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EXTENDED EXPONENTIATED WEIBULL DISTRIBUTION AND ITS APPLICATIONS

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1. Introduction

The three-parameter exponentiated Weibull (EW) distribution introduced by Mudholkar and Srivastava (1993) as an extension of the Weibull family, is a very flexible class of probability distribution functions. The applications of the EW distribution in reliability and survival studies were illustrated by Mudholkar et al. (1995). The EW distribution has the cumulative distribution function (CDF)

$$F(x;\alpha,\lambda,\gamma) = \left(1 - e^{-\lambda x^{\gamma}}\right)^{\alpha}, \quad x > 0,$$

and the associated probability density function (PDF) as

$$f(x;\alpha,\lambda,\gamma) = \alpha\gamma\lambda x^{\gamma-1} e^{-\lambda x^{\gamma}} (1 - e^{-\lambda x^{\gamma}})^{\alpha-1}, \quad x > 0,$$

where $\alpha > 0$ and $\gamma > 0$ are shape parameters and $\lambda > 0$ is the scale parameter. Gupta and Kundu (1999) considered a special case of the EW distribution when

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$\gamma = 1$, and called it as the generalized exponential (GE) distribution. The GE distribution has received a considerable amount of attention in recent years. The readers are referred to a review article by Gupta and Kundu (2007) for a current account on the generalized exponential distribution and a book length treatment of different exponentiated distributions by Al-Hussaini and Ahsanullah (2015).

Mudholkar et al. (1996) presented a three-parameter generalized Weibull (GW) family that contains distributions with unimodal and bathtub shaped hazard rates. They showed that the distributions in this family are analytically tractable and computationally manageable. The modeling and analysis of survival data using this family of distributions have been discussed and illustrated in terms of a lifetime data set and the results of a two-arm clinical trial. The CDF of the GW distribution is the following

$$F(y;\alpha,\lambda,\delta) = 1 - \left(1 - \delta(\lambda y)^{1/\alpha}\right)^{1/\delta},$$

where $0 < y < \infty$ for $\delta \le 0$ and $0 < y < \dfrac{1}{\lambda\delta^{\alpha}}$ for $\delta > 0$.

Recently, Gupta and Kundu (2011) introduced a three-parameter extended GE (EGE) distribution by adding a shape parameter to a GE distribution. The EGE distribution contains many well known distributions such as exponential, GE, uniform, generalized Pareto and Pareto distributions as special cases. Interestingly, the EGE distribution has increasing, decreasing, unimodal and bathtub shaped hazard rate functions similar to the EWE distribution. The EGE distribution has the following CDF and PDF,

$$F(y;\alpha,\beta,\lambda) = \begin{cases} (1-(1-\beta\lambda y)^{1/\beta})^{\alpha}, & \beta \neq 0, \\ (1-e^{-\lambda y})^{\alpha}, & \beta = 0, \end{cases}$$

and

$$f(y;\alpha,\beta,\lambda) = \begin{cases} \alpha\lambda(1-\beta\lambda y)^{1/\beta-1}(1-(1-\beta\lambda y)^{1/\beta})^{\alpha-1}, & \beta \neq 0, \\ \alpha\lambda e^{-\lambda y}(1-e^{-\lambda y})^{\alpha-1}, & \beta = 0, \end{cases}$$

respectively, when $0 < y < \infty$ for $\beta \le 0$ and $0 < y < \frac{1}{\beta\lambda}$ for $\beta > 0$.

It may be mentioned that EW, GW or EGE are very flexible class of distributions. Their hazard functions can take variety of shapes. For EW and EGE distributions, the hazard functions can be increasing, decreasing, bathtub shaped or unimodal functions depending on the shape parameters. Although, the hazard functions can be very flexible, they cannot take decreasing-increasing-decreasing (DID) shapes. In many practical situations, it is observed that the hazard function can take DID shapes. Not too many lifetime distributions, at least not known to the authors, the hazard functions can take five different types, namely increasing, decreasing, bathtub shaped, unimodal and DID shaped.

The main aim of this paper is to extend the EGE distribution to a four-parameter distribution by adding a new shape parameter. It contains EW and

EGE as special cases. The new distribution is capable of modeling bathtub-shaped, upside-down bathtub (unimodal), increasing, decreasing and decreasing-increasing-decreasing (DID) hazard rate functions which are widely used in engineering for repairable systems. Hence, it can be used quite effectively for analyzing lifetime data of different types.

The rest of the paper is organized as follows. In Section 2, we introduce the EEW distribution and outline some sub-models of the distribution and then, discuss some of its properties. Some related issues are discussed in Section 3. In Section 4, we provide the estimation procedures and the asymptotic distributions of the estimators. Simulation results and the analysis of a data set are provided in Section 5, and finally conclusions arrive in Section 6.

2. Definition and some properties

In this section we formally define the extended exponentiated Weibull family of distributions. We observe that several well known distributions can be obtained as special cases of the proposed distribution. We also derive different properties and different measures of the proposed distribution in this section.

2.1. *The Extended Exponentiated Weibull Distribution*

The random variable Y is said to have an extended exponentiated Weibull (EEW) distribution, if the CDF of the random variable Y, denoted by $F(y;\alpha,\beta,\gamma,\lambda)$, is given by

$$F(y;\alpha,\beta,\gamma,\lambda)=\begin{cases}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha}, & \beta\neq 0,\\ (1-e^{-\lambda y^{\gamma}})^{\alpha}, & \beta=0,\end{cases} \tag{1}$$

where $\alpha,\gamma,\lambda>0$ and $-\infty<\beta<\infty$. Here, $0<y<\infty$ if $\beta\leq 0$ and $0<y<\frac{1}{(\beta\lambda)^{1/\gamma}}$ if $\beta>0$. The PDF of Y can be expressed as

$$f(y;\alpha,\beta,\gamma,\lambda)=\begin{cases}\alpha\gamma\lambda y^{\gamma-1}(1-\beta\lambda y^{\gamma})^{1/\beta-1}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha-1}, & \beta\neq 0,\\ \alpha\gamma\lambda y^{\gamma-1}e^{-\lambda y^{\gamma}}(1-e^{-\lambda y^{\gamma}})^{\alpha-1}, & \beta=0.\end{cases} \tag{2}$$

A random variable X follows the EEW distribution with parameters α,β,γ and λ is denoted by $X\sim EEW(\alpha,\beta,\gamma,\lambda)$.

Several well known distribution functions can be obtained as special cases of the EEW distribution depending on the values of α,β,γ. The details are presented below.

(i) For $\gamma=1$, the EEW distribution reduces to the EGE introduced and studied by Gupta and Kundu (2011).

(ii) For $\gamma = 1,\ \beta = 0,\ \alpha \neq 1$, the EEW distribution reduces to the GE distribution introduced by Gupta and Kundu (2007).

(iii) For $\gamma = 1,\ \beta = 1$ and $\alpha = 1$, the EEW distribution reduces to the Uniform distribution with CDF
$$F(y;\lambda) = \lambda y.$$

(iv) The Pareto distribution with CDF
$$F(y;\beta,\lambda) = 1 - (1-\beta\lambda y)^{1/\beta},$$
is a special case of the EEW distribution for $\gamma = 1,\ \alpha = 1$ and $\beta > 0$.

(v) For $\gamma = 1$ and $\alpha = 1$ $b = 1, \theta \to 0^+$, the EEW distribution reduces to the generalized Pareto (GP) distribution with CDF
$$F(y;\beta,\lambda) = \begin{cases} 1-(1-\beta\lambda y)^{1/\beta}, & \beta \neq 0, \\ 1 - e^{-\lambda y}, & \beta = 0. \end{cases}$$

(vi) For $\gamma = 1,\ \beta \neq 0,\ \alpha \neq 1$, the EEW distribution reduces to the GW distribution introduced and analyzed by Mudholkar et al. (1996).

(vii) For $\gamma \neq 1,\ \alpha = 1$ and $\beta = 0$, the EEW distribution reduces to the Weibull distribution with CDF
$$F(y;\gamma,\lambda) = 1 - e^{-\lambda y^{\gamma}}.$$

(viii) For $\gamma \neq 1,\ \alpha \neq 1$ and $\beta = 0$, the EEW distribution reduces to the EW distribution proposed by Nassar and Eissa (2003).

(ix) For $\gamma = 2,\ \alpha \neq 1$ and $\beta = 0$, the EEW distribution reduces to the Burr X distribution with CDF
$$F(y;\alpha,\lambda) = (1 - e^{-\lambda y^2})^{\alpha}.$$

2.1.1. Different shapes of EEW probability density and hazard function

In this section we provide different shapes of the PDFs and hazard functions of the EEW distribution. Because of the complicated nature of the PDFs and hazard functions, it is difficult to obtain the shapes of the PDFs and hazard functions analytically. The following observations have been made graphically. For $\beta < 0$, the shape of the PDF of the EEW distribution can be decreasing or unimodal depending on different values of parameters. For $\beta > 0$, the shape of the PDFs of the EEW distribution can be bath-tub shaped and increasing depending on the different values of the shape parameters. The PDFs of the EEW for different values of α, β and γ, where $\lambda = 1$ are plotted in Figure 1.

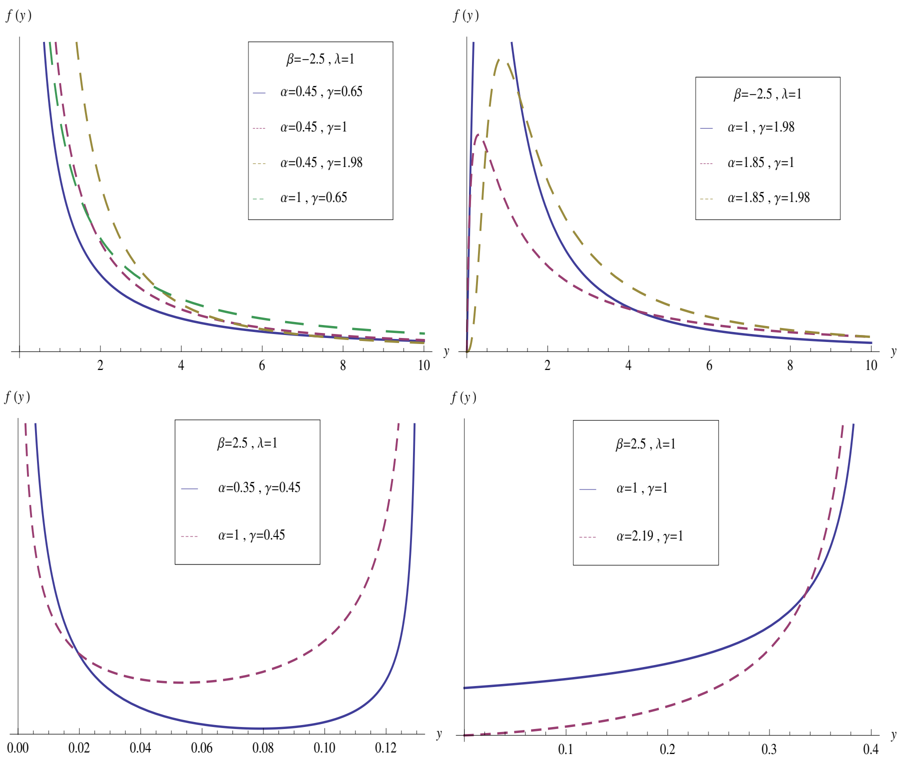

Figure 1 – The PDFs of the EEW for different values of α, β and γ with $\lambda = 1$

The hazard rate and survival function of the EEW distribution are given by, respectively

$$h(y;\alpha,\beta,\gamma,\lambda) = \begin{cases} \dfrac{\alpha\gamma\lambda y^{\gamma-1}(1-\beta\lambda y^{\gamma})^{1/\beta-1}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha-1}}{1-(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha}}, & \beta \neq 0, \\ \dfrac{\alpha\gamma\lambda y^{\gamma-1}e^{-\lambda y^{\gamma}}(1-e^{-\lambda y^{\gamma}})^{\alpha-1}}{1-(1-e^{-\lambda y^{\gamma}})^{\alpha}}, & \beta = 0, \end{cases}$$

and

$$s(y;\alpha,\beta,\gamma,\lambda) = \begin{cases} 1-(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha}, & \beta \neq 0, \\ 1-(1-e^{-\lambda y^{\gamma}})^{\alpha}, & \beta = 0. \end{cases} \tag{3}$$

The hazard function of the EEW distribution can take different shapes, namely increasing, decreasing, DID, unimodal and bathtub shaped. Figure 2 provides the hazard functions of the EEW distribution for different values of α, β, γ, and λ.

Because of the complicated nature of the hazard function, we could not establish the above results in its full generality, but the following results can be

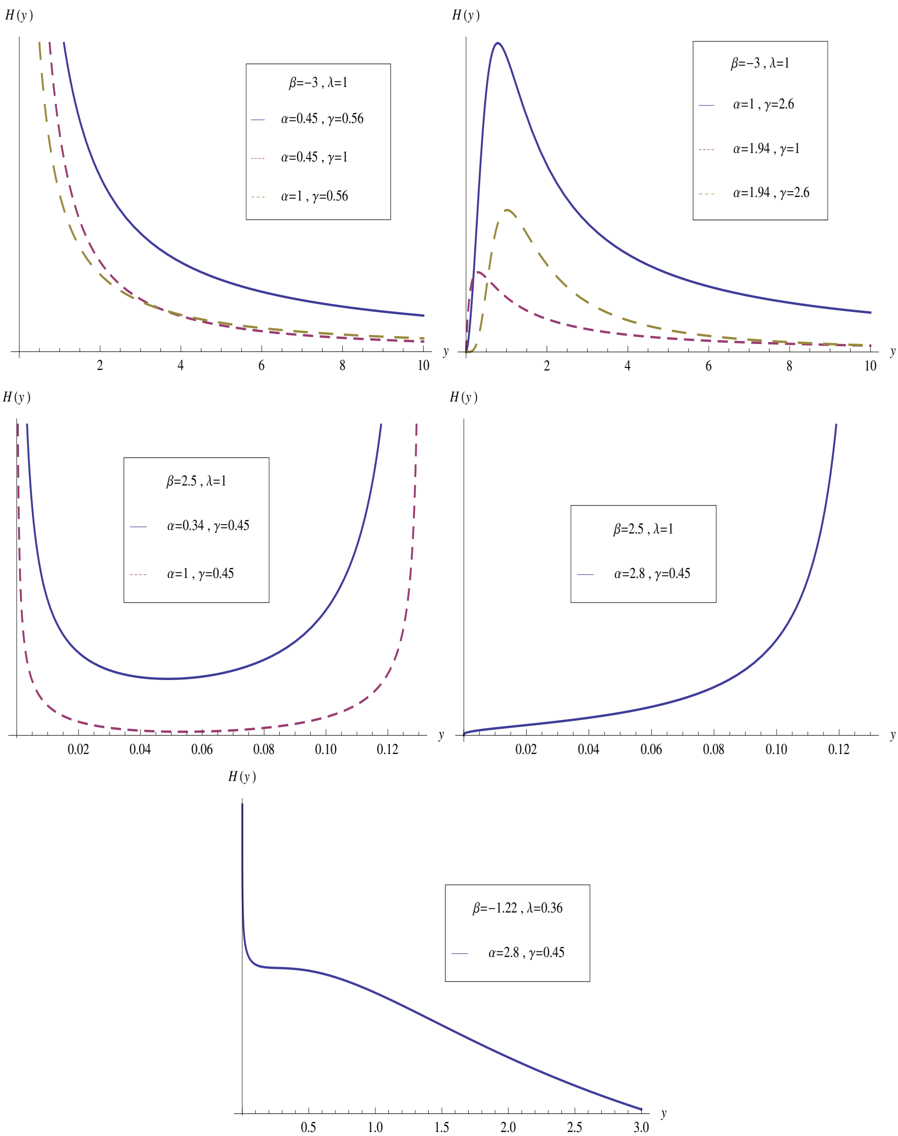

Figure 2 – The hazard rate of the EEW for different values of α, β and γ with $\lambda = 1$

established. In all these cases without loss of generality, we have assumed that $\lambda = 1$.

THEOREM 1. *If* $\gamma = 1$ *and* $\beta < 0$*, then for* *(a)* $\alpha > 1$, *the hazard function of EEW is an unimodal shaped,*

(b) $0 < \alpha < 1$ *it is a decreasing function.*

PROOF. See Theorem 1 of Gupta and Kundu (2011).

THEOREM 2. *If* $\gamma = 1$ *and for*

(a) $\beta > 1$, $\alpha > 1$, *the hazard function of EEW is an increasing function,*

(b) $0 < \alpha < 1$, $0 < \beta < 1$, *it is a bathtub shaped.*

PROOF. See Theorem 2 of Gupta and Kundu (2011).

THEOREM 3. *If* $0 < \alpha < 1$, $\beta < 0$ *and* $0 < \gamma < 1$, *then the hazard function of EEW is a decreasing function.*

PROOF. See in the Appendix.

THEOREM 4. *If* $\alpha > 1$, $\beta > 1$ *and* $\gamma > 1$, *then the hazard function of EEW is an increasing function.*

PROOF. See in the Appendix.

THEOREM 5. *If* $\alpha > 1$, $\beta < 0$ *and* $\gamma > 1$, *then the hazard function of EEW is an unimodal function.*

PROOF. See in the Appendix.

The quantile of a distribution plays an important role for any lifetime distribution. In case of the EEW distribution, it is observed that the p-th quantile can be obtained in explicit form. Hence, if we have maximum likelihood estimators (MLEs) of the unknown parameters of a EEW distribution, the MLE of the corresponding p-th quantile estimator also can be easily obtained.

The p-th quantile of the EEW distribution is given by

$$Q(p;\alpha,\beta,\gamma,\lambda) = \begin{cases} (\frac{1}{\beta\lambda}(1-(1-p^{1/\alpha})^{\beta}))^{1/\gamma}, & \beta \neq 0, \\ (-\frac{1}{\lambda}\log(1-p^{1/\alpha}))^{1/\gamma}, & \beta = 0. \end{cases}$$

When $p = \frac{1}{2}$, the median of EEW is

$$M = \begin{cases} (\frac{1}{\beta\lambda}(1-(1-2^{-1/\alpha})^{\beta}))^{1/\gamma}, & \beta \neq 0, \\ (-\frac{1}{\lambda}\log(1-2^{1/\alpha}))^{1/\gamma}, & \beta = 0. \end{cases} \tag{4}$$

The mode of EEW distribution cannot be obtained in explicit form. It has to be obtained by solving non-linear equation, and it is not pursued here.

2.2. Moments and Moment Generating Function

The moment and moment generating functions play important roles for analyzing any distributions functions. The moment generating function characterizes the distribution function. Although, we could not obtain the moments in explicit forms, they can be obtained as infinite summation of beta functions. Different moments can be easily calculated using any standard mathematical softwares. The k-th moments and the k-th central moments of the EEW distribution can be obtained by using the expansion $(1-z)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} (-1)^i z^i$, where α is not integer. We have the following results:

$$\mu'_k = E(Y^k) = \begin{cases} \sum_{i=0}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^i \alpha}{\beta^{1+k/\gamma} \lambda^{k/\gamma}} Be(1+k/\gamma, (i+1)/\beta), & \beta > 0, \\ \sum_{i=0}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^i \alpha}{(-\beta)^{1+k/\gamma} \lambda^{k/\gamma}} Be(1+k/\gamma, -k/\gamma - (i+1)/\beta), & \beta < 0, \\ \sum_{i=0}^{\infty} \binom{\alpha-1}{i} \frac{(-1)^i \alpha}{\lambda^{k/\gamma}} (i+1)^{-(1+k/\gamma)} \Gamma(1+k/\gamma), & \beta = 0, \end{cases} \tag{5}$$

where $Be(.,.)$ is beta function. Using $\mu_k = E(Y - \mu'_k)^k = \sum_{i=0}^{k} \binom{k}{j} \mu'_j (-\mu'_1)^{k-j}$, the k-th central moments can be obtained as

$$\mu_k = \begin{cases} \sum_{j=0}^{k} \sum_{i=0}^{\infty} \binom{k}{j} \binom{\alpha-1}{i} \frac{(-1)^{k+i-j} \alpha}{\beta^{1+j/\gamma} \lambda^{j/\gamma}} Be(1+j/\gamma, (i+1)/\beta) {\mu'_1}^{k-j}, & \beta > 0, \\ \sum_{j=0}^{k} \sum_{i=0}^{\infty} \binom{k}{j} \binom{\alpha-1}{i} \frac{(-1)^{k+i-j} \alpha}{(-\beta)^{1+j/\gamma} \lambda^{j/\gamma}} Be(1+j/\gamma, -j/\gamma - (i+1)/\beta) {\mu'_1}^{k-j}, & \beta < 0, \\ \sum_{j=0}^{k} \sum_{i=0}^{\infty} \binom{k}{j} \binom{\alpha-1}{i} \frac{(-1)^{k+i-j} \alpha}{\lambda^{j/\gamma}} (i+1)^{-(1+j/\gamma)} \Gamma(1+j/\gamma) {\mu'_1}^{k-j}, & \beta = 0. \end{cases}$$

Using the moments of different orders, moment generating function of EEW can be easily obtained. If all the moments of a random variable exist, then the moment generating function of Y can be written as $M_Y(t) = E(e^{tY}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(Y^k)$. Thus, we have

$$M_Y(t) = \begin{cases} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^k}{k!} \binom{\alpha-1}{i} \frac{(-1)^i \alpha}{\beta^{1+k/\gamma} \lambda^{k/\gamma}} Be(1+k/\gamma, (i+1)/\beta), & \beta > 0, \\ \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^k}{k!} \binom{\alpha-1}{i} \frac{(-1)^i \alpha}{(-\beta)^{1+k/\gamma} \lambda^{k/\gamma}} Be(1+k/\gamma, -k/\gamma - (i+1)/\beta), & \beta < 0, \\ \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^k}{k!} \binom{\alpha-1}{i} \frac{(-1)^i \alpha}{\lambda^{k/\gamma}} (i+1)^{-(1+k/\gamma)} \Gamma(1+k/\gamma), & \beta = 0. \end{cases}$$

3. Some Related Issues

3.1. Order Statistics

The CDF and PDF of the first order statistics are given by

$$F_{1:n}(y) = \begin{cases} 1-(1-(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha})^{n}, & \beta \neq 0, \\ 1-(1-(1-e^{-\lambda y^{\gamma}})^{\alpha})^{n}, & \beta = 0, \end{cases}$$

and

$$f_{1:n}(y) = \begin{cases} n\alpha\gamma\lambda y^{\gamma-1}(1-\beta\lambda y^{\gamma})^{1/\beta-1}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha-1} & \beta \neq 0, \\ \times(1-(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha})^{n-1}, & \\ n\alpha\gamma\lambda y^{\gamma-1}e^{-\lambda y^{\gamma}}(1-e^{-\lambda y^{\gamma}})^{\alpha-1}(1-(1-e^{-\lambda y^{\gamma}})^{\alpha})^{n-1}, & \beta = 0, \end{cases}$$

respectively. The CDF and PDF of the largest order statistics are given, respectively, by

$$F_{n:n}(y) = \begin{cases} (1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{n\alpha}, & \beta \neq 0, \\ (1-e^{-\lambda y^{\gamma}})^{n\alpha}, & \beta = 0, \end{cases}$$

and

$$f_{n:n}(y) = \begin{cases} n\alpha\gamma\lambda y^{\gamma-1}(1-\beta\lambda y^{\gamma})^{1/\beta-1}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{n\alpha-1}, & \beta \neq 0, \\ n\alpha\gamma\lambda y^{\gamma-1}e^{-\lambda y^{\gamma}}(1-e^{-\lambda y^{\gamma}})^{n\alpha-1}, & \beta = 0. \end{cases}$$

The CDF and PDF of r-th order statistics, where $1 \leq r \leq n$ and $\beta \neq 0$, are given by

$$F_{r:n}(y) = \sum_{i=0}^{\infty}\sum_{j=r}^{n}\binom{n-j}{i}\binom{n}{j}(-1)^{i}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha(i+j)},$$

$$f_{r:n}(y) = \frac{n!\alpha\gamma\lambda y^{\gamma-1}}{(r-1)!(n-r)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{n-r}{i}\binom{\alpha(i+r)-1}{j}(-1)^{i+j}(1-\beta\lambda y^{\gamma})^{\frac{j+1}{\beta}-1},$$

respectively.

3.2. Mean Deviations

For a random variable Y with the PDF $f(y)$, CDF $F(y)$, mean $\mu = E(Y)$ and $M = Median(Y)$, the mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1 = E|Y-\mu| = 2\mu F(\mu) - 2I(\mu),$$

and

$$\delta_2 = E|Y-M| = \mu - 2I(M),$$

respectively, where $I(t) = \int_0^t yf(y)dy$.

THEOREM 6. *The mean deviation functions of the EEW distribution are*

$$\delta_1 = 2\mu(1-(1-\beta\lambda\mu^\gamma)^{1/\beta})^\alpha - 2\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}\mu^{\gamma+1}(1-\beta\lambda\mu^\gamma)^{\frac{i+1}{\beta}}$$
$$\times {}_2F_1(1, 1+\frac{1}{\gamma}+\frac{i+1}{\beta}; 2+\frac{1}{\gamma}; \beta\lambda\mu^\gamma),$$

and

$$\delta_2 = \mu - 2\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}M^{\gamma+1}(1-\beta\lambda M^\gamma)^{\frac{i+1}{\beta}}$$
$$\times {}_2F_1(1, 1+\frac{1}{\gamma}+\frac{i+1}{\beta}; 2+\frac{1}{\gamma}; \beta\lambda M^\gamma),$$

where $\beta < 0$*,* μ *is the mean of the EEW distribution given in (5) ,* M *is the median of the EEW distribution given in (4), and the Hypergeometric function* ${}_2F_1$ *can be expressed as* ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$*, where* c *does not equal to 0, -1, -2, ..., and*

$$(q)_n = \begin{cases} 1 & if\ n = 0 \\ q(q+1)...(q+n-1) & if\ n > 0. \end{cases}$$

PROOF.

$$F(y) = (1-(1-\beta\lambda y^\gamma)^{\frac{1}{\beta}})^\alpha,$$

and

$$\begin{aligned} f(y) &= \alpha\gamma\lambda y^{\gamma-1}(1-\beta\lambda y^\gamma)^{1/\beta-1}(1-(1-\beta\lambda y^\gamma)^{1/\beta})^{\alpha-1} \\ &= \alpha\gamma\lambda(1-\beta\lambda y^\gamma)^{\frac{1}{\beta}-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^\gamma)^{\frac{i}{\beta}} \\ &= \alpha\gamma\lambda y^{\gamma-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^\gamma)^{\frac{i+1}{\beta}-1}. \end{aligned}$$

so,

$$\begin{aligned} I(\mu) &= \int_0^\mu yf(y)dy \\ &= \alpha\gamma\lambda\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\int_0^\mu y^\gamma(1-\beta\lambda y^\gamma)^{\frac{i+1}{\beta}-1}dy \\ &= \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}\mu^{\gamma+1}(1-\beta\lambda\mu^\gamma)^{\frac{i+1}{\beta}} \\ &\quad \times {}_2F_1(1, 1+\frac{1}{\gamma}+\frac{i+1}{\beta}; 2+\frac{1}{\gamma}; \beta\lambda\mu^\gamma) \end{aligned}$$

Thus, we have

$$\delta_1 = 2\mu(1-(1-\beta\lambda y^{\gamma})^{\frac{1}{\beta}})^{\alpha} - 2\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}\mu^{\gamma+1}(1-\beta\lambda\mu^{\gamma})^{\frac{i+1}{\beta}}$$
$$\times {}_2F_1(1,1+\frac{1}{\gamma}+\frac{i+1}{\beta};2+\frac{1}{\gamma};\beta\lambda\mu^{\gamma}),$$

and

$$\delta_2 = \mu - 2\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}M^{\gamma+1}(1-\beta\lambda M^{\gamma})^{\frac{i+1}{\beta}}$$
$$\times {}_2F_1(1,1+\frac{1}{\gamma}+\tfrac{i+1}{\beta};2+\tfrac{1}{\gamma};\beta\lambda M^{\gamma}).$$

3.3. Probability Weighted Moment

The probability weighted moments (PWMs) makes use of the analytical relationship among the parameters and the so-called PWMs of probability distribution in calculating magnitudes for the parameters. Although, probability weighted moments are useful to characterize a distribution. We use the PWMs to derive estimates of the parameters and quantiles of the probability distribution. Estimates based on PWMs are often considered to be superior to standard moment-based estimates. These concepts used when maximum likelihood estimates (MLE), are unavailable or difficult to compute; e.g. see Greenwood et al. (1979); Hosking (1986); Harvey et al. (2017) and Tarko (2018). The PWMs are defined by

$$\tau_{s,r} = E(Y^sF(Y)^r) = \begin{cases} \int_0^{\infty} y^sF(y)^rf(y)dy, & \beta<0, \\ \int_0^{\frac{1}{(\beta\lambda)^{1/\gamma}}} y^sF(y)^rf(y)dy, & \beta>0, \end{cases}$$

where r and s are positive integers and $F(y)$ and $f(y)$ are the CDF and PDF of distribution. The following theorem gives the PWMs of the EEW distribution.

THEOREM 7. *The PWMs of the EEW distribution are*

$$\tau_{s,r} = \begin{cases} \sum_{i=0}^{\infty}\binom{r\alpha}{i}(-1)^i\frac{(-\beta\lambda)^{\frac{s-1}{\gamma}}\Gamma(\frac{s+1}{\gamma})\Gamma(-\frac{\beta+s\beta+i\gamma}{\beta\lambda})}{\gamma\Gamma(-\frac{i}{\beta})}, & \beta<0, \\ \sum_{i=0}^{\infty}\binom{r\alpha}{i}(-1)^i\frac{(-\beta\lambda)^{\frac{s-1}{\gamma}}\Gamma(\frac{i+\beta}{\beta})\Gamma(-\frac{s+1}{\lambda})}{\gamma\Gamma(-\frac{i}{\beta}+\frac{1+s+r}{\gamma})}, & \beta>0, \end{cases}$$

PROOF. Here, we only give the proof for $\beta>0$, in which case $0<y<\frac{1}{(\beta\lambda)^{1/\gamma}}$. Proof of the other case is similar. We have

$$\tau_{s,r} = \int_0^{\frac{1}{(\beta\lambda)^{1/\gamma}}} \alpha\gamma\lambda y^{s+\gamma-1}(1-\beta\lambda y^{\gamma})^{1/\beta-1}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{(r+1)\alpha-1}dy.$$

By using the expansion $(1-z)^{\alpha} = \sum_{i=0}^{\infty} \binom{\alpha}{i}(-1)^i z^i$, we have

$$\begin{aligned} \tau_{s,r} &= \sum_{i=0}^{\infty} \binom{(r+1)\alpha - 1}{i} (-1)^i \int_0^{\frac{1}{(\beta\lambda)^{1/\gamma}}} \alpha\gamma\lambda y^{s+\gamma-1}(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}-1} \\ &= \sum_{i=0}^{\infty} \binom{(r+1)\alpha - 1}{i} (-1)^i \frac{\alpha\lambda(\beta\lambda)^{-\frac{s+\gamma}{\gamma}}\Gamma(\frac{1+i}{\beta})\Gamma(\frac{s+\gamma}{\gamma})}{\Gamma(\frac{i+1}{\beta} - \frac{s+\gamma}{\gamma})}, \end{aligned}$$

where α, β, γ, and s are positive.

By use of PWMs, we can obtain the mean and variance of the distribution according to

$$E(Y) = \tau_{1,0}, \quad \text{and} \quad Var(Y) = \tau_{2,0} - \tau_{1,0}^2.$$

3.4. Bonferroni and Lorenz curve

The Bonferroni and Lorenz curves are fundamental tools for analyzing data arising in economics and reliability. Also, these curves have many applications in other fields like demography, insurance and medicine; e.g. see Bonferroni (1930); Gail and Gastwirth (1978); Giorgi and Crescenzi (2001); Shanker et al. (2017) and Arnold and Sarabia (2018). The Lorenz curve is a function of the cumulative proportion of ordered individuals mapped onto the corresponding cumulative proportion of their size. The Bonferroni curve is given by

$$B_F(F(y)) = \frac{1}{\mu F(y)} \int_0^y t f(t)dt,$$

or equivalently given by $B_F(p) = \frac{1}{\mu p}\int_0^p F^{-1}(t)dt$, where $p = F(y)$ and $F^{-1}(t) = \inf\{y : F(y) \geq t\}$. Also, the Lorenz curve is given by

$$L_F(F(y)) = F(y).B_F(F(y)) = \frac{1}{\mu}\int_0^y t f(t)dt,$$

where $\mu = E(Y)$.

THEOREM 8. *The Bonferroni and Lorenz curve of the EEW distribution are given by, respectively*

$$\begin{aligned} B_F(F(y)) &= \frac{1}{\mu F(y)} \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \frac{\alpha\gamma\lambda}{\gamma+1} y^{\gamma+1}(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}} \\ &\quad \times {}_2F_1(1, 1+\frac{1}{\gamma}+\frac{i+1}{\beta}; 2+\frac{1}{\gamma}; \beta\lambda y^{\gamma}), \end{aligned}$$

and

$$L_F(F(y)) = \frac{1}{\mu}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}y^{\gamma+1}(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}} \times {}_2F_1(1,1+\frac{1}{\gamma}+\frac{i+1}{\beta};2+\frac{1}{\gamma};\beta\lambda y^{\gamma}),$$

where $\beta \neq 0$, $\mu = E(Y)$ *is given in (5) and* $F(y)$ *is given in (1).*

Proof. Such as the proof of theorem 2, we have

$$F(y) = (1-(1-\beta\lambda y^{\gamma})^{\frac{1}{\beta}})^{\alpha},$$

and

$$\begin{aligned} f(y) &= \alpha\gamma\lambda y^{\gamma-1}(1-\beta\lambda y^{\gamma})^{1/\beta-1}(1-(1-\beta\lambda y^{\gamma})^{1/\beta})^{\alpha-1} \\ &= \alpha\gamma\lambda(1-\beta\lambda y^{\gamma})^{\frac{1}{\beta}-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^{\gamma})^{\frac{i}{\beta}} \\ &= \alpha\gamma\lambda y^{\gamma-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}-1}. \end{aligned}$$

Thus, we have

$$B_F(F(y)) = \frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}y^{\gamma+1}(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}} \times {}_2F_1(1,1+\frac{1}{\gamma}+\frac{i+1}{\beta};2+\frac{1}{\gamma};\beta\lambda y^{\gamma})$$

and

$$L_F(F(y)) = \frac{1}{\mu}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}y^{\gamma+1}(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}} \times {}_2F_1(1,1+\frac{1}{\gamma}+\frac{i+1}{\beta};2+\frac{1}{\gamma};\beta\lambda y^{\gamma}).$$

3.5. Rényi and Shannon entropy

In information theory, entropy is a measure of the uncertainty associated with a random variable. The Shannon entropy is a measure of the average information content one is missing when one doesn't know the value of the random variable. A useful generalization of Shannon entropy is the Rényi entropy. The Rényi entropy is important in ecology and statistics. It is also important in quantum information,

where it can be used as a measure of entanglement; e.g. see Shannon (1948); Renyi (1961); Seo and Kang (1970); Kayal and Kumar (2013); Kayal et al. (2015) and Kang et al. (2012).
The Rényi and Shannon entropy of the EEW distribution are given, respectively, by

$$
\begin{aligned}
I_R(r) &= \frac{1}{1-r}\log\int_y f^r(y)dy \\
&= \begin{cases} \frac{1}{1-r}\log\sum_{i=0}^{\infty}\binom{r(\alpha-1)}{i}(-1)^i \frac{(-\beta\lambda)^{\frac{r-r\gamma-1}{\gamma}}(\alpha\gamma\lambda)^{\gamma}\Gamma(-\frac{\gamma+1}{\beta}+\frac{\gamma-1}{\gamma})}{\gamma\Gamma(-\frac{r-r\beta+i}{\beta})}, & \beta<0, \\ \frac{1}{1-r}\log\sum_{i=0}^{\infty}\binom{r(\alpha-1)}{i}(-1)^i\frac{1}{1+r(\gamma-1)}(\alpha\gamma\lambda)^{\gamma}(\beta\lambda)^{-\frac{1+r(\gamma-1)}{\gamma}} & \\ \times {}_2F_1(-\frac{i+r-r\beta}{\beta}, \frac{1+r(\gamma-1)}{\gamma}; \frac{1+r(\gamma-1)+\gamma}{\gamma}; \beta\lambda((\beta\lambda)^{-1/\gamma})^{\gamma}), & \beta>0, \end{cases}
\end{aligned}
$$

and

$$
E(-\log f(y)) = \log(\alpha\gamma\lambda) - \frac{\alpha-1}{\alpha} + (1-\gamma)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\binom{i}{j}(-1)^j}{i}{\mu'_1}^j + (1-\frac{1}{\beta})\sum_{i=0}^{\infty}\frac{(\beta\lambda)^i}{i}\mu'_{i\gamma},
$$

where $\mu'_k = E(Y^k)$ is given in (5).

3.6. Residual life function

Given that a component survives up to time $t \geq 0$, the residual life function is the period beyond t, and defined by the conditional variable $Y-t|Y>t$. We obtain the r-th order moment of the residual life of the EEW distribution via the general formula

$$
m_r(t) = E[(Y-t)^r|Y>t],
$$

in two cases; $\beta<0$ and $\beta>0$. Let $\beta<0$, thus the r-th residual life is given by

$$
\begin{aligned}
m_r(t) &= \frac{1}{s(t)}\int_t^{\infty}(y-t)^r f(y)dy = \frac{1}{s(t)}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\binom{\alpha-1}{i}\binom{r}{j}(-1)^{r+i-j} \\
&\quad \times\frac{t^{\gamma+\frac{(i+1)\gamma}{\beta}}\alpha\gamma(-\beta\gamma)^{\frac{i+1}{\beta}}}{j\beta+\gamma+i\gamma}\ {}_2F_1(\frac{\beta+i-1}{\beta}, -\frac{j\beta+\gamma+i\gamma}{\beta\gamma}; 1-\frac{i+1}{\beta}-\frac{j}{\beta}; \frac{t-\gamma}{\beta\lambda}),
\end{aligned}
$$

and for $\beta>0$, the r-th residual life is given by

$$
\begin{aligned}
m_r(t) = \frac{1}{s(t)}\int_t^{\frac{1}{(\beta\lambda)^{1/\gamma}}}(y-t)^r f(y)dy &= \frac{1}{s(t)}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\binom{\alpha-1}{i}\binom{r}{j}(-1)^{r+i-j}\frac{\alpha\gamma t^{\gamma-j}}{j\beta+\gamma+i\gamma} \\
&\times[-e^{-i\pi\frac{j\beta+\gamma+i\gamma}{\beta\gamma}}(-\beta\lambda)^{-j}\ {}_2F_1(\frac{\beta-i-1}{\beta}, -\frac{j\beta+\gamma+i\gamma}{\beta\gamma}; 1-\frac{i+1}{\beta}-\frac{j}{\beta}; e^{i\pi}) \\
&+t^{\frac{j\beta+\gamma+i\gamma-\beta\gamma}{\beta}}(-\beta\lambda)^{\frac{i+1}{\beta}}\ {}_2F_1(\frac{\beta-i-1}{\beta}, -\frac{j\beta+\gamma+i\gamma}{\beta\gamma}; -\frac{j\beta+\gamma+i\gamma-\beta\gamma}{\beta\gamma}; \frac{t^{-\gamma}}{\beta\lambda})],
\end{aligned}
$$

where $s(t)$ (the survival function of Y) is given in (3).
The mean residual life (MRL) function is a helpful tool in model building, and it has been used for both parametric and nonparametric building. It is very important in any lifetime modeling as it characterizes the distribution. Lifetime can exhibit increasing MRL (IMRL) or decreasing MRL (DMRL) shaped. MRL functions that first increasing (decreasing) and then decreasing (increasing) are usually called upside-down bathtub (bathtub) shaped, UMRL (BMRL). the relationship between the behavior of the two functions of a distribution was studied by many authors such as Ghitany (1998), Mi (1995), Park (1985), Shanbhag (1970) and Tang et al. (1999). The MRL function for the EEW distribution obtains by setting $r = 1$ in the above equations and it is given in the following theorem. Although, the MRL function of a EEW cannot be obtained in explicit form, it can be obtained in terms of infinite series.

THEOREM 9. *The MRL function of the EEW distribution is*

$$\begin{aligned} m_1(t) &= \frac{1}{s(t)}[\sum_{i}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma t^{\frac{\beta+\gamma+i\gamma}{\beta}}(-\beta\lambda)^{\frac{i+1}{\beta}}}{\beta+\gamma+i\gamma} \\ &\quad \times {}_2F_1(\frac{\beta-i-1}{\beta},-\frac{\beta+\gamma+i\gamma}{\beta\gamma};\frac{\gamma-1}{\gamma}-\frac{i+1}{\beta};\frac{t^{-\gamma}}{\beta\lambda}) \\ &\quad -\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{t\alpha(1-\beta\lambda t^{\gamma})^{\frac{i+1}{\beta}}}{i+1}], \end{aligned}$$

where $\beta < 0$, *and*

$$\begin{aligned} m_1(t) = &\frac{1}{s(t)}[\sum_{i}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}t^{\gamma+1}(1-\beta\lambda t^{\gamma})^{\frac{i+1}{\beta}} \\ &\times {}_2F_1(1,1+\frac{1}{\gamma}+\frac{i+1}{\beta};2+\frac{1}{\gamma};\beta\lambda t^{\gamma}) \\ &-\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{t\alpha}{i+1}(1-\beta\lambda t^{\gamma})^{\frac{i+1}{\beta}}]. \end{aligned}$$

Proof. when $\beta < 0$, we have

$$\begin{aligned} m_1(t) &= \frac{1}{s(t)}\int_t^{\infty}(y-t)f(y)dy = \frac{1}{s(t)}[\int_t^{\infty} yf(y)dy - \int_t^{\infty} tf(y)dy] \\ &= \frac{1}{s(t)}[\int_t^{\infty}\alpha\gamma\lambda y^{\gamma}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}-1}dy \\ &\quad -t\int_0^{\infty}\alpha\gamma\lambda y^{\gamma-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}-1}dy] \\ &= \frac{1}{s(t)}[\sum_i^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma t^{\frac{\beta+\gamma+i\gamma}{\beta}}(-\beta\lambda)^{\frac{i+1}{\beta}}}{\beta+\gamma+i\gamma} \\ &\quad \times{}_2F_1(\frac{\beta-i-1}{\beta}, -\frac{\beta+\gamma+i\gamma}{\beta\gamma}; \frac{\gamma-1}{\gamma}-\frac{i+1}{\beta}; \frac{t^{-\gamma}}{\beta\lambda}) \\ &\quad -\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{t\alpha(1-\beta\lambda t^{\gamma})^{\frac{i+1}{\beta}}}{i+1}], \end{aligned}$$

and when $\beta > 0$, we have

$$\begin{aligned} m_1(t) &= \frac{1}{s(t)}\int_t^{1/(\beta\lambda)^{\frac{1}{\gamma}}}(y-t)f(y)dy = \frac{1}{s(t)}[\int_t^{1/(\beta\lambda)^{\frac{1}{\gamma}}} yf(y)dy - \int_t^{1/(\beta\lambda)^{\frac{1}{\gamma}}} tf(y)dy] \\ &= \frac{1}{s(t)}[\int_t^{1/(\beta\lambda)^{\frac{1}{\gamma}}}\alpha\gamma\lambda y^{\gamma}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}-1}dy \\ &\quad -t\int_0^{1/(\beta\lambda)^{\frac{1}{\gamma}}}\alpha\gamma\lambda y^{\gamma-1}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i(1-\beta\lambda y^{\gamma})^{\frac{i+1}{\beta}-1}dy] \\ &= \frac{1}{s(t)}[\sum_i^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{\alpha\gamma\lambda}{\gamma+1}t^{\gamma+1}(1-\beta\lambda t^{\gamma})^{\frac{i+1}{\beta}} \\ &\quad \times{}_2F_1(1, 1+\frac{1}{\gamma}+\frac{i+1}{\beta}; 2+\frac{1}{\gamma}; \beta\lambda t^{\gamma}) - \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\frac{t\alpha}{i+1}(1-\beta\lambda t^{\gamma})^{\frac{i+1}{\beta}}]. \end{aligned}$$

On the other hand, we analogously discuss the reversed residual life and some of its properties. The reversed residual life can be defined as the conditional random variable $t - Y|Y \le t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. Also in reliability, the mean reversed residual life and the ratio of two consecutive moments of reversed residual life characterize the distribution uniquely.

The r-th order moment of the reversed residual life for the EEW distribution can be obtained via the general formula

$$\mu_r(t) = E[(t-y)^r|Y \le t],$$

in two cases; $\beta < 0$ and $\beta > 0$. Let $\beta < 0$, thus the r-th reversed residual life is given by

$$\mu_r(t) = \frac{1}{F(t)} \int_0^t (t-y)^r f(y)dy = \frac{1}{F(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{r} \binom{\alpha-1}{i} \binom{r}{j} (-1)^{r+i-j} \alpha e^{-\frac{i\pi(\gamma+r-j)}{\gamma}}$$
$$\times [\frac{t^j (\beta\lambda)^{\frac{j-r}{\gamma}} \Gamma(\frac{j-r}{\gamma} + \frac{i+1}{\beta}) \Gamma(\frac{\gamma+r-j}{\gamma})}{\beta \Gamma(\frac{\beta-i-1}{\beta})} + \frac{\gamma(\beta\lambda)^{\frac{i+1}{\beta}}}{r\beta - j\beta + \gamma + i\gamma} e^{\frac{i\pi(r\beta - j\beta + \gamma + i\gamma)}{\beta\gamma}}$$
$$\times t^{r+\frac{(i+1)\gamma}{\beta}} {}_2F_1(\frac{\beta-i-1}{\beta}, \frac{j-r}{\gamma} - \frac{j+1}{\beta}; \frac{\gamma+j-r}{\gamma} - \frac{i+1}{\beta}; \frac{t^{-\gamma}}{\beta\lambda})], \quad (6)$$

and for $\beta > 0$ the r-th reversed residual life is given by

$$\mu_r(t) = \frac{1}{F(t)} \int_0^{\frac{1}{(\beta\lambda)^{1/\gamma}}} (t-y)^r f(y)dy$$
$$= \frac{1}{F(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{r} \binom{\alpha-1}{i} \binom{r}{j} (-1)^{r+i-j} \frac{\alpha\lambda t^j (\beta\lambda)^{-\frac{\gamma+r-j}{\gamma}} \Gamma(\frac{i+1}{\beta}) \Gamma(\frac{\gamma+r-j}{\gamma})}{\Gamma(\frac{i+1}{\beta} + \frac{\gamma+r-j}{\gamma})},$$

where $t > \frac{1}{(\beta\lambda)^{1/\gamma}}$, and for $t < \frac{1}{(\beta\lambda)^{1/\gamma}}$, the r-th reversed residual life is same as the equation (6).
The mean and second moment of the reversed residual life of the EEW distribution can be obtained by setting $r = 1, 2$ in above equations. Also, by using $\mu_1(t)$ and $\mu_2(t)$, we obtained the variance of the reversed residual life of the EEW distribution.

4. Parametric inference

In this section, we consider the parametric inference of the unknown parameters α, β, γ and λ of the EEW distribution. The parametric inferences will be discussed based on likelihood method in two situations; censored and full data.

4.1. Likelihood method based on censored data

In most of survival analysis and reliability studies, the censored data are often encountered. Let Y_i be the random variable from EEW distribution with the parameters vector $\Theta = (\alpha, \beta, \gamma, \lambda)$. A simple random censoring procedure is one in which each element i is assumed to have a lifetime Y_i and a censoring time C_i, where Y_i and C_i are independent random variables. Suppose that the data including n independent observations $y_i = \min(Y_i, C_i)$ for $i = 1, \cdots, n$. The distribution of C_i does not depend on any of the unknown parameters of Y_i. The censored log-likelihood $\ell_c(\Theta)$ is given by

$$\ell_c(\Theta) = r\log\alpha + r\log\gamma + r\log\lambda + (\gamma-1)\sum_{i\in F}\log y_i + (\frac{1}{\beta}-1)\sum_{i\in F}\log(1-\beta\lambda y_i^{\gamma})$$
$$+(\alpha-1)\sum_{i\in F}\log(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}) + \sum_{i\in C}\log[1-(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta})^{\alpha}],$$

where r is the number of failures and F and C denote the uncensored and censored sets of observations, respectively.

By differentiating the log-likelihood function with respect to α, β, γ and λ, respectively, components of score vector $U(\Theta) = (\frac{\partial \ell_c(\Theta)}{\partial\alpha}, \frac{\partial \ell_c(\Theta)}{\partial\beta}, \frac{\partial \ell_c(\Theta)}{\partial\gamma}, \frac{\partial \ell_c(\Theta)}{\partial\lambda})$ are derived as

$$\frac{\partial \ell_c(\Theta)}{\partial\alpha} = \frac{r}{\alpha} + \sum_{i\in F}\log\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)$$
$$-\sum_{i\in C}\frac{\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha}\log\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)}{1-\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha}}$$

$$\frac{\partial \ell_c(\Theta)}{\partial\beta} = -\frac{1}{\beta^2}\sum_{i\in F}\log(1-\beta\lambda y_i^{\gamma}) - \left(\frac{1}{\beta}-1\right)\sum_{i\in F}\frac{\lambda y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}}$$
$$-(\alpha-1)\sum_{i\in F}\frac{(1-\beta\lambda y_i^{\gamma})^{1/\beta}\left(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})}\right)}{1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}}$$
$$+\sum_{i\in C}\frac{\alpha(1-\beta\lambda y_i^{\gamma})^{1/\beta}\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha-1}\left(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})}\right)}{1-\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha}}$$

$$\frac{\partial \ell_c(\Theta)}{\partial\gamma} = \frac{r}{\gamma} - \left(\frac{1}{\beta}-1\right)\sum_{i\in F}\frac{\beta\lambda y_i^{\gamma}\log(y_i)}{1-\beta\lambda y_i^{\gamma}} + (\alpha-1)\sum_{i\in F}\frac{\lambda y_i^{\gamma}\log(y_i)(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}}$$
$$+\sum_{i\in F}\log(y_i) - \sum_{i\in C}\frac{\alpha\lambda y_i^{\gamma}\log(y_i)(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha-1}}{1-\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha}}$$

$$\frac{\partial \ell_c(\Theta)}{\partial\lambda} = \frac{r}{\lambda} + \left(\frac{1}{\beta}-1\right)\sum_{i\in F}-\frac{\beta y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}} + (\alpha-1)\sum_{i\in F}\frac{y(i)^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}}$$
$$-\sum_{i\in C}\frac{\alpha y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha-1}}{1-\left(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}\right)^{\alpha}}.$$

4.2. Likelihood method based on complete data

Let $y_1, ..., y_n$ be the random sample of size n from EEW distribution. The log-likelihood function is given by

$$\ell(\alpha,\beta,\gamma,\lambda) = n\log\alpha + n\log\gamma + n\log\lambda + (\gamma-1)\sum_{i=1}^{n}\log y_i + (\frac{1}{\beta}-1)\sum_{i=1}^{n}\log(1-\beta\lambda y_i^{\gamma}) + (\alpha-1)\sum_{i=1}^{n}\log(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta}).$$

The first derivatives of the log-likelihood function with respect to α, β, γ and λ are given in Appendix, and the maximum likelihood estimators (MLEs) of parameters can be obtained by maximizing this function. For given β, γ and λ, the MLE of α can be obtained as

$$\hat{\alpha}(\beta,\gamma,\lambda) = -\frac{n}{\sum_{i=1}^{n}\log(1-(1-\beta\lambda y_i^{\gamma})^{1/\beta})}.$$

By maximizing the profile log-likelihood function $\ell(\hat{\alpha}(\beta,\gamma,\lambda),\beta,\gamma,\lambda)$, with respect to β, γ, λ, the MLEs of β, γ and λ can be obtained. Now we will discuss the asymptotic properties of the MLEs in two situations; $\beta < 0$ and $\beta > 0$.

The Regular Case: When $\beta < 0$, the situation is exactly the same as the generalized Weibull case discussed by Mudholkar et al. (1996). In this case EEW satisfies all the regularity properties of the parametric family. Then asymptotically, as $n \to \infty$,

$$\sqrt{n}(\hat{\Theta}-\Theta) \longrightarrow N_4(0, I^{-1}(\Theta)),$$

where $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$, $\Theta = (\alpha, \beta, \gamma, \lambda)$, N_4 denotes the four-variate normal distribution and $I(\Theta)$ is the Fisher information matrix. The observed information matrix is

$$I_n(\Theta) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\gamma} & I_{\alpha\lambda} \\ I_{\alpha\beta} & I_{\beta\beta} & I_{\beta\gamma} & I_{\beta\lambda} \\ I_{\alpha\gamma} & I_{\beta\gamma} & I_{\gamma\gamma} & I_{\gamma\lambda} \\ I_{\alpha\lambda} & I_{\beta\lambda} & I_{\gamma\lambda} & I_{\lambda\lambda} \end{bmatrix},$$

where

$$I_{\theta_i\theta_j} = \frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}, \quad i,j = 1,2,3,4$$

and they are provided in the Appendix.

The Non-Regular Case: When $\beta > 0$, we propose a reparametrization of $\alpha, \beta, \gamma, \lambda$ as $(\alpha, \beta, \gamma, \phi)$, where $\phi = \frac{1}{(\beta\lambda)^{\frac{1}{\gamma}}}$, then $\lambda = \frac{\phi^{-\gamma}}{\beta}$. The PDF (2) and CDF (1) can be written as

$$f(y;\alpha,\beta,\gamma,\phi) = \alpha\gamma\frac{\phi^{-\gamma}}{\beta}y^{\gamma-1}(1-(\frac{y}{\phi})^{\gamma})^{\frac{1}{\beta}-1}(1-(1-(\frac{y}{\phi})^{\gamma})^{\frac{1}{\beta}})^{\alpha-1}, \quad (7)$$

and

$$F(y;\alpha,\beta,\gamma,\phi) = (1-(1-(\frac{y}{\phi})^{\gamma})^{\frac{1}{\beta}})^{\alpha},$$

respectively, for $0 < y < \phi$ and 0 otherwise. The corresponding quantile function becomes

$$Q(y;\alpha,\beta,\gamma,\phi) = (\phi^{\gamma}(1-(1-u^{\frac{1}{\alpha}})^{\beta}))^{\frac{1}{\gamma}}.$$

Based on a random sample $y_1, ..., y_n$ from (7), the MLEs can be obtained by maximizing the log-likelihood function:

$$\ell(\alpha,\beta,\gamma,\phi) = n\log\alpha + n\log\gamma - n\gamma\log\phi - n\log\beta + (\gamma-1)\sum_{i=1}^{n}\log y_{(i)} + (\frac{1}{\beta}-1)\sum_{i=1}^{n}\log(1-(\frac{y_{(i)}}{\phi})^{\gamma}) + (\alpha-1)\sum_{i=1}^{n}\log(1-(1-(\frac{y_{(i)}}{\phi})^{\gamma})^{\frac{1}{\beta}}). \quad (8)$$

It is immediate from (8) that for fixed $0 < \alpha < 1$, $0 < \beta < 1$ and $\gamma > 0$, as $\phi \downarrow y_{(n)}$, $\ell(\alpha,\beta,\gamma,\phi) \to \infty$. Thus, in this case the MLEs do not exist.

To estimate the unknown parameters, first estimate the parameter ϕ by its consistent estimator $\tilde{\phi} = y_{(n)}$. The modified log-likelihood function based on the remaining $(n-1)$ observations after ignoring the largest observation and replacing ϕ by $\tilde{\phi} = y_{(n)}$ is

$$\begin{aligned}\ell(\alpha,\beta,\gamma,\tilde{\phi}) &= (n-1)\log\alpha + (n-1)\log\gamma - (n-1)\gamma\log y_{(n)} - (n-1)\log\beta \\ &\quad + (\gamma-1)\sum_{i=1}^{n-1}\log y_{(i)} + (\frac{1}{\beta}-1)\sum_{i=1}^{n-1}\log(1-(\frac{y_{(i)}}{y_{(n)}})^{\gamma}) \\ &\quad + (\alpha-1)\sum_{i=1}^{n-1}\log(1-(1-(\frac{y_{(i)}}{y_{(n)}})^{\gamma})^{\frac{1}{\beta}}).\end{aligned}$$

The modified MLE of α and λ, for fixed β and γ, can be obtained as

$$\tilde{\alpha}(\beta,\gamma) = -\frac{n-1}{1-(1-(\frac{y_{(i)}}{y_{(n)}})^{\gamma})^{\frac{1}{\beta}}}$$

and

$$\tilde{\lambda} = \frac{y_{(n)}^{-\gamma}}{\beta},$$

respectively. Therefore, in this case the modified MLE of β and λ can be obtained by solving the optimization problem from the modified log-likelihood function of β and γ.

For the propose of statistical inference, an understanding of the distribution of $\tilde{\phi}$ is necessary. This is given in the following theorem.

THEOREM 10. *i) The marginal distribution of* $\tilde{\phi} = y_{(n)}$ *is given by*

$$p(\tilde{\phi} \leq t) = (1 - (1 - (\frac{t}{\phi})^{\gamma})^{\frac{1}{\beta}})^{n\alpha}.$$

ii) Asymptotically as $n \to \infty$,

$$n^{\beta}((\frac{y_{(n)}}{\phi})^{\gamma} - 1) \to -X^{\beta},$$

where $X \sim Exp(\alpha)$, *with mean* $\frac{1}{\alpha}$.

PROOF. i)

$$p(\tilde{\phi} \leq t) = p(Y_{(n)} \leq t) = (p(Y \leq t))^n = (1 - (1 - (\frac{t}{\phi})^{\gamma})^{\frac{1}{\beta}})^{n\alpha}.$$

ii)

$$Y_{(n)} \stackrel{d}{=} Q(U_{(n)}) = \phi(1 - (1 - U_{(n)}^{\frac{1}{\alpha}})^{\beta})^{\frac{1}{\gamma}},$$

so

$$\frac{Y_{(n)}}{\phi} = (1 - (1 - U_{(n)}^{\frac{1}{\alpha}})^{\beta})^{\frac{1}{\gamma}},$$

then we have

$$n^{\beta}((\frac{Y_{(n)}}{\phi})^{\gamma} - 1) = -(n(1 - U_{(n)}^{\frac{1}{\alpha}}))^{\beta}.$$

Hence,

$$\begin{aligned} p(n(1 - U_{(n)}^{\frac{1}{\alpha}}) \leq t) &= p(1 - U_{(n)}^{\frac{1}{\alpha}} \leq \frac{t}{n}) = p(-U_{(n)}^{\frac{1}{\alpha}} \leq \frac{t}{n} - 1) = p(U_{(n)}^{\frac{1}{\alpha}} \geq 1 - \frac{t}{n}) \\ &= p(U_{(n)} \geq (1 - \frac{t}{n})^{\alpha}) = 1 - ((1 - \frac{t}{n})^{\alpha})^n = 1 - (1 - \frac{t}{n})^{n\alpha}, \end{aligned}$$

as $n \longrightarrow \infty$, we have

$$p(n(1 - U_{(n)}^{\frac{1}{\alpha}}) \leq t) \longrightarrow 1 - e^{-\alpha t}.$$

Thus

$$n^{\beta}((\frac{y_{(n)}}{\phi})^{\gamma} - 1) \to -X^{\beta}, \qquad where \quad X \sim Exp(\alpha).$$

5. SIMULATION EXPERIMENTS AND DATA ANALYSIS

5.1. Simulation Experiments

In this section, we perform some simulation studies, just to verify how the MLEs work for different sample sizes and different parameter values for the proposed

EEW model. The results are obtained from 1000 Monte Carlo replications from simulations carried out using the software R.
We have used the following parameter sets:
Model 1: $\alpha = 1, \beta = 2, \gamma = 1.5, \lambda = 1$,
Model 2: $\alpha = 2.35, \beta = 2, \gamma = 1.21, \lambda = 1$,
Model 3: $\alpha = 0.8, \beta = -0.50, \gamma = 0.8, \lambda = 1$,
Model 4: $\alpha = 1.57, \beta = 0.81, \gamma = 2.5, \lambda = 1.5$,
Model 5: $\alpha = 0.48, \beta = -0.78, \gamma = 0.35, \lambda = 1$
Model 6: $\alpha = 1.35, \beta = 1.57, \gamma = 1.49, \lambda = 1.87$
and different sample sizes, namely: $n = 50, 100, 150, 200, 300$ and 400.
We report the average estimates and the associated square root of mean squared errors (RMSEs). The results are presented in Table 1. From the results presented in Table 1 the following points are quite clear. (i) It is quite clear that the MLEs are working quite well. As the sample size increases the standard deviation and the square root of mean squared errors decrease. (ii) This verifies the consistency properties of the MLEs. For all practical purposes, MLEs can be used quite effectively for estimating the unknown parameters of the proposed EEW model.

TABLE 1
The MLEs, Std and RMSE

n	$(\alpha, \beta, \gamma, \lambda)$	MLE				Std				RMSE			
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$
50	(1.00, 2.00, 1.50, 1.00)	1.076	1.769	1.069	1.909	0.153	0.277	0.135	0.488	0.171	0.361	0.452	1.032
	(2.35, 2.00, 1.21, 1.00)	2.073	1.918	1.801	4.709	0.921	0.303	1.084	7.311	0.962	0.314	1.235	8.198
	(0.80, -0.50, 0.80, 1.00)	1.798	-0.927	1.354	1.072	2.215	1.599	1.248	0.852	2.430	1.655	1.366	0.855
	(1.57, 0.81, 2.50, 1.50)	1.370	0.817	1.274	5.896	0.463	0.139	0.342	3.284	0.504	0.139	1.273	5.487
	(0.48, -0.78, 0.35, 1.00)	1.796	-0.821	1.075	0.776	4.409	1.196	0.867	0.828	4.601	1.197	1.131	0.858
	(1.35, 1.57, 1.49, 1.87)	1.302	1.480	1.188	4.462	0.344	0.241	0.272	2.672	0.347	0.258	0.406	3.723
100	(1.00, 2.00, 1.50, 1.00)	1.060	1.832	1.022	1.701	0.091	0.194	0.051	0.213	0.109	0.256	0.480	0.733
	(2.35, 2.00, 1.21, 1.00)	2.124	1.971	1.510	2.913	0.657	0.189	0.889	3.783	0.695	0.191	0.938	4.239
	(0.80, -0.50, 0.80, 1.00)	1.509	-0.471	0.993	1.050	1.288	0.665	0.660	0.648	1.470	0.665	0.688	0.650
	(1.57, 0.81, 2.50, 1.50)	1.347	0.815	1.217	5.162	0.330	0.101	0.263	1.963	0.397	0.101	1.310	4.155
	(0.48, -0.78, 0.35, .1.00)	0.882	-0.898	1.027	0.757	0.840	1.207	0.710	0.559	0.931	1.212	0.884	0.610
	(1.35, 1.57, 1.49, 1.87)	1.278	1.528	1.133	3.873	0.228	0.199	0.231	2.269	0.239	0.204	0.424	3.027
150	(1.00, 2.00, 1.50, 1.00)	1.034	1.900	1.035	1.664	0.073	0.170	0.075	0.248	0.081	0.197	0.471	0.709
	(2.35, 2.00, 1.21, 1.00)	2.103	1.968	1.442	2.523	0.586	0.186	0.786	2.950	0.636	0.188	0.820	3.320
	(0.80, -0.50, 0.80, 1.00)	1.237	-0.537	1.040	1.950	0.987	0.563	0.599	0.553	1.079	0.565	0.645	0.555
	(1.57, 0.81, 2.50, 1.50)	1.333	0.825	1.213	5.051	0.300	0.084	0.265	1.863	0.383	0.086	1.314	4.010
	(0.48, -0.78, 0.35, 1.00)	0.998	-0.817	0.991	0.779	1.338	0.851	0.625	0.614	1.435	0.852	0.795	0.653
	(1.35, 1.57, 1.49, 1.87)	1.269	1.524	1.119	3.632	0.214	0.134	0.197	1.403	0.229	0.141	0.420	2.253
200	(1.00, 2.00, 1.50, 1.00)	1.031	1.911	1.018	1.612	0.059	0.145	0.041	0.153	0.066	0.170	0.484	0.631
	(2.35, 2.00, 1.21, 1.00)	2.178	1.998	1.293	1.816	0.504	0.178	0.563	1.292	0.533	0.178	0.569	1.529
	(0.80, -0.50, 0.80, 1.00)	1.148	-0.467	0.964	0.931	0.739	0.442	0.427	0.468	0.817	0.443	0.458	0.473
	(1.57, 0.81, 2.50, 1.50)	1.339	0.818	1.217	5.049	0.307	0.076	0.256	1.721	0.385	0.077	1.308	3.944
	(0.48, -0.78, 0.35, 1.00)	0.654	-0.755	0.983	0.644	0.372	0.651	0.474	0.376	0.411	0.652	0.677	0.517
	(1.35, 1.57, 1.49, 1.87)	1.271	1.544	1.127	3.633	0.201	0.132	0.211	1.379	0.216	0.135	0.419	2.239
300	(1.00, 2.00, 1.50, 1.00)	1.023	1.932	1.022	1.599	0.044	0.119	0.043	0.138	0.050	0.137	0.480	0.615
	(2.35, 2.00, 1.21, 1.00)	2.202	1.994	1.279	1.788	0.485	0.137	0.536	1.342	0.507	0.137	0.540	1.556
	(0.80, -0.50, 0.80, 1.00)	1.079	-0.463	0.971	0.916	0.606	0.356	0.385	0.426	0.668	0.358	0.421	0.432
	(1.57, 0.81, 2.50, 1.50)	1.348	0.815	1.208	4.990	0.300	0.058	0.259	1.675	0.373	0.058	1.317	3.871
	(0.48, -0.78, 0.35, 1.00)	0.636	-0.776	0.977	0.648	0.349	0.607	0.414	0.359	0.382	0.607	0.632	0.503
	(1.35, 1.57, 1.49, 1.87)	1.241	1.567	1.134	3.590	0.185	0.111	0.204	1.265	0.214	0.111	0.410	2.135
400	(1.00, 2.00, 1.50, 1.00)	1.022	1.941	1.013	1.575	0.037	0.100	0.039	0.120	0.043	0.116	0.488	0.587
	(2.35, 2.00, 1.21, 1.00)	2.216	1.984	1.215	1.641	0.409	0.120	0.462	1.103	0.431	0.121	0.462	1.276
	(0.80, -0.50, 0.80, 1.00)	1.047	-0.456	0.970	0.910	0.551	0.330	0.349	0.362	0.604	0.333	0.388	0.373
	(1.57, 0.81, 2.50, 1.50)	1.353	0.820	1.186	4.772	0.293	0.055	0.236	1.417	0.364	0.056	1.335	3.565
	(0.48, -0.78, 0.35, 1.00)	0.614	-0.697	0.942	0.622	0.277	0.396	0.317	0.304	0.308	0.404	0.544	0.485
	(1.35, 1.57, 1.49, 1.87)	1.273	1.566	1.112	3.464	0.163	0.780	0.194	1.224	0.180	0.079	0.425	2.010

5.2. Data Analysis

In this part, we fit the EEW distribution to the real data set and also compare the fitted EEW with some sub-models such as: the EW, W, EGE, GE and Exponential distributions, to show the superiority of the EEW distribution. In fact, it is observed that empirical hazard function of the data indicates that the data are coming from a lifetime distribution which has a DID shaped hazard function and the proposed distribution provides the best fit than many existing lifetime distributions. Therefore, the proposed distribution provides another option to a practitioner to use it for data analysis purposes. The results are obtained by using the function *optim* form package *stats4* in R.

As a data set, we consider the 101 data points represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 70% stress level until all had failed, so that we have complete data with exact times of failure, which are shown by Andrews and Herzberg (1985). Cooray and Ananda (2008) used this data in fitting generalization of the half-normal distribution.

The TTT plot of this data set in Figure 3 display a decreasing-increasing-decreasing (DID) hazard rate function. The MLEs of the parameters, -2log-likelihood, AIC (Akaike Information Criterion), the Kolmogorov-Smirnov test statistic (K-S), the Anderson-Darling test statistic (AD), the Cramér-von Mises test statistic (CM) and Durbin-Watson test statistic (DW) are displayed in Table 2. The CM and DW test statistics are described in details in Chen and Balakrishnan (1995) and Watson (1961), respectively. In general, the smaller the values of KS, AD, CM and WA, the better the fit to the data. From the values of these statistics, we conclude that the EEW distribution provides a better fit to this data than the EW, W, E

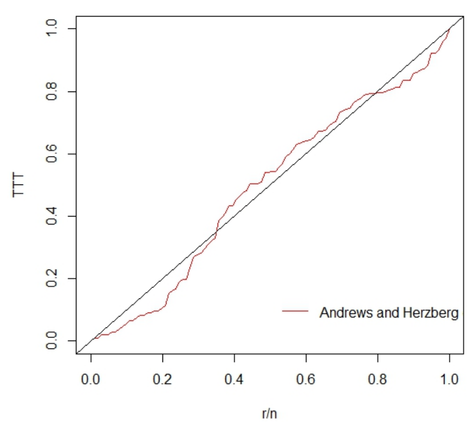

Figure 3 – TTT Plots of Andrews and Herzberg data.

TABLE 2
MLEs, KS, p-value, -2 Log L, AD, CM, DW and AIC statistics for Andrews and Herzberg data.

Dist	MLE	K-S	AIC	p-value	-2 Log L	AD	CM	DW
EEW	$\hat{\alpha} = 0.1004, \hat{\beta} = -3.4749$ $\hat{\gamma} = 6.9363, \hat{\lambda} = 0.0081$	0.0588	205.7	0.8760	197.7	0.4550	0.1575	0.1486
EGE	$\hat{\alpha} = 0.8766, \hat{\beta} = -0.0178$ $\hat{\lambda} = 0.9113$	0.0909	211.6	0.3745	205.6	1.0560	0.2722	0.2634
GE	$\hat{\alpha} = 0.8663, \hat{\lambda} = 0.8883$	0.0887	209.6	0.4041	205.6	1.0209	0.2634	0.2566
EW	$\hat{\alpha} = 0.7930, \hat{\gamma} = 1.0604$ $\hat{\lambda} = 0.8113$	0.0844	211.6	0.4674	205.6	0.9553	0.2473	0.2434
E	$\hat{\lambda} = 0.9758$	0.0888	209.0	0.4035	207.0	1.2481	0.2469	0.2457
W	$\hat{\gamma} = 0.9259, \hat{\lambda} = 1.0094,$	0.0965	210.0	0.3033	206.0	1.1221	0.2789	0.2715

Plots of the estimated PDF and CDF of the EEW, EW, W, EGE, GE and Exponential models fitted to the data set corresponding to Table 2, are given in Figures 4 and 5. Also, Figure 6 is displayed the QQ-plot of the EEW model to the data set. These plots suggest that the EEW distribution is superior to the other distributions in terms of model fitting.

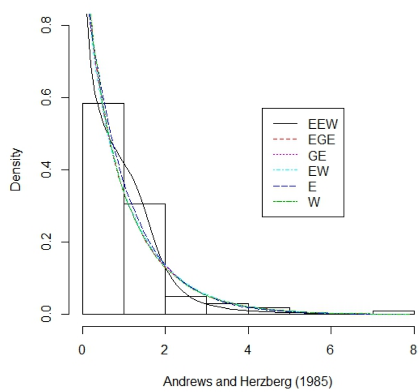

Figure 4 – Plots of fitted PDF and CDF of EEW, EGE, GE, EW, Exponential and Weibull models for Andrews and Herzberg data.

6. Conclusions and Some Future Work

We propose the Extended Exponentiated Weibull (EEW) distribution to generalized the Extended Generalized Exponential (EGE) distribution by adding a new

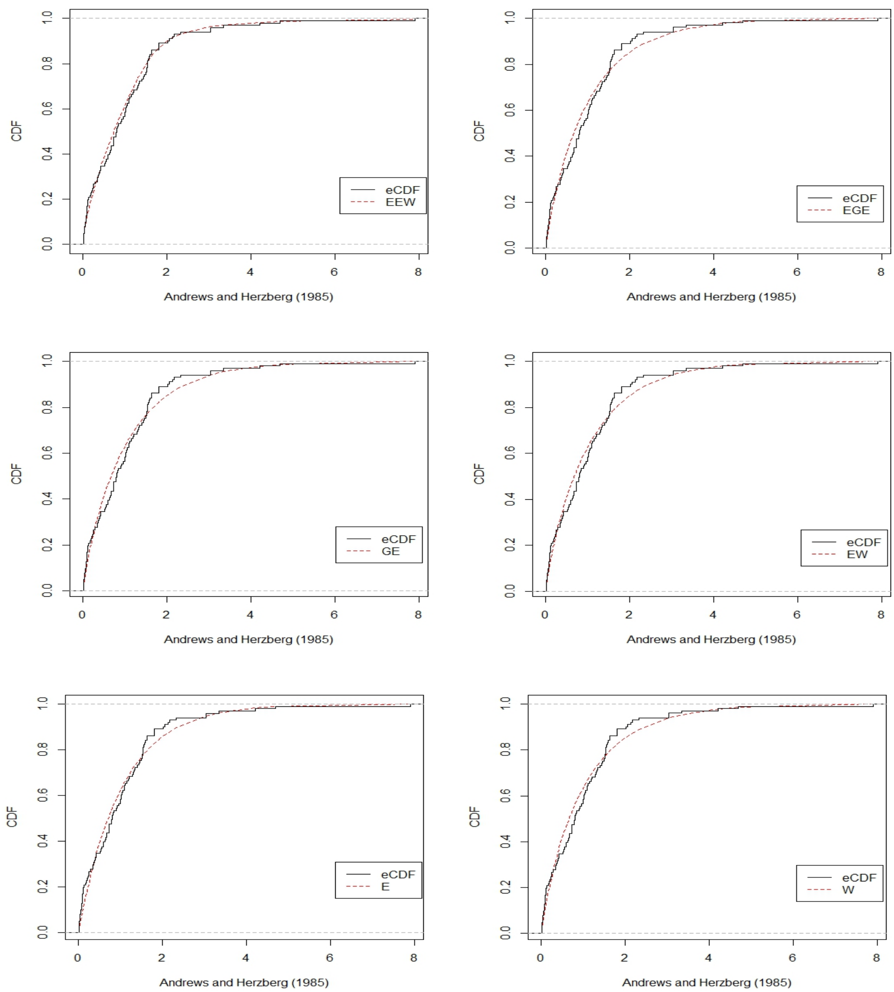

Figure 5 – Estimated distribution function versus the empirical distribution from the fitted EEW, EGE, GE, EW, E and Weibull models for Andrews and Herzberg data.

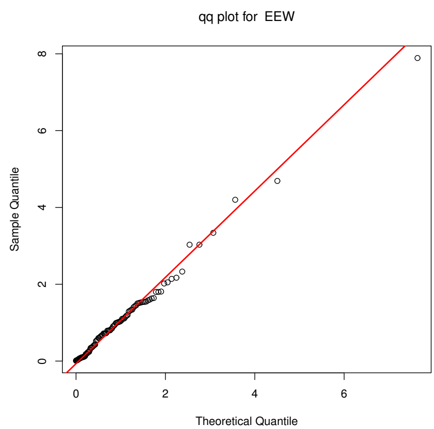

Figure 6 – QQ-plot of EEW model for Andrews and Herzberg data.

shape parameter. The PDF of the EEW distribution can take various shapes depending on it's parameter values. The hazard rate function of the EEW distribution can take i) increasing, ii) decreasing, iii) unimodal, iv) bathtub and v) decreasing-increasing-decreasing (DID) shaped, depending on it's parameter values. Therefore, it is quite flexible and can be used effectively in modeling survival data and reliability problems. Application of the EEW distribution to the real data set is given to show that the new distribution provides consistently better fit than the some of its sub-models. Now a natural question is how to choose the correct model, i.e. whether we should choose the full four-parameter model or one of the sub models. As we have mentioned before, we may use some testing of hypothesis namely whether the shape parameters take some specific values as indicated in Section 2, or we can use some of the information theoretic criteria to choose the correct model.

It should be mentioned that in this paper we have mainly discussed about the classical inference of the unknown parameters. It will be interesting to develop the Bayesian inference of the unknown parameters. One may think of taking independent gamma priors of α, γ and δ and normal priors on β. It is expected that the Bayes estimates or the associated highest posterior density credible intervals cannot be obtained in closed form. One may try to use importance sampling method or Markov Chain Monte Carlo methods to compute Bayes estimates and to construct highest posterior density credible intervals. Another important development can be to provide inference both classical and Bayesian when we have censored data. More work is needed along these directions.

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APPENDIX

A. HAZARD FUNCTION

To prove Theorems 3, 4 and 5 we need the following lemma.
LEMMA 1: Let U be a non-negative absolutely continuous random variable with the PDF, CDF and hazard function as f_U, F_U and h_U, respectively. For $\theta > 0$, let us define $V = U^\theta$. Then the shape of the hazard function of V will be the same as the shape of the function $g(u) = h_U(u)u^{1-\theta}$, for $u > 0$.
PROOF: If we denote the PDF, CDF and the hazard function of V as $f_V(\cdot)$, $F_V(\cdot)$ and $h_V(\cdot)$, respectively, then after some calculations, it can be that for $v > 0$

$$h_V(v) = \frac{1}{\theta} h_U(v^{1/\theta}) v^{(1/\theta)(1-\theta)}.$$

Since $v^{1/\theta}$ is an increasing function and it increases from zero to infinity as v increases from zero to infinity, for $\theta > 0$, the result follows. □

PROOF OF THEOREM 3: First observe that if $V \sim \text{EEW}(\alpha, \beta, \gamma, \lambda)$, then $U = V^\gamma \sim \text{EGE}(\alpha, \beta, \lambda)$. Now let us use Lemma 1 with $\theta = 1/\gamma$. Using part (b) of Theorem 1 and Lemma 1, the result immediately follows. □

PROOF OF THEOREM 4: Using part (a) of Theorem 4 and Lemma 1, the result immediately follows. □

PROOF OF THEOREM 5: From Theorem 3, it follows that if $U \sim \text{EGE}(\alpha, \beta, \lambda)$, then $h_U(u)$ is unimodal. If for $u > 0$, $g(u) = h_U(u)u^{1-1/\gamma}$, then

$$g'(u) = h'_U(u)u^{1-1/\gamma} + \frac{(1-1/\gamma)h_U(u)}{u^{1/\gamma}}.$$

Therefore, the sign of $g'(u)$ will be the same as the sign of

$$p(u) = h'_U(u)u + (1-1/\gamma)h_U(u).$$

Since $h_U(u) \to 0$ as $u \to \infty$, and $\gamma > 1$, then $p(u)$ changes sign only once. Hence, $h_V(v)$ is also unimodal. □

B. The first derivatives of the log-likelihood function

The first derivatives of the log-likelihood function with respect to α, β, γ and λ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}), \\
\frac{\partial \ell}{\partial \beta} &= -\frac{\sum_{i=1}^{n} \log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - (\frac{1}{\beta}-1)\sum_{i=1}^{n} \frac{\lambda y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}} \\
&\quad + (\alpha-1)\sum_{i=1}^{n} \frac{(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}(\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} + \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})})}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}, \\
\frac{\partial \ell}{\partial \gamma} &= \frac{n}{\gamma} + \sum_{i=1}^{n} \log(y_i) - (\frac{1}{\beta}-1)\sum_{i=1}^{n} \frac{\beta\lambda \log(y_i) y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}} \\
&\quad + (\alpha-1)\sum_{i=1}^{n} \frac{\lambda \log(y_i) y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}, \\
\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - (\frac{1}{\beta}-1)\sum_{i=1}^{n} \frac{\beta y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}} + (\alpha-1)\sum_{i=1}^{n} \frac{y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}.
\end{aligned}$$

The elements of the observed information matrix are

$$I_{\alpha\alpha} = \frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2},$$

$$I_{\beta\beta} = \frac{\partial^2 \ell}{\partial \beta^2} = \frac{2\sum_{i=1}^{n} \log(1-\beta\lambda y_i^{\gamma})}{\beta^2} + (\frac{1}{\beta}-1)\sum_{i=1}^{n} -\frac{\lambda^2 y_i^{2\gamma}}{(1-\beta\lambda y_i^{\gamma})^2}$$

$$-\frac{2\sum_{i=1}^{n} -\frac{\lambda y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}}}{\beta^2} + (\alpha-1)\sum_{i=1}^{n}(-\frac{(1-\beta\lambda y_i^{\gamma})^{\frac{2}{\beta}}(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})})^2}{(1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}})^2}$$

$$-\frac{(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}(\frac{2\log(1-\beta\lambda y_i^{\gamma})}{\beta^3} - \frac{\lambda^2 y_i^{2\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})^2} + \frac{2\lambda y_i^{\gamma}}{\beta^2(1-\beta\lambda y_i^{\gamma})})}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}$$

$$-\frac{(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})^2})^2}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}),$$

$$I_{\gamma\gamma} = -\frac{n}{\gamma^2} + \frac{\partial^2 \ell}{\partial \gamma^2} = (\frac{1}{\beta}-1)\sum_{i=1}^{n}(-\frac{\beta^2\lambda^2 \log(y_i)^2 y_i^{2\gamma}}{(1-\beta\lambda y_i^{\gamma})^2} - \frac{\beta\lambda \log(y_i)^2 y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}})$$

$$+(\alpha-1)\sum_{i=1}^{n}(-\frac{\lambda^2 \log(y_i)^2 y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{2}{\beta}-2}}{(1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}})^2}$$

$$-\frac{(\frac{1}{\beta}-1)\beta\lambda^2 \log(y_i)^2 y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-2}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}} + \frac{\lambda \log(y_i)^2 y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}),$$

$$I_{\lambda\lambda} = \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} + (\frac{1}{\beta}-1)\sum_{i=1}^{n} -\frac{\beta^2 y_i^{2\gamma}}{(1-\beta\lambda y_i^{\gamma})^2}$$

$$+(\alpha-1)\sum_{i=1}^{n}(-\frac{y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{2}{\beta}-2}}{(1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}})^2} - \frac{(\frac{1}{\beta}-1)\beta y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-2}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}),$$

$$I_{\alpha\beta} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} -\frac{(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})})}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}},$$

$$I_{\alpha\gamma} = \frac{\partial^2 \ell}{\partial \alpha \partial \gamma} = \sum_{i=1}^{n} \frac{\lambda \log(y_i) y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}},$$

$$I_{\alpha\lambda} = \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \frac{y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}},$$

$$I_{\beta\gamma} = \frac{\partial^2 \ell}{\partial \beta \partial \gamma} = -\frac{\sum_{i=1}^{n} -\frac{\beta\lambda \log(y_i) y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}}}{\beta^2} + (\frac{1}{\beta}-1)\sum_{i=1}^{n}(-\frac{\beta\lambda^2 \log(y_i) y_i^{2\gamma}}{(1-\beta\lambda y_i^{\gamma})^2} - \frac{\lambda \log(y_i) y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}})$$

$$+(\alpha-1)\sum_{i=1}^{n}(\frac{\lambda \log(y_i) y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{2}{\beta}-1}(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})})}{(1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}})^2}$$

$$+\frac{\lambda^2 \log(y_i) y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-2}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}$$

$$+\frac{\lambda \log(y_i) y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}(-\frac{\log(1-\beta\lambda y_i^{\gamma})}{\beta^2} - \frac{\lambda y_i^{\gamma}}{\beta(1-\beta\lambda y_i^{\gamma})})}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}),$$

$$I_{\beta\lambda} = \frac{\partial^2 \ell}{\cdots} = \frac{\sum_{i=1}^{n} \frac{\beta y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}}}{\cdots} + (\frac{1}{\cdots}-1)\sum^{n}(-\frac{\beta\lambda y_i^{2\gamma}}{\cdots} - \frac{y_i^{\gamma}}{\cdots})$$

$$I_{\gamma\lambda} = \frac{\partial^2 \ell}{\partial\gamma\partial\lambda} = (\frac{1}{\beta}-1)\sum_{i=1}^{n}(-\frac{\beta^2\lambda\log(y_i)y_i^{2\gamma}}{(1-\beta\lambda y_i^{\gamma})^2} - \frac{\beta\log(y_i)y_i^{\gamma}}{1-\beta\lambda y_i^{\gamma}})$$
$$+(\alpha-1)\sum_{i=1}^{n}(-\frac{\lambda\log(y_i)y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{2}{\beta}-2}}{(1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}})^2}$$
$$-\frac{(\frac{1}{\beta}-1)\beta\lambda\log(y_i)y_i^{2\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-2}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}} + \frac{\log(y_i)y_i^{\gamma}(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}-1}}{1-(1-\beta\lambda y_i^{\gamma})^{\frac{1}{\beta}}}).$$

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Summary

In this paper, we introduce a univariate four-parameter distribution. Several known distributions like exponentiated Weibull or extended generalized exponential distribution can be obtained as special case of this distribution. The new distribution is quite flexible and can be used quite effectively in analysing survival or reliability data. It can have a decreasing, increasing, decreasing-increasing-decreasing (DID), upside-down bathtub (unimodal) and bathtub-shaped failure rate function depending on its parameters. We provide a comprehensive account of the mathematical properties of the new distribution. In particular, we derive expressions for the moments, mean deviations, Rényi and Shannon entropy. We discuss maximum likelihood estimation of the unknown parameters of the new model for complete sample using the profile and modified likelihood functions. One empirical application of the new model to real data are presented for illustrative purposes.

Keywords: Probability weighted moments; Rényi and Shannon entropy; Extended generalized exponential distribution; Regular family of distributions