Meta-Analysis of a Step-Stress Experiment Under Weibull Distribution

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Abstract

In this article we mainly focus on the meta-analysis of several simple step-stress experimental data sets, when the lifetime of the items at each experiment follow Weibull distribution. It is assumed that independent data sets are obtained from s simple step-stress experiments. It is further assumed that the lifetime of the experimental units follow two parameter Weibull distribution with different shape and scale parameters at different stress levels. The classical and Bayesian inference of the model parameters have been provided. Since the closed form solution of maximum likelihood estimators of the model parameters do not exist, asymptotic properties of the estimators have been used to construct confidence intervals. On the other hand Gibbs sampling technique has been used to obtain the Bayes estimates and the associated credible intervals of the model parameters. Extensive simulation experiments have been performed to assess the performance of the proposed methods, and the analyses of two data sets have been presented for illustrative purpose.

Key Words Step-stress Life-tests; Meta-analysis; Maximum Likelihood Estimator; Bayes Estimator; Confidence Interval; Credible Interval.

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1 Introduction

Designing of a life testing experiment of highly reliable experimental units is a challenging task. Due to consumer's demand, products are becoming more reliable and hence it is required to design the experiment in such a way that it yields more failure time data in a shorter period of time. The accelerated life testing (ALT) experiment has been widely used to reduce the experimental time. In an ALT experiment the mean life times of the experimental units are reduced by exposing the units to a higher stress level. Key references on ALT model are Nelson [19] and Bagdonavicius and Nikulin [1]. Depending on the type of the experimental units, several stress factors like temperature, voltage, etc. can be used during the experiment. A particular type of ALT experiment is known as the step-stress life testing (SSLT) experiment. In a SSLT experiment, the experiment starts with a certain number of experimental units at an initial stress level S_1 and then it increases to its next level S_2, either at a pre-specified time or after a pre-specified number of failures, and so on. If there are only two stress levels, then the experiment is called the simple step-stress experiment. An extensive amount of work has been done in analyzing data obtained from a simple step-stress experiment, see for example the monograph by Kundu and Ganguly [17] in this respect.

In general different lifetime distributions are assumed at different stress levels. Therefore, we need a model to connect the distributions under different stress levels. The most popular model in this regard is the cumulative exposure model (CEM). Another popular model mainly for the Weibull lifetime distribution has been proposed by Khamis and Higgins [14], and it is popularly known as the Khamis-Higgings model or KHM. Several works have been done to analyze the data obtained from a step stress life testing experiment assuming CEM or KHM assumption, see for example Balakrishnan et al. [3], Balakrishnan et al. [4], Mitra et al. [18], Xiong and Milliken [23], Drop et al. [9], Drop and Mazzuchi [8], Balakrishnan et al. [2], Samanta and Kundu [22], Samanta et al. [20], Samanta et al. [21] and the references cited therein.

Most of the works on step-stress model available in the literature is based on a single sample. But one may have independent data sets from several experiments with similar experimental units. The sample size, stress changing time and even the termination time of the experiment may be different for different step-stress experiments. A meta-analysis approach is proposed by Kateri et al. [11] to develop inference on the model parameters based on the data obtained from different Type-II censored step-stress experiments. It is assumed that the lifetime of the experimental units follow one-parameter exponential distribution at each stress level. In case of a single sample, the maximum likelihood estimates (MLEs) of the unknown parameters exist only when at least one failure is observed at each stress level. In case of a multi-sample, the MLEs exist if at each stress level at least one failure is observed in one of the samples. Therefore, in case of a multi-sample problem, a weaker condition is sufficient for the existence of MLEs. The inference for multi-sample exponential step-stress model under Type-I censoring is provided by Kateri et al. [12], see also Kateri et al. [13] and Bedbur et al. [5] in this respect.

Most of the meta-analysis works on step-stress experiment is done by assuming exponential lifetime distribution and the inferences are provided based on the classical setup. The main aim of this paper is to extend the results when the lifetime of the experimental units under different stress levels follow Weibull distribution with different shape and scale parameters. The distributions under different stress levels are connected through generalized KHM assumptions. This model is analytically more tractable than the CEM. We have considered both the classical and Bayesian inference of the model parameters when the data are Type-II censored. Although, we have considered only Type-II censored data, the methods can be extended for other censoring schemes also.

The MLEs of the unknown parameters cannot be obtained in closed form. They have to be obtained numerically by solving more than one non-linear equations. Under certain regularity condition, it can be shown that the MLEs are unique. Moreover, the proposed condition can be easily verified from the observed data sets. Due to non-existence of the closed form of MLEs, it is not possible to construct the exact confidence intervals (CIs).

Therefore we propose to use asymptotic CIs based on the observed Fisher information matrix and assuming the asymptotic normality property of the MLEs. We have also considered the Bayesian inference of the model parameters under squared error loss function. The Bayes estimates (BEs) and the associated credible intervals (CRIs) cannot be obtained explicitly. We propose to use Gibbs sampling technique to compute BEs and the associated CRIs. Extensive simulation study has been performed to assess the performances of the proposed methods, and the analyses of two data sets have been presented to see the effectiveness of the proposed model.

The rest of the article is organized as follows. The model assumptions and the likelihood function are provided in Section 2. The classical and Bayesian inferences have been developed in Section 3 and Section 4, respectively. Simulation results and the data analyses have been presented in Section 5. Finally we have concluded the article in Section 6. All the proofs of the necessary results are provided in the Appendix.

2 Model Assumption and Likelihood Function

It is assumed that s independent life testing experiments start at the initial stress level S_1 with $n_1, \ldots, n_s$ number of identical units. It is further assumed that for the k-th ($k = 1, \ldots, s$) experiment the stress level changes from S_1 to its next level S_2 at time τ_k and the experiment terminates as soon as the pre-specified r_k-th ($\leq n_k$) failure occurs. Therefore, the data obtained from the k-th experiment will be of the form:

$$t_{1:n_k} \leq \ldots \leq t_{r_{1k}:n_k} \leq \tau_k \leq t_{r_{1k}+1:n_k} \leq \ldots \leq t_{r_k:n_k},$$

where r_{1k} ($0 \leq r_{1k} \leq r_k$) is the number of failures at the first stress level. Without loss of generality, it is assumed that out of s independent samples, the first s_1 samples have zero failures at the second stress level, the next s_2 samples have zero failures at the first stress level and the remaining $s - s_1 - s_2$ samples have at least one failure at both the stress levels.

The lifetime distribution of the experimental units at stress level S_i $(i = 1, 2)$ is assumed to be a Weibull distribution with the shape parameter $\alpha_i > 0$ and the scale parameter $\theta_i > 0$. We will denote this distribution by $WE(\alpha_i,\, \theta_i)$. The probability density function (PDF) of the experimental unit at the stress level S_i $(i = 1, 2)$ is given by

$$f(t; \alpha_i, \theta_i) = \alpha_i \theta_i t^{\alpha_i - 1} e^{-\theta_i t^{\alpha_i}}, \qquad 0 < t < \infty, \quad \alpha_i > 0, \quad \theta_i > 0,$$

zero, otherwise. Now to connect the distribution functions at the two stress levels, we have used a generalized form of the KHM. It is assumed that the hazard function, when stress changes at the time τ, is of the form

$$h(t) = \begin{cases} \alpha_1 \theta_1 t^{\alpha_1 - 1} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_2 t^{\alpha_2 - 1} & \text{if} \quad \tau < t < \infty. \end{cases}$$

Hence, the cumulative hazard function and the survival function are, respectively,

$$H(t) = \begin{cases} \int_0^t \alpha_1 \theta_1 u^{\alpha_1 - 1} du & = \theta_1 t^{\alpha_1} & \text{if} \quad 0 < t \leq \tau \\ \int_0^\tau \alpha_1 \theta_1 u^{\alpha_1 - 1} du + \int_\tau^t \alpha_2 \theta_2 u^{\alpha_2 - 1} du & = \theta_1 \tau^{\alpha_1} + \theta_2 (t^{\alpha_2} - \tau^{\alpha_2}) & \text{if} \quad \tau < t < \infty, \end{cases}$$

and

$$S(t) = e^{-H(t)} = \begin{cases} e^{-\theta_1 t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ e^{-\theta_2 (t^{\alpha_2} - \tau^{\alpha_2}) - \theta_1 \tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty. \end{cases} \tag{1}$$

Hence the PDF of the lifetimes is given by

$$f_1(t) = \begin{cases} \alpha_1 \theta_1 t^{\alpha_1 - 1} e^{-\theta_1 t^{\alpha_1}} & \text{if} \quad 0 < t \leq \tau \\ \alpha_2 \theta_2 t^{\alpha_2 - 1} e^{-\theta_2 (t^{\alpha_2} - \tau^{\alpha_2}) - \theta_1 \tau^{\alpha_1}} & \text{if} \quad \tau < t < \infty. \end{cases} \tag{2}$$

We will denote the set of all unknown model parameters as $\eta = (\alpha_1,\, \theta_1,\, \alpha_2,\, \theta_2)^\top$. Since the data are coming from s independent Type-II censored simple step stress life testing experiment, therefore the data obtained from k-th step-stress experiment will be one of the following forms:

(a) $t_{1:n} \leq \ldots \leq t_{r_k:n} \leq \tau$

(b) $\tau \leq t_{1:n} \leq \ldots \leq t_{r_k:n}$

(c) $t_{1:n} \leq \ldots \leq t_{r_{1k}:n} \leq \tau \leq t_{r_{1k}+1:n} \leq \ldots \leq t_{r_k:n}$.

If the data of the k-th sample is of the form (a), then the likelihood contribution of the sample is given by

$$L_1^{(k)}(\mathbf{t^{(k)}};\boldsymbol{\eta}) = \frac{n_k!}{r_k!}\theta_1^{r_k}\alpha_1^{r_k}[\prod_{i=1}^{r_k} t_{i:n_k}^{\alpha_1-1}]e^{-\theta_1[\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}+(n_k-r_k)t_{r_k:n_k}^{\alpha_1}]}.$$

Here, $\mathbf{t^{(k)}}$ denotes the data set from the k-th experiment. Again if the data of the k-th sample is of the form (b), then the likelihood contribution of the sample is given by

$$L_2^{(k)}(\mathbf{t^{(k)}};\,\boldsymbol{\eta}) = \frac{n_k!}{r_k!}\theta_2^{r_k}\alpha_2^{r_k}[\prod_{i=1}^{r_k} t_{i:n_k}^{\alpha_2-1}]e^{-\theta_2[\sum_{i=1}^{r_k}(t_{i:n_k}^{\alpha_2}-\tau_k^{\alpha_2})+(n_k-r_k)(t_{r_k:n_k}^{\alpha_2}-\tau_k^{\alpha_2})]}e^{-\theta_1 n_k\tau_k^{\alpha_1}}.$$

For case (c), the likelihood contribution of the k-th sample is given by

$$\begin{aligned} L_{12}^{(k)}(\mathbf{t^{(k)}};\,\boldsymbol{\eta}) &= \tfrac{n_k!}{r_{1k}!(r_k-r_{1k})!}\theta_1^{r_{1k}}\theta_2^{r_k-r_{1k}}\alpha_1^{r_{1k}}\alpha_2^{r_k-r_{1k}}[\textstyle\prod_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1-1}][\prod_{i=r_{1k}+1}^{r_k} t_{i:n_k}^{\alpha_2-1}] \\ &\quad e^{-\theta_1[\sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}+(n_k-r_{1k})\tau_k^{\alpha_1}]}e^{-\theta_2[\sum_{i=r_{1k}+1}^{r_k}(t_{i:n_k}^{\alpha_2}-\tau_k^{\alpha_2})+(n_k-r_k)(t_{r_k:n_k}^{\alpha_2}-\tau_k^{\alpha_2})]}. \end{aligned}$$

Since we assume that the first s_1 samples are of the form (a), next s_2 samples are of the form (b) and the remaining $s-s_1-s_2$ samples are of the form (c), therefore, the combined likelihood of the data obtained from s different step-stress experiments is given by

$$\begin{aligned} L(\mathbf{t};\,\boldsymbol{\eta}) &= \textstyle\prod_{i=1}^{s_1} L_1^{(i)}(\mathbf{t};\,\boldsymbol{\eta})\prod_{i=s_1+1}^{s_1+s_2} L_2^{(i)}(\mathbf{t};\,\boldsymbol{\eta})\prod_{i=s_1+s_2+1}^{s} L_{12}^{(i)}(\mathbf{t};\,\boldsymbol{\eta}) \\ &= C\times\theta_1^{\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}}\theta_2^{\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})} \\ &\quad\times\alpha_1^{\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}}\alpha_2^{\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})} \\ &\quad\times(\textstyle\prod_{k=1}^{s_1}\prod_{i=1}^{r_k} t_{i:n_k}^{\alpha_1-1})\times(\prod_{k=s_1+s_2+1}^{s}\prod_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1-1})\times(\prod_{k=s_1+1}^{s_1+s_2}\prod_{i=1}^{r_k} t_{i:n_k}^{\alpha_2-1}) \\ &\quad\times(\textstyle\prod_{k=s_1+s_2+1}^{s}\prod_{i=r_{1k}+1}^{r_k} t_{i:n_k}^{\alpha_2-1})\times e^{-\theta_1 P(\alpha_1)}\times e^{-\theta_2 Q(\alpha_2)}, \end{aligned}$$

where, $\boldsymbol{t}$ denotes the combined data, and

$$\begin{aligned} C &= \prod_{k=1}^{s_1+s_2} \frac{n_k!}{r_k!} \times \prod_{k=s_1+s_2+1}^{s} \frac{n_k!}{r_{1k}!(r_k-r_{1k})!}, \\ P(\alpha_1) &= \sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1} + \sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1} + \sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1} \\ &\quad + \sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} + \sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1}, \\ Q(\alpha_2) &= \sum_{k=s_1+1}^{s_1+s_2}\sum_{i=1}^{r_k}(t_{i:n_k}^{\alpha_2}-\tau_k^{\alpha_2}) + \sum_{k=s_1+1}^{s_1+s_2}(n_k-r_k)(t_{r_k:n_k}^{\alpha_2}-\tau_k^{\alpha_2}) \\ &\quad + \sum_{k=s_1+s_2+1}^{s}\sum_{i=r_{1k}+1}^{r_k}(t_{i:n_k}^{\alpha_2}-\tau_k^{\alpha_2}) + \sum_{k=s_1+s_2+1}^{s}(n_k-r_k)(t_{r_k:n_k}^{\alpha_2}-\tau_k^{\alpha_2}). \end{aligned}$$

In the next two sections we provide the classical and Bayesian inferences of the unknown parameters based on the above likelihood function.

3 Classical Inference

3.1 Maximum Likelihood Estimation

The combined log-likelihood of the data is given by

$$\begin{aligned} l(\mathbf{t};\boldsymbol{\eta}) &= \ln(C) + \Big(\sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k}\Big)\ln(\theta_1) + \\ &\Big(\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})\Big)\ln(\theta_2) + \Big(\sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k}\Big)\ln(\alpha_1) \\ &+\Big(\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})\Big)\ln(\alpha_2) + (\alpha_1-1)\sum_{k=1}^{s_1}\sum_{i=1}^{r_k}\ln(t_{i:n_k}) \\ &+(\alpha_1-1)\sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}}\ln(t_{i:n_k}) + (\alpha_2-1)\sum_{k=s_1+1}^{s_1+s_2}\sum_{i=1}^{r_k}\ln(t_{i:n_k}) \\ &+(\alpha_2-1)\sum_{k=s_1+s_2+1}^{s}\sum_{i=r_{1k}+1}^{r_k}\ln(t_{i:n_k}) - \theta_1 P(\alpha_1) - \theta_2 Q(\alpha_2). \end{aligned} \tag{3}$$

For given α_1 and α_2, the MLEs of θ_1 and θ_2 can be obtained by maximizing the equation (3) with respect to θ_1 and θ_2 respectively. The MLEs of θ_1 and θ_2 can be obtained by differentiating (3) with respect to θ_1 and θ_2 and equating them to zero. Hence, the MLEs of θ_1 and θ_2, for given α_1 and α_2, are given by

$$\widehat{\theta}_1(\alpha_1) = \frac{\sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k}}{P(\alpha_1)} \quad \text{and} \quad \widehat{\theta}_2(\alpha_2) = \frac{\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})}{Q(\alpha_2)}, \tag{4}$$

and clearly they are unique. Now the MLEs of α_1 and α_2 can be obtained by maximizing the profile log-likelihood of α_1 and α_2. Replacing θ_1 and θ_2 in equation (3) by $\widehat{\theta}_1(\alpha_1)$ and $\widehat{\theta}_2(\alpha_2)$, respectively, we have the profile log-likelihood of α_1 and α_2. Hence, the profile log-likelihood of α_1 and α_2 is given by

$$\begin{aligned} l_1(\mathbf{t};\, \alpha_1, \alpha_2) \;\propto\;& (\textstyle\sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k})[\ln(\alpha_1) - \ln(P(\alpha_1))] + \\ & (\textstyle\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s} (r_k - r_{1k}))[\ln(\alpha_2) - \ln(Q(\alpha_2))] + \\ & (\alpha_1 - 1)\textstyle\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} \ln(t_{i:n_k}) + (\alpha_1 - 1)\sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}} \ln(t_{i:n_k}) + \\ & (\alpha_2 - 1)\textstyle\sum_{k=s_1+1}^{s_1+s_2}\sum_{i=1}^{r_k} \ln(t_{i:n_k}) + (\alpha_2 - 1)\sum_{k=s_1+s_2+1}^{s}\sum_{i=r_{1k}+1}^{r_k} \ln(t_{i:n_k}) \\ =\;& l_{11}(\mathbf{t};\, \alpha_1) + l_{12}(\mathbf{t};\, \alpha_2). \end{aligned} \tag{5}$$

The MLEs of α_1 and α_2 can be obtained by solving following two non-linear equations

$$\begin{aligned} & \big(\textstyle\sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k}\big)\big[\frac{1}{\alpha_1} - \frac{P'(\alpha_1)}{P(\alpha_1)}\big] + \\ & \textstyle\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} \ln(t_{i:n_k}) + \sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}} \ln(t_{i:n_k}) \;=\; 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} & \big(\textstyle\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s} (r_k - r_{1k})\big)\big[\frac{1}{\alpha_2} - \frac{Q'(\alpha_2)}{Q(\alpha_2)}\big] + \\ & \textstyle\sum_{k=s_1+1}^{s_1+s_2}\sum_{i=1}^{r_k} \ln(t_{i:n_k}) + \sum_{k=s_1+s_2+1}^{s}\sum_{i=r_{1k}+1}^{r_k} \ln(t_{i:n_k}) \;=\; 0, \end{aligned} \tag{7}$$

where, $P'(\alpha_1)$ is the first order partial derivative of $P(\alpha_1)$ with respect to α_1 and $Q'(\alpha_2)$ is the first order partial derivative of $Q(\alpha_2)$ with respect to α_2.

Now in Lemma 1 we show that the unique solution of the equation (6) always exists and in Lemma 2 we show that the unique solution of equation (7) exists under certain regularity condition. Moreover, the condition can be easily verified from the observed data.

Lemma 1. *The unique solution of equation (6) always exists which maximizes (5) with respect to* α_1.

Proof. See Appendix A.1. □

Lemma 2. *Let us define*

$$u(\alpha_2) \quad = \quad \frac{1}{\alpha_2^2} + \frac{Q(\alpha_2)Q''(\alpha_2) - [Q'(\alpha_2)]^2}{[Q(\alpha_2)]^2}, \tag{8}$$

where $Q'(\alpha_2)$ and $Q''(\alpha_2)$ are respectively first order and second order partial derivatives of $Q(\alpha_2)$ with respect to α_2. If $u(\alpha_2) > 0$, for all $\alpha_2 > 0$, then the unique solution of equation (7) exists which maximizes (5) with respect to α_2.

Proof. See Appendix A.2. □

3.2 Asymptotic Confidence Interval

Since the closed form of MLEs do not exist, it is not possible to obtain the exact distribution of MLEs and hence the exact CIs. We propose to use asymptotic CIs of the MLEs. An asymptotic confidence interval of $\boldsymbol{\eta} = (\alpha_1, \theta_1, \alpha_2, \theta_2)^\top$ can be constructed by assuming asymptotic normality of its MLEs. We have used the observed Fisher information matrix for the construction of asymptotic CIs. The observed Fisher information matrix is given by

$$F \quad = \quad ((f_{ij})) = ((-\frac{\partial^2 l(\mathbf{t}; \boldsymbol{\eta})}{\partial\eta_i \partial\eta_j})).$$

The elements of the Fisher information matrix are given in Appendix A.3. The asymptotic distribution of $\widehat{\eta} = (\widehat{\alpha}_1, \widehat{\theta}_1, \widehat{\alpha}_2, \widehat{\theta}_2)^\top$ is given by $\widehat{\eta} - \eta \sim N_4(0, F^{-1})$. Therefore $100(1-\alpha)\%$ asymptotic CI of η_i is given by

$$\left[\widehat{\eta}_i \pm z_{1-\frac{\alpha}{2}} \sqrt{V_{ii}}\right],$$

where V_{ii} is $(i,i)^{th}$ element of the matrix F^{-1}. Though F is a 4×4 matrix, it can be partitioned as two block diagonal matrices, each of order 2×2. Therefore F^{-1} can easily be obtained analytically.

4 Bayesian Inference

In this section we have considered the Bayesian inference of the model parameters. To compute the Bayes estimates and the associated credible intervals we have assumed independent gamma priors on the unknown parameters. Let us denote a gamma distribution with parameters $a > 0$ and $b > 0$ by $GA(a, b)$. The PDF of $GA(a, b)$ is given by

$$f(x|a, b) = \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1}; \qquad x > 0, a > 0, b > 0,$$

zero, otherwise. For $a_i > 0$ and $b_i > 0$ $(i = 0, 1, 2, 3)$, we assume the following prior distributions: $\alpha_1 \sim GA(a_0, b_0)$, $\theta_1 \sim GA(a_1, b_1)$, $\alpha_2 \sim GA(a_2, b_2)$, and $\theta_2 \sim GA(a_3, b_3)$. It is further assumed that the priors are independently distributed. Hence, the joint prior distribution is given by

$$\widetilde{\pi}(\alpha_1, \theta_1, \alpha_2, \theta_2) \propto e^{-a_0\alpha_1}\alpha_1^{b_0-1}e^{-a_1\theta_1}\theta_1^{b_1-1}e^{-a_2\alpha_2}\alpha_2^{b_2-1}e^{-a_3\theta_2}\theta_2^{b_3-1}. \tag{9}$$

Therefore the joint posterior distribution of $(\alpha_1, \theta_1, \alpha_2, \theta_2)$ is given by

$$\begin{aligned}\pi(\alpha_1, \theta_1, \alpha_2, \theta_2|data) \propto\ & e^{-a_0\alpha_1}\alpha_1^{b_0+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}-1}\left(\prod_{k=1}^{s_1}\prod_{i=1}^{r_k} t_{i:n_k}\right)^{\alpha_1-1} \\ & \left(\prod_{k=s_1+s_2+1}^{s}\prod_{i=1}^{r_{1k}} t_{i:n_k}\right)^{\alpha_1-1} e^{-\theta_1(a_1+P(\alpha_1))}\theta_1^{b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}-1} \\ & e^{-a_2\alpha_2}\alpha_2^{b_2+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})-1}\left(\prod_{k=s_1+1}^{s_1+s_2}\prod_{i=1}^{r_k} t_{i:n_k}\right)^{\alpha_2-1} \\ & \left(\prod_{k=s_1+s_2+1}^{s}\prod_{i=r_{1k}+1}^{r_k} t_{i:n_k}\right)^{\alpha_2-1} e^{-\theta_2(a_3+Q(\alpha_2))} \\ & \theta_2^{b_3+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})-1}.\end{aligned}$$

The Bayes estimates (BEs) under squared error loss function is the mean of the posterior distribution. In general, an explicit form of the BEs of the unknown parameters under the squared error loss function cannot be obtained. Hence we propose to use Gibbs sampling technique to obtain the Bayes estimates and the associated credible intervals. Note that the joint posterior distribution can be written as

$$\pi(\alpha_1, \theta_1, \alpha_2, \theta_2|data) \propto \pi_1(\alpha_1)\pi_2(\theta_1|\alpha_1)\pi_3(\alpha_2)\pi_4(\theta_2|\alpha_2),$$

where

$$\begin{aligned}
\pi_1(\alpha_1) &\propto e^{-a_0\alpha_1}\alpha_1^{b_0+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}-1}\left[a_1+P(\alpha_1)\right]^{-(b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k})} \\
&\quad\times\left(\prod_{k=1}^{s_1}\prod_{i=1}^{r_k} t_{i:n_k}\right)^{\alpha_1-1}\times\left(\prod_{k=s_1+s_2+1}^{s}\prod_{i=1}^{r_{1k}} t_{i:n_k}\right)^{\alpha_1-1}, \\
\pi_2(\theta_1|\alpha_1) &\propto \frac{\left[a_1+P(\alpha_1)\right]^{(b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k})}}{\Gamma(b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k})}e^{-\theta_1(a_1+P(\alpha_1))}\theta_1^{b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}-1}, \\
\pi_3(\alpha_2) &\propto e^{-a_2\alpha_2}\alpha_2^{b_2+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})-1} \\
&\quad\times\left[a_3+Q(\alpha_2)\right]^{-(b_3+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k}))} \\
&\quad\times\left(\prod_{k=s_1+1}^{s_1+s_2}\prod_{i=1}^{r_k} t_{i:n_k}\right)^{\alpha_2-1}\times\left(\prod_{k=s_1+s_2+1}^{s}\prod_{i=r_{1k}+1}^{r_k} t_{i:n_k}\right)^{\alpha_2-1}, \\
\pi_4(\theta_2|\alpha_2) &\propto \frac{\left[a_3+Q(\alpha_2)\right]^{(b_3+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k}))}}{\Gamma(b_3+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k}))}e^{-\theta_2(a_3+Q(\alpha_2))} \\
&\quad\times\theta_2^{b_3+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})-1}.
\end{aligned}$$

For given α_1, the posterior distribution of θ_1 is Gamma with the shape parameter $b_1 + \sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k}$ and the scale parameter $a_1 + P(\alpha_1)$. Again for a given α_2, the posterior distribution of θ_2 is gamma with the shape parameter $b_3+\sum_{k=s_1+1}^{s_1+s_2} r_k+\sum_{k=s_1+s_2+1}^{s}(r_k-r_{1k})$ and the scale parameter $a_3 + Q(\alpha_2)$. In Lemma 3 we show that $\pi_1(\alpha_1)$ is a log-concave density function. In Lemma 4 we provide a condition under which $\pi_3(\alpha_2)$ is a log-concave density function. Moreover, the condition can be easily verified from the observed data sets. Therefore, we can generate α_1 and α_2 easily using the method proposed by Devroye [7]. Alternatively, one can use the ratio-of-uniform method introduced by Kinderman and Monahan [15] to generate α_1 and α_2. Once we generate α_1 and α_2, we can generate θ_1 for given α_1 and θ_2 for given α_2 very easily. We need the following results for further development.

Lemma 3. $\pi_1(\alpha_1)$ *is a log-concave density function.*

Proof. See Appendix A.4. □

Lemma 4. *Let*

$$\begin{aligned} u_2(\alpha_2) &= \frac{b_2 + \sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s}(r_k - r_{1k}) - 1}{\alpha_2^2} \\ &\quad + \left(b_3 + \sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s} (r_k - r_{1k})\right) \frac{a_3 Q''(\alpha_2) + Q(\alpha_2)Q''(\alpha_2) - (Q'(\alpha_2))^2}{[a_3 + Q(\alpha_2)]^2}, \end{aligned}$$

where $Q'(\alpha_2)$ and $Q''(\alpha_2)$ are respectively first and second order parial derivatives of $Q(\alpha_2)$ with respect to α_2. If $u_2(\alpha_2) \geq 0$ then $\pi_3(\alpha_2)$ is a log-concave density function.

Proof. See Appendix A.5. □

We propose the following algorithm to compute BEs of the unknown parameters and the associated CRIs.

Algorithm 1

Step 1. Generate α_1 and α_2 from $\pi_1(\alpha_1)$ and $\pi_3(\alpha_2)$, respectively, using the method proposed by Devroye [7] or Kundu [16]. Alternatively, one can use the ratio-of-uniform method introduced by Kinderman and Monahan [15] to generate α_1 and α_2. Though the ratio-of-uniform method do not need the log-concavity property of α_1 and α_2, it might leads to the higher number of rejection during the sample generation.

Step 2. For a given α_1 generate θ_1 from $GA(a_1 + P(\alpha_1), b_1 + \sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k})$ and for a given α_2 generate θ_2 from $GA(a_3 + Q(\alpha_2), b_3 + \sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s} (r_k - r_{1k}))$.

Step 3. Repeat Step 1 and Step 2, M times to obtain $(\alpha_1^1,\, \theta_1^1,\, \alpha_2^1,\, \theta_2^1)\,, \ldots, (\alpha_1^M,\, \theta_1^M,\, \alpha_2^M,\, \theta_2^M)$.

Step 4. BEs of α_1, θ_1, α_2, and θ_2 with respect to squared error loss function are given by

$$\begin{aligned} \widehat{\alpha}_{1(B)} &= \tfrac{1}{M} \textstyle\sum_{k=1}^{M} \alpha_1^k, \qquad \widehat{\theta}_{1(B)} = \tfrac{1}{M} \textstyle\sum_{k=1}^{M} \theta_1^k, \\ \widehat{\theta}_{2(B)} &= \tfrac{1}{M} \textstyle\sum_{k=1}^{M} \theta_2^k, \qquad \widehat{\alpha}_{2(B)} = \tfrac{1}{M} \textstyle\sum_{k=1}^{M} \alpha_2^k, . \end{aligned}$$

Step 5. To obtain the credible interval of α_1, we order $\alpha_1^1, \ldots, \alpha_1^M$ as $\alpha_1^{(1)} < \ldots < \alpha_1^{(M)}$. Then $100(1-\alpha)\%$ symmetric credible interval of α_1 is given by $(\alpha_1^{([\frac{\alpha}{2}M])}, \alpha_1^{([(1-\frac{\alpha}{2})M])})$, where $[x]$ is the largest integer less than or equal to x.

Step 6. To construct $100(1-\alpha)\%$ highest posterior density (HPD) credible interval of α_1, consider the set of credible intervals $(\alpha_1^{(j)}, \alpha_1^{([j+(1-\alpha)M])})$, $j = 1, \ldots, [\alpha M]$. Therefore, a $100(1-\alpha)\%$ HPD credible interval of α_1 is $(\alpha_1^{(j^*)}, \alpha_1^{([j^*+(1-\alpha)M])})$, where j^* is such that

$$\alpha_1^{([j^*+(1-\alpha)M])} - \alpha_1^{(j^*)} < \alpha_1^{([j+(1-\alpha)M])} - \alpha_1^{(j)}$$

for all $j = 1 \ldots [\alpha M]$.

Following the method of Step 5 and Step 6 we can obtain the symmetric and HPD credible intervals for other parameters.

5 Simulation Study and Data Analysis

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5.1 Simulation Study

In this section we have performed extensive simulation study to evaluate the behavior of the proposed methods. Throughout the simulation study we have considered $s = 4$, i.e., the data are coming from four different step-stress experiments. For brevity, we have taken $n_k = n$ and $\tau_k = \tau$ for all $k = 1, \ldots, s$, i.e., the sample sizes and the stress changing time are same for all the four step-stress experiments. The experiments are Type-II censored and we have considered $r_1 = r_2 = 60\%$ of n and $r_3 = r_4 = 75\%$ of n. The simulation study has been performed for different values of n and τ. The true parameter values are $\alpha_1 = 1.2$, $\theta_1 = 1.4$, $\alpha_2 = 1.5$, and $\theta_2 = 1.8$. In case of Bayesian inference we consider both, non-informative and informative prior. For non-informative prior, as suggested by Congdon [6], the hyper parameters are taken as $a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0.0001$. In case of informative prior the hyper parameters are chosen in such a way that the prior mean becomes equal to the true parameter value. In this case, $\frac{b_0}{a_0} = 1.2$, $\frac{b_1}{a_1} = 1.4$, $\frac{b_2}{a_2} = 1.5$, $\frac{b_3}{a_3} = 1.2$. In Bayesian simulation with informative prior we consider $a_0 = 0.3$, $b_0 = 0.36$,

$a_1 = 0.3$, $b_1 = 0.42$, $a_2 = 0.2$, $b_2 = 0.3$, $a_3 = 0.2$ and $b_3 = 0.36$. We have provided the MLEs and BEs along with their mean square errors (MSEs) of the model parameters (See, Table 1, Table 3 and Table 6). We have also provided the average lengths (ALs) and the coverage percentages (CPs) of asymptotic CIs, symmetric CRIs and highest posterior density (HPD) CRIs (See, Table 2, 4, 5, 7 and 8). All the simulation results are based on 5000 replications. In case of Bayesian inference we have taken $M = 10000$. All the computations have been on R environment. The codes can be obtained from the author on request.

Some of the points are very clear from the simulation results. It is observed that in all the cases as the sample size increases, the biases and the MSEs decrease. It indicates the consistency properties of MLEs and BEs. Again for fixed sample size and fixed termination time if we increase the stress changing time τ then the estimates of the parameters under first stress level, i.e. α_1 and θ_1 becomes better and on the other hand the estimates of α_2 and θ_2 becomes worse in terms of their respective MSEs. If we observe the performance of the asymptotic CIs and the symmetric and HPD CRIs, their performances are also quite satisfactory. The coverage percentages are very close to the corresponding nominal value, the average length of the CIs/CRIs decrease with the increase of the sample size.

Now if we compare the performances of MLEs and BEs (Table 1,Table 3 and Table 6), the BEs provides better results in terms of lower MSEs. It is also observed from Table 2, Table 4, Table 5, Table 7 and Table 8 that the 95% asymptotic CIs provides shorter length for α_1 and θ_1 than the 95% symmetric and HPD CRIs. Again for α_2 and θ_2 the performance of 95% HPD CRIs are better than the 95% symmetric CRIs and asymptotic CIs.

If we compare the BEs based on non-informative and informative priors (Table 3 and Table 6), the MSEs of Bayes estimates using informative prior is lower than that of using non-informative prior. In both the cases the CPs of symmetric and HPD CRIs are close to the nominal value but ALs of CRIs are lower in case of informative prior (see Table 4, 5, 7 and 8).

Table 1: Average estimates (AEs) and MSEs of MLEs based on 5000 simulation ($\alpha_1 = 1.2$, $\theta_1 = 1.4$, $\alpha_2 = 1.5$, $\theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	0.3	(12, 12, 15, 15)	1.2580	0.0788	1.6056	0.5380	1.7345	0.5009	2.1083	0.5442
20	0.4	(12, 12, 15, 15)	1.2364	0.0505	1.4948	0.2150	1.9760	1.2152	2.2780	1.2990
30	0.3	(18, 18, 22, 22)	1.2363	0.0448	1.5267	0.2719	1.6499	0.3470	2.0198	0.2821
30	0.4	(18, 18, 22, 22)	1.2278	0.0315	1.4701	0.1285	1.8457	0.8802	2.1776	0.8339
40	0.3	(24, 24, 30, 30)	1.2276	0.0325	1.4902	0.1815	1.6050	0.2290	1.9442	0.1356
40	0.4	(24, 24, 30, 30)	1.2204	0.0242	1.4488	0.0913	1.7093	0.5554	2.0674	0.5202
50	0.3	(30, 30, 38, 38)	1.2150	0.0257	1.4600	0.1337	1.5793	0.1760	1.9017	0.0803
50	0.4	(30, 30, 38, 38)	1.2168	0.0187	1.4407	0.0696	1.6612	0.4262	2.0039	0.3633
80	0.3	(48, 48, 60, 60)	1.2110	0.0151	1.4388	0.0761	1.5578	0.1131	1.8678	0.0425
80	0.4	(48, 48, 60, 60)	1.2108	0.0114	1.4217	0.0413	1.6060	0.2679	1.9322	0.1572

Table 2: ALs and CPs of 95% asymptotic CIs based on 5000 simulation ($\alpha_1 = 1.2$, $\theta_1 = 1.4$, $\alpha_2 = 1.5$, $\theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AL	CP	AL	CP	AL	CP	AL	CP
20	0.3	(12, 12, 15, 15)	1.0120	95.16	2.5455	94.04	2.5023	95.02	2.3481	97.90
20	0.4	(12, 12, 15, 15)	0.8479	95.02	1.7011	94.62	3.6227	95.56	4.2112	97.04
30	0.3	(18, 18, 22, 22)	0.8079	95.38	1.9193	94.40	2.1582	94.84	1.7599	98.42
30	0.4	(18, 18, 22, 22)	0.6847	95.40	1.3473	95.08	3.1807	95.28	3.4272	97.36
40	0.3	(24, 24, 30, 30)	0.6940	95.44	1.6041	94.94	1.8237	95.24	1.2639	98.06
40	0.4	(24, 24, 30, 30)	0.5893	94.76	1.1444	94.70	2.6826	95.64	2.5310	97.14
50	0.3	(30, 30, 38, 38)	0.6134	94.92	1.3901	94.58	1.6008	94.92	1.0301	97.38
50	0.4	(30, 30, 38, 38)	0.5249	95.26	1.0135	95.14	2.3717	95.26	2.0080	97.02
80	0.3	(48, 48, 60, 60)	0.4827	95.44	1.0734	95.50	1.3029	95.14	0.7589	97.14
80	0.4	(48, 48, 60, 60)	0.4130	95.24	0.7874	94.88	1.9880	95.12	1.2965	96.98

Table 3: AEs and MSEs of BEs based on 5000 simulation using non-informative prior ($\alpha_1 = 1.2$, $\theta_1 = 1.4$, $\alpha_2 = 1.5$, $\theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	0.3	(12, 12, 15, 15)	1.2464	0.0694	1.7546	1.1965	1.4747	0.2717	2.2478	0.5911
20	0.4	(12, 12, 15, 15)	1.2220	0.0449	1.5400	0.2391	1.5652	0.3805	2.4719	0.8999
30	0.3	(18, 18, 22, 22)	1.2176	0.0421	1.5723	0.3348	1.4304	0.2206	2.1348	0.2856
30	0.4	(18, 18, 22, 22)	1.2131	0.0294	1.4860	0.1335	1.5146	0.3445	2.3862	0.6631
40	0.3	(24, 24, 30, 30)	1.2151	0.0302	1.5363	0.2137	1.4024	0.1766	2.0650	0.1788
40	0.4	(24, 24, 30, 30)	1.2056	0.0224	1.4519	0.0905	1.4469	0.2694	2.2822	0.4390
50	0.3	(30, 30, 38, 38)	1.2053	0.0240	1.4418	0.1219	1.4078	0.1456	2.0088	0.1170
50	0.4	(30, 30, 38, 38)	1.2032	0.0176	1.4028	0.0547	1.4076	0.2300	2.2214	0.3387
80	0.3	(48, 48, 60, 60)	1.2027	0.0148	1.4609	0.0847	1.4143	0.1092	1.9361	0.0580
80	0.4	(48, 48, 60, 60)	1.2045	0.0109	1.4300	0.0434	1.3860	0.1853	2.1355	0.2234

Table 4: ALs and CPs of 95% symmetric CRIs based on 5000 simulation using non-informative prior ($\alpha_1 = 1.2,\ \theta_1 = 1.4,\ \alpha_2 = 1.5,\ \theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AL	CP	AL	CP	AL	CP	AL	CP
20	0.3	(12, 12, 15, 15)	1.1266	97.16	3.5921	96.84	2.4275	97.12	2.5604	94.36
20	0.4	(12, 12, 15, 15)	0.9535	97.58	2.0645	96.72	3.1058	98.56	3.4577	92.86
30	0.3	(18, 18, 22, 22)	0.9142	97.20	2.4229	97.04	2.1560	96.88	2.0172	94.70
30	0.4	(18, 18, 22, 22)	0.7891	97.78	1.6064	97.24	2.8479	98.08	3.0029	92.40
40	0.3	(24, 24, 30, 30)	0.7991	97.78	2.0098	97.50	1.9039	96.26	1.6403	94.36
40	0.4	(24, 24, 30, 30)	0.6903	97.86	1.3569	97.42	2.5033	97.42	2.5574	92.28
50	0.3	(30, 30, 38, 38)	0.7263	98.06	1.6934	98.16	1.7176	96.18	1.3555	95.02
50	0.4	(30, 30, 38, 38)	0.6288	98.16	1.1841	98.40	2.2489	96.72	2.2750	92.54
80	0.3	(48, 48, 60, 60)	0.5792	98.12	1.3339	97.76	1.4698	96.28	0.9937	95.52
80	0.4	(48, 48, 60, 60)	0.5057	98.50	0.9553	98.20	1.9947	96.38	1.8666	91.82

Table 5: ALs and CPs of 95% HPD CRIs based on 5000 simulation using non-informative prior ($\alpha_1 = 1.2,\ \theta_1 = 1.4,\ \alpha_2 = 1.5,\ \theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AL	CP	AL	CP	AL	CP	AL	CP
20	0.3	(12, 12, 15, 15)	1.1132	96.96	3.1742	96.82	2.3155	94.34	2.3793	97.80
20	0.4	(12, 12, 15, 15)	0.9449	97.40	1.9417	96.56	2.8597	96.40	3.1878	97.94
30	0.3	(18, 18, 22, 22)	0.9065	97.14	2.2422	96.92	2.0818	93.62	1.8891	98.26
30	0.4	(18, 18, 22, 22)	0.7839	97.60	1.5401	97.24	2.6480	95.22	2.7733	97.46
40	0.3	(24, 24, 30, 30)	0.7936	97.74	1.8960	97.54	1.8614	92.62	1.5390	97.96
40	0.4	(24, 24, 30, 30)	0.6865	97.70	1.3133	97.26	2.3615	94.46	2.3588	97.26
50	0.3	(30, 30, 38, 38)	0.7220	97.90	1.6134	97.36	1.6956	92.84	1.2724	97.90
50	0.4	(30, 30, 38, 38)	0.6258	98.04	1.1521	97.58	2.1500	93.16	2.0837	97.36
80	0.3	(48, 48, 60, 60)	0.5766	98.06	1.2930	97.64	1.4611	94.22	0.9398	97.62
80	0.4	(48, 48, 60, 60)	0.5038	98.42	0.9379	98.20	1.9326	92.72	1.6919	97.40

Table 6: AEs and MSEs of BEs based on 5000 simulation using informative prior ($\alpha_1 = 1.2,\ \theta_1 = 1.4,\ \alpha_2 = 1.5,\ \theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	0.3	(12, 12, 15, 15)	1.2195	0.0523	1.6030	0.3999	1.4283	0.1964	2.1697	0.3178
20	0.4	(12, 12, 15, 15)	1.2127	0.0391	1.5093	0.1892	1.4653	0.1821	2.3087	0.5456
30	0.3	(18, 18, 22, 22)	1.2183	0.0366	1.5608	0.2651	1.4235	0.1859	2.1002	0.2216
30	0.4	(18, 18, 22, 22)	1.2102	0.0275	1.4698	0.1172	1.4327	0.1921	2.2720	0.4238
40	0.3	(24, 24, 30, 30)	1.2065	0.0279	1.5028	0.1810	1.4215	0.1612	2.0391	0.1491
40	0.4	(24, 24, 30, 30)	1.2029	0.0204	1.4507	0.0879	1.4205	0.1935	2.2135	0.3242
50	0.3	(30, 30, 38, 38)	1.2057	0.0220	1.4877	0.1387	1.4240	0.1418	1.9964	0.1075
50	0.4	(30, 30, 38, 38)	1.2068	0.0166	1.4474	0.0677	1.4084	0.1892	2.1804	0.2729
80	0.3	(48, 48, 60, 60)	1.2019	0.0137	1.4501	0.0767	1.4176	0.1074	1.9328	0.0581
80	0.4	(48, 48, 60, 60)	1.2001	0.0102	1.4229	0.0403	1.3903	0.1682	2.1071	0.1838

Table 7: ALs and CPs of 95% symmetric CRIs based on 5000 simulation using informative prior ($\alpha_1 = 1.2$, $\theta_1 = 1.4$, $\alpha_2 = 1.5$, $\theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AL	CP	AL	CP	AL	CP	AL	CP
20	0.3	(12, 12, 15, 15)	1.0602	97.70	2.8484	97.76	2.2872	97.86	2.2100	95.44
20	0.4	(12, 12, 15, 15)	0.9301	98.10	1.9422	97.74	2.7118	99.24	2.8763	93.76
30	0.3	(18, 18, 22, 22)	0.8920	98.02	2.2808	97.90	2.1041	96.86	1.8690	94.92
30	0.4	(18, 18, 22, 22)	0.7801	98.06	1.5579	97.60	2.5578	98.72	2.5621	92.96
40	0.3	(24, 24, 30, 30)	0.7818	97.74	1.9008	97.56	1.8850	96.54	1.5463	94.98
40	0.4	(24, 24, 30, 30)	0.6824	97.96	1.3335	97.24	2.3777	98.40	2.3060	93.14
50	0.3	(30, 30, 38, 38)	0.7088	98.06	1.6858	97.66	1.7216	96.48	1.3198	94.58
50	0.4	(30, 30, 38, 38)	0.6209	98.42	1.1999	97.82	2.2147	97.68	2.1227	93.08
80	0.3	(48, 48, 60, 60)	0.5760	98.40	1.3101	98.30	1.4561	96.44	0.9690	94.96
80	0.4	(48, 48, 60, 60)	0.5019	98.72	0.9423	98.00	1.9706	96.86	1.7623	92.90

Table 8: ALs and CPs of 95% HPD CRIs based on 5000 simulation using informative prior ($\alpha_1 = 1.2$, $\theta_1 = 1.4$, $\alpha_2 = 1.5$, $\theta_2 = 1.8$).

			α_1		θ_1		α_2		θ_2	
n	τ	(r_1, r_2, r_3, r_4)	AL	CP	AL	CP	AL	CP	AL	CP
20	0.3	(12, 12, 15, 15)	1.0502	97.40	2.5988	96.88	2.1875	94.52	2.0871	98.30
20	0.4	(12, 12, 15, 15)	0.9224	97.84	1.8375	97.42	2.5305	97.60	2.6811	97.54
30	0.3	(18, 18, 22, 22)	0.8853	97.88	2.1314	97.56	2.0387	93.44	1.7620	97.98
30	0.4	(18, 18, 22, 22)	0.7751	97.84	1.4966	97.40	2.3972	96.32	2.3856	97.22
40	0.3	(24, 24, 30, 30)	0.7769	97.46	1.8012	97.16	1.8504	93.06	1.4571	97.82
40	0.4	(24, 24, 30, 30)	0.6787	98.00	1.2921	96.94	2.2562	95.02	2.1310	97.24
50	0.3	(30, 30, 38, 38)	0.7049	97.94	1.6119	97.46	1.7017	94.04	1.2441	97.78
50	0.4	(30, 30, 38, 38)	0.6179	98.32	1.1686	97.98	2.1245	94.50	1.9473	97.74
80	0.3	(48, 48, 60, 60)	0.5735	98.40	1.2709	98.20	1.4486	94.38	0.9206	97.62
80	0.4	(48, 48, 60, 60)	0.5000	98.78	0.9254	97.88	1.9186	93.76	1.5981	97.74

5.2 Data Analysis

5.2.1 Simulated Data Set

Here we have combined two Type-II censored simulated data set and analyzed them for illustrative purpose. We have taken data (v) and (vi) presented in Table 1 of Kateri et al. [11]. Both the data are simulated from exponential simple step-stress experiment with $\theta_1 = e^{2.5}$ and $\theta_2 = e^{1.5}$, where θ_1 and θ_2 are mean of the exponential distribution at first and second stress level respectively. For both the data set, sample size is 35 and the experiment terminated as soon as 26-th failure occurs. For the first data set stress changes at time 5 and for the second data set stress changes at time 8. The observed failure times are as follows:

Table 9: Simulated data set.

Data Set-1	0.053	0.22	0.92	1.58	1.70	2.55	2.72	3.12	3.85
	4.51	4.53	4.90	5.45	5.48	5.48	5.70	5.74	5.98
	6.36	7.27	7.73	7.74	8.01	9.54	9.67	9.89	
Data set-2	0.053	1.11	1.20	1.67	2.32	2.57	3.19	4.51	4.67
	4.98	5.48	5.54	5.72	5.86	5.91	6.15	6.82	7.10
	7.32	8.14	8.23	8.25	8.62	9.51	10.05	11.00	

We have analyzed these data sets assuming lifetime distribution to be Weibull. Plot (a) of Figure 1 shows that the data satisfies the condition for the existence of unique MLE of α_2. The maximum likelihood estimates of α_1, θ_1, α_2, θ_2 are 1.1344, 14.5325, 0.8356 and 3.1008 respectively. The asymptotic CIs of the model parameters are given in Table 10. We have also obtained the Bayes estimates and the associated credible interval of the data. In Bayesian analysis we have used the same values of hyper parameters as used in simulation section. The posterior density of α_2 is not log-concave since the data does not satisfy the condition stated in Lemma 4 (see plot (b) of Figure 1). Therefore we have used the ratio-of-uniform method to generate α_2 from the posterior distribution. Below is the algorithm to generate α_2 using ratio-of-uniform method.

Algorithm 2

Step 1. Generate U_1 and U_2 from $uniform(0,1)$ distribution.

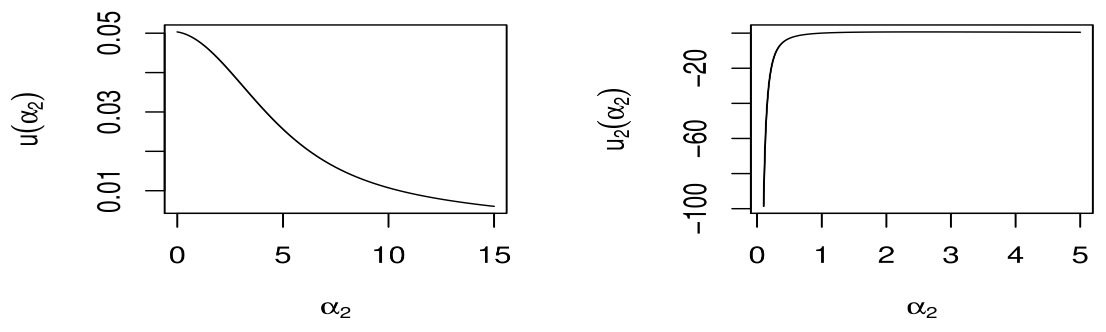
Step 2. Check whether $U_1 < \sqrt{\frac{\pi_3(U_2/U_1)}{\int_0^\infty \pi_3(\alpha_2)d\alpha_2}}$ or not.

Step 3. If $U_1 < \sqrt{\frac{\pi_3(U_2/U_1)}{\int_0^\infty \pi_3(\alpha_2)d\alpha_2}}$ then $\alpha^* = \frac{U_2}{U_1}$ has the density function $\frac{\pi_3(\alpha_2)}{\int_0^\infty \pi_3(\alpha_2)d\alpha_2}$, otherwise repeat Step 1 - Step 2.

The Bayes estimates of α_1, θ_1, α_2, θ_2 are 1.1197, 15.9471, 0.8199 and 6.5164. The CRIs are given in Table 11. The Bayes estimates and the CRIs are based on 10000 replication.

Table 10: Asymptotic CIs of simulated data set.

	α_1		θ_1		α_2		θ_2	
Level	LL	UL	LL	UL	LL	UL	LL	UL
90%	0.8242	1.4448	5.5026	23.5623	0.0000	2.4600	0.0000	19.3895
95%	0.7636	1.5053	3.7407	25.3243	0.0000	2.7770	0.0000	22.5678
99%	0.6482	1.6207	0.3820	28.6829	0.0000	3.3812	0.0000	28.6265



(a) Condition on Lemma 2 (b) Condition on Lemma 4

Figure 1: Plot of the necessary conditions in Lemma 2 and in Lemma 4 for the simulated data.

Table 11: Symmetric and HPD CRIs of simulated data set.

		α_1		θ_1		α_2		θ_2	
CRI	Level	LL	UL	LL	UL	LL	UL	LL	UL
	90%	0.7995	1.5163	7.5511	31.2723	0.3329	1.5975	0.4445	29.5346
Symmetric	95%	0.7558	1.5815	6.8405	35.8485	0.3166	1.6943	0.3918	37.3491
	99%	0.6775	1.7174	5.6766	46.9107	0.3030	1.7736	0.3146	52.0242
	90%	0.7814	1.4911	6.1180	27.3206	0.3001	1.4448	0.2460	19.1945
HPD	95%	0.7428	1.5620	5.4610	31.6585	0.3002	1.5976	0.2460	29.5453
	99%	0.6670	1.6973	4.6635	41.9503	0.3001	1.7541	0.2188	46.4900

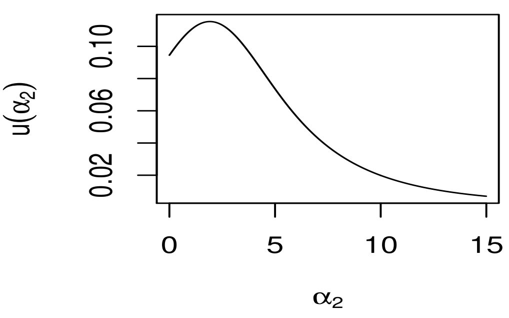
5.2.2 Swimming Performance of Fish Data Set

In this subsection we have analyzed a step-stress data taken from Greven et al. [10]. A multiple step-stress experiment was performed on two group of fishes. Group-1 and Group-2 consisted of fourteen and fifteen fishes respectively. Two group of fishes were swum at initial flow rate 15 cm/sec and then flow rate was increased at time 110, 130, 150 and 170 minutes. The time at which a fish could not maintain its position is recorded as the failure time. The observed failure times for two groups are as follows:

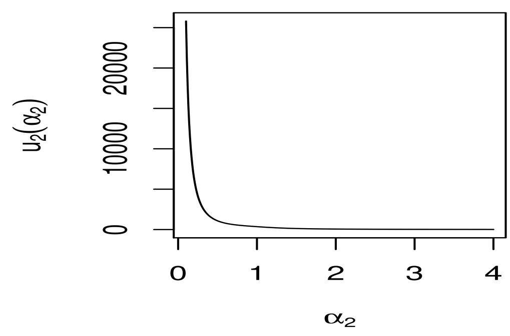
Table 12: Fish data set.

Group-1	83.50	91.00	91.00	97.00	107.00	109.50	114.00	115.41
	128.61	133.53	138.58	140.00	152.08	155.10		
Group-2	91.00	93.00	94.00	98.20	115.81	116.00	116.50	117.25
	126.75	127.50	154.33	159.50	164.00	184.14	188.33	

Though both the data sets are obtained from multiple step-stress experiment, we have combined some stress levels for our analysis. It is assumed that there are only two stress levels and the stress change occurred at the time 110 minutes. We have subtracted 80 from each data points and then analyzed the data. This data also satisfy the condition of unique MLE of α_2 (see plot (a) of Figure 2). The maximum likelihood estimates of α_1, θ_1, α_2, θ_2 are 1.4084, 0.0035, 1.7945 and 0.0008 respectively. The asymptotic CIs of the model parameters are given in Table 13.



(a) Condition on Lemma 2



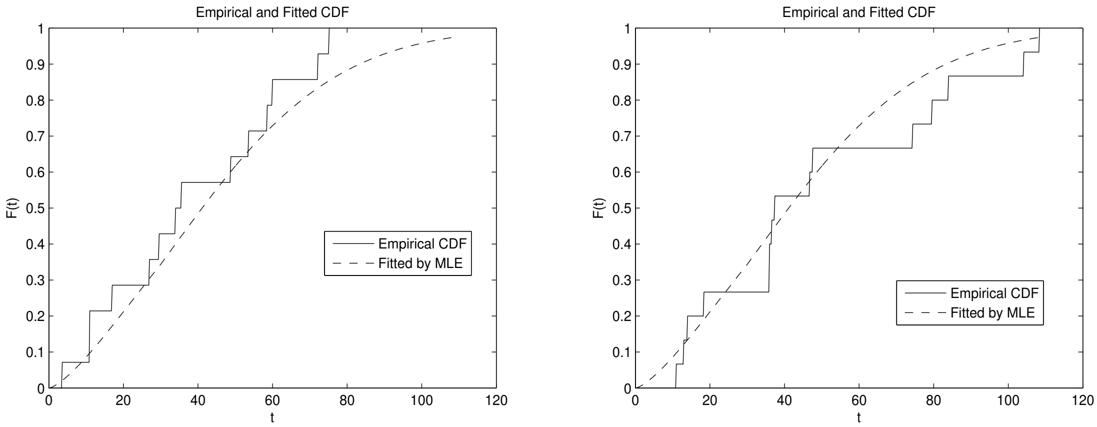
(b) Condition on Lemma 4

Figure 2: Plot of the necessary conditions in Lemma 2 and in Lemma 4 for the Fish data.

Table 13: Asymptotic CIs of swimming performance data set.

	α_1		θ_1		α_2		θ_2	
Level	LL	UL	LL	UL	LL	UL	LL	UL
90%	0.7203	2.0965	0.0000	0.0120	0.6864	2.9025	0.0000	0.0049
95%	0.5861	2.2307	0.0000	0.0136	0.4702	3.1187	0.0000	0.0057
99%	0.3302	2.4866	0.0000	0.0167	0.0581	3.5308	0.0000	0.0073

Now we have performed the Kolmogorov-Smirnov (KS) test to check the goodness of fit of the propose model to each group of the fish data. We have performed the test separately for group-1 and group-2. The MLEs of the model parameters fit both the group of fish data very nicely. The KS distance and the p-value of the test using MLEs of the model parameters for group-1 fish data are 0.1512 and 0.8603 respectively and for group-2 fish data they are 0.1818 and 0.6402 respectively. Hence for both the data we accept the hypothesis that the CDF fitted by the MLEs and the empirical distribution function are same. In Figure 3, we provide the fitted and the empirical CDFs for both the data sets.



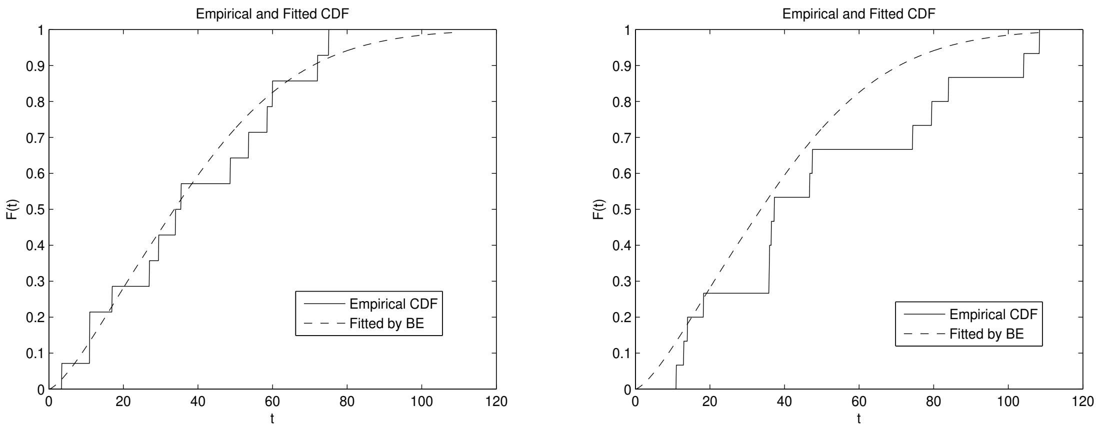
(a) Group - 1 (b) Group - 2

Figure 3: Plot of Empirical CDF and Fitted CDF using MLEs for Group-1 and Group-2 Fish data.

Now we consider the Bayesian analysis of the data using informative priors. For informative prior we have used the information from MLEs and the asymptotic variances of MLEs. For example, since the MLE of α_1 is 1.4084 and the asymptotic variance of MLE of α_1 is 0.17602, we have chosen a_0 and b_0 in such a way that $\frac{b_0}{a_0} \approx 1.4048$ and $\frac{b_0}{a_0^2} \approx 0.17602$. We have taken $a_0 = 8$, $b_0 = 11.26$, $a_1 = 201$, $b_1 = 0.70395$ $a_2 = 135$, $b_2 = 240$, $a_3 = 370$, and $b_3 = 0.27$. Plot (b) of Figure 2 shows that the posterior density of α_2 is log-concave. The Bayes estimates of α_1, θ_1, α_2, θ_2 are 1.406787, 0.004889, 1.776146 and 0.001137 respectively. The CRIs are given in Table 14. The Bayes estimates and the CRIs are based on 10000 replication. Next we have performed the KS test for group-1 and group-2 fish data. The model fitted by Bayes estimates using informative prior fits both the data. The KS distance and p-value of the test for group-1 fish data are respectively 0.1373 and 0.9222. For group-2 fish data the KS distance and p-value are 0.2664 and 0.1980 respectively. Hence for both the data we accept the hypothesis that the fitted CDF and the empirical CDF are same at 5% level of significance. The graphical representation of fitted CDF using Bayes estimates based on informative prior and the empirical CDF is shown in Figure 4.

Table 14: Symmetric and HPD CRIs of swimming performance data set based on informative prior.

		α_1		θ_1		α_2		θ_2	
CRI	Level	LL	UL	LL	UL	LL	UL	LL	UL
	90%	1.0092	1.9352	0.00058	0.01241	1.5367	2.0525	0.000231	0.002759
Symmetric	95%	0.9620	2.0174	0.00042	0.01459	1.5149	2.0808	0.000194	0.003130
	99%	0.8621	2.1952	0.00023	0.01998	1.4729	2.1456	0.000142	0.003994
	90%	0.9825	1.9005	0.00013	0.01033	1.5335	2.0489	0.000142	0.002364
HPD	95%	0.9399	1.9865	0.00008	0.01243	1.5106	2.0754	0.000117	0.002785
	99%	0.8522	2.1793	0.00006	0.01743	1.4653	2.1332	0.000077	0.003695



(a) Group - 1

(b) Group - 2

Figure 4: Plot of Empirical CDF and Fitted CDF using Bayes estimate based on informative prior for Group-1 and Group-2 Fish data.

6 Conclusion

In this paper we have considered the meta-analysis of Type-II censored step-stress experiment based on the assumption that the lifetime of experimental units follow two parameter Weibull distribution. The distributions under different stress level are connected through generalized KH model. We provide classical and Bayesian inference of the model parameters. Due to non-existence of close form of MLEs we propose to use asymptotic confidence interval. Bayes estimates under squared error loss function are obtained using Gibbs sampling technique. Extensive simulation study shows that the performance of the proposed methods are quite satisfactory. The analysis of two data sets indicate that the model works quite well.

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References

[1] Bagdonavicius, V. B. and Nikulin, M. *Accelerated life models: modeling and statistical analysis*. Chapman and Hall CRC Press, Boca Raton, Florida, 2002.

[2] Balakrishnan, N., Beutner, E., and Kateri, M. Order restricted inference for exponential step-stress models. *IEEE Transactions on Reliability*, 58:132–142, 2009.

[3] Balakrishnan, N., Kundu, D., Ng, H. K. T., and Kannan, N. Point and interval estimation for a simple step-stress model with Type-II censoring. *Journal of Quality Technology*, 9:35–47, 2007.

[4] Balakrishnan, N., Rasouli, A., and Farsipour, A. S. Exact likelihood inference based on an unified hybrid censoring sample from the exponential distribution. *Journal of Statistical Computation and Simulation*, 78:475–488, 2008.

[5] Bedbur, S., Kamps, U., and Kateri, M. Meta-analysis of general step-stress experiments under repeated Type-II censoring. *Applied Mathematical Modelling*, 39:2261–2275, 2015.

[6] Congdon, P. *Applied Bayesian Modeling*. John Wiley and Sons, New York, 2003.

[7] Devroye, L. A simple algorithm for generating random variables with log-concave density. *Computing*, 33:247–257, 1984.

[8] Drop, J. R. and Mazzuchi, T. A. A general Bayes exponential inference model for accelerated life testing. *Journal of Statistical Planning and Inference*, 119:55–74, 2004.

[9] Drop, J. R., Mazzuchi, T. A., Fornell, G. E., and Pollock, L. R. A Bayes approach to step-stress accelerated life testing. *IEEE Transactions on Reliability*, 45:491–498, 1996.

[10] Greven, S., Bailer, A. J., Kupper, L. L., Muller, K. E., and Craft, J. L. A parametric model for studying organism fitness step stress experiments. *Biometrics*, 60:793–799, 2004.

[11] Kateri, M., Kamps, U., and Balakrishnan, N. A meta-analysis approach for step-stress experiments. *Journal of Statistical Planning and Inference*, 139:2907–2919, 2009.

[12] Kateri, M., Kamps, U., and Balakrishnan, N. Multi-sample simple step-stress experiment under time constraints. *Statistica Neerlandica*, 64:77–96, 2010.

[13] Kateri, M., Kamps, U., and Balakrishnan, N. Optimal allocation of change points in simple step-stress experiments under Type-II censoring. *Computational Statistics and Data Analysis*, 55:236–247, 2011.

[14] Khamis, I. H. and Higgins, J. J. A new model for step-stress testing. *IEEE Transactions on Reliability*, 47:131–134, 1998.

[15] Kinderman, A. J. and Monahan, F. J. Computer generation of random variables using the ratio of uniform deviates. *ACM Trans. Math. Software*, 3(3):257–260, 1977.

[16] Kundu, D. Bayesian inference and life testing plan for Weibull distribution in presence of progressive censoring. *Technometrics*, 50:144–154, 2008.

[17] Kundu, D. and Ganguly, A. *Analysis of Step-Stress Model: Existing Methods and Recent Developments*. Elsevier/Academic Press, Singapore, 2017.

[18] Mitra, S., Ganguly, A., Samanta, D., and Kundu, D. On simple step-stress model for two-parameter exponential distribution. *Statistical Methodology*, 15:95–114, 2013.

[19] Nelson, W. B. Accelerated life testing: step-stress models and data analysis. *IEEE Transactions on Reliability*, 29:103–108, 1980.

[20] Samanta, D., Ganguly, A., Gupta, A., and Kundu, D. On classical and bayesian order restricted inference for multiple exponential step stress model. *Statistics*, 53:177–195, 2019.

[21] Samanta, D., Gupta, A., and Kundu, D. Analysis of Weibull step-stress model in presence of competing risk. *IEEE Transactions on Reliability*, 68:420–438, 2019.

[22] Samanta, D. and Kundu, D. Order restricted inference of a multiple step-stress model. *Computational Statistics and Data Analysis*, 117:62–75, 2018.

[23] Xiong, C. and Milliken, G. A. Step-stress life testing with random stress changing times for exponential data. *IEEE Transactions on Reliability*, 48:141–148, 1999.

A Appendix

A.1 Proof of lemma 1

$$\frac{\partial^2 l_1(\mathbf{t};\alpha_1,\alpha_2)}{\partial\alpha_1^2} = -\Big(\sum_{k=1}^{s_1} r_k + \sum_{k=s_1+s_2+1}^{s} r_{1k}\Big)\left[\frac{1}{\alpha_1^2} + \frac{P(\alpha_1)P''(\alpha_1)-[P'(\alpha_1)]^2}{[P(\alpha_1)]^2}\right],$$

where $P'(\alpha_1)$ and $P''(\alpha_1)$ are respectively first order and second order partial differentiation of $P(\alpha_1)$ with respect to α_1. To show $\frac{\partial^2 l_1(\mathbf{t};\alpha_1,\alpha_2)}{\partial\alpha_1^2} \leq 0$, it is sufficient to show $P(\alpha_1)P''(\alpha_1) - [P'(\alpha_1)]^2 \geq 0$. Note that $P(\alpha_1)P''(\alpha_1) - [P'(\alpha_1)]^2$ can be expressed as $\sum_{i=1}^{15} A_i$. Now we will provide the explicit form of A_i's $(i = 1,\ldots,15)$ and we will show that each $A_i \geq 0$.

$$\begin{aligned}
A_1 &= \textstyle\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 - \Big(\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k})\Big)^2\\
&\geq 0 \text{ (By Cauchy-Schwarz inequality)},\\
A_2 &= \textstyle\sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1}\sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1}(\ln(t_{r_k:n_k}))^2\\
&\quad -\Big(\textstyle\sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1}\ln(t_{r_k:n_k})\Big)^2\\
&\geq 0 \text{ (By Cauchy-Schwarz inequality)},
\end{aligned}$$

$$
\begin{aligned}
A_3 &= \sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1} \sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1}(\ln(\tau_k))^2 - \left(\sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1}\ln(\tau_k)\right)^2 \\
&\geq 0 \text{ (By Cauchy-Schwarz inequality)}, \\
A_4 &= \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \\
&\quad -\left(\sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k})\right)^2 \\
&\geq 0 \text{ (By Cauchy-Schwarz inequality)}, \\
A_5 &= \sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1} \sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1}(\ln(\tau_k))^2 \\
&\quad -\left(\sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1}\ln(\tau_k)\right)^2 \\
&\geq 0 \text{ (By Cauchy-Schwarz inequality)}, \\
A_6 &= \sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1} \sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1}(\ln(t_{r_k:n_k}))^2 \\
&\quad +\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1} \\
&\quad -2\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k}) \sum_{k=1}^{s_1}(n_k-r_k)t_{r_k:n_k}^{\alpha_1}\ln(t_{r_k:n_k}) \\
&= \sum_{l=1}^{s_1}\sum_{k=1}^{s_1}\sum_{i=1}^{r_k}(n_l-r_l)t_{i:n_k}^{\alpha_1}t_{r_l:n_l}^{\alpha_1}[\ln(t_{i:n_k})-\ln(t_{r_l:n_l})]^2 \\
&\geq 0, \\
A_7 &= \sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1} \sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1}(\ln(\tau_k))^2 + \sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1} \\
&\quad -2\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k}) \sum_{k=s_1+1}^{s_1+s_2} n_k\tau_k^{\alpha_1}\ln(\tau_k) \\
&= \sum_{l=s_1+1}^{s_1+s_2}\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} n_l t_{i:n_k}^{\alpha_1}\tau_l^{\alpha_1}[\ln(t_{i:n_k})-\ln(\tau_l)]^2 \\
&\geq 0, \\
A_8 &= \sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1} \sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \\
&\quad +\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \\
&\quad -2\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k}) \sum_{k=s_1+s_2+1}^{s}\sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k}) \\
&= \sum_{l=s_1+s_2+1}^{s}\sum_{j=1}^{r_{1l}}\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}t_{j:n_l}^{\alpha_1}[\ln(t_{i:n_k})-\ln(t_{j:n_l})]^2 \\
&\geq 0, \\
A_9 &= \sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1} \sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1}(\ln(\tau_k))^2 \\
&\quad +\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1} \\
&\quad -2\sum_{k=1}^{s_1}\sum_{i=1}^{r_k} t_{i:n_k}^{\alpha_1}\ln(t_{i:n_k}) \sum_{k=s_1+s_2+1}^{s}(n_k-r_{1k})\tau_k^{\alpha_1}\ln(\tau_k) \\
&= \sum_{l=s_1+s_2+1}^{s}\sum_{k=1}^{s_1}\sum_{i=1}^{r_k}(n_l-r_{1l})t_{i:n_k}^{\alpha_1}\tau_l^{\alpha_1}[\ln(t_{i:n_k})-\ln(\tau_l)]^2 \\
&\geq 0,
\end{aligned}
$$

$$
\begin{aligned}
A_{10} &= \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} \sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1}(\ln(\tau_k))^2 \\
&\quad + \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1}(\ln(t_{r_k:n_k}))^2 \sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1} \\
&\quad -2 \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} \ln(t_{r_k:n_k}) \sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1} \ln(\tau_k) \\
&= \textstyle\sum_{l=s_1+1}^{s_1+s_2} \sum_{k=1}^{s_1}(n_k - r_k)n_l t_{r_k:n_k}^{\alpha_1} \tau_l^{\alpha_1}[\ln(t_{r_k:n_k}) - \ln(\tau_l)]^2 \\
&\geq 0, \\
A_{11} &= \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \\
&\quad + \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1}(\ln(t_{r_k:n_k}))^2 \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \\
&\quad -2 \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} \ln(t_{r_k:n_k}) \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \ln(t_{i:n_k}) \\
&= \textstyle\sum_{l=s_1+s_2+1}^{s} \sum_{k=1}^{s_1} \sum_{i=1}^{r_{1l}}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} t_{i:n_l}^{\alpha_1}[\ln(t_{r_k:n_k}) - \ln(t_{i:n_l})]^2 \\
&\geq 0, \\
A_{12} &= \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} \sum_{k=s_1+s_2+1}^{s}(n_k - r_{1k})\tau_k^{\alpha_1}(\ln(\tau_k))^2 \\
&\quad + \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1}(\ln(t_{r_k:n_k}))^2 \sum_{k=s_1+s_2+1}^{s}(n_k - r_{1k})\tau_k^{\alpha_1} \\
&\quad -2 \textstyle\sum_{k=1}^{s_1}(n_k - r_k)t_{r_k:n_k}^{\alpha_1} \ln(t_{r_k:n_k}) \sum_{k=s_1+s_2+1}^{s}(n_k - r_{1k})\tau_k^{\alpha_1} \ln(\tau_k) \\
&= \textstyle\sum_{l=s_1+s_2+1}^{s} \sum_{k=1}^{s_1}(n_k - r_k)(n_l - r_{1l})t_{r_k:n_k}^{\alpha_1} \tau_l^{\alpha_1}[\ln(t_{r_k:n_k}) - \ln(\tau_l)]^2 \\
&\geq 0, \\
A_{13} &= \textstyle\sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1} \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1}(\ln(t_{i:n_k}))^2 \\
&\quad + \textstyle\sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1}(\ln(\tau_k))^2 \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \\
&\quad -2 \textstyle\sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1} \ln(\tau_k) \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \ln(t_{i:n_k}) \\
&= \textstyle\sum_{l=s_1+s_2+1}^{s} \sum_{k=s_1+1}^{s_1+s_2} \sum_{i=1}^{r_{1l}} n_k \tau_k^{\alpha_1} t_{i:n_l}^{\alpha_1}[\ln(\tau_k) - \ln(t_{i:n_l})]^2 \\
&\geq 0, \\
A_{14} &= \textstyle\sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1} \sum_{k=s_1+s_2+1}^{s}(n_k - r_{1k})\tau_k^{\alpha_1}(\ln(\tau_k))^2 \\
&\quad + \textstyle\sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1}(\ln(\tau_k))^2 \sum_{k=s_1+s_2+1}^{s}(n_k - r_{1k})\tau_k^{\alpha_1} \\
&\quad -2 \textstyle\sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1} \ln(\tau_k) \sum_{k=s_1+s_2+1}^{s}(n_k - r_{1k})\tau_k^{\alpha_1} \ln(\tau_k) \\
&= \textstyle\sum_{l=s_1+s_2+1}^{s} \sum_{k=s_1+1}^{s_1+s_2} n_k \tau_k^{\alpha_1}(n_l - r_{1l})\tau_l^{\alpha_1}[\ln(\tau_k) - \ln(\tau_l)]^2 \\
&\geq 0,
\end{aligned}
$$

$$
\begin{aligned}
A_{15} &= \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \sum_{k=s_1+s_2+1}^{s} (n_k - r_{1k}) \tau_k^{\alpha_1} (\ln(\tau_k))^2 \\
&\quad + \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} (\ln(t_{i:n_k}))^2 \sum_{k=s_1+s_2+1}^{s} (n_k - r_{1k}) \tau_k^{\alpha_1} \\
&\quad -2 \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} t_{i:n_k}^{\alpha_1} \ln(t_{i:n_k}) \sum_{k=s_1+s_2+1}^{s} (n_k - r_{1k}) \tau_k^{\alpha_1} \ln(\tau_k) \qquad (10) \\
&= \sum_{l=s_1+s_2+1}^{s} \sum_{k=s_1+s_2+1}^{s} \sum_{i=1}^{r_{1k}} (n_l - r_{1l}) t_{i:n_k}^{\alpha_1} \tau_l^{\alpha_1} [\ln(t_{i:n_k}) - \ln(\tau_l)]^2 \\
&\geq 0.
\end{aligned}
$$

A.2 Proof of lemma 2

$$
\frac{\delta^2 l_1(\mathbf{t}; \alpha_1, \alpha_2)}{\delta \alpha_2^2} = -\left(\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{k=s_1+s_2+1}^{s} (r_k - r_{1k}) \right) u(\alpha_2).
$$

If $u(\alpha_2) > 0$, for all $\alpha_2 > 0$, then $\frac{\delta^2 l_1(\mathbf{t};\alpha_1,\alpha_2)}{\delta \alpha_2^2} < 0$ and hence an unique solution of equation (7) exists which maximizes (5) with respect to α_2.

A.3 Elements of Fisher information Matrix

$$
\begin{aligned}
f_{11} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \alpha_1^2} = \frac{\sum_{k=1}^{s_1} r_k + \sum_{s_1+s_2+1}^{s} r_{1k}}{\alpha_1^2} + \theta_1 P''(\alpha_1), \\
f_{12} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \alpha_1 \partial \theta_1} = P'(\alpha_1) = f_{21}, \\
f_{13} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \alpha_1 \partial \alpha_2} = 0 = f_{31}, \\
f_{14} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \alpha_1 \partial \theta_2} = 0 = f_{41}, \\
f_{22} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \theta_1^2} = \frac{\sum_{k=1}^{s_1} r_k + \sum_{s_1+s_2+1}^{s} r_{1k}}{\theta_1^2}, \\
f_{23} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \theta_1 \partial \alpha_2} = 0 = f_{32}, \\
f_{24} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \theta_1 \partial \theta_2} = 0 = f_{42}, \\
f_{33} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \alpha_2^2} = \frac{\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{s_1+s_2+1}^{s} (r_k - r_{1k})}{\alpha_2^2} + \theta_2 Q''(\alpha_2), \\
f_{34} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \alpha_2 \partial \theta_2} = Q'(\alpha_2) = f_{43}, \\
f_{44} &= -\frac{\partial^2 l(\mathbf{t};\boldsymbol{\eta})}{\partial \theta_2^2} = \frac{\sum_{k=s_1+1}^{s_1+s_2} r_k + \sum_{s_1+s_2+1}^{s} (r_k - r_{1k})}{\theta_2^2}.
\end{aligned}
$$

A.4 Proof of lemma 3

$$\pi_1(\alpha_1) \propto e^{-a_0\alpha_1}\alpha_1^{b_0+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}-1}\Big[a_1+P(\alpha_1)\Big]^{-(b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k})}$$
$$\times\Big(\prod_{k=1}^{s_1}\prod_{i=1}^{r_k} t_{i:n_k}\Big)^{\alpha_1-1}\times\Big(\prod_{k=s_1+s_2+1}^{s}\prod_{i=1}^{r_{1k}} t_{i:n_k}\Big)^{\alpha_1-1},$$

Therefore,

$$\frac{\partial^2 \ln\pi_1(\alpha_1)}{\partial\alpha_1^2} = -\frac{b_0+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k}-1}{\alpha_1^2} - \frac{(b_1+\sum_{k=1}^{s_1} r_k+\sum_{k=s_1+s_2+1}^{s} r_{1k})[a_1P''(\alpha_1)+P(\alpha_1)P''(\alpha_1)-(P'(\alpha_1))^2]}{\Big[a_1+P(\alpha_1)\Big]^2}.$$

Since $a_1P''(\alpha_1) \geq 0$ and we have already shown in Appendix A.1 that $P(\alpha_1)P''(\alpha_1) - (P'(\alpha_1))^2 \geq 0$. Therefore $\frac{\partial^2 \ln\pi_1(\alpha_1)}{\partial\alpha_1^2} \leq 0$. Hence $\pi_1(\alpha_1)$ is a log-concave density function.

A.5 Proof of lemma 4

$$\frac{\partial^2 \ln\pi_3(\alpha_2)}{\partial\alpha_2^2} = -u_2(\alpha_2).$$

Therefore, if $u_2(\alpha_2) \geq 0$ then $\frac{\partial^2 \ln\pi_3(\alpha_2)}{\partial\alpha_2^2} \leq 0$ and hence $\pi_3(\alpha_2)$ is a log-concave density function.