpha]$ we use grid search to find the optimal value of τ. The next theorem shows that Algorithm B is finite, i.e., we can find an optimal sampling plan in a finite number of search steps.

Theorem 4. *Assuming $0 < \xi \leq \xi^*$, let us denote $R_B(n,r,\tau,\xi') = min_{\xi \leq \xi^*} R_B(n,r,\tau,\xi)$ for some fixed n (≥ 1) and τ. Further, let n_0 be the optimal sample size. Then,*

$$n_0 \leq \min\left\{\frac{C_r}{C_s-r_s}, \frac{a_0+a_1\mu_1+\ldots+a_k\mu_k}{C_s-r_s}, \frac{R_B(n,r,\tau,\xi')}{C_s-r_s}\right\}$$

and $r_0 \leq n_0$.

Proof. Proof is similar to Theorem 2. □

3 Comparisons with existing Bayesian sampling plan

Lin et al. [18] proposed the BSP (n_B, τ_B, δ_B) for Type-I censoring and Liang and Yang [14] proposed BSP $(n_B, r_B, \tau_B, \delta_B)$ for Type-I hybrid censoring using a quadratic loss function. In this section we show that for more general loss functions, the optimum DSP can be obtained without any additional effort as compared to BSP.

3.1 Higher degree polynomial loss function

In Section 3 we have shown that for a higher degree polynomial loss function, Bayes risk of the DSP can be easily computed for Type-I and Type-I hybrid censoring. In this section we show that the Bayes risk of BSP for higher degree polynomial loss function cannot be expressed in explicit form. The optimal sampling plans obtained are "approximate optimal" plans if $g(\lambda)$, cost due to acceptance, is an approximation of the true acceptance cost. So it is necessary to have a better approximation of the true cost for better results. Therefore, it is meaningful to study what happens when $g(\lambda)$ is a higher degree polynomial loss function. Bayes decision function (see, Lin et al. [18] and Liang and Yang [14]) for BSP is given as

$$\delta_B(\mathbf{x}) = \begin{cases} 1, & \text{if } \phi_\pi(m,z) \le C_r \\ 0, & otherwise, \end{cases}$$

where for Type-I censoring

$$z = \sum_{i=1}^{m} x_i + (n-m)\tau,$$

and for Type-I hybrid censoring

$$z = \begin{cases} \sum_{i=1}^{m} x_i + (n-m)\tau, & \text{if } m = 1,2\ldots r-1 \\ \sum_{i=1}^{r} x_i + (n-r)x_r, & \text{if } m = r, \end{cases}$$

with

$$\phi_\pi(m,z) = \int_0^\infty g(\lambda)\pi(\lambda|m,z)d\lambda.$$

Since the prior distribution of λ is $G(a,b)$, it is well known that the posterior distribution of λ is also gamma, viz.,

$$\pi(\lambda|m,z) \sim G(m+a, z+b).$$

Now when $g(\lambda) = a_0 + a_1\lambda + \ldots + a_k\lambda^k$ in loss function (7) then,

$$\phi_\pi\big(m,z\big) = \int_0^\infty g(\lambda)\pi\big(\lambda|m,z\big)d\lambda = a_0 + \sum_{j=1}^{k} a_j \frac{(m+a)\ldots(m+a+j-1)}{(z+b)^j}.$$

Thus to find the closed form of the decision function we need to obtain the set

$$A = \{z;\ z \geq 0, \phi_\pi\big(m,z\big) \leq C_r\},$$

and to construct A, we need to obtain the set of $z \geq 0$, such that

$$h_1(z) = a_0 + \sum_{j=1}^{k} a_j \frac{(m+a)\ldots(m+a+j-1)}{(z+b)^j} \leq C_r, \tag{14}$$

which is equivalent to finding $z \geq 0$, such that,

$$h_2(z) = (C_r - a_0)\big(z+b\big)^k - \sum_{j=1}^{k} a_j(m+a)\ldots(m+a+j-1)\big(z+b\big)^{k-j} \geq 0. \tag{15}$$

Note that for $k = 2$,

$$h_2(z) = (C_r - a_0)\big(z+b\big)^2 - a_1(m+a)\big(z+b\big) - a_2(m+a)(m+a+1) \geq 0.$$

It can easily be shown that if $D_n(m)$ is the only real root or $D_n(m)$ is the maximum real root of $h_2(z) = 0$, then the Bayes decision function will take the following form:

$$\delta_B(\mathbf{x}) = \begin{cases} 1, & \text{if } \ z \geq D_n(m) - b \\ 0, & otherwise. \end{cases} \tag{16}$$

It is well known that there is no algebraic solution to polynomial equations of degree five or higher (see chapter 5, Herstein [7]). So closed form of the Bayes risk of BSP cannot be determined for five or higher degree polynomial loss functions. This difficulty does not arise in computation of the proposed optimum DSP, because the decision function does not depend on the form of loss function. So for higher degree polynomial loss functions, applying the DSP is easier than the BSP.

3.2 Non-polynomial loss function

In Section 3.1 we discussed the limitations of BSP for polynomial loss function where DSP can be obtained quite easily. In this section, we discuss that if we have a non-polynimial loss function, then the obtaining DSP is easier as compared to BSP. To illustrate this, we consider a very simple non-polynomial loss function

$$L(\delta(\mathbf{x}),\lambda)=\begin{cases} nC_s-(n-M)r_s+\tau C_\tau+a_0+a_1\lambda+a_2\lambda^{5/2}, & \text{if } \delta(\mathbf{x})=1 \\ nC_s-(n-M)r_s+\tau C_\tau+C_r, & \text{if } \delta(\mathbf{x})=0, \end{cases} \quad (17)$$

where $g(\lambda)=a_0+a_1\lambda+a_2\lambda^{5/2}$, which is an increasing function in λ. The Bayes risk of DSP for the non-polynomial loss function (17) is computed by similar a approach as in Section 2 and is given by :
For Type -I censoring

$$R_B(n,\tau,\xi)=n(C_s-r_s)+r_sE(M)+\tau C_\tau+a_0+a_1\mu_1+a_2\frac{\Gamma(a+\frac{5}{2})}{\Gamma(a)b^{\frac{5}{2}}}+\sum_{l=0}^{2}C_l\frac{b^a}{\Gamma(a)}$$
$$\times\left[\frac{\Gamma(a+p_l)I_{(\frac{b+n\tau}{a}<\xi)}}{(b+n\tau)^{(a+p_l)}}+\sum_{m=1}^{n}\sum_{j=0}^{m}(-1)^j\binom{n}{m}\binom{m}{j}\frac{\Gamma(a+p_l)I_{S^*_{j,m,a,b}}(m,a+p_l)}{((m+a)\tau_{j,m,a,b})^{a+p_l}}\right],$$

where the expression of $E(M)$ and $I_{S^*_{j,m,a,b}}(m,a+p_l)$ are given in the proof of the Theorem 1 in the Appendix. For Type -I hybrid censoring,

$$R_B(n,r,\tau,\xi) = n(C_s-r_s)+E(M)r_s+E(\tau^*)C_\tau+a_0+a_1\mu_1+a_2\frac{\Gamma(a+\frac{5}{2})}{\Gamma(a)b^{\frac{5}{2}}}$$
$$+\sum_{l=0}^{2}C_l\frac{b^a}{\Gamma(a)}\left[\frac{\Gamma(a+p_l)}{(b+n\tau)^{(a+p_l)}}I_{(\frac{b+n\tau}{a}<\xi)}+\sum_{m=1}^{r-1}\sum_{j=0}^{m}\binom{m}{n}\binom{m}{j}(-1)^jR_{p_l,j,m}\right.$$
$$\left.+R_{p_l,r-n,r}+\sum_{k=1}^{r}\binom{n}{r}\binom{r-1}{k-1}(-1)^k\frac{r}{(n-r+k)}R_{p_l,k,r}\right],$$

where the expressions for $E(M)$, $E(\tau^*)$, C_l and $R_{p_l,j,m}$ are defined in the proof of the Theorem 3 in the Appendix and

$$p_l=\begin{cases}0, & \text{if } l=0\\ 1, & \text{if } l=1\\ \frac{5}{2}, & \text{if } l=2.\end{cases}$$

Now, in case of BSP, when $g(\lambda)=a_0+a_1\lambda+a_2\lambda^{5/2}$, then by Section 3.1

$$\phi_\pi(m,z)=\int_0^\infty g(\lambda)\pi(\lambda|m,z)d\lambda=a_0+\frac{a_1(m+a)}{(z+b)}+\frac{a_2\Gamma(m+a+\frac{5}{2})}{\Gamma(m+a)(z+b)^{\frac{5}{2}}}.$$

So to find a closed form of the decision function we need to obtain the set A as defined in Section 3.1. Note that to construct A, we need to obtain the set of $z\geq 0$, such that

$$h_1(z)=a_0+\frac{a_1(m+a)}{(z+b)}+\frac{a_2\Gamma(m+a+\frac{5}{2})}{\Gamma(m+a)(z+b)^{\frac{5}{2}}}\leq C_r,$$

which is equivalent to finding $z\geq 0$, such that,

$$h_2(z) = (C_r - a_0)\Gamma(m+a)\big(z+b\big)^{\frac{5}{2}} - a_1(m+a)\Gamma(m+a)\big(z+b\big)^{\frac{3}{2}} - a_2\Gamma(m+a+\frac{5}{2}) \geq 0.$$

It is obvious that finding the closed form solution of the non polynomial equation $h_2(z) = 0$ is not possible. So, for the given very simple non-polynomial loss function the Bayes risk of BSP cannot be obtained analytically but the Bayes risk of DSP is obtained quite easily.

Hence, we see that in general BSP cannot be obtained easily for all type of functional forms of the loss function but we can obtain the DSP for such cases quite easily because we have to compute just the expected value of the loss function.

4 Numerical Results

In this section, we focus on comparing the optimum DSP with the sampling plan of Lam [12], Lin et al.[16, 17] and with the BSP proposed by Lin et al. [18] and Liang and Yang [14]. We also obtained the optimum DSP for fifth degree polynomial loss function and for non polynomial loss function proposed in Section 3.2. Let n_1^*, τ_b, ξ^* from Theorem 2, denote the upper bound for optimal value of n, τ and ξ under Type-I censoring, i.e,

$$0 \leq n_0 \leq n_1^*, \quad 0 \leq \tau_0 \leq \tau_b \quad and \quad 0 < \xi_0 \leq \xi^*.$$

Since τ and ξ are continuous and n is an integer, therefore, to obtain numerical results we take a grid size of 0.0125 for τ and 0.0015 for ξ. Then by applying the Algorithm A, we obtain the optimum DSP (n_0, τ_0, ξ_0) for Type-I censoring.

Similarly, let n_2^* and ξ^* from Theorem 4 denote the upper bound for optimal value of n and ξ under Type-I hybrid censoring, i.e,

$$0 \leq n_0 \leq n_2^*, \quad r_0 \leq n_0, \quad 0 \leq \xi_0 \leq \xi^*$$

and optimal value of $\tau \in [0, \tau_\alpha]$ where τ_α is determined by (13) for given value of a and b. Here we use $\alpha = 0.01$. To obtain numerical results we take a grid size of 0.0015 for ξ and for τ we take the grid size of 0.0125. Then we apply Algorithm B to obtain the optimum DSP for Type-I hybrid censoring. The results presented in Tables 1-7 are obtained by using a program written in *R*.

4.1 Numerical results for quadratic loss function

In Section 2 we have obtained the Bayes risk of DSP for a k^{th} order polynomial loss function. For comparison with Lam [12], Lin et al. [16, 17], Lin et al. [18] and Liang and Yang [14] we assume $k = 2$. They choose this value of k because a quadratic polynomial makes computation easy and straightforward.

For comparison with Lam [12] and Lin et al. [16, 17] we fix the values of coefficients $a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_s = 0.5,\ C_\tau = 0,\ r_s = 0$ and $C_r = 30$ which they have used to obtain the sampling plan and we take $\xi^* = 3$. In Table 1, comparison is shown by varying the hyper-parameters a and b and keeping others fixed. From

Table 1: Numerical comparison with sampling plan of Lam [12] and Lin et al.[16, 17] for different values of a and b.

Scheme	a	b	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0	a	b	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0
DSP	2.5	0.8	24.8419	4	1.3125	0.3255	1.5	0.8	16.5825	3	0.7000	0.3330
LAM			24.9367	3	0.7077	0.3539			16.6233	3	0.5262	0.2631
Lin et al.[16, 17]			24.9893	4	0.6808	0.3404			16.7533	3	0.5262	0.2631
DSP	2.5	1.0	21.7081	4	1.1125	0.3255	2.0	0.8	21.1398	4	1.1500	0.3270
LAM			21.7640	3	0.5483	0.2742			21.2153	3	0.6051	0.3026
Lin et al.[16, 17]			21.8515	4	0.5819	0.2910			21.2875	4	0.6051	0.3026
DSP	3.0	0.8	27.5581	3	1.1625	0.3270	2.5	0.6	27.7267	3	1.2125	0.3285
LAM			27.6136	3	0.8170	0.4085			27.7834	3	0.8537	0.4268
Lin et al.[16, 17]			27.6521	3	0.8170	0.4085			29.8193	3	0.8537	0.4268
DSP	3.5	0.8	29.2789	2	1.0125	0.3285	10.0	3.0	29.5166	2	0.7875	0.3165
LAM			29.2789	2	1.0037	0.5019			29.5166	2	0.7928	0.3964
Lin et al.[16, 17]			29.3642	2	1.0037	0.5019			29.5959	2	0.8194	0.4097

Table-1 it is clear that Bayes risk of the optimum DSP is less than or equal to the Bayes risk of the sampling plan of Lam [12] and Lin et al.[16, 17].

For comparison with the BSP proposed by Lin et al. [18] and Liang and Yang [14] we choose the same standard set of parameter values and coefficients used by Lin et al. [18] and Liang and Yang [14], which are as follows, for Type-I censoring, $a = 2.5,\ b = 0.8,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_s = 0.5,\ r_s = 0,\ C_t = 0.5,\ C_r = 30,\ \xi^* = 3$ and for Type-I hybrid censoring, $a = 2.5,\ b = 0.8,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_s = 0.5,\ r_s = 0.3,\ C_t = 0.5,\ C_r = 30,\ \xi^* = 3$. Comparison of the Bayes risk of the BSP and the optimum DSP is given in Tables 2 and 3 by varying a, b, C_s, C_t and C_r one at a time and keeping others fixed.

In Table 2, $R_B(n_B,\tau_B,\delta_B)$ denotes the Bayes risk of the BSP and $R_B(n_0,\tau_0,\xi_0)$ denotes the Bayes risk of the optimum DSP (n_0,τ_0,ξ_0). Similarly, in Table 3 the Bayes risk of the BSP is denoted by $R_B(n_B,r_B,\tau_B,\delta_B)$ and $R_B(n_0,r_0,\tau_0,\xi_0)$ denotes the Bayes risk of the optimum DSP (n_0,r_0,τ_0,ξ_0). In case of Type-I hybrid censoring, the Bayes risk of BSP includes a complicated integral which is computed by simulation techniques. So the Bayes risk for BSP in case of Type-I hybrid censoring is an approximation of the exact Bayes risk. From Tables 2 and 3 it is clear that performance of the optimum DSP is as good as the BSP.

In Type-I censoring, the Bayes risk of the DSP is a function of sampling plan (n,τ,ξ) in which n takes discrete values and others are continuous. Theorem 2 states

Table 2: Numerical comparison between DSP and BSP for Type-I censoring.

		BSP		DSP				BSP		DSP		
a	b	$R_B(n_B,\tau_B,\delta_B)$	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0	C_s	$R_B(n_B,\tau_B,\delta_B)$	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0
2.5	0.8	25.7777	25.2777	3	0.7250	0.3285	0.3	24.4279	24.4282	5	0.7750	0.3240
2.5	1.0	22.0361	22.0361	3	0.5625	0.3285	0.7	25.8777	25.8777	3	0.7250	0.3285
3.5	0.8	29.7131	29.7131	2	0.8125	0.3285	2.0	27.9535	27.9535	1	0.3750	0.4095
C_t							C_r					
0.1		24.9446	24.9446	4	0.9625	0.3255	20	19.3292	19.3293	2	0.8750	0.4275
0.7		25.4197	25.4197	3	0.7000	0.3285	50	32.2081	32.2085	5	0.5625	0.2400
2.0		26.1439	26.1439	4	0.3875	0.3285	100	35.5937	35.5937	0	0.0000	0.0000

Table 3: Numerical comparison between DSP and BSP for Type-I hybrid censoring.

		BSP[1]		DSP					BSP[1]		DSP			
a	b	$R_B(n_B,r_B,\tau_B,\delta_B)$	$R_B(n_0,r_0,\tau_0,\xi_0)$	n_0	r_0	τ_0	ξ_0	C_s	$R_B(n_B,r_B,\tau_B,\delta_B)$	$R_B(n_0,r_0,\tau_0,\xi_0)$	n_0	r_0	τ_0	ξ_0
2.5	1.0	21.6762	21.6772	4	4	0.5625	0.3270	0.5	24.9678	24.9625	4	4	0.7500	0.3255
2.5	0.8	24.9678	24.9625	4	4	0.7500	0.3255	0.7	25.5932	25.5953	3	3	0.7250	0.3285
3.0	0.8	27.7339	27.7370	3	3	0.8750	0.3270	2.0	27.7681	27.7681	1	1	0.3750	0.3975
C_t								C_r						
0.1		24.7344	24.7312	4	4	0.8375	0.3255	20	19.0392	19.0383	2	2	0.9375	0.4275
0.7		25.0531	25.0533	5	4	0.4500	0.3255	30	24.9678	24.9625	4	4	0.7500	0.3255
2.0		25.4595	25.4552	4	3	0.3750	0.3285	50	31.7324	31.7326	6	6	0.3625	0.2385

[1] Bayes risk of BSP is obtained by simulation.

that the optimal value of n is bounded above, so it is sufficient to provide the contour plot of Bayes risk with respect to τ and ξ in Fig. 1 for different values of n to show that it has unique minimum. We provide the plots for standard set of coefficients for which minimum Bayes risk is $R_B(3, 0.7250, 0.3285) = 25.2777$. It is clear from the plots that Bayes risk first decreases then increases as n increases which ensures that Bayes risk has a unique minimum w.r.t n, τ and ξ.

For Type-I hybrid censoring, the Bayes risk is a function of the sampling plan (n, r, τ, ξ) in which n and r take discrete values and others are continuous. Since optimal values of n and r are bounded above (Theorem 4), so for different values of n and r, we provide the contour plot of Bayes risk with respect to τ and ξ in Fig. 2. We provide the plots for standard set of coefficients for which minimum Bayes risk is $R_B(4, 4, 0.7500, 0.3255) = 24.9678$. In this case also the Bayes risk has unique minimum and as n increases the Bayes risk first decreases then increases. The contour plot can also be used for predicting the range which includes the optimal values of τ and ξ by which we can reduce significantly the effort in the search for the optimum plan.

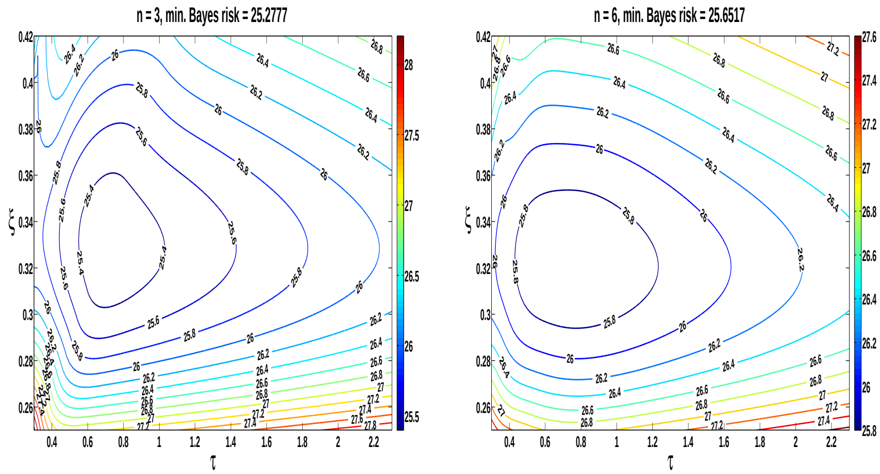

Fig. 1: Contour plot of Bayes risk with standard set of coefficient $a_0 = 2, a_1 = 2,_2 = 2, C_s = 0.5, r_s = 0, C_\tau = 0.5, C_r = 30$, and $a = 2.5, b = 0.8$ for Type-I censoring

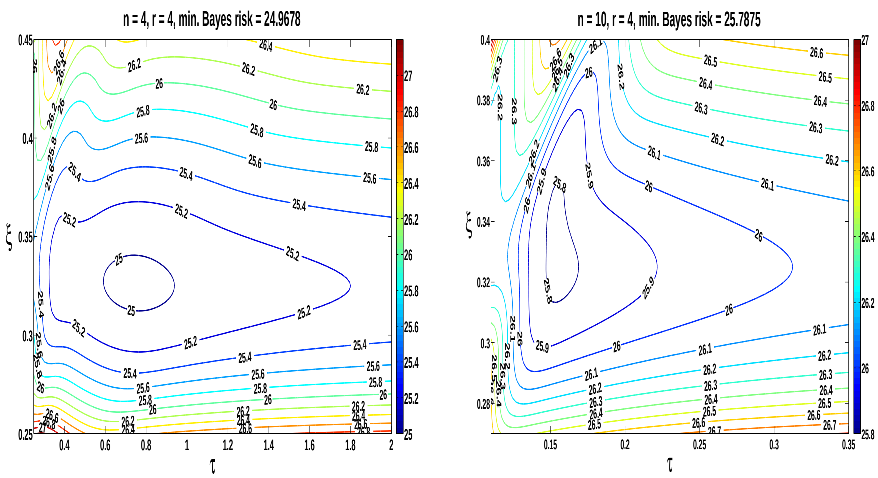

Fig. 2: Contour plot of Bayes risk with standard set of coefficients $a_0 = 2, a_1 = 2, a_2 = 2, C_s = 0.5, r_s = 0.3, C_\tau = 0.5, C_r = 30$, and $a = 2.5, b = 0.8$ for Type-I hybrid censoring

4.2 Numerical results for Fifth degree polynomial and Non polynomial loss function

For better approximation of optimal sampling plan we assume that loss function is of order $k = 5$ in (8). To obtain optimum DSP, following parameter and coefficient values are used: $a = 3.0,\ b = 2.5,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ a_3 = 2,\ a_4 = 2,\ a_5 = 2,\ C_s = 0.5,\ r_s = 0,\ C_\tau = 0.5,\ C_r = 30,\ \xi^* = 3$ for Type-I censoring. For Type-I hybrid censoring, we use following sets of coefficients: $a = 3.0,\ b = 2.5,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ a_3 = 2,\ a_4 = 2,\ a_5 = 2,\ C_s = 0.5,\ r_s = 0.3,\ C_\tau = 0.5,\ C_r = 30,\ \xi^* = 3$. Then varying the parameters a and b or one coefficient out of C_s, C_τ

Table 4: Minimum Bayes risk and corresponding optimum DSP for fifth degree polynomial loss function for Type-I censoring.

a	b	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0	C_s	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0
1.5	0.8	12.7301	3	1.1875	0.9750	0.4	24.2273	7	1.6750	0.8525
3.0	2.5	24.8613	6	1.7125	0.8650	0.7	25.8920	4	1.9875	0.9000
3.0	3.0	22.4152	6	1.5750	0.8650	1.0	26.9562	3	2.1125	0.9125
C_τ						C_r				
0.3		24.4827	6	2.1125	0.8625	30	24.8613	6	1.7125	0.8650
1.0		25.6103	6	1.3500	0.8750	50	33.0079	8	1.7375	0.7300
1.5		26.2238	6	1.0750	0.8850	100	45.7463	12	1.6000	0.5900

and C_r at a time, and setting others fixed, we obtain the minimum Bayes risk and the optimum DSP which is given in Table 4 and 5. We observe in Table 4 and 5 that as costs C_s, C_τ and C_r increase, the minimum Bayes risk increases. It is also observed that as cost per unit inspection C_s increases the optimal value of n_0 and r_0 decreases and the optimal value of τ_0 and ξ_0 increases. If rejection cost C_r increases then n_0

Table 5: Minimum Bayes risk and corresponding optimum DSP for fifth degree polynomial loss function for Type-I hybrid censoring.

a	b	$R_B(n_0,r_0,\tau_0,\xi_0)$	n_0	r_0	τ_0	ξ_0	C_s	$R_B(n_0,r_0,\tau_0,\xi_0)$	n_0	r_0	τ_0	ξ_0
1.5	0.8	26.2979	5	4	1.6375	0.9300	0.4	23.4093	10	7	0.9500	0.8450
3.0	2.5	24.2369	7	6	1.4125	0.8600	0.7	25.4201	5	4	1.3625	0.8900
3.0	3.0	20.7241	7	6	1.1875	0.8625	1.5	27.6478	2	2	1.9500	0.9475
C_τ							C_r					
0.1		23.7217	6	6	2.4125	0.8575	20	18.6112	4	3	1.4625	1.0275
1.0		24.6729	7	5	1.0250	0.8725	50	32.2362	10	9	1.2125	0.7225
1.5		25.0037	8	5	0.8125	0.8725	100	44.6304	14	13	1.0125	0.5875

and r_0 increases and the optimal value of τ_0 and ξ_0 decreases. Therefore, for fifth degree polynomial loss function results presented in Tables 4 and 5 have similar behaviour with those obtained from the quadratic loss function and they are quite acceptable in terms of n_0, r_0, τ_0 and ξ_0.

To obtain the optimum DSP for non polynomial loss function given in Section 3.2, we use the following standard set of parameter values and coefficients for Type-I censoring: $a = 2.5,\ b = 0.8,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_r = 30,\ C_s = 0.5,\ r_s = 0,\ C_\tau = 0.5,\ \xi^* = 3$. Similarly, for Type-I hybrid censoring: $a = 2.5,\ b = 0.8,\ a_0 = 2,\ a_1 = 2,\ a_2 = 2,\ C_r = 30,\ C_s = 0.5,\ r_s = 0.3,\ C_\tau = 0.5,\ \xi^* = 3$, i.e., the standard set of parameter values and coefficients are used. The minimum Bayes risk and optimum DSP is obtained in Tables 6 and 7 by varying parameters a and b or one coefficient out of C_s, C_τ and C_r at a time and setting others fixed. It is observed from Tables 6 and 7 that as costs C_s, C_τ and C_r increase, the minimum Bayes risk increases. The optimal values of n_0 and r_0 decrease when C_s increases and when C_r increases, then n_0 and r_0 increase. The value of ξ_0 increases when cost C_s increases and decreases when cost C_r increases. So, for given non polynomial loss function also, the results in Tables 6 and 7 are quite reasonable and acceptable.

Table 6: Minimum Bayes risk and corresponding optimum DSP for non polynomial loss function for Type-I censoring.

a	b	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0	C_s	$R_B(n_0,\tau_0,\xi_0)$	n_0	τ_0	ξ_0
2.5	0.8	27.5536	4	1.0625	0.4185	0.3	26.6463	6	0.9250	0.4110
2.5	1.0	24.9524	4	1.0000	0.4185	0.7	28.1649	3	1.0875	0.4245
3.0	0.8	29.6926	2	1.0750	0.4275	2.0	29.9411	1	0.7250	0.6000
C_t						C_r				
0.1		27.0722	4	1.3875	0.4170	30	27.5536	4	1.0625	0.4185
0.7		27.7546	4	0.9500	0.4200	50	38.4987	6	0.9250	0.3225
2.0		28.6323	4	0.5625	0.4230	100	54.6498	9	0.7250	0.2325

Table 7: Minimum Bayes risk and corresponding optimum DSP for non polynomial loss function for Type-I hybrid censoring.

a	b	$R_B(n_0,r_0,\tau_0,\xi_0)$	n_0	r_0	τ_0	ξ_0	C_s	$R_B(n_0,r_0,\tau_0,\xi_0)$	n_0	r_0	τ_0	ξ_0
2.5	1.0	24.5869	5	4	0.6000	0.4170	0.5	27.2090	4	4	1.2125	0.4170
2.5	0.8	27.2090	4	4	1.2125	0.4170	0.7	27.8214	3	3	1.1750	0.4230
3.0	0.8	29.3249	2	2	1.2000	0.4275	2.0	29.6794	1	1	0.7500	0.4530
C_t							C_r					
0.1		26.9124	4	4	1.3375	0.4170	20	19.7253	1	1	1.0750	0.5685
0.7		27.3031	5	4	0.6875	0.4170	30	27.2090	4	4	1.2125	0.4170
2.0		27.7698	5	3	0.4125	0.4230	50	38.1061	7	6	0.5750	0.3225

5 Conclusion

In this paper, we have shown that a sampling plan can be obtained using the Bayes estimator of the mean lifetime θ. This estimator always exists for both Type-I and Type-I hybrid censoring. We have developed a methodology for finding the DSP using a decision function which is based on the Bayes estimator of θ. We propose an algorithm to find the optimum DSP. It is shown numerically that the optimum DSP is better than the sampling plans of Lam [12] and Lin et al. [16, 17] and as good as the BSP proposed by Lin et al. [18] and Liang and Yang [14] for Type-I censoring and Type-I hybrid censoring. Further, we have generalized the existing work for higher degree polynomial loss function and for a specific choice of the non-polynomial loss function, which cannot be handled easily by the BSP. Thus, we see that the DSP is applicable to a wider class of loss functions than the BSP.

Acknowledgements The authors would like to thank the reviewers for their careful review and valuable suggestions which led to the improvement of the presentation of the paper.

Appendix

Proof of Theorem 1

To derive the Bayes risk we consider the decision function (6) and loss function (8) with $\tau^* = \tau$. Then from (9) Bayes risk under Type-I censoring is,

$$\begin{aligned} R_B(n,\tau,\xi) &= n(C_s - r_s) + r_s E(M) + \tau C_\tau + a_0 + a_1\mu_1 + \ldots + a_k\mu_k \\ &\quad + \sum_{l=0}^{k} C_l \frac{b^a}{\Gamma(a)} \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} P(\hat{\theta}_B < \xi|\lambda)\, d\lambda, \end{aligned} \tag{18}$$

where C_l is defined as

$$C_l = \begin{cases} C_r - a_l, & \text{if} \quad l = 0 \\ -a_l, & \text{if} \quad l = 1,2,\ldots,k. \end{cases} \tag{19}$$

Using (10) and Lemma 1 we get

$$\begin{aligned} &\int_0^\infty \lambda^{a+l-1} e^{-\lambda b} P(\hat{\theta}_B < \xi|\lambda)\, d\lambda \\ &= \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} p S_\lambda(\xi)\, d\lambda + \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} (1-p) H_\lambda(\xi)\, d\lambda \\ &= \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} e^{-n\lambda\tau}\, I_{\left(\frac{b+n\tau}{a} < \xi\right)}\, d\lambda + \sum_{m=1}^{n} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} (-1)^j \end{aligned}$$

$$\times \frac{(m+a)^m}{\Gamma(m)} \int_0^\infty \int_{\tau_{j,m,a,b}}^{\xi} \lambda^{a+l+m-1} e^{-\lambda\{(m+a)x\}} \left(x-\tau_{j,m,a,b}\right)^{m-1} dx\, d\lambda$$

$$= \frac{\Gamma(a+l)}{(b+n\tau)^{(a+l)}} I_{\left(\frac{b+n\tau}{a}<\xi\right)} + \sum_{m=1}^{n}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j$$

$$\times \frac{\Gamma(a+l+m)}{\Gamma(m)(m+a)^{(a+l)}} \int_0^{\xi-\tau_{j,m,a,b}} \frac{y^{m-1}}{\{\tau_{j,m,a,b}+y\}^{a+l+m}} dy.$$

$$= \frac{\Gamma(a+l)}{(b+n\tau)^{(a+l)}} I_{\left(\frac{b+n\tau}{a}<\xi\right)} + \sum_{m=1}^{n}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j$$

$$\times \frac{\Gamma(a+l)}{((m+a)\tau_{j,m,a,b})^{(a+l)}} \frac{\Gamma(a+l+m)}{\Gamma(m)\Gamma(a+l)} \int_0^{\frac{\xi-\tau_{j,m,a,b}}{\tau_{j,m,a,b}}} \frac{z^{m-1}}{(1+z)^{a+l+m}} dz.$$

Now taking a transformation $z = u/(1-u)$, we have

$$\int_0^{C^*_{j,m,a,b}} \frac{z^{m-1}}{(1+z)^{a+l+m}} dz = \int_0^{S^*_{j,m,a,b}} u^{m-1}(1-u)^{a+l-1} du = B_{S^*_{j,m,a,b}}(m, a+l),$$

where $C^*_{j,m,a,b} = \frac{\xi-\tau_{j,m,a,b}}{\tau_{j,m,a,b}}$, $S^*_{j,m,a,b} = \frac{C^*_{j,m,a,b}}{1+C^*_{j,m,a,b}}$ and

$$B_x(\alpha,\beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1} du, \quad 0 \le x \le 1, \tag{20}$$

is the incomplete beta function. Let us denote the cumulative distribution function of beta as $I_x(\alpha,\beta) = B_x(\alpha,\beta)/B(\alpha,\beta)$. Then Bayes risk is finally obtained as

$$R_B(n,\tau,\xi) = n(C_s - r_s) + r_s E(M) + \tau C_\tau + a_0 + a_1\mu_1 + \ldots + a_k\mu_k + \sum_{l=0}^{k} C_l \frac{b^a}{\Gamma(a)} \Bigg[\frac{\Gamma(a+l) I_{\left(\frac{b+n\tau}{a}<\xi\right)}}{(b+n\tau)^{(a+l)}} + \sum_{m=1}^{n}\sum_{j=0}^{m} (-1)^j \binom{n}{m}\binom{m}{j} \frac{\Gamma(a+l) I_{S^*_{j,m,a,b}}(m,a+l)}{((m+a)\tau_{j,m,a,b})^{a+l}} \Bigg], \tag{21}$$

where

$$E(M) = E_\lambda\{E(M|\lambda)\}.$$

Since for each $m = 1,2,\ldots,n$, probability mass function of M given λ is

$$P(M=m|\lambda) = \binom{n}{m}(1-e^{-\lambda\tau})^m (e^{-\lambda\tau})^{n-m} = \sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j e^{-(n-m+j)\lambda\tau},$$

therefore,

$$E(M) = \sum_{m=1}^{n}\sum_{j=0}^{m} m \binom{n}{m}\binom{m}{j}(-1)^j \frac{b^a}{(b+(n-m+j)\tau)^a}.$$

Proof of Theorem 2

Note that the Bayes risk can be written as,

$$R_B(n,\tau,\xi) = n(C_s - r_s) + \tau C_\tau + E(M) r_s \\ + E_\lambda \big\{ (a_0 + a_1\lambda + \ldots + a_k\lambda^k) P(\hat{\theta}_B \geq \xi|\lambda) + C_r P(\hat{\theta}_B < \xi|\lambda) \big\}.$$

Now we know that $a_0 + a_1\lambda + \ldots + a_k\lambda^k \geq 0$ and C_r the rejection cost is non negative. Therefore, if (n_0,τ_0,ξ_0) is the optimal sampling plan then the corresponding Bayes risk is

$$R_B(n_0,\tau_0,\xi_0) \geq n_0(C_s - r_s) + \tau_0 C_\tau. \tag{22}$$

Now when $\xi = \infty$ we reject the batch without sampling and the corresponding Bayes risk is given by $R_B(0,0,\infty) = C_r$. When $\xi = 0$ we accept the batch without sampling and corresponding Bayes risk is given by $R_B(0,0,0) = a_0 + a_1\mu_1 + \ldots + a_k\mu_k$. Then the optimal Bayes risk is

$$R_B(n_0,\tau_0,\xi_0) \leq min\big\{R_B(0,0,0), R_B(0,0,\infty), R_B(n,\tau,\xi')\big\}. \tag{23}$$

Hence from equations (22) and (23) we have

$$n_0(C_s - r_s) + \tau_0 C_\tau \leq min\big\{R_B(0,0,0), R_B(0,0,\infty), R_B(n,\tau,\xi')\big\},$$

from where it follows that

$$n_0 \leq min\left\{\tfrac{C_r}{C_s - r_s}, \tfrac{a_0 + a_1\mu_1 + \ldots + a_k\mu_k}{C_s - r_s}, \tfrac{R_B(n,\tau,\xi')}{C_s - r_s}\right\},$$
$$\tau_0 \leq min\left\{\tfrac{C_r}{C_\tau}, \tfrac{a_0 + a_1\mu_1 + \ldots + a_k\mu_k}{C_\tau}, \tfrac{R_B(n,\tau,\xi')}{C_\tau}\right\}.$$

Proof of Theorem 3

To derive the Bayes risk for Type-I hybrid censoring, we consider the decision function (6) and loss function (8) with $\tau^* = min\{X_{(r)}, \tau\}$. Then from (9)

$$R_B(n,r,\tau,\xi) = n(C_s - r_s) + E(M) r_s + E(\tau^*) C_\tau + a_0 + a_1\mu_1 + \ldots + a_k\mu_k \\ + \sum_{l=0}^{k} C_l \frac{b^a}{\Gamma(a)} \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} P(\widehat{\theta}_B \leq \xi|\lambda) d\lambda, \tag{24}$$

where μ_i and C_l are defined earlier. Let $\xi^* = min\{\frac{b}{a} + \frac{n\tau}{a}, \xi\}$, then define

$$R_{l,j,m} = \int_0^\infty \int_0^{\xi^*} \lambda^{a+l-1} e^{-\lambda(b+\tau(n-m+j))} \pi\Big(y - \tau_{j,m,a,b}; m, (m+a)\lambda\Big) dy\, d\lambda$$

$$
\begin{aligned}
&= \frac{(m+a)^m}{\Gamma(m)} \int_0^\infty \int_{\tau_{j,m,a,b}}^{\xi^*} \lambda^{a+l+m-1} e^{-\lambda\{(m+a)y\}} \big(y-\tau_{j,m,a,b}\big)^{m-1} dy\, d\lambda \\
&= \frac{(m+a)^m}{\Gamma(m)} \int_0^{\xi^*-\tau_{j,m,a,b}} \frac{v^{m-1}\Gamma(a+l+m)}{\big((m+a)\tau_{j,m,a,b}+(m+a)v\big)^{a+l+m}} dv \\
&= \frac{\Gamma(a+l+m)}{\Gamma(m)(m+a)^{a+l}\tau_{j,m,a,b}^{a+l+m}} \int_0^{\xi^*-\tau_{j,m,a,b}} \frac{v^{m-1}}{\Big(1+\frac{v}{\tau_{j,m,a,b}}\Big)^{a+l+m}} dv \\
&= \frac{\Gamma(a+l)}{((m+a)^{a+l}\tau_{j,m,a,b})^{a+l}} \frac{\Gamma(a+l+m)}{\Gamma(m)\Gamma(a+l)} \int_0^{\frac{\xi^*-\tau_{j,m,a,b}}{\tau_{j,m,a,b}}} \frac{z^{m-1}}{(1+z)^{a+l+m}} dz.
\end{aligned}
$$

Taking similar transformation as in Type-I censoring and using the incomplete beta function, we obtain

$$
R_{l,j,m} = \frac{\Gamma(a+l)}{((m+a)\tau_{j,m,a,b})^{a+l}} I_{S^*_{j,m,a,b}}(m,a+l). \tag{25}
$$

Using (10), Lemma 2 in (24) and relation (25) we get

$$
\begin{aligned}
&\int_0^\infty \lambda^{a+l-1} e^{-\lambda b} P(\widehat{\theta}_B \le \xi|\lambda) d\lambda \\
&= \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} p S_\lambda(\xi)\, d\lambda + \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} (1-p) H_\lambda(\xi)\, d\lambda \\
&= \int_0^\infty \lambda^{a+l-1} e^{-\lambda b} e^{-n\lambda\tau}\, I_{(\frac{b+n\tau}{a}<\xi)}\, d\lambda + \sum_{m=1}^{r-1}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j \\
&\quad \times \int_0^\infty \int_0^{\xi^*} \lambda^{a+l-1} e^{-\lambda\{b+\tau(n-m+j)\}} \pi\Big(y-\tau_{j,m,a,b}; m, (m+a)\lambda\Big) dy\, d\lambda \\
&+ \int_0^\infty \int_0^{\xi^*} \lambda^{a+l-1} e^{-\lambda b} \pi\Big(y-\frac{b}{r+a}; r, (r+a)\lambda\Big) dy\, d\lambda + \sum_{j=1}^{r} \binom{n}{r}\binom{r-1}{j-1} \\
&\quad \times \frac{(-1)^j r}{(n-r+j)} \int_0^\infty \int_0^{\xi^*} \lambda^{a+l-1} e^{-\lambda\{b+\tau(n-r+j)\}} \pi\Big(y-\tau_{j,r,a,b}; r, (r+a)\lambda\Big) dy\, d\lambda \\
&= \frac{\Gamma(a+l)}{(b+n\tau)^{(a+l)}} I_{(\frac{b+n\tau}{a}<\xi)} + \sum_{m=1}^{r-1}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j R_{l,j,m} + R_{l,r-n,r} \\
&\qquad\qquad + \sum_{j=1}^{r} \binom{n}{r}\binom{r-1}{j-1}(-1)^j \frac{r}{(n-r+j)} R_{l,j,r}.
\end{aligned}
$$

Thus Bayes risk of DSP (n,r,τ,ξ) under Type-I hybrid censoring is given by

$$
\begin{aligned}
&R_B(n,r,\tau,\xi) = n(C_s-r_s) + E(M)r_s + E(\tau^*)C_\tau + a_0 + a_1\mu_1 + \ldots + a_k\mu_k \\
&+ \sum_{l=0}^{k} C_l \frac{b^a}{\Gamma(a)} \Bigg\{ \frac{\Gamma(a+l)}{(b+n\tau)^{(a+l)}} I_{(\frac{b+n\tau}{a}<\xi)} + \sum_{m=1}^{r-1}\sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j R_{l,j,m} + R_{l,r-n,r}
\end{aligned}
$$

$$+\sum_{j=1}^{r}\binom{n}{r}\binom{r-1}{j-1}(-1)^j\frac{r}{(n-r+j)}R_{l,j,r}\Bigg\},\qquad (26)$$

where

$$E(M) = \sum_{m=1}^{r-1}\sum_{j=0}^{m} m\binom{n}{m}\binom{m}{j}\frac{(-1)^j b^a}{(b+(n-m+j)\tau)^a} + \sum_{i=r}^{n}\sum_{j=0}^{i} r\binom{n}{i}\binom{i}{j}\frac{(-1)^j b^a}{(b+(n-i+j)\tau)^a},$$

$$E(\tau^*) = r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\Bigg\{\frac{b}{(n-j)^2(a-1)} - \frac{tb^a}{(n-j)((n-j)\tau+b)^a} - \frac{b^a}{(n-j)^2(a-1)((n-j)\tau+b)^{a-1}}\Bigg\} + \sum_{i=r}^{n}\sum_{j=0}^{i}\tau\binom{n}{i}\binom{i}{j}(-1)^j\frac{b^a}{(b+(n-i+j)\tau)^a}.$$

For computation of $E(M)$ and $E(\tau^*)$ see Liang and Yang [14].

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