Exact Inference of a Simple Step Stress Model with Hybrid Type-II Stress Changing Time

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Abstract

In this article we consider a simple step stress model for exponentially distributed lifetime units. As failure rate is lower at the initial stress level therefore, it is important to pay more attention on the stress changing time. Here we consider a simple step stress model where the stress level changes either after a prefixed time or after a prefixed number of failures, whichever occurs later. It ensures a prefixed minimum number of failures at the first stress level and also sets up a control on the expected experimental time. We have obtained the maximum likelihood estimators of the model parameters along with their exact distributions. The monotonicity properties of the maximum likelihood estimators have been established here, and it can be used to construct the exact confidence intervals of the unknown parameters. We provide approximate and bias corrected accelerated bootstrap confidence intervals of the model parameters. We also define an optimality criteria and based on that obtain an optimal stress changing time for a given sample size. Finally an extensive simulation study has been performed to asses the performance of the proposed methods and provide the analyses of two data sets for illustrative purpose.

Key Words Step-stress Life-tests; maximum likelihood estimator; approximate confidence interval; bias corrected accelerated bootstrap confidence interval; optimality.

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1 Introduction

The objective of the life testing experiment includes the determination of the expected life of the experimental units at a design condition, the percentage of failing on warranty, the reliability of product on a stressed situation, the dependency of the product life on manufacturing and some other related variables etc. The statistical analysis of any life testing data requires sufficient observations; which becomes very difficult now a days due to high reliability of the products. The accelerated life testing (ALT) is an useful technique to reduce the experimental time and yield failures quickly. One way to reduce the experimental time is to perform the experiment on a stressed conditions. The stress factors that usually used in an ALT experiment are temperature, voltage, humidity, load, pressure etc. A constant stress can be employed to all the units throughout the experiment. Another way to perform the ALT experiment is to increase the stress levels at some prespecified time. Such an ALT experiment is called the step stress life testing (SSLT) experiment. In a SSLT experiment certain number of identical units are put into an experiment and after each prespecified length of time, the stress level is increased to its next level. For simplicity many works on SSLT experiments, such as, Balakrishnan and Xie [9, 8], Mitra et al. [19] have been done by taking only two stress levels. Such a model is called the simple step stress model. Recently Samanta et al. [21] have worked on the order restricted inference of a simple step stress model. For more recent developments on step stress model see the monograph by Kundu and Ganguly [18]. The analysis of the data from SSLT experiment needs a model. Such a model relates the lifetime distributions between different stress levels. The widely used model in this purpose is the cumulative exposure model (CEM) as proposed by Sedyakin [23]. This model relates the distributions under constant stress levels to the distribution under step stress experiment by assuming a Markov property, where the remaining life of units depends only on the current cumulative fraction. This model has been studied extensively by Nelson [20], Bagdonavicius and Nikulin [2] and Ismail [15] and see the references cited therein. An extensive review work on step stress experiment assuming CEM has been done by Balakrishnan [4]. Another step stress model is the tampered failure rate model proposed by Bhattacharyya and Soejoeti [10].

The reliability experimenters are mainly interested on the parameter estimates at the normal operating conditions which is usually the first stress level of the experiment. Therefore, more attention needs to be paid on the stress changing time. In most of the SSLT experiment the stress changes either after a prefixed time or after a prefixed number of failures. In the first case there may be a situation where the number of observed failures at the first stress level is zero or very few. In that case the inference on the unknown parameter(s) may not be very reliable. In the second case the stress changing time is random, which has been first considered by Xiong and Milliken [24]. Later this model has been studied by several authors, see for example Kundu and Balakrishnan [16] and Ganguly and Kundu [13]. But in this case the prefixed number of failures may occur too early, where as the experimenter can afford to give some more time at the first stress level which will yield more information about the first stress level. This two problems can be taken care by considering hybrid Type-II stress changing time, i.e., if τ is a prefixed time and r is a prefixed number then $\tau^* = max\{\tau, t_{r:n}\}$, is our stress changing time, where $t_{r:n}$ is the $r-$th failure time. This ensures a minimum of r failures at the first stress level and also allows the experimenter to continue the experiment with the initial stress at least time τ. Again to control the total experimental time, r and τ should be selected very carefully. Hence, the experimenter can prefixed the experimental time and choose the best r and τ based on some optimality criteria, among all sets of pair (r, τ) for which the expected experimental time is less than the prefixed time. Balakrishnan [4] in an open problem suggested considering inference procedure based on hybrid stress changing time point. The Type-II hybrid stress changing time ensures sufficient failure data at the initial stress level and also it has a control on the expected time of the experiment.

Though the more importance is given on the inference of the parameters at the first stress level by considering the hybrid type stress changing time, the primary objective of an ALT experiment is to reduce the experimental time. Therefore, the selection of the prefixed time τ and the prefixed number r should be done very carefully and they cannot be arbitrary. Hence, it is necessary to provide an optimality criteria and the selection of τ and r should be based on this criteria. Several criteria, such as, A-optimality, D-optimality, optimality by minimizing Bayes risk have been discussed in the literature. Some key references on step stress optimality are Bai et al. [3], Alhadeed and Yang [1], Balakrishnan and Han [5] and the references cited therein for some recent developments.

In this paper we consider a simple step stress model with exponential failure time distribution at different stress levels under the cumulative exposure model assumption. The main objective of this work is to consider the Type-II hybrid stress changing time and draw the related inferences. We have obtained the maximum likelihood estimators (MLEs) of the model parameters and provide the exact distributions of the MLEs. To construct the exact confidence intervals, stochastic monotonicity properties of the MLEs are required which has been established here using the approach used by Balakrishnan and Iliopoulos [6] for different censoring schemes. We have provided an algorithm to construct the bias adjusted bootstrap confidence intervals. We also provide an optimality criteria, and based on that obtain an optimal stress changing time. Extensive simulation study and the analyses of two data sets have been performed for illustrative purpose.

The rest of the article is organized as below. Model assumptions and the MLEs are given in Section 2. In Section 3 we have obtained the exact distributions of the MLEs. Construction of different confidence intervals are provided in Section 4. In Section 5 the simulation results and the analyses of two data sets are provided. An optimality criteria and an algorithm for choosing an optimal scheme based on this criteria are presented in Section 6. Finally the paper has been concluded in Section 7. The proofs of all the theorems and lemmas used are presented in Appendix.

2 Notations and Model Assumption

2.1 Notations

ALT: accelerated life testing.

CDF: cumulative distribution function.

CEM: cumulative exposure model.

MGF: moment generating function.

MLE: maximum likelihood estimator.

PDF: probability density function.

SSLT: step stress life testing.

$\exp(\theta)$: exponential random variable with PDF:$\frac{1}{\theta}e^{-\frac{x}{\theta}}; x > 0$

$\exp(\mu, \theta)$: exponential random variable with PDF:$\frac{1}{\theta}e^{-\frac{x-\mu}{\theta}}; x > \mu$

$p(\theta) : 1 - (1 - e^{-\frac{\tau}{\theta}})^n$.

$c_{dj}(\theta) : (-1)^j \binom{n}{d} \binom{d}{j} e^{-\frac{\tau}{\theta}(n-d+j)},\ j = 0, 1, \ldots, d;\ d = 0, 1, \ldots, n-1.$

$f_{GA}(x, \alpha, p) : \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1},\ x > 0, \alpha > 0, p > 0.$

$\mu_{dj} : \frac{\tau(n-d+j)}{r},\ j = 0, 1, \ldots, d;\ d = 0, 1, \ldots, r-1.$

$\mu'_{dj} : \frac{\tau(n-d+j)}{d},\ j = 0, 1, \ldots, d;\ d = r, r+1, \ldots, n-1.$

$c_d(\theta) : \binom{n}{d} (1 - e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau}{\theta}(n-d)},\ d = 0, 1, \ldots, n.$

$\Gamma(z, \alpha, p) : \int_0^z f_{GA}(x, \alpha, p) dx,\ z > 0.$

2.2 Model Assumptions

Suppose n identical units are put into a reliability experiment. We consider a simple step stress model with stress levels s_1 and s_2. The experiment starts with the initial stress level

s_1, a given prefixed time τ and a prefixed number r. The stress level changes to s_2 at a random time $\tau^* = max\{\tau, t_{r:n}\}$, where $t_{r:n}$ is the $r-th$ ordered failure time. Since the expected lifetime is higher in first stress level, therefore to ensure minimum of r number of failures, we use τ^* as the stress changing time. The experiment continues till the last failure occurs. Let D be the number of failures before the time point τ and d be the observed value of D. Therefore, the observed failure data must be from one of the following forms:

(a) $\{t_{1:n} < ... < t_{r:n} < ... < t_{n:n} < \tau\}$ if $t_{r:n} \leq t_{n:n} \leq \tau$,

(b) $\{t_{1:n} < ... < t_{r:n} < ... < t_{d:n} < \tau < t_{d+1:n} < ... < t_{n:n}\}$ if $t_{r:n} \leq \tau < t_{n:n}$,

(c) $\{t_{1:n} < ... < t_{d:n} < \tau < t_{d+1:n} < ... < t_{r:n} < ... < t_{n:n}\}$ if $\tau < t_{r:n}$.

In case (a), there is no failure occurred at the stress level s_2, since all the failures occurred before stress changing time τ. Now we provide the likelihood function for all the three cases. In case of (a), the likelihood function of the observed data is

$$L(\theta_1; data) = \frac{n!}{\theta_1^n} e^{-\frac{\sum_{i=1}^n t_{i:n}}{\theta_1}}. \tag{1}$$

In case of (b), based on the assumptions of the CEM, the joint PDF of $\{t_{1:n} < ... < t_{r:n} < ... < t_{d:n} < \tau < t_{d+1:n} < ... < t_{n:n}\}$ can be written as follows. The joint PDF of $\{t_{1:n} < ... < t_{r:n} < ... < t_{d:n} < \tau\}$ is the first d order statistics from a sample of size n from $\exp(\theta_1)$ which has failed before τ, and the rest $n-d$ did not fail before τ. Further, the conditional PDF of $\{t_{d+1:n} < ... < t_{n:n}\}$ given $\{t_{1:n} < ... < t_{r:n} < ... < t_{d:n} < \tau\}$ is the order statistics from a sample of size $n-d$ from $\exp(\tau, \theta_2)$. Therefore, the likelihood of the data set is given by

$$L_1(\theta_1, \theta_2; data) = \frac{c_1}{\theta_1^d} e^{-\frac{D_1^b}{\theta_1}} \times \frac{c_2}{\theta_2^{n-d}} e^{-\frac{D_2^b}{\theta_2}} = \frac{n!}{\theta_1^d \theta_2^{n-d}} e^{-\frac{D_1^b}{\theta_1} - \frac{D_2^b}{\theta_2}}, \tag{2}$$

where $c_1 = n(n-1)\ldots(n-d+1)$, $c_2 = (n-d)!$, $D_1^b = \sum_{i=1}^d t_{i:n} + (n-d)\tau$ and $D_2^b = \sum_{i=d+1}^n (t_{i:n} - \tau) = \sum_{i=d+1}^n t_{i:n} - (n-d)\tau$. Similarly, in case (c), the likelihood of the data set is

$$L_2(\theta_1, \theta_2; data) = \frac{n!}{\theta_1^r \theta_2^{n-r}} e^{-\frac{D_1^c}{\theta_1} - \frac{D_2^c}{\theta_2}}, \tag{3}$$

where $D_1^c = \sum_{i=1}^{r} t_{i:n} + (n-r)t_{r:n}$ and $D_2^c = \sum_{i=r+1}^{n} t_{i:n} - (n-r)t_{r:n}$.

Let us define $N_1 = max\{r, D\}$ and n_1 be the observed value of N_1. Hence, the combined likelihood function of (b) and (c) can be written as

$$L(\theta_1, \theta_2; data) = \frac{n!}{\theta_1^{n_1}\theta_2^{n-n_1}} e^{-\frac{D_1}{\theta_1} - \frac{D_2}{\theta_2}}, \tag{4}$$

where $D_1 = \sum_{i=1}^{n_1} t_{i:n} + (n-n_1)\tau^*$ and $D_2 = \sum_{i=n_1+1}^{n} t_{i:n} - (n-n_1)\tau^*$.

Note that the MLEs of both θ_1 and θ_2 exist only if $N_1 \le n-1$, and in this case the MLEs of θ_1 and θ_2 are obtained by maximizing (4) and is given by

$$\widehat{\theta}_1 = \frac{D_1}{n_1} \quad \text{and} \quad \widehat{\theta}_2 = \frac{D_2}{n-n_1}.$$

3 Exact Distribution of MLEs

In this section we consider the exact conditional distribution of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ conditioning on $A = \{N_1 \le n-1\}$. To obtain the exact distributions of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ we will use the uniqueness property of the moment generating function. The conditional probability density function (PDF) of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are given in Theorem 1 and Theorem 2, respectively.

Theorem 1. *The PDF of $\widehat{\theta}_1$ conditioning on the event A is given by*

$$f_{\hat{\theta}_1|n_1\in A}(t_1) = \frac{1}{p(\theta_1)}\Bigg[\sum_{d=0}^{r-1}\sum_{j=0}^{d} c_{dj}(\theta_1) f_{GA}(t_1-\mu_{dj}, \tfrac{r}{\theta_1}, r) + \sum_{d=r}^{n-1}\sum_{j=0}^{d} c_{dj}(\theta_1) f_{GA}(t_1-\mu'_{dj}, \tfrac{d}{\theta_1}, d)\Bigg].$$

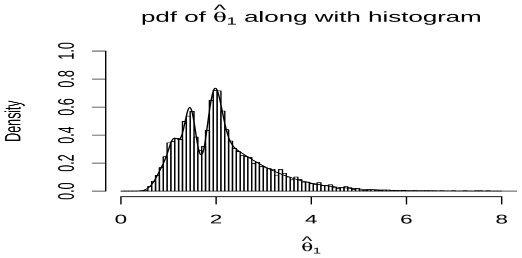
Proof. See Appendix A.1. □

Theorem 2. *The PDF of $\widehat{\theta}_2$ conditioning on A is given by*

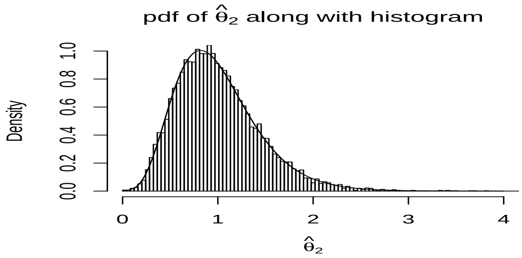
$$f_{\hat{\theta}_2|n_1\in A}(t_2) = \frac{1}{p(\theta_1)}\Bigg[\sum_{d=0}^{r-1} c_d(\theta_1) f_{GA}(t_2, \tfrac{n-r}{\theta_2}, n-r) + \sum_{d=r}^{n-1} c_d(\theta_1) f_{GA}(t_2, \tfrac{n-d}{\theta_2}, n-d)\Bigg].$$

Proof. See Appendix A.2. □

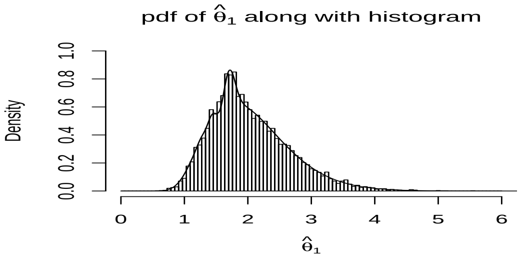
The PDF of $\widehat{\theta}_1$ is the generalized mixture of gamma distributions and the PDF of $\widehat{\theta}_2$ is the mixture of gamma distributions. Since the exact distributions are quite complicated, we provide plots of the PDF. The shape of the PDF of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with the histogram for different values of n, r and τ are given in Figure 1. In each case the PDF plot and the histogram match quite well. It has been observed that the distribution of $\widehat{\theta}_1$ is multimodal, specially for small sample size and $\widehat{\theta}_2$ has an unimodal density function. The PDF of $\widehat{\theta}_1$ gradually becomes unimodal as sample size increases.



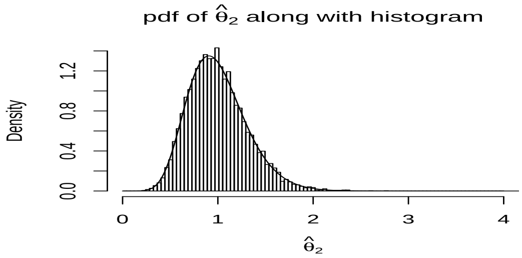
(a) $n = 10$, $r = 4$, $\tau = 1$



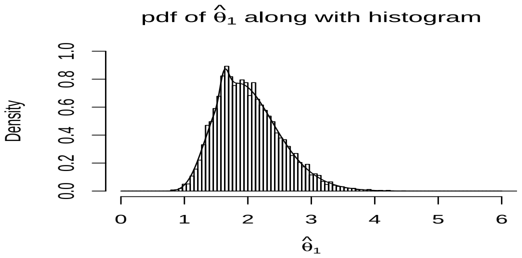
(b) $n = 10$, $r = 4$, $\tau = 1$



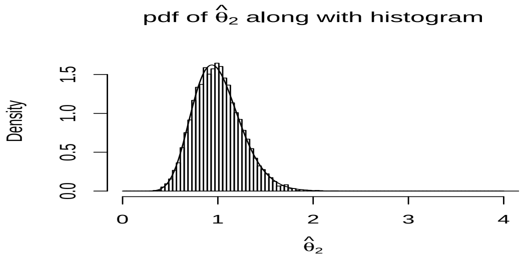
(c) $n = 20$, $r = 9$, $\tau = 1$



(d) $n = 20$, $r = 9$, $\tau = 1$



(e) $n = 30$, $r = 14$, $\tau = 1$



(f) $n = 30$, $r = 14$, $\tau = 1$

Figure 1: PDF plot of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ along with the histogram for $\theta_1 = 2$ and $\theta_2 = 1$.

4 Different Types of Confidence Interval

In this section we consider the construction of two type of confidence intervals (CIs) *viz.* approximate and bootstrap confidence interval. In case of bootstrap confidence interval the performance of percentile bootstrap was not satisfactory and hence we consider bias corrected accelerated (BCa) bootstrap confidence intervals.

4.1 Approximate Confidence Interval

In this subsection we will explain the method of constructing the approximate CIs of θ_1 and θ_2. The Approximate CIs of θ_1 and θ_2 can be constructed based on the conditional CDF of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. It is to be noted that the following two lemmas are necessary for the construction of approximate CIs.

Lemma 1. *For any $x > 0$, $P_{\theta_1|n_1 \in A}(\hat{\theta}_1 \leq x)$ is monotonically decreasing function of θ_1.*

Proof. See Appendix A.3. □

Lemma 2. *For any $x > 0$, $P_{\theta_2|n_1 \in A}(\hat{\theta}_2 \leq x)$ is monotonically decreasing function of θ_2.*

Proof. See Appendix A.4. □

A very common method of constructing the approximate CIs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ is pivoting the CDF of MLE, which requires the above two monotonicity property of CDF of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. Balakrishnan and Iliopoulos [6, 7] provide a formal proof of the monotonicity of CDFs of MLEs using three monotonicity lemmas. Here also we have used the same approach to prove the above two lemmas. This property ensures the invertability of the CDFs. Several authors including Chen and Bhattacharya [11], Kundu and Basu [17], Childs et al. [12] have explained the method of constructing approximate CIs by assuming the monotonicity property of CDFs of MLEs. Let θ_{iL} and θ_{iU} be the lower and upper limit of a $100(1-\alpha)\%$ approximate CI of $\widehat{\theta}_i$ $(i = 1, 2)$. If $\widehat{\theta}_1^{obs}$ is the observed value of $\widehat{\theta}_1$ then using the Lemma 1, θ_{1L} and θ_{1U} can be obtained by solving the following equations:

$$\begin{aligned} 1 - \tfrac{\alpha}{2} \; = \; & \tfrac{1}{p(\theta_{1L})} \Bigg[\textstyle\sum_{d=0}^{r-1} \sum_{j=0}^{d} c_{dj}(\theta_{1L}) \Gamma(\widehat{\theta}_1^{obs} - \mu_{dj}, \tfrac{r}{\theta_{1L}}, r) \\ & + \textstyle\sum_{d=r}^{n-1} \sum_{j=0}^{d} c_{dj}(\theta_{1L}) \Gamma(\widehat{\theta}_1^{obs} - \mu'_{dj}, \tfrac{d}{\theta_{1L}}, d) \Bigg], \end{aligned} \tag{5}$$

$$\frac{\alpha}{2} = \frac{1}{p(\theta_{1U})}\Big[\sum_{d=0}^{r-1}\sum_{j=0}^{d} c_{dj}(\theta_{1U})\Gamma(\widehat{\theta}_1^{obs} - \mu_{dj}, \tfrac{r}{\theta_{1U}}, r) + \sum_{d=r}^{n-1}\sum_{j=0}^{d} c_{dj}(\theta_{1U})\Gamma(\widehat{\theta}_1^{obs} - \mu'_{dj}, \tfrac{d}{\theta_{1U}}, d)\Big]. \tag{6}$$

Now let the observed value of $\widehat{\theta}_2$ be $\widehat{\theta}_2^{obs}$. The equations for obtaining θ_{2L} and θ_{2U} are

$$1 - \frac{\alpha}{2} = \frac{1}{p(\widehat{\theta}_1^{obs})}\Big[\sum_{d=0}^{r-1} c_d(\widehat{\theta}_1^{obs})\Gamma(\widehat{\theta}_2^{obs}, \tfrac{n-r}{\theta_{2L}}, n-r) + \sum_{d=r}^{n-1} c_d(\widehat{\theta}_1^{obs})\Gamma(\widehat{\theta}_2^{obs}, \tfrac{n-d}{\theta_{2L}}, n-d)\Big], \tag{7}$$

$$\frac{\alpha}{2} = \frac{1}{p(\widehat{\theta}_1^{obs})}\Big[\sum_{d=0}^{r-1} c_d(\widehat{\theta}_1^{obs})\Gamma(\widehat{\theta}_2^{obs}, \tfrac{n-r}{\theta_{2U}}, n-r) + \sum_{d=r}^{n-1} c_d(\widehat{\theta}_1^{obs})\Gamma(\widehat{\theta}_2^{obs}, \tfrac{n-d}{\theta_{2U}}, n-d)\Big]. \tag{8}$$

In Lemma 3 we will show that the solution of equations 5 and 6 and equations 7 and 8 always exist, i.e, approximate CIs of θ_i $(i = 1, 2)$ always exists.

Lemma 3. *For $i = 1, 2$, and fixed $x > 0$,*

$$\lim_{\theta_i \to \infty} P_{\theta_i}(\widehat{\theta}_i \le x | n_1 \in A) = 0$$

and

$$\lim_{\theta_i \to 0} P_{\theta_i}(\widehat{\theta}_i \le x | n_1 \in A) = 1.$$

Proof. See Appendix A.5. □

Hence using the above lemmas we can construct the approximate CIs of model parameters by solving the corresponding above equations. The performance of approximate CIs are evaluated through simulation study in section 6.

4.2 Bootstrap Confidence Interval

The exact conditional distribution of MLEs are quite complicated and hence the construction of the approximate confidence intervals are not very easy. Another alternative is to consider bootstrap CIs. In this subsection we will provide an algorithm to construct the

BCa bootstrap CIs of θ_1 and θ_2.

Algorithm 1

1. For a given n, r, τ and the original sample $\{t_{1:n}, \ldots, t_{n:n}\}$, obtain $\widehat{\theta}_1$ and $\widehat{\theta}_2$, the MLEs of θ_1 and θ_2 respectively.

2. Simulate a sample of size n, say, $\{t^*_{1:n}, \ldots, t^*_{n_1:n}, t^*_{n_1+1:n}, \ldots, t^*_{n:n}\}$ from (??) with parameters $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

3. Using the new sample obtained in the previous step, estimate the MLEs of θ_1 and θ_2, say $\widehat{\theta}_1^1$ and $\widehat{\theta}_2^1$, respectively.

4. Repeat steps 2-3, B times and obtain $\widehat{\theta}_1^1, \ldots, \widehat{\theta}_1^B$ and $\widehat{\theta}_2^1, \ldots, \widehat{\theta}_2^B$.

5. To construct $100(1-\alpha)\%$ BCa bootstrap CI of θ_1, arrange $\hat{\theta}_1^1, \ldots, \hat{\theta}_1^B$ in ascending order and let denote the ordered MLEs as $\hat{\theta}_1^{(1)}, \ldots, \hat{\theta}_1^{(B)}$.

   A two sided $100(1-\alpha)\%$ BCa bootstrap confidence interval for θ_1 is $(\hat{\theta}_1^{([\alpha_1 B])}, \hat{\theta}_1^{([\alpha_2 B])})$, where, $\alpha_1 = \Phi\{\widehat{z}_0 + \frac{\widehat{z}_0 + z_{1-\alpha/2}}{1-a(\widehat{z}_0 + z_{1-\alpha/2})}\}$ and $\alpha_2 = \Phi\{\widehat{z}_0 + \frac{\widehat{z}_0 + z_{\alpha/2}}{1-a(\widehat{z}_0 + z_{\alpha/2})}\}$, $[x]$ denotes the largest integer less than or equals to x. Here $\Phi(\cdot)$ denotes the CDF of the standard normal distribution, z_α is the upper α-point of the standard normal distribution, and

$$\widehat{z}_0 = \Phi^{-1}\left\{\frac{\#\text{ of } \hat{\theta}_1^{(j)} < \widehat{\theta}_1}{B}\right\}.$$

   An estimate of the acceleration factor a is

$$\widehat{a} = \frac{\sum_{i=1}^{n_1}\left[\widehat{\theta}_1^{(\cdot)} - \widehat{\theta}_1^{(i*)}\right]^3}{6\left\{\sum_{i=1}^{n_1}\left[\widehat{\theta}_1^{(\cdot)} - \widehat{\theta}_1^{(i*)}\right]^2\right\}^{3/2}},$$

   where $\widehat{\theta}_1^{(i*)}$ is the MLE of θ_1 based on the original sample with the i-th observation deleted, and

$$\widehat{\theta}_1^{(\cdot)} = \frac{1}{n_1}\sum_{i=1}^{n1}\widehat{\theta}_1^{(i*)}.$$

6. A two sided $100(1-\alpha)\%$ BCa bootstrap confidence interval of θ_2 can be constructed following the same way as explained in Step 5 for θ_1.

5 Simulation and Data Analysis

5.1 Simulation

In this subsection the model performance has been assessed by an extensive simulation study. Simulation results are provided for two sets of parameter values and for different (τ, r) combinations. In the first set we have considered $\{\theta_1 = 2.0,\ \theta_2 = 1.0\}$, whereas in the second set $\{\theta_1 = 4.0,\ \theta_2 = 1.5\}$. In the second set of parameter assumptions the rate of increase of the stress factor is higher than that of the first set. All the simulation results are based on 5000 replications. The MLEs of parameters along with the mean square errors (MSEs) are given in Tables 1 and 2. The average length (AL) and the coverage percentage (CP) of the 95% approximate and bias adjusted bootstrap confidence intervals are also provided in Table 1 and 2.

From the simulation results it has been observed the MLEs of θ_1 and θ_2 converge to the respective true parameter values and MSEs also decrease as the sample size n increases. In all the cases the coverage percentages of both, approximate and bias adjusted bootstrap, confidence intervals are very close to the nominal value. The average length of CIs also decreases as n increases and coverage percentages remains close to the nominal value. Again in most of the cases, for fixed n and τ if we increase r the MSEs and the ALs of CIs of θ_1 gradually decreases and those for θ_2 increases slowly. Comparing both the confidence intervals, BCa bootstrap CIs provide shorter length than the approximate CIs, even for a small sample size. Therefore, for satisfactory performance of BCa CIs and its computational simplicity we propose to use BCa confidence intervals.

5.2 Data Analysis

In this subsection we have analyzed one simulated data set and a real data set for illustrative purposes. In both the analyses we have provided the MLEs, and the associated 90%, 95%

Table 1: AE, MSEs and 95% different CIs of MLEs based on 5000 simulation ($\theta_1 = 2, \theta_2 = 1$).

n	r	τ	Parameter	AE	MSE	AL-Approx	CP-Approx	AL-Boot BCa	CP-Boot BCa
10	4	1.0	θ_1	2.1354	0.8360	6.2211	95.26	4.0452	96.70
			θ_2	1.0005	0.1987	3.0105	96.12	2.1518	94.92
10	4	1.5	θ_1	2.1988	0.9056	5.5606	94.82	4.0784	94.88
			θ_2	1.0002	0.2385	4.4763	96.70	2.4969	94.88
10	5	1.0	θ_1	2.0498	0.7198	5.1540	94.90	3.7108	94.42
			θ_2	0.9974	0.2089	3.2411	96.30	2.2713	94.36
10	5	1.5	θ_1	2.1169	0.7024	4.9512	94.86	3.6436	96.02
			θ_2	0.9990	0.2612	4.6311	96.64	2.5927	94.38
10	6	1.0	θ_1	2.0075	0.6127	4.4104	95.30	3.3871	93.78
			θ_2	1.0015	0.2669	3.7236	95.24	2.4699	93.96
10	6	1.5	θ_1	2.0620	0.6392	4.4068	94.70	3.3903	95.78
			θ_2	0.9883	0.2805	5.0018	96.62	2.7196	93.86
20	7	1.0	θ_1	2.1211	0.4959	3.6816	95.08	2.9209	96.28
			θ_2	1.0021	0.0864	1.4381	95.82	1.3368	95.36
20	7	1.5	θ_1	2.1283	0.4953	3.1285	94.84	2.8372	94.58
			θ_2	1.0070	0.1162	1.8036	95.72	1.5605	95.30
20	9	1.0	θ_1	2.0344	0.3951	3.1907	94.96	2.6440	95.44
			θ_2	0.9981	0.1004	1.5149	95.20	1.3884	95.30
20	9	1.5	θ_1	2.0915	0.3989	2.9930	95.02	2.5622	95.46
			θ_2	0.9987	0.1159	1.8127	96.00	1.5915	95.24
20	11	1.0	θ_1	1.9964	0.3522	2.7917	94.68	2.4180	93.56
			θ_2	0.9921	0.1105	1.6652	95.60	1.4974	94.22
20	11	1.5	θ_1	2.0399	0.3394	2.7576	94.40	2.3793	95.84
			θ_2	0.9959	0.1227	1.9121	96.02	1.6361	94.94
30	11	1.0	θ_1	2.0519	0.3078	2.6566	94.80	2.3089	96.30
			θ_2	0.9994	0.0601	1.0836	95.00	1.0426	95.12
30	11	1.5	θ_1	2.0986	0.3079	2.3522	95.02	2.2246	94.78
			θ_2	1.0015	0.0713	1.2680	95.94	1.2030	95.72
30	14	1.0	θ_1	2.0168	0.2652	2.3863	94.74	2.1062	95.12
			θ_2	1.0014	0.0649	1.1467	95.50	1.0879	95.06
30	14	1.5	θ_1	2.0644	0.2521	2.2667	94.92	2.0241	95.88
			θ_2	1.0060	0.0731	1.2994	96.00	1.2178	95.10
30	17	1.0	θ_1	1.9961	0.2260	2.1184	95.20	1.9375	93.78
			θ_2	1.0016	0.0771	1.2739	95.22	1.1935	94.42
30	17	1.5	θ_1	2.0285	0.2225	2.1111	95.00	1.8839	95.98
			θ_2	1.0029	0.0811	1.3692	95.72	1.2656	95.22

and 99% approximate and BCa bootstrap CIs.

5.2.1 Simulated Data

Here we have simulated a data set of size 25 from a simple step stress CEM in which stress level changes at the time $\tau = 1$ or after 12-th failure, whichever occurs later. The true values

Table 2: AE, MSEs and 95% different CIs of MLEs based on 5000 simulation ($\theta_1 = 4.0, \theta_2 = 1.5$).

n	r	τ	Parameter	AE	MSE	AL-Approx	CP-Approx	AL-Boot BCa	CP-Boot BCa
10	4	2.0	θ_1	4.2421	3.4314	12.3163	94.78	8.2032	96.68
			θ_2	1.4952	0.4205	4.5028	96.16	3.2360	94.48
10	4	2.5	θ_1	4.3704	3.5606	11.8808	94.50	8.1313	95.86
			θ_2	1.4882	0.4913	5.3980	96.50	3.5024	94.50
10	5	2.0	θ_1	4.1122	2.9338	10.3426	95.02	7.3214	94.32
			θ_2	1.4915	0.4741	4.8874	96.04	3.3932	94.26
10	5	2.5	θ_1	4.1448	2.7487	10.0776	95.42	7.3189	95.70
			θ_2	1.5061	0.5273	5.8606	96.60	3.6722	95.08
10	6	2.0	θ_1	4.0034	2.5089	8.7900	95.08	6.8543	93.66
			θ_2	1.5249	0.5998	5.7331	95.50	3.6836	93.00
10	6	2.5	θ_1	4.0051	2.3341	8.6995	95.28	6.7796	94.76
			θ_2	1.5047	0.6067	6.4342	96.14	3.8998	92.96
20	7	2.0	θ_1	4.1810	1.7903	7.2294	95.66	5.8985	96.18
			θ_2	1.5040	0.2042	2.1689	95.26	1.9886	95.58
20	7	2.5	θ_1	4.2694	2.0023	6.8006	95.42	5.8066	95.22
			θ_2	1.4898	0.2191	2.3592	95.90	2.1354	95.40
20	9	2.0	θ_1	4.0590	1.6075	6.3607	95.06	5.2236	95.60
			θ_2	1.4985	0.2096	2.2799	95.78	2.0753	95.08
20	9	2.5	θ_1	4.1144	1.4784	6.1902	95.20	5.1505	95.96
			θ_2	1.5083	0.2384	2.4791	95.96	2.2273	95.74
20	11	2.0	θ_1	4.0226	1.4162	5.6275	95.18	4.8157	94.40
			θ_2	1.5043	0.2426	2.5186	95.52	2.2494	94.66
20	11	2.5	θ_1	4.0330	1.3417	5.5745	95.14	4.7868	95.22
			θ_2	1.4830	0.2615	2.6200	95.80	2.3275	95.08
30	11	2.0	θ_1	4.1400	1.2433	5.3763	94.72	4.5952	96.00
			θ_2	1.5089	0.1265	1.6310	95.54	1.5559	95.68
30	11	2.5	θ_1	4.1567	1.2154	4.9943	95.06	4.5145	94.44
			θ_2	1.4971	0.1455	1.7448	95.80	1.6613	95.06
30	14	2.0	θ_1	4.0458	1.0393	4.7879	95.84	4.1537	94.82
			θ_2	1.4968	0.1456	1.7116	95.28	1.6495	95.00
30	14	2.5	θ_1	4.0686	0.9962	4.6670	95.08	4.1059	96.56
			θ_2	1.5069	0.1541	1.8134	95.54	1.7114	95.44
30	17	2.0	θ_1	4.0014	0.9333	4.2464	94.86	3.8631	94.30
			θ_2	1.5031	0.1712	1.9114	95.36	1.7989	94.48
30	17	2.5	θ_1	4.0268	0.8746	4.2511	95.12	3.8492	94.48
			θ_2	1.4954	0.1721	1.9483	95.66	1.8396	94.76

of θ_1 and θ_2 are, respectively, 2 and 1. The observed failure times at the first stress level are 0.0092, 0.1191, 0.1542, 0.2493, 0.6141, 0.6690, 0.6800, 0.8591, 0.9800, 1.2436, 1.2712, 1.3114 From the data it is observed that the stress change occurred after 12-th failure. The remaining thirteen units are failed at second stress level and the actual time to failures are 1.3201, 1.3611, 1.7060, 1.7609, 1.8639, 1.8755, 1.8879, 2.1300, 2.2135, 2.2459, 2.7384, 4.5292, 5.6471. The MLEs of θ_1 and θ_2 are 2.1007 and 1.0947, respectively. The lower limit (LL) and the upper limit (UL) of 90%, 95% and 99% approximate and BCa bootstrap confidence intervals are provided in Table 3.

Table 3: Approximate and BCa bootstrap CIs for the unknown parameters based on simulated data.

	θ_1				θ_2			
	Approximate		BCa Bootstrap		Approximate		BCa Bootstrap	
Level	LL	UL	LL	UL	LL	UL	LL	UL
90%	1.3845	3.6406	1.3567	3.3181	0.7298	1.8607	0.7244	1.8158
95%	1.2808	4.0655	1.2651	3.6109	0.6764	2.0702	0.6604	2.0070
99%	1.1067	5.0988	1.0745	4.2790	0.5863	2.5792	0.5885	2.4168

5.2.2 Real Data

Here we have analyzed a simple step stress data taken from Han and Kundu [14] where a simple step stress experiment was performed on 31 solar lighting devices. The experiment was started at the normal operating temperature $293K$ and then it increases to $353K$ to reduce the experimental time. Here we assume $r = 12$ and and stress changing time is $max\{5, t_{12:31}\}$. The observed failure times, obtained from this experiment are 0.14, 0.783, 1.324, 1.582, 1.716, 1.794, 1.883, 2.293, 2.660, 2.674, 2.725, 3.085, 3.924, 4.396, 4.612, 4.892, 5.002, 5.022, 5.082, 5.112, 5.147, 5.238, 5.244, 5.247, 5.305, 5.337, 5.407, 5.408, 5.445, 5.483, 5.717. Clearly the stress changing time is $\tau = 5$ and the number of failure at the first stress level is sixteen. We have analyzed this data assuming the exponential simple step stress model. The maximum likelihood estimates of θ_1 and θ_2 are 7.2177 and 0.2797, respectively. The 90%, 95% and 99% approximate and BCa bootstrap confidence intervals are given in Table 4. To check the goodness of fit to the above data we have performed the Kolmogorov-Smirnov (K-S) test. The value of the K-S statistic is 0.109 and the p-value of the test is 0.817 which indicates a good fit of the model to the solar lighting devices data. The plot of empirical and fitted CDF has been shown in Figure 2.

Table 4: Approximate and BCa bootstrap CI for the unknown parameters.

	θ_1				θ_2			
	Approximate		BCa Bootstrap		Approximate		BCa Bootstrap	
Level	LL	UL	LL	UL	LL	UL	LL	UL
90%	4.8832	11.2861	4.9243	11.0310	0.1917	0.4546	0.1799	0.4255
95%	4.5573	12.3933	4.5165	11.8541	0.1783	0.5026	0.1681	0.4636
99%	4.0001	15.0016	3.9567	13.6566	0.1550	0.6208	0.1388	0.5584

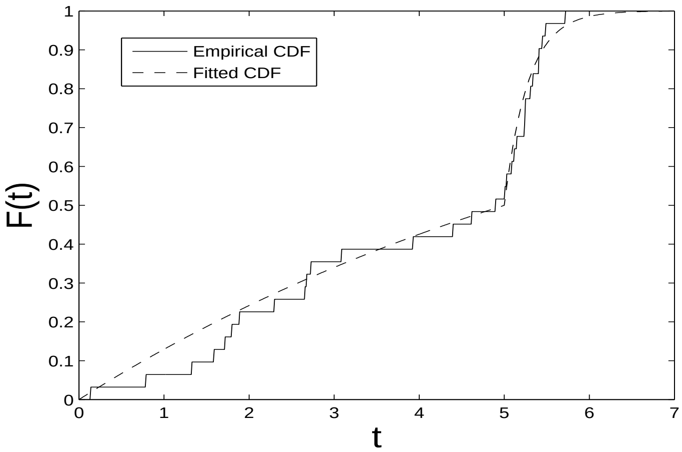


Figure 2: Empirical and the fitted CDF for real data.

6 Optimality of Test Plan

The objective of this section is to propose a criterion and choose the best step stress scheme based on this criterion. The primary aim of increasing stress level is to reduce the experimental time and gather more information in shorter time period. Suppose the experiment starts with n units having mean life θ_1. If the experimenter does not increase the stress level then the expected experimental time would be $E(X_{n:n})$, where $X_{n:n}$ is the $n-th$ ordered failure of the experiment. Now let the experimenter decide to run the experiment maximum of $\delta\%$ of $E(X_{n:n})$ by increasing the stress level. If $X_1, \ldots, X_n$ follow exponential distribution with mean θ_1 then the expectation of $n-th$ ordered statistics is given by

$$E(X_{n:n}) = I_n = \int_0^\infty \frac{nx}{\theta_1} e^{-\frac{x}{\theta_1}} (1-e^{-\frac{x}{\theta_1}})^{n-1} dx = n\theta_1 \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{(n-k)^2} \binom{n-1}{k}. \tag{9}$$

Let $T = \delta\% \times I_n$ is the maximum time which the experimenter can afford in a simple SSLT experiment. Therefore, among all the simple SSLT experiment having expected experimental time less than or equals to T, the experimenter wants to choose τ and r which gives the minimum $\phi(\tau, r)$, where

$$\phi(\tau, r) = \frac{MSE(\widehat{\theta}_1)}{\theta_1^2} + \frac{MSE(\widehat{\theta}_2)}{\theta_2^2}. \tag{10}$$

The expected experimental time of a simple SSLT experiment is the expectation of the n-th ordered statistics of a sample of size n from the simple SSLT experiment. The explicit form of the expected experimental time is given in Appendix A.6. Here, instead of minimizing the sum of two MSEs we have normalized the MSEs, so that the objective function becomes unit free. The similar optimality criterion, considering the sum of the coefficient of variations, has been used by Samanta et al. [22] to obtain the optimal SSLT. Consider the following algorithm to obtain the optimal choice of τ and r from all the step stress experiments having maximum experimental time T and for fixed θ_1 and θ_2.

Algorithm 2

1. For a given n and θ_1 obtain the value of T.
2. Calculate $E(T_{n:n})$, the expected experimental time, using the equation (31) of Appendix A.6.
3. If the expected experimental time is less than or equals to T then calculate $\phi(\tau, r)$.
4. Replicate Step 1 - Step 3 for all possible values of r and $\tau \in (0, T)$. The grid points of τ can be taken as $\{0.01, 0.02, \ldots, T\}$.
5. Choose τ and r for which $\phi(\tau, r)$ is minimum.

An optimal choice of τ and r for different sample size and for two sets of parameter values as considered for simulation, are given in Table 5. It has been observed that the optimal value of r is approximately the 50% of total sample size and for a given set of parameter the optimal value of τ increases as the sample size increases.

Now consider a step stress experiment as conducted for solar lighting devices. For a given sample size and for $\theta_1 = 7.2177$, $\theta_2 = 0.2797$, the MLEs of θ_1 and θ_2 of the real data set,

Table 5: Optimal choice of τ and r.

Parameters	n	$\delta = 60$			$\delta = 70$		
		T	Optimal τ	Optimal r	T	Optimal τ	Optimal r
$\theta_1 = 2,\ \theta_2 = 1$	10	3.51	0.92	4	4.10	0.53	5
	20	4.32	0.99	8	5.03	0.79	9
	30	4.79	1.14	14	5.59	1.08	15
$\theta_1 = 4,\ \theta_2 = 1.5$	10	7.03	1.75	4	8.20	1.65	5
	20	8.63	2.05	10	10.07	1.95	9
	30	9.59	2.48	14	11.19	2.29	15

respectively we have performed an optimality test to obtain the optimal τ and r. First we consider $n = 31$, the solar lighting devices data set size and $\delta = 21$ which gives $T = 6.1$, since all the observations are less than 6.1. In this setup an optimal value of τ and r is 4.17 and 15 respectively. In future, if the sample size and the expected experimental time are given then one can obtain an optimal design for SSLT experiment. Some of the optimal design of SSLT experiment for solar lighting devices are provided in Table 6.

Table 6: Optimal choice of τ and r taking the MLEs of real data set as parameter values ($\theta_1 = 7.2177,\ \theta_2 = 0.2797$).

n	$\delta = 21$			$\delta = 40$			$\delta = 60$		
	T	Optimal τ	Optimal r	T	Optimal τ	Optimal r	T	Optimal τ	Optimal r
20	5.45	3.94	9	10.39	4.16	10	15.58	3.18	9
31	6.10	4.17	15	11.63	3.92	14	17.44	3.40	15
40	6.49	3.86	18	12.35	4.18	18	18.53	4.20	20

7 Conclusion

An exponential simple step stress model emphasizing the inference of the parameter under first stress level has been considered in this article. To ensure a minimum number of failure Type-II hybrid type model is adopted on the stress changing time. We have obtained the MLEs of model parameters along with their conditional probability distributions. Approximate confidence intervals have been obtained using cumulative distribution functions of MLEs. Due to complicated nature of conditional distribution bias adjusted bootstrap confidence intervals are also obtained here. From simulation results it is observed that the performance of parameter estimates and the performance of both the confidence intervals are quite satisfactory. We provide an algorithm to obtain optimal step stress scheme among all the schemes having expected experimental time is less than or equals to a given duration. Therefore for a given sample size and for a given experimental time and when the more importance is given to the parameters under first stress level, this model assumptions and the related inferential technique is relevant to use. It is well known that although exponential distribution has been used quite extensively in different life testing experiment it has its own limitations. Due to this reason more general lifetime distributions like Weibull distribution or generalized exponential distribution will be more useful in practice. It will be interesting to develop inference procedure for these lifetime distributions. They are more challenging problems. More work is needed on this direction.

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A Appendix

A.1 Proof of Theorem 1

Consider the conditional CDF of $\widehat{\theta}_1$ conditioning on the event A=$\{n_1 \leq n-1\}$

$$\begin{aligned}
& P(\widehat{\theta}_1 \leq x | n_1 \in A) \\
= & \tfrac{1}{P(n_1 \in A)} P(\widehat{\theta}_1 \leq x, n_1 \in A) \\
= & \tfrac{1}{P(n_1 \in A)} \left[\textstyle\sum_{d=0}^{r-1} P(\widehat{\theta}_1 \leq x, n_1 \in A, D=d) + \sum_{d=r}^{n-1} P(\widehat{\theta}_1 \leq x, n_1 \in A, D=d) \right] \\
= & \tfrac{1}{P(n_1 \in A)} \left[\textstyle\sum_{d=0}^{r-1} P(\widehat{\theta}_1 \leq x | D=d) P(D=d) + \sum_{d=r}^{n-1} P(\widehat{\theta}_1 \leq x | D=d) P(D=d) \right]
\end{aligned}$$

For $d = 0, 1, \ldots r-1$, the conditional MGF of $\widehat{\theta}_1$ conditioned on the event $\{D = d\}$ is derived below.

$$\begin{aligned}
& M_1(\omega) \\
= & E[e^{\omega \widehat{\theta}_1} | D = d] \\
= & E[e^{\frac{\omega}{r}\{\sum_{i=1}^{r-1} t_{i:n} + (n-r+1) t_{r:n}\}} | D = d] \\
= & \tfrac{n!}{P(D=d)(n-r)!\theta_1^r} \int_\tau^\infty \int_\tau^{t_{r:n}} \cdots \int_\tau^{t_{d+2}} \int_0^\tau \\
& \int_0^{t_{d:n}} \cdots \int_0^{t_{2:n}} e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})[\sum_{i=1}^{r-1} t_{i:n} + (n-r+1) t_{r:n}]} dt_{1:n} \ldots dt_{d:n} dt_{d+1:n} \ldots dt_{r:n} \\
= & \frac{n! (\frac{1}{\theta_1} - \frac{\omega}{r})^{-d} (1 - e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})\tau})^d e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})(r-d-1)\tau}}{P(D=d)(n-r)!d!\theta_1^r} \int_\tau^\infty e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})(n-r+1) t_{r:n}} \\
& \int_\tau^{t_{r:n}} \cdots \int_\tau^{t_{d+2}} e^{-(\frac{1}{\theta_1} - \frac{\omega}{r}) \sum_{i=d+1}^{r-1} (t_{i:n} - \tau)} dt_{d+1:n} \ldots dt_{r:n} \\
= & \frac{n! (\frac{1}{\theta_1} - \frac{\omega}{r})^{-(r-1)} (1 - e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})\tau})^d e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})(r-d-1)\tau}}{P(D=d)(n-r)!d!(r-d-1)!\theta_1^r} \int_\tau^\infty e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})(n-r+1) t_{r:n}} [1 - e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})(t_{r:n} - \tau)}]^{r-d-1} dt_{r:n} \\
= & \tfrac{1}{P(D=d)} \tbinom{n}{d} (1 - \tfrac{\omega\theta_1}{r})^{-r} e^{-(\frac{1}{\theta_1} - \frac{\omega}{r})(n-d)\tau} [1 - e^{(\frac{1}{\theta_1} - \frac{\omega}{r})\tau}]^d \\
= & \tfrac{1}{P(D=d)} \textstyle\sum_{j=0}^{d} \tbinom{n}{d} \tbinom{d}{j} (-1)^j e^{-\frac{\tau}{\theta_1}(n-d+j)} (1 - \tfrac{\omega\theta_1}{r})^{-r}
\end{aligned}$$

Again for $d = r, r+1, \ldots, n-1$, the conditional MGF of $\widehat{\theta}_1$ conditioned on the event $\{D = d\}$ is derived as follows.

$$
\begin{aligned}
&M_2(\omega) \\
= \;& \frac{n!}{P(D=d)(n-d)!\theta_1^d} \int_0^{\tau} \int_0^{t_{d:n}} \ldots \int_0^{t_{2:n}} e^{-(\frac{1}{\theta_1} - \frac{\omega}{d})[\sum_{i=1}^{d} t_{i:n} + (n-d)\tau]} dt_{1:n} \ldots dt_{d:n} \\
= \;& \frac{n! e^{-(\frac{1}{\theta_1} - \frac{\omega}{d})(n-d)\tau}}{P(D=d)(n-d)! d! \theta_1^d} \left[1 - e^{-(\frac{1}{\theta_1} - \frac{\omega}{d})\tau}\right]^d \left(\frac{1}{\theta_1} - \frac{\omega}{d}\right)^{-d} \\
= \;& \frac{1}{P(D=d)} \sum_{j=0}^{d} \binom{n}{d} \binom{d}{j} (-1)^j e^{-(\frac{1}{\theta_1} - \frac{\omega}{d})(n-d+j)\tau} \left(1 - \frac{\omega\theta_1}{d}\right)^{-d}
\end{aligned}
$$

Therefore using the uniqueness property of MGF, the CDF of $\widehat{\theta}_1$ is obtained as

$$
\begin{aligned}
P(\widehat{\theta}_1 \le x | n_1 \in A) \;&= \frac{1}{p(\theta_1)} \Bigg[\sum_{d=0}^{r-1} \sum_{j=0}^{d} c_{dj}(\theta_1) \Gamma(x - \mu_{dj}, \tfrac{r}{\theta_1}, r) \\
&\quad + \sum_{d=r}^{n-1} \sum_{j=0}^{d} c_{dj}(\theta_1) \Gamma(x - \mu'_{dj}, \tfrac{d}{\theta_1}, d) \Bigg].
\end{aligned} \tag{11}
$$

A.2 Proof of Theorem 2

Consider the conditional CDF of $\widehat{\theta}_2$ conditioning on the event A=$\{n_1 \le n-1\}$

$$
\begin{aligned}
&P(\widehat{\theta}_2 \le x | n_1 \in A) \\
= \;& \frac{1}{P(n_1 \in A)} P(\widehat{\theta}_2 \le x, n_1 \in A) \\
= \;& \frac{1}{P(n_1 \in A)} \left[\sum_{d=0}^{r-1} P(\widehat{\theta}_2 \le x, n_1 \in A, D = d) + \sum_{d=r}^{n-1} P(\widehat{\theta}_2 \le x, n_1 \in A, D = d) \right] \\
= \;& \frac{1}{P(n_1 \in A)} \left[\sum_{d=0}^{r-1} P(\widehat{\theta}_2 \le x | D = d) P(D = d) + \sum_{d=r}^{n-1} P(\widehat{\theta}_2 \le x | D = d) P(D = d) \right].
\end{aligned}
$$

Now consider the conditional MGF of $\widehat{\theta}_2$ for $d = 0, 1, \ldots r-1$,

$$
\begin{aligned}
M_3(\omega) \;&=\; E[e^{\omega\widehat{\theta}_2} | D = d] \\
&=\; \frac{n!(1-e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau(r-d-1)}{\theta_1}}}{P(D=d) d! (r-d-1)! \theta_1 \theta_2^{n-r}} \int_{\tau}^{\infty} [1 - e^{-\frac{t_{r:n} - \tau}{\theta_1}}]^{r-d-1} e^{-\frac{(n-r+1)t_{r:n}}{\theta_1}} dt_{r:n} \\
&\quad \int_{t_{r:n}}^{\infty} \int_{t_{r:n}}^{t_{n:n}} \ldots \int_{t_{r:n}}^{t_{r+2}} e^{-(\frac{1}{\theta_2} - \frac{\omega}{n-r}) \sum_{i=r+1}^{n}} (t_{i:n} - t_{r:n}) dt_{r+1:n} \ldots dt_{n:n} \\
&=\; \frac{n!(1-e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau(r-d-1)}{\theta_1}} (\frac{1}{\theta_2} - \frac{\omega}{n-r})^{-(n-r)}}{P(D=d) d! (r-d-1)! (n-r)! \theta_1 \theta_2^{n-r}} \int_{\tau}^{\infty} [1 - e^{-\frac{t_{r:n} - \tau}{\theta_1}}]^{r-d-1} e^{-\frac{(n-r+1)t_{r:n}}{\theta_1}} dt_{r:n} \\
&=\; \frac{n!(1-e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau(r-d-1)}{\theta_1}} (\frac{1}{\theta_2} - \frac{\omega}{n-r})^{-(n-r)} e^{-\frac{\tau(n-r+1)}{\theta_1}} B(n-r+1, r-d)}{P(D=d) d! (r-d-1)! (n-r)! \theta_2^{n-r}} \\
&=\; \frac{1}{P(D=d)} \binom{n}{d} (1 - e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau(n-d)}{\theta_1}} (1 - \frac{\omega\theta_2}{n-r})^{-(n-r)}.
\end{aligned}
$$

Similarly the conditional MGF of $\widehat{\theta}_2$ for $d = r, r+1, \ldots, n-1$ is derived as follows.

$$\begin{aligned}
M_3(\omega) &= E[e^{\omega\widehat{\theta}_2}|D=d] \\
&= E[e^{\frac{\omega}{n-d}\sum_{i=d+1}^{n}(t_{i:n}-\tau)}|D=d] \\
&= \frac{n!(1-e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau(n-d)}{\theta_1}}}{P(D=d)d!\theta_2^{(n-d)}}\int_{\tau}^{\infty}\ldots\int_{\tau}^{t_{d+3:n}}\int_{\tau}^{t_{d+2:n}} e^{-(\frac{1}{\theta_2}-\frac{\omega}{n-d})\sum_{i=d+1}^{n}(t_{i:n}-\tau)}dt_{d+1:n}\ldots dt_{n:n} \\
&= \frac{1}{P(D=d)}\binom{n}{d}(1-e^{-\frac{\tau}{\theta_1}})^d e^{-\frac{\tau(n-d)}{\theta_1}}(1-\frac{\omega\theta_2}{n-d})^{-(n-d)}.
\end{aligned}$$

Therefore using the uniqueness property of MGF, the CDF of $\widehat{\theta}_2$ is obtained as

$$P(\widehat{\theta}_2 \le x|n_1 \in A) = \frac{1}{p(\theta_1)}\Bigg[\sum_{d=0}^{r-1} c_d(\theta_1)\Gamma(x, \tfrac{n-r}{\theta_2}, n-r) + \sum_{d=r}^{n-1} c_d(\theta_1)\Gamma(x, \tfrac{n-d}{\theta_2}, n-d)\Bigg]. \tag{12}$$

A.3 Proof of lemma 1

To establish that $P_{\theta_1}(\widehat{\theta}_1 \le x|n_1 \in A)$ is a decreasing function of θ_1, we use the three monotonic lemmas as given in Balakrishnan and Iliopoulos [6, 7]. In our case we investigate whether each of the three lemmas holds true or not. For $x > 0$, the distribution function of $\widehat{\theta}_1$ can be written as,

$$\begin{aligned}
P_{\theta_1}(\widehat{\theta}_1 \le x|n_1 \in A) =& \sum_{d=0}^{n-1} P[\widehat{\theta}_1 \le x|D=d, n_1 \in A]P(D=d|n_1 \in A) \\
=& \sum_{d=0}^{r-1} P[\widehat{\theta}_1 \le x|D=d, n_1 \in A]P(D=d|n_1 \in A) \\
&+ \sum_{d=r}^{n-1} P[\widehat{\theta}_1 \le x|D=d, n_1 \in A]P(D=d|n_1 \in A).
\end{aligned} \tag{13}$$

Note that for $d = 0, 1, \ldots, r-1$, the event $\{D=d\} \subset \{n_1 \in A\}$ and for $d = r, r+1, \ldots, n-1$, the event $\{D=d\} = \{n_1 \in A\}$. Thus equation (13) becomes,

$$\begin{aligned}
P_{\theta_1}(\widehat{\theta}_1 \le x|n_1 \in A) =& \frac{1}{P(n_1 \in A)}\Big[\sum_{d=0}^{r-1} P[\widehat{\theta}_1 \le x|D=d]P(D=d) \\
&+ \sum_{d=r}^{n-1} P[\widehat{\theta}_1 \le x|D=d]P(D=d)\Big]
\end{aligned} \tag{14}$$

or equivalently,

$$P_{\theta_1}(\widehat{\theta}_1 > x|n_1 \in A) = \frac{1}{P(n_1 \in A)}\Big[\sum_{d=0}^{r-1} P[\widehat{\theta}_1 > x|D = d]P(D = d) + \sum_{d=r}^{n-1} P[\widehat{\theta}_1 > x|D = d]P(D = d)\Big]. \quad (15)$$

This is the same representation as given in Balakrishnan and Iliopoulos [6, 7]. Note that $P(n_1 \in A) = 1-(1-e^{-\frac{T}{\theta}})^n$ which is increasing function of θ_1 and hence $\frac{1}{P(n_1 \in A)}$ is decreasing function of θ_1.

Lemma (M-1)

Case-1: $d \in \{0, 1, \ldots, r-1\}$

$\widehat{\theta}_1 = \frac{\sum_{i=1}^{r-1} X_{i:n} + (n-r+1)X_{r:n}}{r}$.

Clearly, $\sum_{i=1}^{r-1} X_{i:n} + (n-r+1)X_{r:n} = \sum_{i=1}^{d} X_{i:n} + \sum_{i=d+1}^{r-1} X_{i:n} + (n-r+1)X_{r:n}$. Before we proceed further it is evident that given $d \in \{0, 1, \ldots, r-1\}$, the random variables $\{X_{i:n}, \ldots, X_{d:n}\} \overset{\text{d}}{=} \{U_{i:d}, \ldots, U_{d:d}\}$ where, $U_1, \ldots, U_d$ are right truncated at time point T, *iid* exponential random variable with mean θ_1 and the random variables $\{X_{d+1:n}, \ldots, X_{r:n}\} \overset{\text{d}}{=} \{V_{1:n-d}, \ldots, V_{r-d:n-d}\}$ where, $V_1, \ldots, V_{n-d}$ are left truncated at time point T, *iid* exponential random variable with mean θ_1. Hence $\sum_{i=1}^{d} X_{i:n} + \sum_{i=d+1}^{r-1} X_{i:n} + (n-r+1)X_{r:n} \overset{\text{d}}{=} \sum_{i=1}^{d} U_{i:d} + \sum_{i=1}^{r-d-1} V_{i:n-d} + (n-r+1)V_{r-d:n-d}$. Since both left and right truncated exponential random variables are stochastically increasing in θ_1, their sum is also stochastically increasing in θ_1 and hence for $d \in \{0, 1, \ldots, r-1\}$,conditional distribution of $\hat{\theta}_1$, given $D = d$ is stochastically increasing in θ_1.

Case-2: $d \in \{r, r+1, \ldots, n-1\}$

$\widehat{\theta}_1 = \frac{\sum_{i=1}^{d} X_{i:n} + (n-d)T}{d}$.

As before it is noted that given $d \in \{r, r+1, \ldots, n-1\}$, $\{X_{1:n}, \ldots, X_{d:n}\} \overset{\text{d}}{=} \{U_{1:d}, \ldots, U_{d:d}\}$ with the same random variable U as defined above. Thus $\sum_{i=1}^{d} X_{i:n} + (n-d)T \overset{\text{d}}{=} \sum_{i=1}^{d} U_{i:d} + (n-d)T$ and this is stochastically increasing in θ_1. Hence for $d \in \{r, r+1, \ldots, n-1\}$,conditional distribution of $\hat{\theta}_1$, given $D = d$ is stochastically increasing in θ_1.

Lemma (M-2)

Case-1: $d \in \{0, 1, \ldots, r-1\}$

Note that for $d \in \{0, 1, \ldots, r-2\}$ it is clear, $(\widehat{\theta}_1 | D = d) - (\widehat{\theta}_1 | D = d+1) = 0$. When $d = r-1$, then $(\widehat{\theta}_1 | D = r-1) - (\widehat{\theta}_1 | D = r) \overset{\text{d}}{=} \frac{\sum_{i=1}^{r-1} X_{i:n} + (n-r+1) X_{r:n}}{r} - \frac{\sum_{i=1}^{r} X_{i:n} + (n-r)T}{r} \overset{\text{d}}{=} \frac{\sum_{i=1}^{r-1} U_{i:r-1} + (n-r+1) V_{1:n-r+1}}{r} - \frac{\sum_{i=1}^{r} U_{i:r} + (n-d)T}{r}$, where $U_{i:r-1}$ and $V_{1:n-r+1}$ are same as defined in Lemma (M-1). Clearly for $i = 1, 2, \ldots, r-1$, $U_{i:r-1} \geq U_{i:r}$ and $U_{r:r} \leq T \leq V_{1:n-r+1}$. Thus $(\widehat{\theta}_1 | D = r-1) - (\widehat{\theta}_1 | D = r) \geq 0$ and hence for $d \in \{0, 1, \ldots, r-1\}$, conditional distribution of $\widehat{\theta}_1$, given $D = d$ is stochastically decreasing in d.

Case-2: $d \in \{r, r+1, \ldots, n-1\}$

$(\widehat{\theta}_1 | D = d) - (\widehat{\theta}_1 | D = d+1) \overset{\text{d}}{=} \frac{\sum_{i=1}^{d} X_{i:n} + (n-d)T}{d} - \frac{\sum_{i=1}^{d+1} X_{i:n} + (n-d-1)T}{d+1} \overset{\text{d}}{=} \frac{\sum_{i=1}^{d} U_{i:n} + (n-d)T}{d} - \frac{\sum_{i=1}^{d+1} U_{i:n} + (n-d-1)T}{d+1} \geq \frac{\sum_{i=1}^{d} U_{i:n} + (n-d)T}{d+1} - \frac{\sum_{i=1}^{d+1} U_{i:n} + (n-d-1)T}{d+1} = \frac{T - U_{d+1:n}}{d+1} \geq 0$. Thus for $d \in \{r, r+1, \ldots, n-1\}$, conditional distribution of $\hat{\theta}_1$, given $D = d$ is stochastically decreasing in d.

Lemma (M-3)

Note that D is a binomial random variable with parameters $n, (1 - exp(-\frac{T}{\theta_1})$. Let us consider $\theta \leq \theta_1^{'}$. Then we have,

$$\frac{P_{\theta_1}(D = d)}{P_{\theta_1^{'}}(D = d)} \propto \left[\frac{exp(\frac{T}{\theta_1}) - 1}{exp(\frac{T}{\theta_1^{'}}) - 1} \right]^d.$$

This is increasing in d. Thus D has the monotone likelihood ratio property with respect to θ_1 and hence D is stochastically decreasing in θ_1. Thus all the three lemmas are established and hence $\sum_{d=0}^{n-1} P[\widehat{\theta}_1 > x | D = d] P(D = d)$ is increasing function of θ_1. Hence $\sum_{d=0}^{n-1} P[\widehat{\theta}_1 < x | D = d] P(D = d)$ is decreasing function of θ_1. Thus using the fact that $\frac{1}{P(n_1 \in A)}$ is decreasing function of θ_1, we have for $\theta_1 \leq \theta_1^{'}$, $P_{\theta_1}(\widehat{\theta}_1 \leq x | n_1 \in A) \geq P_{\theta_1^{'}}(\widehat{\theta}_1 \leq x | n_1 \in A)$.

A.4 Proof of lemma 2

Note that,

$$\begin{aligned} \Gamma(x, \tfrac{n-r}{\theta_2}, n-r) &= \frac{(\frac{n-r}{\theta_2})^{n-r}}{\Gamma(n-r)} \int_0^x e^{-\frac{(n-r)t}{\theta_2}} t^{n-r-1} dt \\ &= \frac{1}{\Gamma(n-r)} \int_0^{(n-r)x/\theta_2} e^{-u} u^{n-r-1} du. \end{aligned}$$

Clearly, the above function is a decreasing function of θ_2. Similarly, for $d = r, r+1, \ldots, n-1$, $\Gamma(x, \frac{n-d}{\theta_2}, n-d)$ is a decreasing function of θ_2. Hence for any $x > 0$, $P(\widehat{\theta}_2 \leq x | n_1 \in A)$ is a decreasing function of θ_2.

A.5 Proof of lemma 3

Proceeding similarly as in Appendix A.4 it is immediate that $\Gamma(x - \mu_{dj}, \frac{r}{\theta_1}, r) = \frac{1}{\Gamma r} \int_0^{\frac{r}{\theta_1}} e^{-u} u^{r-1} du$ $\to 0$ as $\theta \to \infty$. Similarly, $\Gamma(x - \mu'_{dj}, \frac{d}{\theta_1}, d) = \frac{1}{\Gamma d} \int_0^{\frac{d}{\theta_1}} e^{-u} u^{d-1} du \to 0$ as $\theta \to \infty$. Also $p(\theta_1) \to 1$ as $\theta_1 \to \infty$. Hence $P(\widehat{\theta}_1 \leq x | n_1 \in A) \to 0$ as $\theta_1 \to \infty$.

When $\theta_1 \to 0$, we have the following observations,

(i) $\Gamma(x - \mu_{dj}, \frac{r}{\theta_1}, r) \to 1$.

(ii) $\Gamma(x - \mu'_{dj}, \frac{d}{\theta_1}, d) \to 1$.

Hence,

$$\lim_{\theta_1 \to 0} P(\widehat{\theta}_1 \leq x | n_1 \in A) = \lim_{\theta_1 \to 0} \frac{1}{p(\theta_1)} \left[\sum_{d=0}^{r-1} \sum_{j=0}^{d} c_{dj}(\theta_1) + \sum_{d=r}^{n-1} \sum_{j=0}^{d} c_{dj}(\theta_1) \right] = \lim_{\theta_1 \to 0} \frac{p(\theta_1)}{p(\theta_1)} = 1.$$

Similar way leads to the establishment of the fact that

$$\lim_{\theta_2 \to \infty} P(\widehat{\theta}_2 \leq x | n_1 \in A) = 0$$

and

$$\lim_{\theta_2 \to 0} P(\widehat{\theta}_2 \leq x | n_1 \in A) = 1.$$

A.6 Expected experimental time of an exponential simple SSLT experiment

Here we want to calculate $E(T_{n:n})$, where $t_{i:n}$ for $i = 1, 2, \ldots, n$ be the i-th order observation coming from the experiment. Note that, for $x > 0$, the distribution function of $T_{n:n}$ can be written as,

$$P(T_{n:n} \leq x) = P(T_{n:n} \leq x, \tau < T_{r:n}) + P(T_{n:n} \leq x, T_{r:n} < \tau) \tag{16}$$

We calculate the two probabilities of right hand side of Equation (16) separately. Note that, for $x < \tau$,

$$P(T_{n:n} \leq x, \tau < T_{r:n}) = P(T_{n:n} \leq x < \tau < T_{r:n}) = 0 \tag{17}$$

and,

$$\begin{aligned} & P(T_{n:n} \leq x, T_{r:n} < \tau) \\ & = P(T_{r:n} < T_{n:n} \leq x < \tau) \\ & = \int_0^x \int_0^{t_n} \dots \int_0^{t_2} n! \Big(\frac{1}{\theta_1}\Big)^n e^{-\frac{1}{\theta_1} \sum_{i=1}^n t_i} dt_1 \dots dt_{n-1} dt_n \\ & = \big[1 - e^{-\frac{x}{\theta_1}}\big]^n. \end{aligned} \tag{18}$$

For $x > \tau$,

$$
\begin{aligned}
&P(T_{n:n} \le x, \tau < T_{r:n}) \\
=&P(\tau < T_{r:n} < T_{n:n} \le x) \\
=&n!\Big(\frac{1}{\theta_1}\Big)^r\Big(\frac{1}{\theta_2}\Big)^{n-r}\Bigg[\int_0^{t_r}\dots\int_0^{t_2} e^{-\frac{1}{\theta_1}\sum_{i=1}^{r-1} t_i} dt_1 \dots dt_{r-1} \int_{t_r}^{x}\dots\int_{t_r}^{t_{r+1}} e^{-\frac{1}{\theta_2}\sum_{i=r+1}^{n}(t_i - t_r)} \\
&dt_{r+1}\dots dt_n\Bigg] e^{-\frac{1}{\theta_1}(n-r+1)t_r} dt_r \\
=&n!\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\frac{1}{(r-1)!(n-r)!}\Bigg[\Big(\frac{1}{\theta_1}\Big)^{r-1}(r-1)!\int_0^{t_r}\dots\int_0^{t_2} e^{-\frac{1}{\theta_1}\sum_{i=1}^{r-1} t_i} dt_1 \dots dt_{r-1} \\
&\int_{t_r}^{x}\dots\int_{t_r}^{t_{r+1}}\Big(\frac{1}{\theta_2}\Big)^{n-r}(n-r)! e^{-\frac{1}{\theta_2}\sum_{i=r+1}^{n}(t_i - t_r)} dt_{r+1}\dots dt_n\Bigg] e^{-\frac{1}{\theta_1}(n-r+1)t_r} dt_r \\
=&r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\int_\tau^x \Big[1 - e^{-\frac{t_r}{\theta_1}}\Big]^{r-1}\Big[1 - e^{-\frac{1}{\theta_2}(x - t_r)}\Big]^{n-r} dt_r \\
=&r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\sum_{i=0}^{r-1}\sum_{j=0}^{n-r} e^{-\frac{j}{\theta_2}x}\binom{r-1}{i}\binom{n-r}{j}(-1)^{i+j}\int_\tau^x e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})t_r} dt_r \\
=&r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\sum_{i=0}^{r-1}\sum_{j=0}^{n-r} e^{-\frac{j}{\theta_2}x}\binom{r-1}{i}\binom{n-r}{j}(-1)^{i+j}\Big(\frac{i}{\theta_1}-\frac{j}{\theta_2}\Big)^{-1}\Big[e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})\tau} - e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})x}\Big]
\end{aligned}
\tag{19}
$$

$$
P(T_{n:n} \le x, T_{r:n} < \tau) = P(T_{r:n} < \tau < T_{n:n} \le x) + P(T_{r:n} < T_{n:n} < \tau < x) \tag{20}
$$

Now,

$$
\begin{aligned}
&P(T_{r:n} < \tau < T_{n:n} \leq x) \\
&= \sum_{d=r}^{n-1} P(T_{r:n} < \tau < T_{n:n} \leq x, D = d) \\
&= \sum_{d=r}^{n-1} n! \Big(\frac{1}{\theta_1}\Big)^d \Big(\frac{1}{\theta_2}\Big)^{n-d} \int_0^{\tau} \dots \int_0^{t_2} e^{-\frac{1}{\theta_1}[\sum_{i=1}^{d} t_i + (n-d)\tau]} dt_1 \dots dt_d \times \\
&\int_{\tau}^{x} \dots \int_{\tau}^{t_{d+2}} e^{-\frac{1}{\theta_2}\sum_{i=d+1}^{n}(t_i-\tau)} dt_{d+1} \dots dt_n \\
&= \sum_{d=r}^{n-1} n! \frac{1}{d!(n-d)!} d! \Big(\frac{1}{\theta_1}\Big)^d \int_0^{\tau} \dots \int_0^{t_2} e^{-\frac{1}{\theta_1}[\sum_{i=1}^{d} t_i + (n-d)\tau]} dt_1 \dots dt_d \times \\
&(n-d)! \Big(\frac{1}{\theta_2}\Big)^{n-d} \int_{\tau}^{x} \dots \int_{\tau}^{t_{d+2}} e^{-\frac{1}{\theta_2}\sum_{i=d+1}^{n}(t_i-\tau)} dt_{d+1} \dots dt_n \\
&= \sum_{d=r}^{n-1} \binom{n}{d} \Big[1 - e^{-\frac{\tau}{\theta}}\Big]^d e^{-\frac{\tau}{\theta_1}(n-d)} \Big[1 - e^{-\frac{x-\tau}{\theta_2}}\Big]^{n-d} \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&P(T_{r:n} < T_{n:n} < \tau < x) \\
&= \int_0^{\tau} \int_0^{t_n} \dots \int_0^{t_2} n! \Big(\frac{1}{\theta_1}\Big)^n e^{-\frac{1}{\theta_1}\sum_{i=1}^{n} t_i} dt_1 \dots dt_{n-1} dt_n \\
&= \big[1 - e^{-\frac{\tau}{\theta_1}}\big]^n. \qquad (22)
\end{aligned}
$$

Thus from equations (17, 18, 19, 21, 22) we find that the distribution function of $T_{n:n}$ at the point $x > 0$ as,

$$
G(x) = P(T_{n:n} \leq x) = \begin{cases} G_1(x) & \text{if } 0 < x < \tau, \\ G_2(x) & \text{if } \tau < x. \end{cases} \qquad (23)
$$

where,

$$
G_1(x) = \big[1 - e^{-\frac{x}{\theta_1}}\big]^n \qquad (24)
$$

and

$$G_2(x) = r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\sum_{i=0}^{r-1}\sum_{j=0}^{n-r} e^{-\frac{j}{\theta_2}x}\binom{r-1}{i}\binom{n-r}{j}\Big(\frac{i}{\theta_1}-\frac{j}{\theta_2}\Big)^{-1}(-1)^{i+j}\times$$
$$\big[e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})\tau} - e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})x}\big] + \sum_{d=r}^{n-1}\binom{n}{d} e^{-\frac{\tau}{\theta_1}(n-d)}\Big[1-e^{-\frac{\tau}{\theta_1}}\Big]^d\Big[1-e^{-\frac{x-\tau}{\theta_2}}\Big]^{n-d} +$$
$$\big[1-e^{-\frac{\tau}{\theta_1}}\big]^n \tag{25}$$

Hence the density function of $T_{n:n}$ at the point $0 < x < \tau$ is obtained as,

$$g(x) = \frac{d}{dx}G(x) = \begin{cases} g_1(x) & \text{if } 0 < x < \tau, \\ g_2(x) & \text{if } \tau < x. \end{cases} \tag{26}$$

where,

$$g_1(x) = \frac{d}{dx}G_1(x) = \frac{n}{\theta_1}\big[1-e^{-\frac{x}{\theta_1}}\big]^{n-1}e^{-\frac{x}{\theta_1}} \tag{27}$$

and

$$\begin{aligned} g_2(x) &= \frac{d}{dx}G_2(x) \\ &= r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\sum_{i=0}^{r-1}\sum_{j=0}^{n-r}\binom{r-1}{i}\binom{n-r}{j}\Big(\frac{i}{\theta_1}-\frac{j}{\theta_2}\Big)^{-1}(-1)^{i+j}\times \\ &\big[\frac{i}{\theta_1}e^{-\frac{i}{\theta_1}x} - \frac{j}{\theta_2}e^{-\frac{j}{\theta_2}x-(\frac{i}{\theta_1}-\frac{j}{\theta_2})\tau}\big] + \sum_{d=r}^{n-1}\binom{n}{d}\frac{n-d}{\theta_2}e^{-\frac{\tau}{\theta_1}(n-d)}\Big[1-e^{-\frac{\tau}{\theta_1}}\Big]^d\times \\ &\Big[1-e^{-\frac{x-\tau}{\theta_2}}\Big]^{n-d-1}e^{-\frac{x-\tau}{\theta_2}}. \end{aligned} \tag{28}$$

Hence, $E(T_{n:n}) = \int_0^\tau xg_1(x)dx + \int_\tau^\infty xg_2(x)dx$. We find these two integrals separately.

$$
\begin{aligned}
&\int_0^{\tau} x g_1(x)dx \\
=&\int_0^{\tau} n\frac{x}{\theta_1}\big[1-e^{-\frac{x}{\theta_1}}\big]^{n-1}e^{-\frac{x}{\theta_1}}dx \\
=&n\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\int_0^{\tau}\frac{x}{\theta_1}e^{-\frac{x}{\theta_1}(i+1)}dx \\
=&\sum_{i=0}^{n-1}\binom{n}{i+1}(-1)^i\frac{\theta_1}{(i+1)^2}\Gamma\big(\frac{\tau}{\theta_1}(i+1),1,2\big)
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&\int_{\tau}^{\infty} x g_2(x)dx \\
=&r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\sum_{i=0}^{r-1}\sum_{j=0}^{n-r}\binom{r-1}{i}\binom{n-r}{j}\Big(\frac{i}{\theta_1}-\frac{j}{\theta_2}\Big)^{-1}(-1)^{i+j}\times \\
&\big[\frac{i}{\theta_1}\int_{\tau}^{\infty}xe^{-\frac{i}{\theta_1}x}dx-\frac{j}{\theta_2}\int_{\tau}^{\infty}xe^{-\frac{j}{\theta_2}x-(\frac{i}{\theta_1}-\frac{j}{\theta_2})\tau}dx\big]+\sum_{d=r}^{n-1}\binom{n}{d}\frac{n-d}{\theta_2}e^{-\frac{\tau}{\theta_1}(n-d)}\times \\
&\Big[1-e^{-\frac{\tau}{\theta_1}}\Big]^d\sum_{i=0}^{n-d-1}\binom{n-d-1}{i}(-1)^i\int_{\tau}^{\infty}xe^{-\frac{x-\tau}{\theta_2}(i+1)}dx \\
=&r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big)\sum_{i=0}^{r-1}\sum_{j=0}^{n-r}\binom{r-1}{i}\binom{n-r}{j}\Big(\frac{i}{\theta_1}-\frac{j}{\theta_2}\Big)^{-1}(-1)^{i+j}\times \\
&\Big[\frac{\theta_1}{i}[1-\Gamma(\frac{\tau i}{\theta_1},1,2)]-e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})}\frac{\theta_2}{j}\big[1-\Gamma(\frac{\tau j}{\theta_2},1,2)\big]\Big]+\sum_{d=r}^{n-1}\binom{n}{d}\frac{n-d}{\theta_2}e^{-\frac{\tau}{\theta_1}(n-d)}\times \\
&\Big[1-e^{-\frac{\tau}{\theta_1}}\Big]^d\sum_{i=0}^{n-d-1}\binom{n-d-1}{i}(-1)^i e^{\frac{\tau}{\theta_2}(i+1)}\frac{\theta_2}{i+1}\big[1-\Gamma(\frac{\tau}{\theta_2}(i+1),1,2)\big].
\end{aligned} \tag{30}
$$

Thus,

$$
\begin{aligned}
&E(T_{n:n}) = \sum_{i=0}^{n-1} \binom{n}{i+1} (-1)^i \frac{\theta_1}{(i+1)^2} \Gamma\Big(\frac{\tau}{\theta_1}(i+1), 1, 2\Big) + \\
&r\binom{n}{r}\Big(\frac{1}{\theta_1}\Big)\Big(\frac{1}{\theta_2}\Big) \sum_{i=0}^{r-1}\sum_{j=0}^{n-r} \binom{r-1}{i}\binom{n-r}{j}\Big(\frac{i}{\theta_1}-\frac{j}{\theta_2}\Big)^{-1}(-1)^{i+j}\times \\
&\Big[\frac{\theta_1}{i}[1-\Gamma(\frac{\tau i}{\theta_1},1,2)] - e^{-(\frac{i}{\theta_1}-\frac{j}{\theta_2})}\frac{\theta_2}{j}\big[1-\Gamma(\frac{\tau j}{\theta_2},1,2)\big]\Big] + \sum_{d=r}^{n-1}\binom{n}{d}\frac{n-d}{\theta_2}e^{-\frac{\tau}{\theta_1}(n-d)}\times \\
&\Big[1-e^{-\frac{\tau}{\theta_1}}\Big]^d \sum_{i=0}^{n-d-1}\binom{n-d-1}{i}(-1)^i e^{\frac{\tau}{\theta_2}(i+1)}\frac{\theta_2}{i+1}\big[1-\Gamma(\frac{\tau}{\theta_2}(i+1),1,2)\big]. \qquad (31)
\end{aligned}
$$