Bayesian Sampling Plan for the Exponential Distribution with Generalized Type - II Hybrid Censoring Scheme

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Abstract

In this paper, a decision-theoretic approach is used to obtain the Bayesian sampling plan (BSP) for the generalized Type-II hybrid censoring scheme when lifetimes of sampled units follow a one-parameter exponential distribution. An efficient loss function is used to decide whether to accept or reject the batch. The BSP is obtained by constructing the closed form of the Bayes decision function. It is observed that the closed form of the Bayes decision function cannot be obtained analytically for any arbitrary loss function. A numerical approach is proposed to determine the optimum BSP for an arbitrary loss function. To illustrate this, we consider a higher degree polynomial loss function and obtain the optimum BSP numerically. Some numerical studies have been conducted to check the performance of the optimum BSP in different cases.

Keywords and Phrases: Bayesian sampling plan, Bayes risk, Decision-theoretic approach, Exponential distribution, Generalized Type-II hybrid censoring.

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1 Introduction

Quality control is an important aspect for manufacturers as it can directly affect their product sales and profits. An acceptance sampling plan is a technique used in the industry by which a decision of acceptance or rejection of a batch is made. An acceptance sampling plan is further characterized in two categories viz., attribute sampling plan and variable sampling plan. Various schemes have been proposed to choose the optimal sampling plan. In a decision-theoretic approach, a sampling plan is determined by making an optimal decision based on some realistic criterion such as maximizing the return or minimizing the risk. So, for an economic point of view, it is more reasonable and is, therefore, widely employed in most of the statistical analyses procedures. Some of the relevant work along these lines are by Hald (1967), Fertig and Mann (1974), Lam (1988), Lam (1994), Lin *et al.* (2002), Huang and Lin (2002), Chen *et al.* (2004), Huang and Lin (2004), Lin *et al.* (2008, 2010), Liang and Yang (2013), Tsai *et al.* (2014), Yang *et al.* (2017) and the reference cited therein.

Most of the work among these are based on an estimator of the mean lifetime and a loss function, which includes the cost per item, cost on time and the cost depends on the decision made. However, if the estimator is not efficient, then it affects the optimal sampling plan which leads to a wrong decision. Therefore, it is reasonable that the cost on the efficiency of an estimator should be considered in the loss function. See, for example, Bhattacharya *et al.* (2014) in this respect. In any life-testing experiment, the quality of a batch is measured by the lifetime of the items, which need to be estimated based on a random sample. Moreover, in most of the life-testing experiments, observations are often censored, i.e., the sampled units are partially observed. Most common and popular censoring schemes are Type-I and Type-II censoring. The mixture of Type-I and Type-II censoring schemes is known as the hybrid censoring scheme.

Now we briefly describe the different hybrid censoring schemes and their generalizations. Suppose $X_1, X_2, \ldots, X_n$ are the lifetimes of sampled units, assumed to be independent and identically distributed (i.i.d.) random variables. The ordered lifetimes are denoted by $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$. In a Type-I hybrid censoring, the experiment is terminated at time $\tau^* = min\{X_{(r)}, \tau\}$, where $X_{(r)}$ stands for the time of the r^{th} failure out of n items for a pre-fixed positive integer $r \leq n$ and τ is a prefixed censoring time. In Type-II hybrid censoring, the experiment is terminated at time $\tau^* = max\{X_{(r)}, \tau\}$. Type-I hybrid censoring has a prefixed maximum experimental time, and it may so happen that very few failures occur during that time, and this affects the efficiency of the estimators. On the other hand, Type-II hybrid censoring ensures r number of failures, but it may take a long time to observe those r failures. Readers are referred to Balakrishnan and Kundu (2013), Gupta and Kundu (1998) and Childs *et al.* (2003) for more information on different hybrid censoring schemes. To overcome the shortcoming of the Type-II hybrid censoring, Chandrasekar *et al.* (2004), proposed the generalized Type-II hybrid censoring scheme (Type-II GHCS). In Type-II GHCS, for a fixed integer $r \in \{1, 2, \ldots, n\}$ and time points $\tau_1, \tau_2 \in (0, \infty)$, with $\tau_1 < \tau_2$, if the r^{th} failure occurs before the time point τ_1, the experiment is terminated at τ_1. If the r^{th} failure occurs between τ_1 and τ_2, the experiment is terminated at $X_{(r)}$. Finally, if the r^{th} failure occurs after τ_2, the experiment is terminated at τ_2. This censoring guarantees that the experiment will be completed by time τ_2.

During the last decade, many authors have designed sampling plans based on censored samples. The optimal sampling plans using a decision-theoretic approach for exponential distribution based on Type-I and Type-II censored samples have been studied by Lam (1990, 1994). Chen *et al.* (2004), Lin *et al.* (2008, 2010), have designed optimal sampling plans when the data are hybrid censored. Among the different decision-theoretic approaches, the BSP is attractive as it minimizes the Bayes risks within a class of sampling plans. For Type-I censored sample Lin *et al.* (2002) have obtained the BSP. Liang and Yang (2013) and Yang *et al.* (2017) have obtained BSP for Type-I, and Type-II hybrid censored samples. Tsai *et al.* (2014) obtained an efficient BSP for the exponential lifetime distribution with Type-I censored samples. Curtailed BSP for exponential distribution based on Type-II censored samples are obtained by Chen *et al.* (2017). Recently, Prajapati *et al.* (2020,

2019) have proposed new decision-theoretic sampling plans (DSPs) for Type-I and Type-I hybrid censored samples which are as good as the BSP in the respective cases. They have also shown that the DSP can be obtained for a larger class of loss functions quite conveniently as compared to the BSP.

So far, the BSP based on Type-II GHCS sample has not been studied in the literature. The goal here is to investigate the problem of designing BSP with Type-II GHCS sample. In this paper, our focus is on obtaining the optimum BSP for Type-II GHCS, when lifetimes of items follow a one-parameter exponential distribution. An efficient loss function is used to construct the Bayes decision function, which includes the cost on mean square error (MSE), cost on per item, cost on time, cost on the action, etc. The BSP is obtained by minimizing the Bayes risk. For Type-II GHCS, maximum likelihood estimator (MLE) may not always exist. The Prajapati *et al.* (2019) propose a suitable estimator which always exists. The Bayes decision function based on the suitable estimator has been obtained, which give advantage in calculation of Bayes risk. The optimum BSP is then obtained by minimizing the Bayes risk for a quadratic loss function. It is noticed that for a general form of the loss function, the Bayes decision function cannot be constructed analytically in a closed form. Therefore, numerical approach is proposed to obtain the Bayes decision function for any arbitrary loss function. For illustration, we consider the higher degree polynomial loss function to obtain the optimum BSP.

The rest of the paper is organized as follows. In Section 2, we propose the decision function based on an estimator of the parameter of the lifetime distribution under Type-II GHCS and a well- defined loss function to obtain the BSP. The Bayes decision function and the Bayes risk of the BSP for quadratic loss are provided in Section 3. In Section 4, we discuss the limitation of BSP and present a finite algorithm to obtain the optimum BSP for a higher degree polynomial loss function. Numerical results for the optimum BSP are provided in Section 5. Finally, we conclude the paper in Section 6. All derivations have been provided in the Appendix.

2 Model Construction and Assumptions

Suppose that the quality of an item is measured by its lifetime and we are given a batch of items to construct the acceptance sampling plan. To analyse the quality of batch of items, a sample of n items is put on life test to observe their failure times. Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of these n items, and it is noted that failed items are not replaced. It is assumed that $X_1, X_2, \ldots, X_n$ are mutually independent and identically follow an exponential distribution with parameter λ with the following density:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \ \lambda > 0,$$

and zero, otherwise. The mean lifetime of an item i.e., $\frac{1}{\lambda}$ is denoted by θ. In Type-II GHCS, let the experiment gets terminated at random time τ^*. Let M_1 and M_2 be the number of failures among n items up to time τ_1 and τ_2, respectively. Then the MLE of λ is given by:

$$\begin{aligned}\widehat{\lambda}_{MLE} &= \begin{cases} \frac{M_1}{\sum_{i=1}^{M_1} X_{(i)} + (n-M_1)\tau_1}, & \text{if} \quad M_1 = r, r+1, \ldots, n \\ \frac{r}{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}, & \text{if} \quad M_1 = 0, 1, \ldots, r-1 \ \& \ M_2 = r \\ \frac{M_2}{\sum_{i=1}^{M_2} X_{(i)} + (n-M_2)\tau_2}, & \text{if} \quad M_2 = 1, \ldots, r-1, \end{cases} \\ &= \frac{M}{\sum_{i=1}^{M} X_{(i)} + (n-M)\tau^*} = \frac{M}{Y(n, r, \tau_1, \tau_2, M)}, \ \forall\, M \geq 1,\end{aligned}$$

where $Y(n, r, \tau_1, \tau_2, M) = \sum_{i=1}^{M} X_{(i)} + (n-M)\tau^*$. Clearly for $M = M_2 = 0$, the MLE does not exist. In place of MLE of λ, we define an estimator which is as follows:

$$\widehat{\lambda} = \begin{cases} 0, & \text{if} \quad M = 0 \\ \widehat{\lambda}_{MLE}, & \text{if} \quad M > 0. \end{cases} \tag{1}$$

It is observed that $\widehat{\lambda}$ has a discrete component at $M = 0$ and a continuous component for $M \geq 1$. Thus, the distribution function of $\widehat{\lambda}$ can be written as

$$P(\widehat{\lambda} \leq x|\lambda) = P(\widehat{\lambda} \leq x|M = 0, \lambda)P(M = 0) + P(\widehat{\lambda} \leq x|M \geq 1, \lambda)P(M \geq 1)$$

$$= I(0 \le x)e^{-n\lambda\tau_2} + P(\widehat{\lambda} \le x | M \ge 1, \lambda)P(M \ge 1)$$
$$= pS(x) + (1-p)H(x), \quad (2)$$

where $p = P(M=0) = e^{-n\lambda\tau_2}$ and

$$S(x) = \begin{cases} 1, & \text{if } x \ge 0 \\ 0, & otherwise, \end{cases} \quad \text{and} \quad H(x) = \begin{cases} \int_0^x h(u)du, & \text{if } \frac{1}{n\tau_2} \le x < \infty \\ 0, & otherwise, \end{cases}$$

$h(u)$ is the PDF of the absolutely continuous part of the distribution function of $\widehat{\lambda}$ which is given in Lemma 2.1.

Lemma 2.1. *The PDF of $\widehat{\lambda}$ under Type-II GHCS when $M \ge 1$, is given by:*

$$\begin{aligned} h(y) &= \frac{1}{1-p}\Bigg[\sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}(-1)^j\frac{e^{-\lambda(n-m_1+j)\tau_1}}{y^2}f_{SG}\Big(\frac{1}{y}-\frac{(n-m_1+j)\tau_1}{m_1};\ m_1,\ m_1\lambda\Big)+\sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1} \\ &\times\sum_{i=0}^{k}\binom{n}{m_1}\binom{n-m_1}{k}\binom{m_1}{j}\binom{k}{i}(-1)^{i+j}\frac{e^{-\lambda[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}}{y^2}f_{SG}\Big(\frac{1}{y}-\frac{[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}{r};\ r,\ r\lambda\Big) \\ &+\sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}(-1)^j\frac{e^{-\lambda(n-m_2+j)\tau_2}}{y^2}f_{SG}\Big(\frac{1}{y}-\frac{(n-m_2+j)\tau_2}{m_2};\ m_2,\ m_2\lambda\Big)\Bigg], \end{aligned}$$

where $p = e^{-n\lambda\tau_2}$ and

$$f_{SG}(y|a,b,c) = \begin{cases} \frac{b^a}{\Gamma(a)}(y-c)^{a-1}e^{-b(y-c)}, & \text{if } y > c \\ 0, & otherwise. \end{cases}$$

Proof. It can be easily obtained using the result of Chandrasekar *et al.* (2004). ∎

Suppose $\delta(\mathbf{X}) = 1$ denotes an action, to accept the batch, and $\delta(\mathbf{X}) = 0$ denotes the action to reject the batch. Then the decision function is defined as:

$$\delta(\mathbf{X}) = \begin{cases} 1, & \text{if } T(\mathbf{X}) \in A \\ 0, & otherwise, \end{cases} \quad (3)$$

where A denotes the acceptance region and $T(\mathbf{X})$ is a suitable estimator of the parameter λ. It is clear that the decision is taken, based on the estimator $T(\mathbf{X})$. In most of the cases, the number of failures observed in the experiment is random, and if significantly low number of failures is observed, it may lead to poor estimate of the estimator $T(\mathbf{X})$, which affects our final decision taken on a batch of items.

We use a loss function incorporating the costs corresponding to sample size, duration of the experiment, action taken on batch and also the MSE of the estimator to deal with the imprecision in the decision. Such a loss function is considered by Bhattacharya *et al.* (2014) to obtain optimum life testing plans. We consider the loss function used by Liang and Yang (2013) and Yang *et al.* (2017), and additionally include the MSE of the estimator $T(\mathbf{X})$ in the cost function. We have considered the following loss function:

$$L(\delta(\mathbf{X}),\lambda) = \begin{cases} nC_s - (n-M)r_s + \tau^* C_\tau + MSE(T(\mathbf{X}))C_v + g(\lambda), & \text{if } \delta(\mathbf{X}) = 1 \\ nC_s - (n-M)r_s + \tau^* C_\tau + MSE(T(\mathbf{X}))C_v + C_r, & \text{if } \delta(\mathbf{X}) = 0, \end{cases} \tag{4}$$

where n is the sample size, M denotes the number of failures in the experiment, τ^* is the duration of the experiment, $MSE(T(\mathbf{X}))$ is the MSE of the estimator $T(\mathbf{X})$, C_s is the inspection cost per item, C_τ is the cost per unit time for conducting the experiment, C_v is the cost associated with the imprecision in the estimator $T(\mathbf{X})$ and C_r is the cost of rejecting the batch. Further $g(\lambda)$ is the cost on accepting the batch, which is positive and an increasing function in λ for $\lambda > 0$. If an item does not fail, it can be reused with a salvage value r_s such that $C_s > r_s \geq 0$. It may be noted that the precision of an estimator is an important component in making any decision. Therefore, it depends on the practitioner to choose C_v compared to other costs. For example, if a practitioner feels that to him/her the precision of the estimator is very important to make a decision whether to accept or reject a batch, then C_v should be large, otherwise it can be small.

Further, under this set up, the sampling plan is given by $(n, r, \tau_1, \tau_2, \delta)$ using the decision function (3), where n is the sample size, r stands for r^{th} failure, τ_1 and τ_2 are censoring times. It is assumed that the parameter λ is a positive random variable, having a prior density $\pi(\lambda)$ over the parameter space $(0, \infty)$. Here we consider a conjugate prior distribution gamma $G(a,b)$ with the following density over the parameter space $(0, \infty)$:

$$\pi(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-\lambda b}, \quad \lambda > 0, \;\; a, b > 0 \tag{5}$$

and zero, otherwise. Here a and b are the hyper parameters. Our aim is to obtain the optimum BSP for Type-II GHCS by determining the optimal decision function under the given loss function (4), which minimizes the Bayes risk among all possible sampling plans.

3 Bayes Decision Function and BSP for Type-II GHCS

For a given $M = m$, let $\mathbf{x} = (x_1, \ldots, x_m)$ denote the observed value of $\mathbf{X} = (X_{(1)}, \ldots, X_{(m)})$. Then, the joint probability density function (PDF) is given by:

$$f(\mathbf{x}|\lambda) = \frac{n!}{(n-m)!}\lambda^m e^{-\lambda y(n,r,\tau_1,\tau_2,m)}, \;\; 0 < x_1 < \ldots < x_m < \tau_2, \; \lambda > 0, \tag{6}$$

where $y(n,r,\tau_1,\tau_2,m) = \sum_{j=1}^{m} x_j + (n-m)\tau^*$. Thus, the posterior PDF of λ given $\mathbf{x}$ is:

$$\pi(\lambda|\mathbf{x}) = \frac{\pi(\lambda) f(\mathbf{x}|\lambda)}{\int_0^\infty \pi(\lambda) f(\mathbf{x}|\lambda) d\lambda} = \frac{\lambda^m e^{-\lambda y(n,r,\tau_1,\tau_2,m)} \pi(\lambda)}{\int_0^\infty \lambda^m e^{-\lambda y(n,r,\tau_1,\tau_2,m)} \pi(\lambda) d\lambda}. \tag{7}$$

Since $(M, Y(n,r,\tau_1,\tau_2,M))$ is a jointly sufficient statistic of λ, the posterior PDF $\pi(\lambda|\mathbf{x})$ depends on $\mathbf{x}$ only through m and $y(n,r,\tau_1,\tau_2,m)$, we may also denote it by $\pi(\lambda|m, y(n,r,\tau_1,\tau_2,m))$. The Bayes risk of the sampling plan $(n,r,\tau_1,\tau_2,\delta)$ w.r.t. the loss function (4) is given by:

$$\begin{aligned} R_B^\delta(n,r,\tau_1,\tau_2) &= n(C_s - r_s) + E(M) r_s + E(\tau^*) C_\tau + E(MSE(T(\mathbf{X}))) C_v \\ &\qquad + E[\delta(\mathbf{X}) g(\lambda) + (1-\delta(\mathbf{X})) C_r] \\ &= n(C_s - r_s) + E(M) r_s + E(\tau^*) C_\tau + E(MSE(T(\mathbf{X}))) C_v + R_\delta(\delta|n,r,\tau_1,\tau_2), \end{aligned} \tag{8}$$

where $R_\delta(\delta|n,r,\tau_1,\tau_2)$ is the risk by the decision function δ, i.e.,

$$R_\delta(\delta|n,r,\tau_1,\tau_2) = E(g(\lambda)) + E[(1-\delta(\mathbf{X}))(C_r - g(\lambda))] = R_1(\delta|n,r,\tau_1,\tau_2) + R_2(\delta|n,r,\tau_1,\tau_2), \tag{9}$$

where $R_1(\delta|n,r,\tau_1,\tau_2) = E(g(\lambda))$ and

$$\begin{aligned} R_2(\delta|n,r,\tau_1,\tau_2) &= E[(1-\delta(\mathbf{X}))(C_r - g(\lambda))] \\ &= E_{\mathbf{X}}[(1-\delta(\mathbf{X}))(C_r - E_{\lambda|\mathbf{x}}[g(\lambda)])] \\ &= E_{\mathbf{X}}[(1-\delta(\mathbf{X}))(C_r - \phi_\pi(m, Y(n,r,\tau_1,\tau_2,m)))]. \end{aligned} \tag{10}$$

Here,

$$\phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) = E_{\lambda|\mathbf{x}}[g(\lambda)] = \int_0^\infty g(\lambda)\pi(\lambda|m, y(n,r,\tau_1,\tau_2,m)) d\lambda \tag{11}$$

denotes the posterior expectation of $g(\lambda)$ given data $\mathbf{X}=\mathbf{x}$. To obtain the Bayes decision function we need to minimize the posterior risk of the decision function which is equivalent to the minimization of $R_2(\delta|n,r,\tau_1,\tau_2)$ w.r.t. δ. For each fixed (n,r,τ_1,τ_2), the Bayes risk $R_B^{\delta}(n,r,\tau_1,\tau_2)$ is minimum when $R_2(\delta|n,r,\tau_1,\tau_2)$ is minimized w.r.t. δ in (9), which can be done by considering two cases:

Case 1: $\phi_\pi(m,y(n,r,\tau_1,\tau_2,m))\leq C_r$:

If $\delta(\mathbf{x})=0$, then $R_2(\delta|n,r,\tau_1,\tau_2)\geq 0$, and if $\delta(\mathbf{x})=1$, then $R_2(\delta|n,r,\tau_1,\tau_2)=0$.

Case 2: $\phi_\pi(m,y(n,r,\tau_1,\tau_2,m))>C_r$:

If $\delta(\mathbf{x})=0$, then $R_2(\delta|n,r,\tau_1,\tau_2)<0$, and if $\delta(\mathbf{x})=1$, then $R_2(\delta|n,r,\tau_1,\tau_2)=0$.

Therefore, for fixed sample size n, positive integer $r\leq n$ and censoring time τ_1 and τ_2, the Bayes decision function, which minimizes the Bayes risk $R_B^{\delta}(n,r,\tau_1,\tau_2)$, among all decision functions is given by:

$$\delta_B=\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2)=\begin{cases}1, & \text{if } \phi_\pi(m,y(n,r,\tau_1,\tau_2,m))\leq C_r\\ 0, & otherwise.\end{cases} \tag{12}$$

Then our aim is to obtain the optimum BSP $(n_B,r_B,\tau_{1B},\tau_{2B},\delta_B)$ which minimizes the Bayes risk among all possible BSPs $(n,r,\tau_1,\tau_2,\delta_B)$.

3.1 Monotonicity of Bayes Decision Function

To present an alternate form of the Bayes decision function δ_B, we investigate the monotonicity associated with it.

Lemma 3.1. *Let $0\leq m,\ m^*\leq n,\ y=y(n,r,\tau_1,\tau_2,m)$ and $y^*=y(n,r,\tau_1,\tau_2,m^*)$. Consider the likelihood ratio:*

$$L_R(\lambda|(m,y),(m^*,y^*))=\frac{\pi(\lambda|m^*,y^*)}{\pi(\lambda|m,y)}$$

provided $\pi(\lambda|m,y)\neq 0$.

(a) If $m=m^*$ and $y<y^*$, then $L_R(\lambda|(m,y),(m^*,y^*))$ is non-increasing in λ.

(b) If $y=y^*$ and $m<m^*$, then $L_R(\lambda|(m,y),(m^*,y^*))$ is non-decreasing in λ.

Proof. From (7) the likelihood ratio is given by:

$$L_R(\lambda|(m,y),(m^*,y^*)) = \frac{\pi(\lambda|m^*,y^*)}{\pi(\lambda|m,y)} = \frac{w(m,y)}{w(m^*,y^*)}\lambda^{m^*-m}e^{-\lambda(y^*-y)},$$

where $w(m,y) = \int_0^\infty \lambda^m e^{-\lambda y}\pi(\lambda)d\lambda = \frac{b^a\Gamma(m+a)}{\Gamma(a)(y+b)^{m+a}}$.

Case *(a)*: When $m = m^*$ and $y < y^*$, then,

$$L_R(\lambda|(m,y),(m^*,y^*)) = \frac{w(m,y)}{w(m,y^*)}e^{-\lambda(y^*-y)},$$

$y^* - y > 0$, which implies that $L_R(\lambda|(m,y),(m^*,y^*))$ is non-increasing in λ.

Case *(b)*: When $y = y^*$ and $m < m^*$, then,

$$L_R(\lambda|(m,y),(m^*,y^*)) = \frac{w(m,y)}{w(m^*,y)}\lambda^{m^*-m},$$

$m^* - m > 0$, which implies that $L_R(\lambda|(m,y),(m^*,y^*))$ is non-decreasing in λ. ∎

Lemma 3.2. *Let $g(\lambda)$ be a positive and increasing function of λ for $\lambda > 0$. Then,*
(a) For fixed $m \geq 0$, $\phi_\pi(m,y(n,r,\tau_1,\tau_2,m))$ is strictly decreasing in $y(n,r,\tau_1,\tau_2,m)$, and for fixed $y(n,r,\tau_1,\tau_2,m) \geq 0$, $\phi_\pi(m,y(n,r,\tau_1,\tau_2,m))$ is strictly increasing in m.
(b) For fixed $m \geq 0$, $\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2)$ is non-decreasing in $y(n,r,\tau_1,\tau_2,m)$, and for fixed $y(n,r,\tau_1,\tau_2,m) \geq 0$, $\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2)$ is non-increasing in m.

Proof. See Appendix. ∎

Let $A(m) = \{y > 0|\phi_\pi(m,y) \leq C_r\}$. Define

$$T_m = \begin{cases} inf\ A(m), & \text{if } A(m) \textit{ is nonempty} \\ \infty, & otherwise. \end{cases} \tag{13}$$

By a discussion analogous to Lemma 2.3 of Liang and Yang (2013), we can obtain the following lemma.

Lemma 3.3.

(1) (a) If $T_m = \infty$, then $\phi_\pi(m,y) > C_r$ for all $y > 0$;

(b) If $T_m = 0$, then $\phi_\pi(m,y) < C_r$ for all $y > 0$; and

(c) If $0 < T_m < \infty$, then

$\phi_\pi(m,y) > C_r$ *for all* $0 < y < T_m$;

$\phi_\pi(m,y) < C_r$ *for all* $y > T_m$; *and*

$\phi_\pi(m,T_m) = C_r$ *if* $y = T_m$.

(2) Let $D = \{m \mid A(m) \text{ is nonempty}\}$. Then, T_m is strictly increasing in m for $m \in D$.

Proof. See Appendix. ∎

3.2 Representation of Bayes decision function in terms of estimator $\hat{\lambda}$

Let

$$T(n,r,\tau_1,\tau_2,m=0) = T_0;\ T(n,r,\tau_1,\tau_2,m) = min\{T_m, n\tau_2\}, \quad for\ m = 1,2,\ldots,n.$$

According to the non-increasing property of $\phi_\pi(m,y(n,r,\tau_1,\tau_2,m))$ with respect to $y(n,r,\tau_1,\tau_2,m)$, the Bayes decision function $\delta_B(|n,r,\tau_1,\tau_2)$, can be expressed as: For each integer $m \geq 0$,

$$\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \ y(n,r,\tau_1,\tau_2,m) \geq T(n,r,\tau_1,\tau_2,m) \\ 0, & otherwise, \end{cases} \tag{14}$$

which can be written in the form of $\hat{\lambda}$,

$$\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \ \hat{\lambda} < c(n,r,\tau_1,\tau_2,m) \\ 0, & otherwise, \end{cases} \tag{15}$$

where for $m \geq 1$, $c(n,r,\tau_1,\tau_2,m) = \dfrac{m}{T(n,r,\tau_1,\tau_2,m)}$ and for $m = 0$, $c(n,r,\tau_1,\tau_2,0) = n\tau_2 - T(n,r,\tau_1,\tau_2,0)$, which is in the form of decision function (3) with $T(\mathbf{X}) = \hat{\lambda}$. Writing the Bayes decision function (15) in terms of $\hat{\lambda}$ is helpful, as the distribution of $\hat{\lambda}$ is known. Prajapati *et al.* (2020, 2019) have shown that the Bayes decision function cannot be obtained analytically for different loss functions. To determine $c(n,r,\tau_1,\tau_2,m)$, we have to obtain T_m.

Note that T_m depends on the number of observations m, and can be evaluated either analytically or numerically, depending on the form of $g(\lambda)$. For many loss functions T_m cannot be obtained analytically. So, for these cases we consider the following scheme to derive the Bayes decision function.

Scheme A:

1. The Bayes decision function (12) can be written as:

$$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) - C_r \le 0 \\ 0, & otherwise. \end{cases}$$

2. For closed form of the Bayes decision function, we need to solve $\phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) - C_r = 0$. From Lemma 3.2 it is clear that for fixed $m \ge 0$, the equation $\phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) - C_r = 0$ has at most one change of sign, i.e., it has only one real root.
3. Obtain the root of the equation $\phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) - C_r = 0$, by any numerical method and denote it by α_m.
4. Then define $T_m = max(0, \alpha_m)$ and Bayes decision function as (15).

Thus, alternative Bayes decision function (15), will be used to obtain the Bayes risk of the BSP. The Bayes risk of the BSP is given by

$$R_B^{\delta_B}(n,r,\tau_1,\tau_2) \;=\; n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(MSE(\hat{\lambda}))C_v + R_\delta(\delta_B|n,r,\tau_1,\tau_2).$$

To compute the Bayes risk, we have to calculate $E(M)$, $E(\tau^*)$, $E(MSE(\hat{\lambda}))$ and $R_\delta(\delta_B|n,r,\tau_1,\tau_2)$. The expressions of $E(M)$, $E(\tau^*)$, and $E(MSE(\hat{\lambda}))$ are given in the Appendix. Therefore, we need to obtain

$$\begin{aligned} R_\delta(\delta_B|n,r,\tau_1,\tau_2) &= E(g(\lambda)) + E[(1-\delta_B(M,Y(n,r,\tau_1,\tau_2,M)|n,r,\tau_1,\tau_2))(C_r - g(\lambda))] \\ &= E(g(\lambda)) + \int_0^\infty [C_r - g(\lambda)] P(\delta_B(M,Y(n,r,\tau_1,\tau_2,M)|n,r,\tau_1,\tau_2) = 0|\lambda)\pi(\lambda)d\lambda. \end{aligned}$$

When $M \ge 0$, from decision function (15)

$$P(\delta_B(M,Y(n,r,\tau_1,\tau_2,M)|n,r,\tau_1,\tau_2) = 0|\lambda) = P\big(\hat{\lambda} \ge c(n,r,\tau_1,\tau_2,M)|\lambda\big),$$

above probability can be obtain quit easily by using distribution of $\hat{\lambda}$, which is given in (2).

3.3 Upper Bounds for optimal value of n and r

In the loss function (4), we have a cost on the sample size and on total time of the experiment. So optimal values of n, r, τ_1 and τ_2 are automatically bounded above. The following theorem gives smaller upper bounds for n and r.

Theorem 3.1. *Let n_B and r_B be the optimal values of n and r, respectively. Then,*

$$n_B \leq min\left\{\frac{E_\lambda[g(\lambda)]}{C_s - r_s}, \frac{C_r}{C_s - r_s}\right\},$$

$0 \leq r_B \leq n_B$.

Proof. Let $(n_B, r_B, \tau_{1B}, \tau_{2B}, \delta_B)$ be the optimum BSP and corresponding Bayes risk is:

$$R_B^{\delta_B}(n_B, r_B, \tau_{1B}, \tau_{2B}) = n_B(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(MSE(\widehat{\lambda}))C_v + R_D(\delta_B|n_B, r_B, \tau_{1B}, \tau_{2B}).$$

Note that $E(M) \geq 0$, $E(\tau^*) \geq 0$, $E(MSE(\widehat{\lambda})) \geq 0$ and $R_D(\delta_B|n_B, r_B, \tau_{1B}, \tau_{2B}) \geq 0$ and all costs are positive. Hence

$$R_B^{\delta_B}(n_B, r_B, \tau_{1B}, \tau_{2B}) \geq n_B(C_s - r_s). \tag{16}$$

Let $(0,0,0,0,1)$ and $(0,0,0,0,0)$ denote the sampling plans when the batch is accepted and rejected, respectively, without sampling. Then the Bayes risk of optimum BSP

$$R_B^{\delta_B}(n_B, r_B, \tau_{1B}, \tau_{2B}) \leq min\{R_B^{\delta_B=1}(0,0,0,0), R_B^{\delta_B=0}(0,0,0,0)\}, \tag{17}$$

where $R_B^{\delta_B=1}(0,0,0,0) = E_\lambda[g(\lambda)]$ is the Bayes risk when we accept the batch without sampling and $R_B^{\delta_B=0}(0,0,0,0) = C_r$ is the Bayes risk when we reject the batch without sampling. Then from (16) and (17) we have

$$n_B(C_s - r_s) \leq min\{E_\lambda[g(\lambda)], C_r\}.$$

Hence, it follows that

$$n_B \leq min\left\{\frac{E_\lambda[g(\lambda)]}{C_s - r_s}, \frac{C_r}{C_s - r_s}\right\}$$

and $0 \leq r_B \leq n_B$. ∎

3.4 Bayes risk of the BSP for quadratic loss

In this section we consider the loss function (4), with $g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$, where $a_0 > 0$, $a_1 > 0$ and $a_2 > 0$. It has been widely used in the literature to obtain the optimal sampling plan, for example, see, Lam (1994); Lam and Choy (1995); Lin *et al.* (2002); Lin *et al.* (2008, 2010); Liang and Yang (2013); Yang *et al.* (2017), as it is analytically tractable, and simplifies the problem to a great extent. If the prior distribution of λ is $G(a,b)$, then it is well known that the posterior distribution of λ is also gamma, viz.,

$$\lambda | m, y(n,r,\tau_1,\tau_2,m) \sim G(m+a, y(n,r,\tau_1,\tau_2,m)+b).$$

Hence, the posterior expectation of $g(\lambda)$ based on (11) is:

$$\begin{aligned}\phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) &= \int_0^\infty g(\lambda)\pi(\lambda|m, y(n,r,\tau_1,\tau_2,m))d\lambda \\ &= a_0 + a_1\frac{(m+a)}{(y(n,r,\tau_1,\tau_2,m)+b)} + a_2\frac{(m+a)(m+a+1)}{(y(n,r,\tau_1,\tau_2,m)+b)^2}.\end{aligned}$$

We pre-specify the parameter values (n,r,τ_1,τ_2) and then construct the Bayes decision function, which is as follows:

$$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } a_0 + a_1\frac{(m+a)}{(y(n,r,\tau_1,\tau_2,m)+b)} + a_2\frac{(m+a)(m+a+1)}{(y(n,r,\tau_1,\tau_2,m)+b)^2} \le C_r \\ 0, & otherwise. \end{cases}$$

The minimum Bayes risk is obtained as:

1. If $a_0 \ge C_r$, then $\phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) > C_r$, for all $(m, y(n,r,\tau_1,\tau_2,m))$ and (n,r,τ_1,τ_2). Thus,

   $$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = 0 \;\forall\; (n,r,\tau_1,\tau_2,m).$$

   In this case, we should take $n_B = 0,\ r_B = 0,\ \tau_{1B} = 0$, and $\tau_{2B} = 0$. The minimum Bayes risk is $R_B^{\delta_B}(n_B, r_B, \tau_{1B}, \tau_{2B}) = R_\delta(\delta_B|n_B, r_B, \tau_{1B}, \tau_{2B}) = C_r$.
2. If $0 < a_0 < C_r$, then Bayes decision function is

   $$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) \le C_r \\ 0, & otherwise. \end{cases}$$

Thus to find the closed form of the decision function we need to obtain the set

$$A(m) = \{z > 0 | \phi_\pi(m,z) \leq C_r\},$$

where $z = y(n,r,\tau_1,\tau_2,m)$ and to construct $A(m)$, we need to obtain the set of $z > 0$, such that

$$h_1(z) = a_0 + a_1 \frac{(m+a)}{(z+b)} + a_2 \frac{(m+a)(m+a+1)}{(z+b)^2} - C_r \leq 0,$$

which is equivalent to finding $z > 0$, such that,

$$h_2(z) = (C_r - a_0)\big(z+b\big)^2 - a_1(m+a)\big(z+b\big) - a_2(m+a)(m+a+1) \geq 0.$$

It can easily be shown that if $D_n(m)$ is the only real root, or $D_n(m)$ is the maximum real root of $h_2(z) = 0$, i.e.,

$$D_n(m) = \frac{a_1(m+a) + \sqrt{a_1^2(m+a)^2 + 4(C_r - a_0)a_2(m+a)(m+a+1)}}{2(C_r - a_0)},$$

then $T_m = max(D_n(m) - b, 0)$. Therefore, the Bayes decision function will take the following form:

$$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \hat{\lambda} < c(n,r,\tau_1,\tau_2,m) \\ 0, & otherwise, \end{cases}$$

where for $m \geq 1$, $c(n,r,\tau_1,\tau_2,m) = \frac{m}{T(n,r,\tau_1,\tau_2,m)}$, $T(n,r,\tau_1,\tau_2,m) = min\{T_m, n\tau_2\}$ for $m = 1,2,\ldots,n$, and for $m = 0$, $c(n,r,\tau_1,\tau_2,0) = n\tau_2 - T(n,r,\tau_1,\tau_2,0)$, $T(n,r,\tau_1,\tau_2,0) = T_0$. Now using the distribution function of $\hat{\lambda}$, the Bayes risk of the BSP $(n,r,\tau_1,\tau_2,\delta_B)$ is:

$$\begin{aligned}
&R_B^{\delta_B}(n,r,\tau_1,\tau_2) \\
&= n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(MSE(\hat{\lambda}))C_v + a_0 + a_1\mu_1 + a_2\mu_2 + \sum_{l=0}^{2} C_l \frac{b^a}{\Gamma(a)} \\
&\times \Bigg[\frac{\Gamma(a+l)\boldsymbol{I}(n\tau_2 < T_0)}{(b+n\tau_2)^{a+l}} + \sum_{m_1=r}^{n}\sum_{j=0}^{m_1} \binom{n}{m_1}\binom{m_1}{j} \frac{(-1)^j\Gamma(a+l)I_{S_{m_1,j}}(m_1, a+l)}{(b+(n-m_1+j)\tau_1)^{a+l}} \\
&+ \sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{k} \binom{n}{m_1}\binom{m_1}{j} \frac{(-1)^j\Gamma(a+l)I_{S_{m_1,k,j,i}}(r,a+l)}{(b+(j+k-i)\tau_1 + (i+n-k-m_1)\tau_2)^{a+l}}
\end{aligned}$$

$$+\sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}\frac{(-1)^j\Gamma(a+l)I_{S_{m_2,j}}(m_2,a+l)}{(b+(n-m_2+j)\tau_2)^{a+l}}\Bigg],$$

where $\mu_i = E(\lambda^i)$ for $i = 1,2$,

$$C_l = \begin{cases} C_r - a_l, & \text{if} \quad l = 0 \\ -a_l, & \text{if} \quad l = 1,2, \end{cases}$$

$$\boldsymbol{I}(n\tau_2 < T_0) = \begin{cases} 1, & \text{if } n\tau_2 < T_0 \\ 0, & otherwise, \end{cases}$$

$$S_{m_1,j} = \frac{C_{m_1,j}}{1+C_{m_1,j}}, \quad C_{m_1,j} = \frac{T(n,r,\tau_1,\tau_2,m_1)-(n-m_1+j)\tau_1}{b+(n-m_1+j)\tau_1},$$

$$S_{m_1,k,j,i} = \frac{C_{m_1,k,j,i}}{1+C_{m_1,k,j,i}}, \quad C_{m_1,k,j,i} = \frac{T(n,r,\tau_1,\tau_2,r)-(j+k-i)\tau_1-(i+n-k-m_1)\tau_2}{b+(j+k-i)\tau_1+(i+n-k-m_1)\tau_2},$$

$$S_{m_2,j} = \frac{C_{m_2,j}}{1+C_{m_2,j}}, \quad C_{m_2,j} = \frac{T(n,r,\tau_1,\tau_2,m_2)-(n-m_2+j)\tau_2}{b+(n-m_2+j)\tau_2},$$

$I_x(\alpha,\beta) = B_x(\alpha,\beta)/B(\alpha,\beta)$ denotes the cumulative distribution function of beta and

$$B_x(\alpha,\beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1}du, \quad 0 \leq x \leq 1,$$

is the incomplete beta function.

Although explicit form of the Bayes risk of the BSP for a quadratic loss function is obtained, the expression is very complicated, and as a result obtaining the optimal values of n, r, τ_1 and τ_2 analytically, is not possible. Hence, we present Algorithm A, which is similar to Yang *et al.* (2017) and Prajapati *et al.* (2020, 2019) for deriving a optimal sampling plan.

Algorithm A:

Algorithm for finding optimum BSP $(n_B, r_B, \tau_{1B}, \tau_{2B}, \delta_B)$

1. For each fixed parameter (n,r,τ_1,τ_2), derive the Bayes decision function δ_B, which minimizes the Bayes risk $R_B^{\delta}(n,r,\tau_1,\tau_2)$ among the class of all decision functions.
2. For each fixed parameter (n,r), find the censoring time $\tau_{1B}(n,r)$ and $\tau_{2B}(n,r)$, which minimize the Bayes risk $R_B^{\delta_B}(n,r,\tau_1,\tau_2)$ among all $\tau_2 > \tau_1 > 0$. That is, $\tau_{1B}(n,r)$ and $\tau_{2B}(n,r)$ are the censoring times such that

$$R_B^{\delta_B}(n,r,\tau_{1B}(n,r),\tau_{2B}(n,r)) = \inf_{\tau_2>\tau_1>0} R_B^{\delta_B}(n,r,\tau_1,\tau_2).$$

3. For each fixed parameter n, find an integer $r_B(n), 0 \leq r_B(n) \leq n$ such that

$$R_B^{\delta_B}(n, r_B(n), \tau_{1B}(n, r_B(n)), \tau_{2B}(n, r_B(n))) = \underset{0 \leq r \leq n}{min} R_B^{\delta_B}(n, r, \tau_{1B}(n, r), \tau_{2B}(n, r)).$$

4. Find an integer $n_B \geq 0$ such that

$$R_B^{\delta_B}(n_B, r_B, \tau_{1B}, \tau_{2B}) = \underset{n \geq 0}{min} R_B^{\delta_B}(n, r_B(n), \tau_{1B}(n, r_B(n)), \tau_{2B}(n, r_B(n))).$$

For simplicity, we use short notation $(n_B, r_B, \tau_{1B}, \tau_{2B}, \delta_B)$ to denote the optimum BSP, where

$$\begin{aligned}
r_B &= r_B(n_B), \\
\tau_{1B} &= \tau_{1B}(n_B, r_B(n_B)), \\
\tau_{2B} &= \tau_{2B}(n_B, r_B(n_B)), \\
\delta_B &= \delta_B(|n_B, r_B(n_B), \tau_{1B}(n_B, r_B(n_B)), \tau_{2B}(n_B, r_B(n_B))).
\end{aligned}$$

4 BSP FOR HIGHER DEGREE POLYNOMIAL LOSS

The justification given by Lam (1990, 1994) for using a quadratic loss function implies that the higher degree polynomial loss function can be a good approximation of true loss function. Therefore, it is meaningful to study what happens when $g(\lambda)$ is a higher degree polynomial function. In this section, we discuss the BSP for the higher degree polynomial loss function. We consider the acceptance cost $g(\lambda) = a_0 + a_1\lambda + \ldots + a_d\lambda^d$, $a_i \geq 0$ for all $i = 0, 1, \ldots, d$. If $g(\lambda)$ is an approximation of the true acceptance cost, then the optimal sampling plans obtained for small values of d are only "approximate optimal" plans. Since the form of the Bayes decision function is:

$$\delta_B(m, y(n, r, \tau_1, \tau_2, m)|n, r, \tau_1, \tau_2) = \begin{cases} 1, & \text{if } \phi_\pi(m, y(n, r, \tau_1, \tau_2, m)) \leq C_r \\ 0, & otherwise, \end{cases}$$

where

$$\phi_\pi\big(m, y(n, r, \tau_1, \tau_2, m)\big) = \int_0^\infty g(\lambda)\pi\big(\lambda|m, y(n, r, \tau_1, \tau_2, m)\big)d\lambda,$$

therefore, when $g(\lambda) = a_0 + a_1\lambda + \ldots + a_d\lambda^d$, then

$$\phi_\pi\big(m, y(n,r,\tau_1,\tau_2,m)\big) = a_0 + \sum_{j=1}^{d} a_j \frac{(m+a)\ldots(m+a+j-1)}{(y(n,r,\tau_1,\tau_2,m)+b)^j}.$$

We have seen, that to find the closed form of the Bayes decision function, we need to obtain the set

$$A(m) = \{z > 0 | \phi_\pi\big(m,z\big) \leq C_r\},$$

and to construct $A(m)$, we need to obtain the set of $z \geq 0$, such that

$$h_1(z) = a_0 + \sum_{j=1}^{d} a_j \frac{(m+a)\ldots(m+a+j-1)}{(z+b)^j} - C_r \leq 0,$$

which is equivalent to finding $z \geq 0$, such that,

$$h_2(z) = (C_r - a_0)\big(z+b\big)^d - \sum_{j=1}^{d} a_j (m+a)\ldots(m+a+j-1)\big(z+b\big)^{d-j} \geq 0. \tag{18}$$

Therefore, to obtain the closed form of the Bayes decision function analytically, we need to solve $h_2(z) = 0$. It is well known that there is no algebraic solution to polynomial equations of degree five or higher (see Herstein (1975)). But it has been shown analytically that the Bayes decision function can be written in the form:

$$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \hat{\lambda} < c(n,r,\tau_1,\tau_2,m) \\ 0, & \textit{otherwise}, \end{cases}$$

where for $m \geq 1$, $c(n,r,\tau_1,\tau_2,m) = \frac{m}{T(n,r,\tau_1,\tau_2,m)}$, $T(n,r,\tau_1,\tau_2,m) = min\{T_m, n\tau_2\}$ for $m = 1,2,\ldots,n$; $m = 0$, $c(n,r,\tau_1,\tau_2,0) = n\tau_2 - T(n,r,\tau_1,\tau_2,0)$, $T(n,r,\tau_1,\tau_2,0) = T_0$ and T_m is defined in (13). To obtain T_m we need to solve $h_1(z) = 0$, and analytically it is not possible to do for $d \geq 5$. Let $C_r > a_0$, then Scheme A can be used to obtain T_m numerically for $d \geq 5$. Thus, the Bayes risk expression for higher degree polynomial loss function can be obtained by similar approach as in the case of quadratic loss function and is given by:

$$\begin{aligned} &R_B^{\delta_B}(n,r,\tau_1,\tau_2) \\ &= n(C_s - r_s) + E(M)r_s + E(\tau^*)C_\tau + E(MSE(\hat{\lambda}))C_v + a_0 + a_1\mu_1 + \ldots + a_d\mu_d \end{aligned}$$

$$+\sum_{l=0}^{d} C_l \frac{b^a}{\Gamma(a)} \left[\frac{\Gamma(a+l)\boldsymbol{I}(n\tau_2 < T_0)}{(b+n\tau_2)^{a+l}} + \sum_{m_1=r}^{n} \sum_{j=0}^{m_1} \binom{n}{m_1} \binom{m_1}{j} \frac{(-1)^j \Gamma(a+l) I_{S_{m_1,j}}(m_1, a+l)}{(b+(n-m_1+j)\tau_1)^{a+l}} \right.$$
$$+ \sum_{m_1=0}^{r-1} \sum_{k=r-m_1}^{n-m_1} \sum_{j=0}^{m_1} \sum_{i=0}^{k} \binom{n}{m_1} \binom{m_1}{j} \frac{(-1)^j \Gamma(a+l) I_{S_{m_1,k,j,i}}(r, a+l)}{(b+(j+k-i)\tau_1+(i+n-k-m_1)\tau_2)^{a+l}}$$
$$\left. + \sum_{m_2=1}^{r-1} \sum_{j=0}^{m_2} \binom{n}{m_2} \binom{m_2}{j} \frac{(-1)^j \Gamma(a+l) I_{S_{m_2,j}}(m_2, a+l)}{(b+(n-m_2+j)\tau_2)^{a+l}} \right],$$

where $\mu_i = E(\lambda^i)$ for $i = 1, \ldots, d$,

$$C_l = \begin{cases} C_r - a_l, & \text{if} \quad l = 0 \\ -a_l, & \text{if} \quad l = 1, \ldots, d \end{cases}$$

Further notations are exactly same as in the case of the quadratic loss function, a special case with $d = 2$.

Algorithm B:

Algorithm for finding optimum BSP $(n_B, r_B, \tau_{1B}, \tau_{2B}, \delta_B)$ *using Scheme A*

1. Let $C_r > a_0$. For each fixed parameter (n, r, τ_1, τ_2), derive the Bayes decision function δ_B to minimize the Bayes risks $R_B^{\delta}(n, r, \tau_1, \tau_2)$ among the class of all decision functions.
2. For each fixed (n, r, τ_1, τ_2), Scheme A is applied to compute the closed form of Bayes decision function by solving $h_1(z) = \phi_\pi(m, z) - C_r = 0$ numerically for all $m \geq 0$. Let $T_m = max(0, \alpha_m)$ in (15), where α_m is the root of equation $h_1(z) = 0$.
3. Now repeat the step 2-4 of Algorithm A.

5 Numerical Results

In this section, numerical studies are conducted to see the performance of the optimum BSP for Type-II GHCS. All numerical computations have been performed using the program written in *R* software, and it can be obtained from the corresponding author on request. We had provided the pseudo code in the Appendix. Results are presented in Tables 1 - 7 and the Bayes risk $R_B^{\delta_B}(n_B, r_B, \tau_{1B}, \tau_{2B})$ of optimum BSP in each table is denoted by $R_B^{\delta_B}$ for brevity.

5.1 Optimum BSP for quadratic loss

We consider the quadratic loss function to check the performance of the optimum BSP. We fix the following values of coefficients and costs for the loss function: $a_0 = 2,\ a_1 = 3,\ a_2 = 4,\ C_s = 1.5,\ r_s = 1.2,\ C_v = 0.5,\ C_\tau = 0.1,\ C_r = 75$, and the hyper parameters are $a = 1.55,\ b = 0.50$. The R software is used to obtain the numerical optimum BSP. Before obtaining the

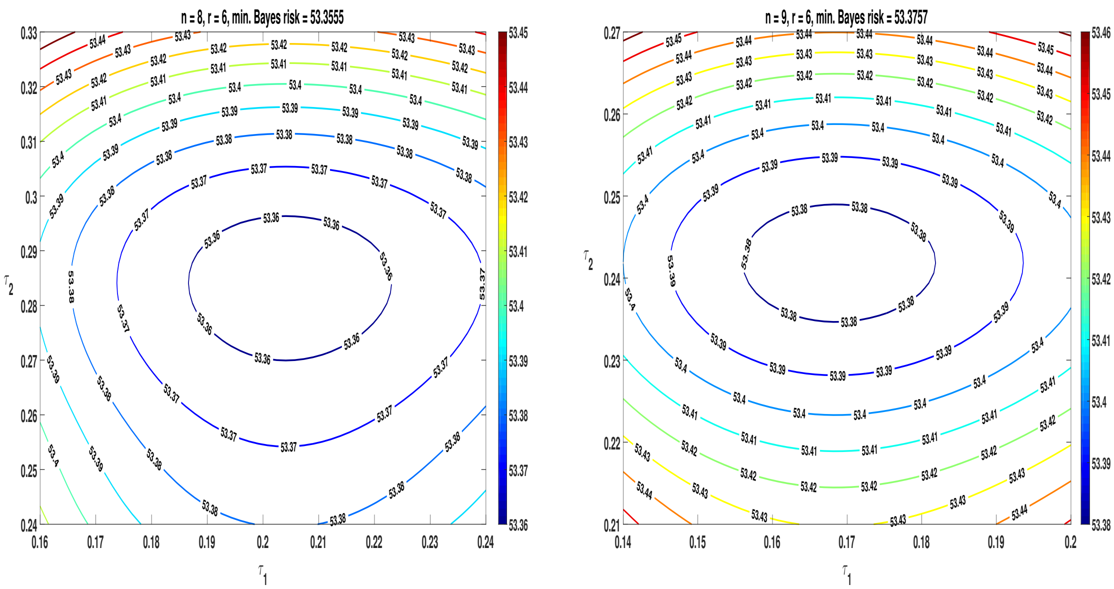

Figure 1: Contour plots of Bayes risk with set of coefficients $a_0 = 2,\ a_1 = 3,\ a_2 = 4,\ C_s = 1.5,\ r_s = 1.2,\ C_v = 0.5,\ C_\tau = 0.1,\ C_r = 75$ and parameters $a = 1.55,\ b = 0.50$.

optimum BSP by numerical optimization method in R, we show graphically that Bayes risk $R_B^{\delta_B}(n, r, \tau_1, \tau_2)$ has a unique minimum. In Fig.1 we fix the value of n and r and draw the contour plots of the Bayes risk w.r.t. τ_1 and τ_2. It is clear from the Fig.1 that Bayes risk has a unique minimum. Numerical results are presented in Tables 1-4. In each table, only two parameters or one coefficient are permitted to vary and others are kept fixed. From Table 1, we see that when $a = 1.55, b = 0.50$, the optimum BSP is $(n_B, r_B, \tau_{1B}, \tau_{2B}, \delta_B) = (8, 6, 0.2041, 0.2843, \delta_B)$, and the associated Bayes risk is 53.3555. We further observe that for fixed b, if a is increasing then the Bayes risk $R_B^{\delta_B}$ is increasing and for fixed a, if b is increasing then the Bayes risk $R_B^{\delta_B}$ is decreasing. If C_s is increasing then the Bayes risk $R_B^{\delta_B}$

Table 1: Minimum Bayes risks and their optimum BSPs when a, b and C_s vary with $a_0 = 2,\ a_1 = 3,\ a_2 = 4,\ r_s = 1.2,\ C_v = 0.5,\ C_\tau = 0.1,\ C_r = 75$

a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
1.55	0.45		57.2735	9	6	0.1625	0.2538			1.3	51.4625	11	6	0.1236	0.1819
1.55	0.50		53.3555	8	6	0.2041	0.2843			1.5	53.3555	8	6	0.2041	0.2843
2.40	0.80	1.5	55.1567	8	8	0.1037	0.2173	1.55	0.50	2.0	56.9243	6	6	0.0100	0.3836
2.50	0.80		57.3335	8	8	0.1166	0.2235			2.5	59.9243	6	6	0.0100	0.3836
4.50	1.25		68.5803	8	5	0.1803	0.2241			3.0	62.6221	5	5	0.0138	0.4946

is increasing and the optimum values of n, r and τ_1 are decreasing. In Table 2, we can see that as coefficients C_τ and C_r increase, the Bayes risk $R_B^{\delta_B}$ is increasing. In Table 3, we

Table 2: Minimum Bayes risks and their optimum BSPs when C_τ and C_r vary with $a = 1.55,\ b = 0.50,\ a_0 = 2,\ a_1 = 3,\ a_2 = 4,\ C_s = 1.5,\ r_s = 1.2,\ C_v = 0.5$

C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
0.05		53.3422	8	6	0.2045	0.2848		60	47.9906	8	5	0.1714	0.2632
0.10		53.3555	8	6	0.2041	0.2843		75	53.3555	8	6	0.2041	0.2843
0.15	75	53.3687	8	6	0.2036	0.2834	0.1	90	57.7290	8	7	0.2506	0.2678
0.50		53.4605	8	6	0.2014	0.2784		100	60.2112	9	7	0.1995	0.2168
1.00		53.5754	9	6	0.1653	0.2381		125	65.2713	9	9	0.1041	0.2165

observe that if the cost on imprecision of estimator is high then we need more samples to get decision on batch. Table 4 represents the optimal sampling plan when the coefficients of the quadratic polynomial vary. From Tables 1-4 it is clear that the performance of the optimum BSP, with respect to various costs, is logical and true to a practical scenario.

Table 3: Minimum Bayes risks and their optimum BSPs when C_v and a_0 vary with $a = 1.55,\ b = 0.50, a_1 = 3,\ a_2 = 4,\ C_s = 1.5,\ r_s = 1.2, C_\tau = 0.1,\ C_r = 75$

C_v	a_0	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	C_v	a_0	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
0.2		51.8132	7	5	0.1580	0.2761		0.5	52.3292	8	6	0.2040	0.2819
1.0		55.2654	10	7	0.2077	0.2226		1.5	53.0143	8	6	0.2042	0.2834
1.5	2.0	56.8266	11	11	0.0998	0.2278	0.5	2.0	53.3555	8	6	0.2041	0.2843
2.0		58.1953	12	12	0.1165	0.2164		3.0	54.0348	8	6	0.2042	0.2857
2.5		59.4396	13	13	0.1116	0.2184		5.0	55.3814	8	6	0.2040	0.2881

Table 4: Minimum Bayes risks and optimum BSPs when a_1 and a_2 vary with $a = 1.55,\ b = 0.50, a_0 = 2,\ C_s = 1.5,\ r_s = 1.2,\ C_v = 0.5,\ C_\tau = 0.1,\ C_r = 75$

a_1	a_2	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	a_1	a_2	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
1.0		50.6040	8	6	0.2041	0.2662		3.0	49.0290	8	6	0.2070	0.2285
2.0		52.0082	8	6	0.2039	0.2751		4.0	53.3555	8	6	0.2041	0.2843
3.0	4.0	53.3555	8	6	0.2041	0.2843	3.0	5.0	56.6034	8	6	0.2040	0.3319
4.0		54.6470	8	6	0.2040	0.2934		7.0	61.1671	8	5	0.1713	0.3250
5.0		55.8799	8	5	0.1715	0.2431		8.0	62.8567	8	5	0.1714	0.3530

5.2 Optimum BSP for higher degree polynomial loss

In this section, we consider the fifth degree polynomial loss function, i.e., $d = 5$. The optimum BSP is obtained by using the numerical Algorithm B. In Tables 5-7, we present the optimum BSP for the following coefficients and costs of the loss function: $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ a_5 = 1,\ C_s = 1.5,\ r_s = 1.2, C_v = 0.5, C_\tau = 0.1, C_r = 75$ and the hyper parameters are $a = 1.55,\ b = 0.80$. In each table, only two parameters or one coefficient are permitted to vary and others are kept fixed. In Table 5, for fixed a, when b is increasing, the Bayes risk $R_B^{\delta_B}$ is decreasing. Similarly, for fixed b, when a is increasing then Bayes risk is increasing. When inspection cost C_s is small, then optimal value of sample size n is large and the Bayes risk $R_B^{\delta_B}$ is small. If C_s is large, then optimal values of n and r are equal, op-

Table 5: Minimum Bayes risks and their optimum BSPs when a, b and C_s vary with $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ a_5 = 1,\ r_s = 1.2,\ C_v = 0.5,\ C_\tau = 0.1,\ C_r = 75$

a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	a	b	C_s	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
1.55	0.45		71.7945	9	6	0.1624	0.7151			1.3	52.6451	11	7	0.1670	0.5635
1.55	0.50		69.0708	8	6	0.2039	0.8785			1.5	54.4660	8	7	0.3036	1.0051
1.55	0.80	1.5	54.4660	8	7	0.3036	1.0051	1.55	0.80	2.0	57.7209	6	6	0.0623	1.5741
2.20	0.80		70.6261	8	6	0.2267	0.8865			3.0	62.8506	5	5	0.0799	1.7853
4.50	2.25		66.7642	7	6	0.3774	1.0601			3.5	65.1017	4	4	0.1092	1.8837

timal value of τ_1 is small and that of τ_2 is large. In Table 6, we see that as C_τ is increasing,

Table 6: Minimum Bayes risks and their optimum BSPs when C_τ and C_r vary with $a = 1.55,\ b = 0.80,\ a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ a_5 = 1,\ C_s = 1.5,\ r_s = 1.2,\ C_v = 0.5$

C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	C_τ	C_r	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
0.00		54.3861	8	7	0.3060	1.0139		60	46.9520	7	6	0.3140	1.0647
0.10		54.4660	8	7	0.3036	1.0051		65	49.5068	7	6	0.3137	1.0597
0.15	75	54.5059	8	7	0.3025	1.0008	0.1	75	54.4660	8	7	0.3036	1.0051
0.50		54.7412	8	6	0.2413	0.7809		90	61.4189	8	7	0.3032	1.0019
1.00		55.0402	8	6	0.2368	0.7709		125	76.0993	10	9	0.2817	0.9163

the Bayes risk $R_B^{\delta_B}$ is increasing and if rejection cost C_r is increasing, then optimal values of sample size n and r are increasing. In Table 7, if cost on MSE, C_v increases, then Bayes risk of optimum BSP increases and optimum values of n and r increase. Similarly, when r_s increases then the Bayes risk of optimum BSP increases. Thus as we had claimed, the optimum BSP and corresponding Bayes risk for fifth degree polynomial loss function can be obtained numerically without much difficulty and the behaviour of the optimum BSP w.r.t costs are logical. So for any value of d the optimum BSP can be obtained.

Table 7: Minimum Bayes risks and their optimum BSPs when C_v and r_s vary with $a = 1.55,\ b = 0.80,\ a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ a_5 = 1,\ C_s = 1.5,\ C_\tau = 0.1,\ C_r = 75$

C_v	r_s	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}	C_v	r_s	$R_B^{\delta_B}$	n_B	r_B	τ_{1B}	τ_{2B}
0.5		54.4660	8	7	0.3036	1.0051		0.2	55.6401	6	6	0.0761	2.2935
1.0		55.2404	9	7	0.2949	0.8013		0.5	55.4022	6	6	0.0476	2.0297
2.0	1.2	56.5746	10	8	0.3562	0.7936	0.5	0.8	55.1324	6	6	0.0422	1.8228
3.0		57.6826	11	8	0.3421	0.6782		1.2	54.4660	8	7	0.3036	1.0051
4.0		58.7129	12	8	0.3272	0.5898		1.4	53.7036	11	7	0.1594	0.5546

6 Concluding remarks

In this paper, we consider the Type-II GHCS scheme to obtain the optimum BSP by decision-theoretic approach. An efficient loss function is proposed which includes mean square error of the estimator to deal with imprecision in the decision, i.e., to get sufficient sample to make better decision on the batch. We proposed to write the Bayes decision function in terms of the suitable estimator, which give advantage in calculation of Bayes risk. Explicit expression of the Bayes risk of the BSP is obtained for quadratic loss function. Finite algorithm is given to arrive at the optimum BSP. A numerical approach is presented to obtain the optimum BSP for any form of the loss function. Numerical results are presented for optimum BSP in terms of the Bayes risk in Tables 1-7 for quadratic and fifth degree loss function. We have considered the one-parameter exponential distribution to illustrate the proposed methodology. In principle, the proposed methodology can be extended to other lifetime distributions. More work is needed along this direction.

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7 Appendix

7.1 Proof of Lemma 3.2

Proof. From Lemma 3.1, the conditional PDF $\pi(\lambda|m, y(n, r, \tau_1, \tau_2, m))$ is a family of densities with MLR in λ, considering $y(n, r, \tau_1, \tau_2, m)$ or m as a fixed quantity.

(a) The proof is similar as given in Lemma 3.4.2 of Lehmann (2005), p.70, which is as follows:

Let for a fixed $m \geq 0$, $y = y(n, r, \tau_1, \tau_2, m)$, $y^* = y(n, r, \tau_1, \tau_2, m)$ and $y < y^*$. Let A and B be the sets such that

$$\begin{aligned} A &= \{\lambda : \pi(\lambda|m, y^*) > \pi(\lambda|m, y) \;\forall\; m > 0\} \\ B &= \{\lambda : \pi(\lambda|m, y^*) \leq \pi(\lambda|m, y) \;\forall\; m > 0\}. \end{aligned}$$

If $g(\lambda)$ is a positive and increasing function of λ and if $a = \underset{A}{sup}\{g(\lambda)\}$ and $b = \underset{B}{inf}\{g(\lambda)\}$ then $b - a > 0$. Now consider

$$\begin{aligned} &\phi_\pi(m, y^*) - \phi_\pi(m, y) \\ &= \int g(\lambda)\pi(\lambda|m, y^*)d\lambda - \int g(\lambda)\pi(\lambda|m, y)d\lambda = \int g(\lambda)[\pi(\lambda|m, y^*) - \pi(\lambda|m, y)]d\lambda \\ &= \int_A g(\lambda)[\pi(\lambda|m, y^*) - \pi(\lambda|m, y)]d\lambda + \int_B g(\lambda)[\pi(\lambda|m, y^*) - \pi(\lambda|m, y)]d\lambda \\ &\leq a\int_A [\pi(\lambda|m, y^*) - \pi(\lambda|m, y)]d\lambda + b\int_B [\pi(\lambda|m, y^*) - \pi(\lambda|m, y)]d\lambda \\ &= -(b-a)\int_A [\pi(\lambda|m, y^*) - \pi(\lambda|m, y)]d\lambda < 0. \end{aligned} \tag{19}$$

The last step follows from the definition of PDF, i.e.,

$$\int_A (\pi(\lambda|m, y^*) - \pi(\lambda|m, y))d\lambda = -\int_B (\pi(\lambda|m, y^*) - \pi(\lambda|m, y))d\lambda.$$

Then (19) implies that

$$\phi_\pi(m, y^*) < \phi_\pi(m, y).$$

Hence, for fixed integer $m \geq 0$, $\phi_\pi(m, y(n, r, \tau_1, \tau_2, m))$ is a strictly decreasing function of $y(n, r, \tau_1, \tau_2, m)$. Similarly, we can prove that $\phi_\pi(m, y(n, r, \tau_1, \tau_2, m))$ is strictly increasing in m

for a fixed $y(n,r,\tau_1,\tau_2,m) \geq 0$.

(b) Follows directly from *(a)* using relation (12), i.e.,

$$\delta_B(m, y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2) = \begin{cases} 1, & \text{if } \phi_\pi(m, y(n,r,\tau_1,\tau_2,m)) \leq C_r \\ 0, & otherwise. \end{cases}$$

For a fixed $m \geq 0$, if $y < y^*$, then from *(a)*

$$\phi_\pi(m, y^*) < \phi_\pi(m, y).$$

Thus, if $\phi_\pi(m,y) > C_r$, then $\delta_B(m,y|n,r,\tau_1,\tau_2) = 0$, but $\delta_B(m,y^*|n,r,\tau_1,\tau_2)$ can take either of the two values 0 or 1. If $\phi_\pi(m,y) \leq C_r$, then $\delta_B(m,y|n,r,\tau_1,\tau_2) = 1$, which implies that $\delta_B(m,y^*|n,r,\tau_1,\tau_2) = 1$. Therefore, in both the cases $\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2)$ is non-decreasing in $y(n,r,\tau_1,\tau_2,m)$. Similarly, we can prove that for fixed $y(n,r,\tau_1,\tau_2,m) \geq 0$, $\delta_B(m,y(n,r,\tau_1,\tau_2,m)|n,r,\tau_1,\tau_2)$ is non-increasing in m. ∎

7.2 Proof of Lemma 3.3

Proof. *(1)* Since T_m is defined as:

$$T_m = \begin{cases} inf\ A(m), & \text{if } A(m) \text{ is nonempty} \\ \infty, & otherwise, \end{cases}$$

therefore, *(a)* if $T_m = \infty$, $A(m)$ is empty set and hence $\forall\ y > 0$, $\phi_\pi(m,y) > C_r$. Similarly, *(b)* if $T_m = 0, A(m)$ is non-empty set, so that $\forall\ y > 0$, $\phi_\pi(m,y) < C_r$. Finally, *(c)* when $0 < T_m < \infty$ and $0 < y < T_m$, then by definition of T_m, we have $y \notin A(m)$, which implies that $\phi_\pi(m,y) > C_r$. If $y > T_m$, then $y \in A(m)$, which implies that $\phi_\pi(m,y) < C_r$ and when $y = T_m$, then $\phi_\pi(m,T_m) = C_r$.

(2) Let $m < m^*$ and m, $m^* \in D$, then

$$\begin{aligned} A(m) &= \{y > 0 : \phi_\pi(m,y) \leq C_r\} \\ A(m^*) &= \{y > 0 : \phi_\pi(m^*,y) \leq C_r\} \end{aligned}$$

and by Lemma 3.2, $\phi_\pi(m,y) < \phi_\pi(m^*,y)$ for all $y > 0$. Therefore, consider $y' > 0$ such that $y' \in A(m^*)$, which implies $\phi_\pi(m^*,y') \leq C_r$, i.e., $\phi_\pi(m,y') \leq C_r$ leading to $y' \in A(m)$. Hence, $A(m^*) \subset A(m)$, which implies that

$$inf\ A(m) < inf\ A(m^*), \text{ i.e., } T_m < T_{m^*}$$

Thus, T_m is strictly increasing in m for $m \in D$. ∎

7.3 Computation of $E(\tau^*)$

Total time of experiment for Type-II GHCS is given as:

$$\tau^* = \begin{cases} \tau_1, & \text{if } X_{(r)} \leq \tau_1 \\ X_{(r)}, & \text{if } \tau_1 < X_{(r)} \leq \tau_2 \\ \tau_2, & \text{if } X_{(r)} > \tau_2, \end{cases}$$

which can be written as:

$$\tau^* = \tau_1 I(X_{(r)} \leq \tau_1) + X_{(r)} I(\tau_1 < X_{(r)} \leq \tau_2) + \tau_2 I(X_{(r)} > \tau_2),$$

where $I(A)$ is an indicator function on set A, such that, if A occurs then $I(A) = 1$, otherwise $I(A) = 0$. Therefore,

$$E(\tau^*|\lambda) = \tau_1 P(X_{(r)} \leq \tau_1|\lambda) + E(X_{(r)} I(\tau_1 < X_{(r)} \leq \tau_2)|\lambda) + \tau_2 P(X_{(r)} > \tau_2|\lambda).$$

The PDF of $X_{(r)}$ is given by:

$$\begin{aligned} f_{X_{(r)}}(y) &= \frac{n!}{(r-1)!(n-r)!}(F_X(y))^{r-1}(1-F_X(y))^{n-r} f_X(y) \\ &= \frac{n!}{(r-1)!(n-r)!}(1-e^{-\lambda y})^{r-1} e^{\lambda(n-r)y} \lambda e^{-\lambda y} \\ &= r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\lambda e^{-\lambda(n-j)y}, \quad y > 0, \ \lambda > 0. \end{aligned}$$

Therefore,

$$P(X_{(r)} \leq \tau_1|\lambda) = \int_0^{\tau_1} f_{X_{(r)}}(y)dy$$

$$
\begin{aligned}
&= r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\int_0^{\tau_1}\lambda e^{-\lambda(n-j)y}dy \\
&= r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\left[\frac{1}{n-j}-\frac{e^{-\lambda(n-j)\tau_1}}{n-j}\right],
\end{aligned}
$$

$$
\begin{aligned}
&E(X_{(r)}I(\tau_1 < X_{(r)} \le \tau_2)|\lambda) \\
&\quad = \int_{\tau_1}^{\tau_2} y f_{X_{(r)}}(y)dy \\
&\quad = r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\int_{\tau_1}^{\tau_2}\lambda e^{-\lambda(n-j)y}dy \\
&\quad = r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\left[\frac{\tau_1 e^{-\lambda(n-j)\tau_1}}{(n-j)}+\frac{e^{-\lambda(n-j)\tau_1}}{\lambda(n-j)^2}-\frac{\tau_2 e^{-\lambda(n-j)\tau_2}}{(n-j)}-\frac{e^{-\lambda(n-j)\tau_2}}{\lambda(n-j)^2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
P(X_{(r)} > \tau_1|\lambda) &= r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\int_{\tau_2}^{\infty}\lambda e^{-\lambda(n-j)y}dy \\
&= r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\frac{e^{-\lambda(n-j)\tau_2}}{n-j}.
\end{aligned}
$$

Hence, expected duration of the experiment is:

$$
\begin{aligned}
E(\tau^*) &= E_\lambda(E(\tau^*|\lambda)) \\
&= \tau_1 r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\left[\frac{1}{n-j}-\frac{b^a}{(n-j)(b+(n-j)\tau_1)^a}\right] \\
&+ r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{(-1)^{r-1-j}}{n-j}\left[\frac{\tau_1 b^a}{(b+(n-j)\tau_1)^a}+\frac{b^a}{(n-j)(a-1)(b+(n-j)\tau_1)^{a-1}}\right. \\
&\qquad \left. -\frac{\tau_2 b^a}{(b+(n-j)\tau_2)^a}-\frac{b^a}{(n-j)(a-1)(b+(n-j)\tau_2)^{a-1}}\right] \\
&\qquad + \tau_2 r\binom{n}{r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\frac{b^a}{(n-j)(b+(n-j)\tau_2)^a}.
\end{aligned}
$$

7.4 Computation of $E(M)$

In Type-II GHCS, number of failures in the experiment is given by:

$$M = \begin{cases} M_1, & \text{if} \quad X_{(r)} \leq \tau_1 \\ r, & \text{if} \quad \tau_1 < X_{(r)} \leq \tau_2 \\ M_2, & \text{if} \quad X_{(r)} > \tau_2. \end{cases}$$

Therefore,

$$E(M) = E_\lambda(E(M|\lambda)),$$

where

$$\begin{aligned} E(M|\lambda) &= \sum_{m_1=r}^{n} m_1 P(M_1 = m_1) + rP(\tau_1 < X_{(r)} < \tau_2) + \sum_{m_2=0}^{r-1} m_2 P(M_2 = m_2) \\ &= \sum_{m_1=r}^{n} m_1 \binom{n}{m_1} (1 - e^{\lambda\tau_1})^{m_1} e^{-\lambda(n-m_1)\tau_1} + r\int_{\tau_1}^{\tau_2} f_{X_{(r)}}(y)dy \\ &\quad + \sum_{m_2=0}^{r-1} m_2 \binom{n}{m_2} (1 - e^{\lambda\tau_2})^{m_2} e^{-\lambda(n-m_2)\tau_2}. \end{aligned}$$

Here

$$\begin{aligned} \int_{\tau_1}^{\tau_2} f_{X_{(r)}}(y)dy &= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \lambda \int_{\tau_1}^{\tau_2} e^{-\lambda(n-j)y} dy \\ &= \frac{n!}{(r-1)!(n-r)!} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \left[\frac{e^{-\lambda(n-j)\tau_1}}{n-j} - \frac{e^{-\lambda(n-j)\tau_2}}{n-j} \right]. \end{aligned}$$

Then

$$\begin{aligned} E(M|\lambda) &= \sum_{m_1=r}^{n} \sum_{j=0}^{m_1} m_1 \binom{n}{m_1} \binom{m_1}{j} (-1)^j e^{-\lambda(n-m_1+j)\tau_1} \\ &\quad + r^2 \sum_{j=0}^{r-1} \binom{n}{r} \binom{r-1}{j} (-1)^{r-1-j} \left[\frac{e^{-\lambda(n-j)\tau_1}}{n-j} - \frac{e^{-\lambda(n-j)\tau_2}}{n-j} \right] \\ &\quad + \sum_{m_2=0}^{r-1} \sum_{j=0}^{m_2} m_2 \binom{n}{m_2} \binom{m_2}{j} (-1)^j e^{-\lambda(n-m_2+j)\tau_2}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
E(M) &= \sum_{m_1=r}^{n}\sum_{j=0}^{m_1} m_1 \binom{n}{m_1}\binom{m_1}{j}(-1)^j \frac{b^a}{(b+(n-m_1+j)\tau_1)^a} \\
&+ r^2 \sum_{j=0}^{r-1}\binom{n}{r}\binom{r-1}{j}\frac{(-1)^{r-1-j}}{n-j}\left[\frac{b^a}{(b+(n-j)\tau_1)^a}-\frac{b^a}{(b+(n-j)\tau_2)^a}\right] \\
&\quad + \sum_{m_2=0}^{r-1}\sum_{j=0}^{m_2} m_2 \binom{n}{m_2}\binom{m_2}{j}(-1)^j \frac{b^a}{(b+(n-m_2+j)\tau_2)^a}.
\end{aligned}
$$

7.5 Computation of $E(MSE(\widehat{\lambda}))$

Since,

$$
E(MSE(\widehat{\lambda})) = E_\lambda(MSE(\widehat{\lambda})), \tag{20}
$$

$$
MSE(\widehat{\lambda}) = E[(\widehat{\lambda}-\lambda)^2|\lambda] = E(\widehat{\lambda}^2|\lambda) + \lambda^2 - 2\lambda E(\widehat{\lambda}|\lambda).
$$

The estimator $\widehat{\lambda}$ is given by:

$$
\widehat{\lambda} = \begin{cases} 0, & \text{if } M = 0 \\ \widehat{\lambda}_{MLE}, & \text{if } M > 0, \end{cases}
$$

Now using the CDF (2) of $\widehat{\lambda}$ and the PDF of absolutely continuous part of the $\widehat{\lambda}$ which is given in Lemma 2.1, we obtain $E(\widehat{\lambda}|\lambda)$ and $E(\widehat{\lambda}^2|\lambda)$, which is given by:

$$
\begin{aligned}
E(\widehat{\lambda}|\lambda) &= \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}\frac{(-1)^j m_1^{m_1}}{\Gamma(m_1)}\int_0^\infty \frac{\lambda^{m_1}u^{m_1-1}e^{-\lambda(m_1u+(n-m_1+j)\tau_1)}}{\left(u+\frac{(n-m_1+j)\tau_1}{m_1}\right)}du \\
&+ \sum_{m_1=0}^{r-1}\sum_{v=r}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{v}\binom{n}{m_1}\binom{n-m_1}{v}\binom{m_1}{j}\binom{v}{i}\frac{(-1)^{i+j}r^r}{\Gamma(r)} \\
&\qquad \times \int_0^\infty \frac{\lambda^r u^{r-1}e^{-\lambda(ru+(j+v-i)\tau_1+(i+n-v-m_1)\tau_2)}}{\left(u+\frac{(j+v-i)\tau_1+(i+n-v-m_1)\tau_2}{r}\right)}du \\
&+ \sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}\frac{(-1)^j m_2^{m_2}}{\Gamma(m_2)}\int_0^\infty \frac{\lambda^{m_2}u^{m_2-1}e^{-\lambda(m_2u+(n-m_2+j)\tau_2)}}{\left(u+\frac{(n-m_2+j)\tau_2}{m_2}\right)}du,
\end{aligned}
$$

$$
\begin{aligned}
E(\widehat{\lambda}^2|\lambda) \;=\;& \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}\frac{(-1)^j m_1^{m_1}}{\Gamma(m_1)}\int_0^\infty \frac{\lambda^{m_1}u^{m_1-1}e^{-\lambda(m_1u+(n-m_1+j)\tau_1)}}{\left(u+\frac{(n-m_1+j)\tau_1}{m_1}\right)^2}du \\
&+\sum_{m_1=0}^{r-1}\sum_{v=r}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{v}\binom{n}{m_1}\binom{n-m_1}{v}\binom{m_1}{j}\binom{v}{i}\frac{(-1)^{i+j}r^r}{\Gamma(r)} \\
&\qquad\times\int_0^\infty \frac{\lambda^r u^{r-1}e^{-\lambda(ru+(j+v-i)\tau_1+(i+n-v-m_1)\tau_2)}}{\left(u+\frac{(j+v-i)\tau_1+(i+n-v-m_1)\tau_2}{r}\right)^2}du \\
&+\sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}\frac{(-1)^j m_2^{m_2}}{\Gamma(m_2)}\int_0^\infty \frac{\lambda^{m_2}u^{m_2-1}e^{-\lambda(m_2u+(n-m_2+j)\tau_2)}}{\left(u+\frac{(n-m_2+j)\tau_2}{m_2}\right)^2}du.
\end{aligned}
$$

Hence from (20) the expected MSE is:

$$
\begin{aligned}
E(MSE(\widehat{\lambda})) &= E_\lambda(E(\widehat{\lambda}^2|\lambda))+E_\lambda(\lambda^2)-2E_\lambda(\lambda E(\widehat{\lambda}|\lambda)) \\
&= E_\lambda(E(\widehat{\lambda}^2|\lambda))+\frac{a(a+1)}{b^2}-2E_\lambda(\lambda E(\widehat{\lambda}|\lambda)),
\end{aligned}
$$

where

$$
\begin{aligned}
&E_\lambda(\lambda E(\widehat{\lambda}|\lambda)) \\
&=\frac{n}{n-1}\frac{a(a+1)}{b^2}+\sum_{\substack{m_1=r\\ (m_1,j):(n-m_1+j)\neq 0}}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}\frac{(-1)^j m_1^{m_1}b^a}{\Gamma(m_1)\Gamma(a)} \\
&\qquad\times\int_0^\infty \frac{u^{m_1-1}\Gamma(a+m_1+1)}{\left(u+\frac{(n-m_1+j)\tau_1}{m_1}\right)\left(m_1u+b+(n-m_1+j)\tau_1\right)^{a+m_1+1}}du \\
&+\sum_{m_1=0}^{r-1}\sum_{v=r}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{v}\binom{n}{m_1}\binom{n-m_1}{v}\binom{m_1}{j}\binom{v}{i}\frac{(-1)^{i+j}r^r b^a}{\Gamma(r)\Gamma(a)} \\
&\times\int_0^\infty \frac{u^{r-1}\Gamma(a+r+1)}{\left(u+\frac{(j+v-i)\tau_1+(i+n-v-m_1)\tau_2}{r}\right)\left(ru+b+(j+v-i)\tau_1+(i+n-v-m_1)\tau_2\right)^{a+r+1}}du \\
&+\sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}\frac{(-1)^j m_2^{m_2}b^a}{\Gamma(m_2)\Gamma(a)} \\
&\qquad\times\int_0^\infty \frac{u^{m_2-1}\Gamma(a+m_2+1)}{\left(u+\frac{(n-m_2+j)\tau_2}{m_2}\right)(m_2u+b+(n-m_2+j)\tau_2)^{a+m_2+l}}du,
\end{aligned}
$$

$$
\begin{aligned}
&E_\lambda(E(\widehat{\lambda}^2|\lambda)) \\
&=\frac{n^2}{(n-1)(n-2)}\frac{a(a+1)}{b^2}+\sum_{\substack{m_1=r\\ (m_1,j):(n-m_1+j)\neq 0}}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}\frac{(-1)^j m_1^{m_1}b^a}{\Gamma(m_1)\Gamma(a)}
\end{aligned}
$$

$$\times \int_0^\infty \frac{u^{m_1-1}\Gamma(a+m_1)}{\left(u+\frac{(n-m_1+j)\tau_1}{m_1}\right)^2\left(m_1u+b+(n-m_1+j)\tau_1\right)^{a+m_1}}du$$

$$+\sum_{m_1=0}^{r-1}\sum_{v=r}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{v}\binom{n}{m_1}\binom{n-m_1}{v}\binom{m_1}{j}\binom{v}{i}\frac{(-1)^{i+j}r^rb^a}{\Gamma(r)\Gamma(a)}$$

$$\times \int_0^\infty \frac{u^{r-1}\Gamma(a+r)}{\left(u+\frac{(j+v-i)\tau_1+(i+n-v-m_1)\tau_2}{r}\right)^2\left(ru+b+(j+v-i)\tau_1+(i+n-v-m_1)\tau_2\right)^{a+r}}du$$

$$+\sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}\frac{(-1)^jm_2^{m_2}b^a}{\Gamma(m_2)\Gamma(a)}$$

$$\times \int_0^\infty \frac{u^{m_2-1}\Gamma(a+m_2)}{\left(u+\frac{(n-m_2+j)\tau_2}{m_2}\right)^2(m_2u+b+(n-m_2+j)\tau_2)^{a+m_2}}du.$$

Note that $E(MSE(\widehat{\lambda}))$ exists when $n>2$.

7.6 Bayes risk of the BSP

Bayes risk of sampling plan $(n,r,\tau_1,\tau_2,\delta_B)$ is given as:

$$\begin{aligned}
R_B^{\delta_B}(n,r,\tau_1,\tau_2) &= n(C_s-r_s)+E(M)r_s+E(\tau^*)C_\tau+E(MSE(\hat{\lambda}))C_v+a_0+a_1\mu_1+a_2\mu_2\\
&+ \int_0^\infty [C_r-a_0-a_1\lambda-a_2\lambda^2]P(\delta_B(M,Y(n,r,\tau_1,\tau_2,M)|n,r,\tau_1,\tau_2)=0|\lambda)\pi(\lambda)d\lambda\\
&= n(C_s-r_s)+E(M)r_s+E(\tau^*)C_\tau+E(MSE(\widehat{\lambda}))C_v+a_0+a_1\mu_1+\ldots+a_k\mu_k\\
&\quad +\sum_{l=0}^{k}C_l\frac{b^a}{\Gamma(a)}\int_0^\infty \lambda^{a+l-1}e^{-\lambda b}P(\delta_B(M,Y(n,r,\tau_1,\tau_2,M)|n,r,\tau_1,\tau_2)=0|\lambda)d\lambda.
\end{aligned}$$

The distribution of $\hat{\lambda}$ given in (2) is used to obtain last integral

$$\begin{aligned}
&\int_0^\infty \lambda^{a+l-1}e^{-\lambda b}P(\delta_B(M,Y(n,r,\tau_1,\tau_2,M)|n,r,\tau_1,\tau_2)=0|\lambda)\,d\lambda\\
=\ &\int_0^\infty \lambda^{a+l-1}e^{-\lambda b}e^{-n\lambda\tau_2}\,\boldsymbol{I}(n\tau_2<T_0)\,d\lambda+\int_0^\infty \lambda^{a+l-1}e^{-\lambda b}P(\hat{\lambda}\geq c(n,r,\tau_1,\tau_2,M)|\lambda,M\geq 1)P(M\geq 1)\,d\lambda\\
=\ &\frac{\Gamma(a+l)\boldsymbol{I}(n\tau_2<T_0)}{(b+n\tau_2)^{(a+l)}}+\int_0^\infty \lambda^{a+l-1}e^{-\lambda b}P(\hat{\lambda}\geq \frac{M}{T(n,r,\tau_1,\tau_2,M)}|\lambda,M\geq 1)P(M\geq 1)\,d\lambda. \qquad (21)
\end{aligned}$$

Using PDF of $\hat{\lambda}$ from Lemma 2.1 in Eq. (21) we get

$$\int_0^\infty \lambda^{a+l-1}e^{-\lambda b}P(\hat{\lambda}\geq \frac{M}{T(n,r,\tau_1,\tau_2,M)}|\lambda,M\geq 1)P(M\geq 1)\,d\lambda$$

$$
\begin{aligned}
= & \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}(-1)^j\frac{(m_1)^{m_1}}{\Gamma(m_1)}\int_0^\infty\int_{\frac{m_1}{T(n,r,\tau_1,\tau_2,m_1)}}^\infty \lambda^{a+m_1+l-1}\frac{e^{-\lambda[\frac{m_1}{x}+b]}}{x^2}\left(\frac{1}{x}-\frac{(n-m_1+j)\tau_1}{m_1}\right)^{m_1-1}dxd\lambda \\
& +\sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{k}\binom{n}{m_1}\binom{n-m_1}{k}\binom{m_1}{j}\binom{k}{i}(-1)^{i+j}\frac{(r)^r}{\Gamma(r)} \\
& \times\int_0^\infty\int_{\frac{r}{T(n,r,\tau_1,\tau_2,r)}}^\infty \lambda^{a+r+l-1}\frac{1}{x^2}\left(\frac{1}{x}-\frac{[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}{r}\right)^{r-1}e^{-\lambda[\frac{r}{x}+b]}dxd\lambda \\
+ & \sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}(-1)^j\frac{(m_2)^{m_2}}{\Gamma(m_2)}\int_0^\infty\int_{\frac{m_2}{T(n,r,\tau_1,\tau_2,m_2)}}^\infty \lambda^{a+m_2+l-1}\frac{1}{x^2}\left(\frac{1}{x}-\frac{(n-m_2+j)\tau_2}{m_2}\right)^{m_2-1}e^{-\lambda[\frac{m_2}{x}+b]}dxd\lambda \\
= & \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}(-1)^j\frac{(m_1)^{m_1}}{\Gamma(m_1)}\int_0^\infty\int_0^{\frac{T(n,r,\tau_1,\tau_2,m_1)}{m_1}} \lambda^{a+m_1+l-1}\left(x-\frac{(n-m_1+j)\tau_1}{m_1}\right)^{m_1-1}e^{-\lambda[m_1x+b]}dxd\lambda \\
& +\sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{k}\binom{n}{m_1}\binom{n-m_1}{k}\binom{m_1}{j}\binom{k}{i}(-1)^{i+j}\frac{(r)^r}{\Gamma(r)} \\
& \times\int_0^\infty\int_0^{\frac{T(n,r,\tau_1,\tau_2,r)}{r}} \lambda^{a+r+l-1}\left(x-\frac{[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}{r}\right)^{r-1}e^{-\lambda[rx+b]}dxd\lambda \\
+ & \sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}(-1)^j\frac{(m_2)^{m_2}}{\Gamma(m_2)}\int_0^\infty\int_0^{\frac{T(n,r,\tau_1,\tau_2,m_2)}{m_2}} \lambda^{a+m_2+l-1}\left(x-\frac{(n-m_2+j)\tau_2}{m_2}\right)^{m_2-1}e^{-\lambda[m_2x+b]}dxd\lambda \\
= & \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}(-1)^j\frac{(m_1)^{m_1}}{\Gamma(m_1)}\int_0^{\frac{T(n,r,\tau_1,\tau_2,m_1)}{m_1}-\frac{(n-m_1+j)\tau_1}{m_1}}\frac{u^{m_1-1}\Gamma(a+m_1+l)}{(m_1u+b+(n-m_1+j)\tau_1)^{a+m_1+l}}du \\
+ & \sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{k}\binom{n}{m_1}\binom{n-m_1}{k}\binom{m_1}{j}\binom{k}{i}(-1)^{i+j}\frac{(r)^r}{\Gamma(r)} \\
& \times\int_0^{\frac{T(n,r,\tau_1,\tau_2,r)}{r}-\frac{[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}{r}}\frac{u^{r-1}\Gamma(a+r+l)}{(ru+b+[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2])^{a+r+l}}du \\
+ & \sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}(-1)^j\frac{(m_2)^{m_2}}{\Gamma(m_2)}\int_0^{\frac{T(n,r,\tau_1,\tau_2,m_2)}{m_2}-\frac{(n-m_2+j)\tau_2}{m_2}}\frac{u^{m_2-1}\Gamma(a+m_2+l)}{(m_2u+b+(n-m_2+j)\tau_2)^{a+m_2+l}}du \\
= & \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}(-1)^j\frac{\Gamma(m_1+a+l)}{\Gamma(m_1)(b+(n-m_1+j)\tau_1)^{a+l}}\int_0^{\frac{T(n,r,\tau_1,\tau_2,m_1)-(n-m_1+j)\tau_1}{b+(n-m_1+j)\tau_1}}\frac{z^{m_1-1}}{(1+z)^{a+m_1+l}}dz \\
+ & \sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{k}\binom{n}{m_1}\binom{n-m_1}{k}\binom{m_1}{j}\binom{k}{i}(-1)^{i+j}\frac{\Gamma(a+r+l)}{\Gamma(r)(b+[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2])^{a+l}} \\
& \times\int_0^{\frac{T(n,r,\tau_1,\tau_2,r)-[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}{b+[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}}\frac{z^{r-1}}{(1+z)^{a+r+l}}dz \\
+ & \sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}(-1)^j\frac{\Gamma(a+m_2+l)}{\Gamma(m_2)(b+(n-m_2+j)\tau_2)^{a+l}}\int_0^{\frac{T(n,r,\tau_1,\tau_2,m_2)-(n-m_2+j)\tau_2}{b+(n-m_2+j)\tau_2}}\frac{z^{m_2-1}}{(1+z)^{a+m_2+l}}dz.
\end{aligned}
$$

Now taking a transformation $z = u/(1-u)$, we have

$$
\int_0^C \frac{z^{m-1}}{(1+z)^{a+l+m}}dz = \int_0^S u^{m-1}(1-u)^{a+l-1}du = B_S(m, a+l),
$$

where $C = C_{m,j}$ or $C_{m_1,k,j,i}$, $C_{m,j} = \frac{T(n,r,\tau_1,\tau_2,m)-(n-m+j)\tau}{b+(n-m+j)\tau}$, $C_{m_1,k,j,i} = \frac{T(n,r,\tau_1,\tau_2,r)-[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}{b+[(j+k-i)\tau_1+(i+n-k-m_1)\tau_2]}$, $S = S_{m,j}$ or $S_{m_1,k,j,i}$, $S_{m,j} = \dfrac{C_{m,j}}{1+C_{m,j}}$, $S_{m_1,k,j,i} = \dfrac{C_{m_1,k,j,i}}{1+C_{m_1,k,j,i}}$ and $B_x(\alpha,\beta)$ and $I_x(\alpha,\beta)$ are as defined earlier. Then the Bayes risk of the BSP $(n,r,\tau_1,\tau_2,\delta_B)$ is given by:

$$\begin{aligned} R_B^{\delta_B}(n,r,\tau_1,\tau_2) &= n(C_s-r_s)+E(M)r_s+E(\tau^*)C_\tau+E(MSE(\hat{\lambda}))C_v+a_0+a_1\mu_1+a_2\mu_2 \\ &+ \sum_{l=0}^{2} C_l \frac{b^a}{\Gamma(a)}\Bigg[\frac{\Gamma(a+l)\boldsymbol{I}(n\tau_2<T_0)}{(b+n\tau_2)^{a+l}} + \sum_{m_1=r}^{n}\sum_{j=0}^{m_1}\binom{n}{m_1}\binom{m_1}{j}\frac{(-1)^j\Gamma(a+l)I_{S_{m_1,j}}(m_1,a+l)}{(b+(n-m_1+j)\tau_1)^{a+l}} \\ &\quad + \sum_{m_1=0}^{r-1}\sum_{k=r-m_1}^{n-m_1}\sum_{j=0}^{m_1}\sum_{i=0}^{k}\binom{n}{m_1}\binom{m_1}{j}\frac{(-1)^j\Gamma(a+l)I_{S_{m_1,k,j,i}}(r,a+l)}{(b+(j+k-i)\tau_1+(i+n-k-m_1)\tau_2)^{a+l}} \\ &\quad\quad + \sum_{m_2=1}^{r-1}\sum_{j=0}^{m_2}\binom{n}{m_2}\binom{m_2}{j}\frac{(-1)^j\Gamma(a+l)I_{S_{m_2,j}}(m_2,a+l)}{(b+(n-m_2+j)\tau_2)^{a+l}}\Bigg]. \end{aligned}$$

7.7 Pseudo Code

Algorithm 1: Pseudo code to obtain optimum BSP

```
Input: a, b, a_0, a_1, a_2, C_s, r_s, C_r, C_v, C_τ  /* Costs, Coefficients and hyper parameters */
Output: n_B, r_B, τ_1B, τ_2B, R_B^{δ_B}  /* Optimum BSP and R_B^{δ_B} */
Algorithm Main:
    /* Let N is an upper bound of n_B */
    for n = 0, 1, ..., N do
        for r = 0, 1, ..., n do
            Function BayesRisk(τ_1, τ_2):
                if τ_1 > 0 && τ_2 > τ_1 then
                    Calculate the Bayes Risk R_B^{δ_B}(n, r, τ_1, τ_2)
                    rb = R_B^{δ_B}(n, r, τ_1, τ_2) /* Bayes Risk */
                else
                    rb = C_r /* Rejection Bayes Risk without sampling */
                return rb
            (i) Function optim() in R is used to minimize BayesRisk(τ_1, τ_2) w.r.t. τ_1 and τ_2.
            (ii) Then store the value of τ_1, τ_2 and corresponding Bayes Risk.
        (i) Minimize the Bayes Risk w.r.t. r.
        (ii) Then store the minimum Bayes Risk and corresponding values of r, τ_1, τ_2.
    (i) Minimize the Bayes Risk w.r.t. n.
    print n_B, r_B, τ_1B, τ_2B, R_B^{δ_B} /* Optimum BSP and corresponding Bayes Risk */
    return 0
```

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