An Extension of the Generalized Exponential Distribution

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Abstract

The two-parameter generalized exponential distribution has been used recently quite extensively to analyze lifetime data. In this paper the two-parameter generalized exponential distribution has been embedded in a larger class of distributions obtained by introducing another shape parameter. Because of the additional shape parameter more flexibility has been introduced in the family. It is observed that the new family is positively skewed, and has increasing, decreasing, unimodal and bathtub shaped hazard functions. It can be observed as a proportional reversed hazard family of distributions. This new family of distributions is analytically quite tractable and it can be used quite effectively to analyze censored data also. Analysis of two data sets are performed and the results are quite satisfactory.

Key Words and Phrases: Generalized exponential distribution; hazard function; reversed hazard function; proportional reversed hazard model; regular family of distributions.

1 Introduction

The two-parameter generalized exponential (GE) distribution has been introduced by Gupta and Kundu [12] and it has the following probability density function (PDF);

$$f_{GE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}; \quad x>0. \tag{1}$$

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Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively. The two-parameter GE distribution has been used quite effectively for analyzing lifetime data. The readers are referred to the recent review article by Gupta and Kundu [13] for a current account on the generalized exponential distribution.

Although, GE distribution can be used quite effectively to analyze a data set which has monotone (increasing/ decreasing) hazard function (HF), but unfortunately it cannot be used if the HF is unimodal or bathtub shaped, similar to the Weibull or gamma distributions. The main aim of this paper is to extend the GE distribution to a three-parameter distribution, with an additional shape parameter. Many well known distributions can be obtained as special cases of the proposed distribution. This new family of distribution functions is always positively skewed, and the skewness decreases as both the shape parameters increase to infinity. Interestingly, the new three-parameter distribution has increasing, decreasing, uni-modal and bathtub shaped HFs. Therefore, it can be used quite effectively for analyzing different types of lifetime data.

The rest of the paper is organized as follows. In section 2 we introduce the model and discuss its properties in Section 3. Statistical inferences are carried out in section 4. Two data sets are analyzed in section 5. In Section 6, we present some simulation results, and finally we provide some generalization and conclude the paper in section 7.

2 Extended GE Family

The extended GE (EGE) family has the distribution function;

$$F(y;\alpha,\beta,\lambda) = \begin{cases} \left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha} & \text{if} \quad \beta \neq 0 \\ \left(1-e^{-\lambda y}\right)^{\alpha} & \text{if} \quad \beta = 0 \end{cases} \tag{2}$$

for $\alpha > 0$, $\lambda > 0$ and $-\infty < \beta < \infty$. The support of the EGE random variable Y in (2) is $(0, \infty)$ if $\beta \leq 0$, and $(0, 1/(\beta\lambda))$ if $\beta > 0$. The following values of the parameters α and β are of particular interest; (i) $\beta = 0$, EGE reduces to GE, (ii) $\beta = 0$, $\alpha = 1$, EGE reduces to exponential, (iii) $\beta = 1$, $\alpha = 1$, EGE reduces to uniform, (iv) $\alpha = 1$, EGE reduces to generalized Pareto, (v) $\alpha = 1$, $\beta < 0$, EGE reduces to Pareto. It may be mentioned that the generalized Pareto distribution has received considerable attention in the recent statistical literature because of its capability to model exceedances over a threshold, see for example Johnson, Kotz and Balakrishnan [16] or Davison and Smith [7]. From now on the three-parameter EGE distribution with parameters α, β and λ will be denoted by EGE(α, β, λ).

The PDF of EGE(α, β, λ) becomes

$$f(y; \alpha, \beta, \lambda) = \begin{cases} \alpha\lambda \left(1 - (1 - \beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha-1} (1 - \beta\lambda y)^{\frac{1}{\beta}-1} & \text{if} \quad \beta \neq 0 \\ \alpha\lambda \left(1 - e^{-\lambda y}\right)^{\alpha-1} e^{-\lambda y} & \text{if} \quad \beta = 0, \end{cases} \tag{3}$$

when $0 < y < \infty$ and $0 < y < 1/(\beta\lambda)$, for $\beta \leq 0$ and $\beta > 0$, respectively. Moreover, its quantile function is

$$Q(u; \alpha, \beta, \lambda) = \begin{cases} \frac{1}{\beta\lambda} \left[1 - (1 - u^{1/\alpha})^{\beta}\right] & \text{if} \quad \beta \neq 0 \\ -\frac{1}{\lambda} \ln\left(1 - u^{1/\alpha}\right) & \text{if} \quad \beta = 0. \end{cases} \tag{4}$$

Clearly, as $\beta \to 0$, the quantile function of the EGE distribution tends to the quantile function of the GE distribution. For $\beta > 0$, the quantile function of the EGE distribution coincides with the quantile function of a transformed beta distribution. For $\beta > 0$, the support is on a finite interval. The shape of the PDF is (i) unimodal if $\alpha > 1$ and $0 < \beta < 1$, (ii) an increasing function if $\alpha > 1$ and $\beta > 1$, (iii) an decreasing function if $0 < \alpha < 1$ and $0 < \beta < 1$, (iv) a bathtub shaped if $0 < \alpha < 1$ and $\beta > 1$. (v) If $\alpha = 1$, for $0 < \beta < 1$ it is a decreasing function and for $\beta > 1$, it is an increasing function.

For $\beta < 0$, it has the support on the whole real line and the shape of the PDF is unimodal if $\alpha > 0$. For $\beta = 0$, it is well known that the PDF is a decreasing function if $0 < \alpha \leq 1$

and it is unimodal if $\alpha > 1$. The PDFs of the EGE for different ranges of α and β when λ =1 are plotted in Figure 1. It is clear that the GE family has been embedded in a larger

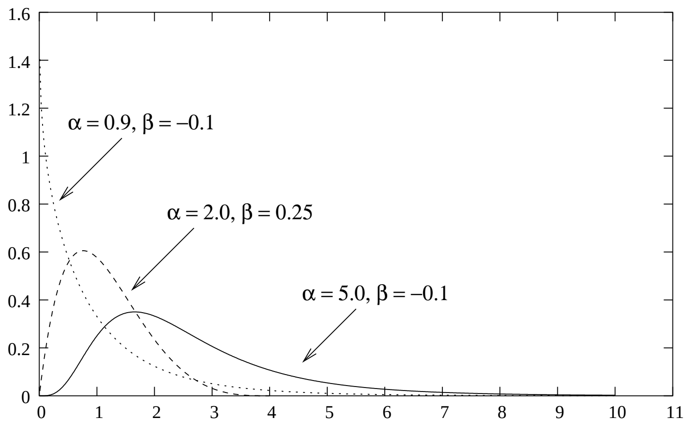

Figure 1: The PDFs of the extended GE for different values of α and β, when $\lambda = 1$.

family, with an additional shape parameter β. Due to this additional shape parameter, more flexibility can be incorporated in the family, which will be useful for data analysis purposes. The EGE model can be seen as a proportional reversed hazard rate model (PRHRM), see for example the recent review article by Gupta and Gupta [11] in this connection. Therefore, several properties of the general PRHRM model can be easily translated for the EGE model. In the next section we discuss different structural properties of the EGE model.

3 Properties

The HF of EGE takes the form

$$h(y;\alpha,\beta,\lambda) = \begin{cases} \dfrac{\alpha\lambda\left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha-1}(1-\beta\lambda y)^{\frac{1}{\beta}-1}}{1-\left(1-(1-\beta\lambda y)^{\frac{1}{\beta}}\right)^{\alpha}} & \text{if} \quad \beta \neq 0 \\ \dfrac{\alpha\lambda\left(1-e^{-\lambda y}\right)^{\alpha-1}e^{-\lambda y}}{1-\left(1-e^{-\lambda y}\right)^{\alpha}} & \text{if} \quad \beta = 0. \end{cases} \qquad (5)$$

It is observed that the HF can take all four different shapes namely (i) increasing, (ii) decreasing, (iii) unimodal and (iv) bathtub. The Figures 2 and 3 provide the HFs of the

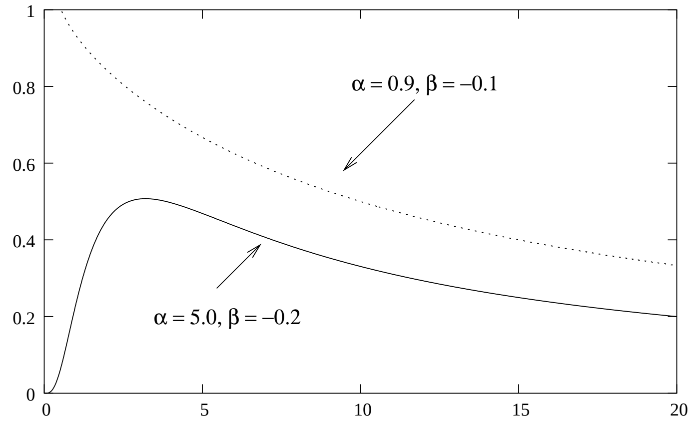

Figure 2: The HFs of the EGE distribution for different values of α and $\beta < 0$.

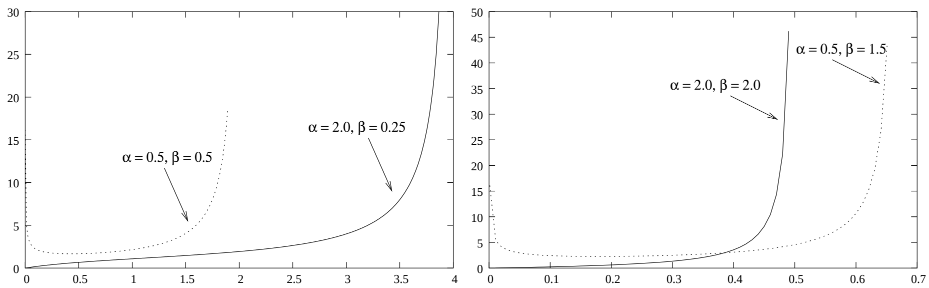

Figure 3: The HFs of the EGE distribution for different values of α and $\beta > 0$.

EGE distributions for different values of α and β, when $\lambda = 1$. We have the following results regarding the shapes of the HFs of the EGE distributions. The proofs are provided in the appendix.

Theorem 1: The HF of the EGE distribution is (a) unimodal if $\alpha > 1$ and $\beta < 0$, (b) a decreasing function if $\alpha < 1$ and $\beta < 0$.

Theorem 2: The HF of the EGE distribution is (a) an increasing function if $\alpha > 1$ and $\beta > 1$, (b) a bathtub shaped if $\alpha < 1$ and $\beta < 1$.

Suppose Y follows EGE$(\alpha, \beta, 1)$, then it can be easily shown that for $\beta \neq 0$ and $k\beta + 1 > 0$

$$E\left[(1-\beta Y)^k\right] = \frac{\Gamma(\alpha+1)\Gamma(k\beta+1)}{\Gamma(\alpha+k\beta+1)}. \tag{6}$$

From (6) it easily follows that

$$E(Y) = \frac{1}{\beta}\left[1 - \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\right] \tag{7}$$

and

$$V(Y) = \frac{1}{\beta^2}\left[\frac{\Gamma(\alpha+1)\Gamma(2\beta+1)}{\Gamma(\alpha+2\beta+1)} - \frac{\Gamma^2(\alpha+1)\Gamma^2(\beta+1)}{\Gamma^2(\alpha+\beta+1)}\right]. \tag{8}$$

Note that when $\alpha = 1$, then $E(Y) = \frac{1}{\beta+1}$, $V(Y) = \frac{1}{(1+\beta)^2(1+2\beta)}$ and they coinside with the corresponding moments of the generalized Pareto distribution, see Johnson, Kotz and Balakrishnan [16]. For $\beta = 0$, the results are available in Gupta and Kundu [12].

The median of the EGE distribution can be easily obtained by substituting $u = 1/2$ in the quantile function 4. For $\alpha < 1$ and $\beta < 1$, the mode of the EGE distribution can be easily seen to be at 0, otherwise it cannot be obtained in explicit form. It can be obtained as root of a non-linear equation.

4 Parametric Inference

In this section we mainly consider the parametric inference of the unknown parameters of the EGE(α, β, λ). It is assumed that we have a sample of size n, from EGE(α, β, λ), say $y_1, \ldots, y_n$. Based on the random sample $y_1, \ldots, y_n$, the maximum likelihood estimators (MLEs) of α, β, λ can be obtained by maximizing the log-likelihood function

$$l(\alpha,\beta,\lambda|data) = n\ln\alpha + n\ln\lambda + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-(1-\beta\lambda y_i)^{\frac{1}{\beta}}\right) + \left(\frac{1}{\beta}-1\right)\sum_{i=1}^{n}\ln(1-\beta\lambda y_i). \tag{9}$$

For given β and λ, the MLE of α, say $\widehat{\alpha}(\beta,\lambda)$ can be obtained as

$$\widehat{\alpha}(\beta,\lambda) = -\frac{n}{\sum_{i=1}^{n}\ln\left(1-(1-\beta\lambda y_i)^{\frac{1}{\beta}}\right)}. \tag{10}$$

The MLEs of β and λ can be obtained by maximizing the profile log-likelihood function $l(\widehat{\alpha}(\beta,\lambda),\beta,\lambda|data)$ with respect to β and λ. Now we will discuss about the asymptotic properties of the MLEs.

REGULAR CASE: Observe that when $\alpha > 0$ and $\beta < 0$, the situation is exactly similar as the generalized Weibull case discussed by Mudholkar, Srivastava and Kollia [21] with $\alpha > 0$ and $\lambda < 0$ according to their notations. It can be verified similarly as in Mudholkar, Srivastava and Kollia [21] that when $\alpha > 0$ and $\beta < 0$, the PDF of EGE(α,β,λ) satisfies all the regularity properties of the parametric family. Therefore, in this case the standard asymptotic normality result holds and it can be stated as

$$\sqrt{n}(\widehat{\phi}-\phi) \longrightarrow N_3\left(0, I^{-1}(\phi)\right), \tag{11}$$

here $\phi = (\alpha,\beta,\lambda)$ and $\widehat{\phi} = (\widehat{\alpha},\widehat{\beta},\widehat{\lambda})$ denotes the MLEs of ϕ, N_3 denotes the trivariate normal distribution and $I(\phi)$ is the expected Fisher information matrix.

NON-REGULAR CASE: Observe that when $\beta > 0$, the support of EGE(α,β,λ) depends on the unknown parameters. For the purpose of statistical inference, when $\beta > 0$, we propose the following re-parameterization of α,β,λ as (α,β,θ), where $\theta = (\beta\lambda)^{-1}$. Therefore, (2) can be written as

$$f(y;\alpha,\beta,\theta) = \frac{\alpha}{\beta\theta}\left(1-\left(1-\frac{y}{\theta}\right)^{\frac{1}{\beta}}\right)^{\alpha-1}\left(1-\frac{y}{\theta}\right)^{\frac{1}{\beta}-1} \tag{12}$$

for $0 < y < \theta$ and 0 otherwise. The corresponding distribution function and the quantile function become

$$F(y;\alpha,\beta,\theta) = \left(1-\left(1-\frac{y}{\theta}\right)^{\frac{1}{\beta}}\right)^{\alpha} \tag{13}$$

and

$$Q(u;\alpha,\beta,\theta) = \theta\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\beta}\right) \tag{14}$$

respectively. In this new parameterization θ is both the scale as well as thresh hold parameter and α and β are both shape parameters.

First, let us observe that based on a random sample $y_1, \cdots, y_n$ from (12), the maximum likelihood estimators can be obtained by maximizing the log-likelihood function

$$l(\alpha,\beta,\theta) = n\ln\alpha - n\ln\beta - n\ln\theta + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-\left(1-\frac{y_{(i)}}{\theta}\right)^{\frac{1}{\beta}}\right) + \left(\frac{1}{\beta}-1\right)\sum_{i=1}^{n}\ln\left(1-\frac{y_{(i)}}{\theta}\right). \tag{15}$$

Here $y_{(1)} < y_{(2)} < \ldots < y_{(n)}$ denote the ordered y_i's. It is immediate from (15) that for fixed $0 < \alpha < 1$, $0 < \beta < 1$, as $\theta \downarrow y_{(n)}$, $l(\alpha,\beta,\theta) \longrightarrow \infty$. Therefore, in this case the MLEs do not exist, and we look for alternative estimators as in Mudholkar, Srivastava and Kollia [21].

To estimate the unknown parameters, the most natural way (see Smith [23]) is to first estimate the threshold parameter θ by its consistent estimator $\widetilde{\theta} = y_{(n)}$. The modified log-likelihood function based on the remaining $(n-1)$ observations after ignoring the largest observation and replacing θ by $\widetilde{\theta} = y_{(n)}$ is

$$\begin{aligned} l(\alpha,\beta,\widetilde{\theta}) \;=\; & (n-1)\ln\alpha - (n-1)\ln y_{(n)} - (n-1)\ln\beta + (\alpha-1)\sum_{i=1}^{n-1}\ln\left(1-\left(1-\frac{y_{(i)}}{y_{(n)}}\right)^{\frac{1}{\beta}}\right) \\ & + \left(\frac{1}{\beta}-1\right)\sum_{i=1}^{n-1}\ln\left(1-\frac{y_{(i)}}{y_{(n)}}\right). \end{aligned} \tag{16}$$

For fixed β, observe that the modified MLE of α can be obtained as

$$\widehat{\alpha}(\beta) = -\frac{n-1}{\sum_{i=1}^{n-1}\ln(1-(1-\frac{y_{(i)}}{y_{(n)}})^{\frac{1}{\beta}})}. \tag{17}$$

Therefore, in this case the modified MLE of β can be obtained by solving a one dimensional optimization problem from the modified profile log-likelihood function of β.

For the purpose of statistical inference, an understanding of the joint distributions of $\widetilde{\theta}$ and the modified likelihood estimators of $\widetilde{\alpha}$ and $\widetilde{\beta}$ obtained from (16) is necessary. It is convenient to describe the joint distribution in terms of the asymptotic marginal of $\widetilde{\theta}$ and the asymptotic conditional distribution of $(\widetilde{\alpha},\widetilde{\beta})$ given $\widetilde{\theta}$. We have the following results, whose proofs are provided in the appendix.

Theorem 3:

(a) The marginal distribution of $\widetilde{\theta} = y_{(n)}$ is given by

$$P[\widetilde{\theta} \leq t] = \left(1 - \left(1 - \frac{t}{\theta}\right)^{\frac{1}{\beta}}\right)^{n\alpha}.$$

(b) Asymptotically as $n \to \infty$,

$$n^{\beta}\left(\frac{Y_{(n)}}{\theta} - 1\right) \to -V^{\beta}, \quad \text{where} \quad V \sim exp(\alpha).$$

Here '$exp(\alpha)$' denotes the exponential distribution with mean $1/\alpha$.

Theorem 4: Given $y_{(n)}$, the conditional asymptotic distribution of $(\tilde{\alpha}, \tilde{\beta})$ obtained by maximizing (16) is a bivariate normal distribution with mean (α, β) and $I^{-1}_{\alpha,\beta}$, the inverse of the Fisher information matrix, as the covariance matrix.

Theorem 5: Asymptotically as $n \to \infty$, the distribution of $(\tilde{\alpha}, \tilde{\beta})$ is (a) bivariate normal, if $\beta > \frac{1}{2}$, (b) bivariate Weibull if $\beta < \frac{1}{2}$, (c) a mixture of normal and Weibull if $\beta = \frac{1}{2}$.

Comments: It may be noted that although for $\beta > 0$, uniform and generalized Pareto distributions can be obtained as special cases of the EGE distribution, the MLEs of the unknown parameters for EGE distribution does not exist. Therefore, the asymptotic properties of the modified maximum likelihood estimators of the EGE distribution are not comparable with the corresponding asymptotic properties of the MLEs of the uniform or generalized Pareto distributions. Moreover, for non-regular case, the results of Smith [23] is not directly applicable for generalized Pareto distribution see Smith [23] (page 88-89), and it has been mentioned that special procedures are needed to develop the asymptotic properties of the MLEs. In case of EGE, the situation is very similar with the corresponding situation of the generalized Weibull distribution of Mudholkar, Srivstava and Kollia [21], and the corresponding asymptotic results are also quite comparable.

5 Data Analysis

Data Set 1: This data set consists of survival times of guinea pigs injected with tubercle bacilli and was originally studied by Bjerkedal [5]. Guinea pigs are known to have high susceptibility to human tuberculosis, which is one of the reasons to choose guinea pigs for such a study. Here we consider only the study where each animals in a single cage are under the same regimen. The regimen number of the common log of the number of bacillary units in 0.5 ml. of challenge solution, *i.e.* regimen 6.6 corresponds to 3.98 $\times 10^6$ bacillary units per 0.5 ml. ($\log(3.98 \times 10^6) = 6.6$). We considered the data for regimen 6.6, and there were 72 observations. The data set is available in Gupta *et al.* [10].

From the preliminary data analysis, the mean, standard deviation and the coefficient of skewness are calculated as 99.82, 80.55 and 1.80 respectively. The skewness measure indicates that the data are positively skewed. The scaled TTT transform of Aarset [1] indicates that the empirical HF is unimodal. Similar claims were made by Gupta *et al.* [10] and Kundu *et al.* [19] while analyzing this data set. Gupta *et al.* [10] used the log-normal distribution and Kundu *et al.* [19] suggested to use the Birnbaum-Saunders distribution for analyzing this data set. It may be mentioned that both log-normal and Birnbaum-Saunders distribution are positively skewed distributions and both have unimodal HFs.

In this case we want to use EGE distribution to analyze this data set. Since EGE has three parameters, and the MLEs cannot be obtained in explicit forms, we need to use some optimization method to compute the MLEs. For any optimization method some initial guesses are required to start the iterative process. Due to the relation (10), we can reduce the three dimensional optimization process to a two dimensional optimization process. It is quite natural to start with the two-dimensional contour plot of (β, λ) to have an idea about the initial guesses of (β, λ). We have used a slightly different approach.

First we have fitted the two-parameter GE distribution by using the standard technique as suggested by Gupta and Kundu [12]. The MLEs of α and λ become 2.4748 and 0.0169. Now to estimate the three-parameter EGE distribution, we take the initial value of λ as λ^0 = 0.0169. It is observed that the profile likelihood function of β, namely $l(\alpha^0, \beta, \lambda^0|data)$ as defined in (9), where $\alpha^0 = \widehat{\alpha}(\beta, \lambda^0)$, as defined in (10) is an unimodal function. and we obtain the initial guess of β as β^0 = -0.05. We use the downhill simplex method as provided in Press *et al.* [22]and obtained the MLEs of α, λ and β as $\widehat{\alpha} = 5.3121$, $\widehat{\lambda} = 0.0382$, $\widehat{\beta}$ = -0.2865. The corresponding 95% confidence intervals are (2.4307, 7.5391), (0.0187, 0.0558) and (-0.4075, -0.0239) respectively.

The following Table 1 provides the Kolmogorov-Smirnov (KS) distances between the empirical distribution function and six different fitted distribution functions namely (i) EGE, (ii) log-normal (LN), (iii) Birnbaum-Saunders (BS), (iv) generalized exponential (GE), (v) exponentiated Weibull (EWE) and (vi) generalized Weibull (GWE), the associated p values and the corresponding log-likelihood values. It may be mentioned that LN, BS and GE have two parameters each, where as EWE and GWE have three parameters each. Moreover, EWE and GWE also can have all four possible shapes of the HFs similarly as the EGE model.

Based on the KS distances, and the log-likelihood values among the two-parameter distributions log-normal provides the best fit, and GE provides the worst fit. Among the three-parameter distributions EGE provides the best fit in this case. Between EGE and LN distributions based on the Akaike Information Criterion (AIC), and also comparing the p values of the KS statistics, we can say that EGE provides a better fit in this case. In Figure 4 we provide the empirical survival function and the best fitted (EGE) and worst fitted (GE) survival functions. It is clear that EGE provides an excellent fit to the Guinea Pigs data.

Data Set 2: Now we present a data analysis of the strength data originally reported by Badar and Priest [2]. The data set represents the strength measured in GPA for single

Distribution	K-S Distance	p	log-likelihood
EGE	0.0912	0.5987	-390.0267
LN	0.0956	0.5263	-395.1656
BS	0.1044	0.4125	-397.2761
GE	0.1157	0.3856	-398.8674
EWE	0.0924	0.5823	-391.1176
GWE	0.0936	0.5810	-393.3419

Table 1: Kolmogorov-Smirnov distances, associated p values and the log-likelihood values for EGE, GE, log-normal, Birnbaum-Saunders (BS), exponentiated Weibull and generalized Weibull distribution functions.

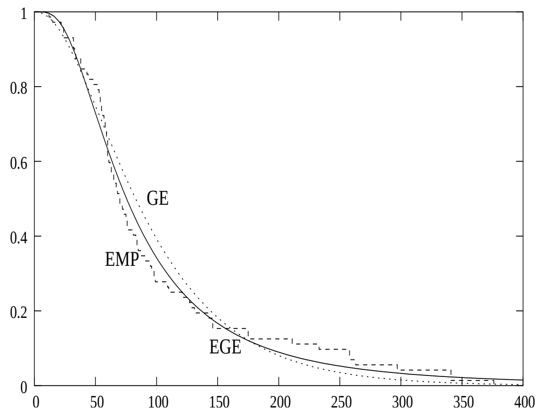

Figure 4: The empirical survival function, fitted survival functions for GE and EGE distributions

carbon fibers and impregnated at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. We have taken the data set of single fibers of 20 mm, with sample size $n = 69$. Surles and Padgett [24] and Kundu and Gupta [17] already analyzed this data set by subtracting 0.75 for all the data points, using generalized Rayleigh and Weibull distributions respectively. We want to use the proposed EGE model to analyze this data set. The histogram of the data indicates that the data are slightly skewed and the scaled TTT transform indicates that the empirical HF is an increasing function.

In this case also first we fit a two-parameter GE distribution and obtain the MLEs of α and λ as 16.6816 and 1.9743. The high value of α indicates that the data are nearly symmetric. Now to fit the EGE model, as before we obtain $\alpha^0 = 16.6816$, $\lambda^0 = 1.9743$ and $\beta^0 = 0.1$ as the initial estimates of α, λ and β respectively. Finally we obtain the MLEs of the unknown parameters as 6.1748, 0.9169 and 0.3493 respectively. The corresponding 95% confidence intervals are (4.1959, 8.1537), (0.5012, 1.3326) and (0.1764, 0.5222) respectively.

Now we provide the Kolmogorov-Smirnov (KS) distances between the empirical distribution function and six different fitted distribution functions namely (i) EGE, (ii) Weibull (WE), (iii) Gamma (GA) and (iv) generalized exponential (GE), (v) generalized Weibull (GWE) and (vi) exponentiated Weibull (EWE), the associated p values and the corresponding log-likelihood values in the following Table 2. It is immediate that based on the KS distances, and the log-likelihood values among the two-parameter distributions Weibull provides the best fit, and GE provides the worst fit. Among the three-parameter distribution GWE performs the best, but the performance of the proposed EGE distribution is very good. Using Akaike Information Criterion (AIC) it can be easily seen that the performance of EGE is better than any of the two-parameter distribution. In Figure 5 we provide the empirical survival function and the best fitted (EGE) and worst fitted (GE) survival functions. It is clear that although, EGE does not perform the best in this case, but it provides an excellent fit to the strength data.

6 Monte Carlo Simulations

In this section we perform some simulation studies, just to verify how the MLEs work for different sample sizes and for different parameter values for the proposed EGE model. All the calculations have been performed using Intel dual core processor and all the program codes are in FORTRAN-77. We have used RAN2 uniform random number generator of Press

Distribution	K-S Distance	p	log-likelihood
EGE	0.0391	0.9541	-43.1765
WE	0.0461	0.8985	-48.8703
GA	0.0501	0.8105	-51.3215
GE	0.0546	0.7906	-52.2987
GWE	0.0381	0.9582	-43.1423
EWE	0.0401	0.9499	-43.2011

Table 2: Kolmogorov-Smirnov distances, associated p values and the log-likelihood values for EGE, Weibull, Gamma, GE, GWE and EWE distribution functions.

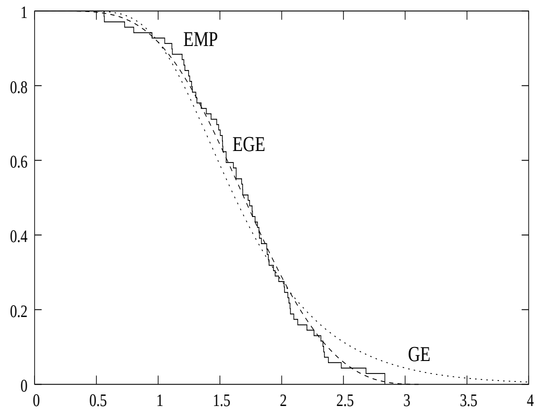

Figure 5: The empirical survival function, fitted survival functions for GE and EGE distributions

et al. [22].

We have used the following parameter sets: Model 1: α = 2.0, λ = 1.0, β = -1.0 and Model 2: α = 2.0, λ = 1.0, β = 1.0, and different sample sizes namely: n = 25, 50, 75 and 100. In all the cases we have calculated the MLEs using the downhill simples method, see Press *et al.* [22], and use the true values as the initial guesses. We replicated the process 1000 times and report the average estimates and the associated mean squared errors (MSEs). The results are presented in Tables 3 and 4.

From the results presented in Tables 3 and 4 the following points are quite clear. It is

Sample Size	α	λ	β
25	1.8162 (1.1125)	0.9471 (0.9561)	-0.7470 (0.1141)
50	2.1385 (0.8397)	1.1707 (0.6772)	-0.9210 (0.0653)
75	2.0957 (0.5257)	1.1117 (0.4579)	-0.9462 (0.0526)
100	2.0110 (0.5081)	1.0114 (0.3996)	-0.9658 (0.0443)

Table 3: Average estimates and the associated MSEs (presented within brackets below) of the MLEs for different sample sizes, when α = 2.0, λ = 1.0, β = -1.0.

quite clear that the MLEs are working quite well. As the sample size increases the average biases and the mean squared errors decrease, it verifies the consistency properties of the MLEs. For all practical purposes MLEs can be used quite effectively for estimating the unknown parameters of the proposed EGE model.

7 Conclusions

In this paper we have proposed a new three-parameter EGE model by embedding the GE model in a larger class of distributions. The proposed EGE model has two shape parameters and one scale parameter, similarly as the exponentiated Weibull model of Mudholkar *et al* [20] or the generalized Weibull model of Mudholkar *et al* [21]. The PDF of EGE model also can take various shapes depending on the shape parameters. Moreover, similarly as the exponentiated Weibull model or generalized Weibull model the HF also can all the four shapes namely (i) increasing, (ii) decreasing, (iii) unimodal or (iv) bathtub shaped, depending on the shape parameters. Therefore, it can be used quite effectively in analyzing various lifetime data. We have proposed to use the maximum likelihood method to estimate

Sample Size	α	λ	β
25	4.7424 (2.3564)	0.8048 (0.5361)	1.3232 (0.3165)
50	4.3153 (1.7015)	0.9069 (0.3899)	1.1633 (0.2401)
75	4.1007 (1.3231)	0.9564 (0.3217)	1.0134 (0.1901)
100	4.0001 (1.1561)	0.9876 (0.2511)	1.0011 (0.1521)

Table 4: Average estimates and the associated MSEs (presented within brackets below) of the MLEs for different sample sizes, when $\alpha = 4.0$, $\lambda = 1.0$, $\beta = 1.0$.

the unknown parameters and the performance of the MLEs are quite satisfactory. We have analyzed two data sets and the proposed EGE model provide very good fit to both the data sets.

Still there exist several open problems mainly involving the numerical issues. It may be mentioned that extensive work has been done related to the numerical issues of the different estimators of the unknown parameters of the generalized Pareto distribution, see for example Castillo and Hadi [6], Grimshaw [8], Hosking and Wallis [14], Hosking, Wallis and Wood [15], Bermudez and Kotz [3, 4] and see the references cited therein. It has been observed that the MLEs may not work properly, even when they exist. Due to these reasons, several algorithms have been proposed to compute the MLEs. Moreover, several alternative estimators have been proposed and their performances have been studied quite extensively using Monte Carlo simulations. Along the same line efficient algorithm to develop the MLEs of the EGE model can also be developed. Moreover, different other estimators may be thought of, and their properties will be worth investigating. Recently the authors, see Kundu and Gupta [18], introduced the bivariate generalized exponential distribution and discuss several of its properties. Along the same line bivariate EGE model also can be developed. More work is needed in these directions.

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Appendix A

Proof of Theorem 1: Without loss of generality it is assumed that $\lambda = 1$ and for this proof only let us assume $\delta = -\frac{1}{\beta} > 0$. In this case the support is the whole positive real line. The density function of the EGE with the parameter α, δ and $\lambda = 1$ can be written as

$$f(y) = \alpha \left(1 - \left(1 + \frac{y}{\delta}\right)^{-\delta}\right)^{\alpha-1} \left(1 + \frac{y}{\delta}\right)^{-\delta-1}, \quad 0 < y < \infty. \tag{18}$$

If we take $Z(y) = 1 + \frac{y}{\delta}$, then

$$f(y) = \frac{\alpha \left(Z(y)^{\delta} - 1\right)^{\alpha-1}}{Z(y)^{\alpha\delta+1}}, \qquad \eta(y) = -\frac{f'(y)}{f(y)} = \frac{Z(y)^{\delta}(1+\delta) - (\alpha\delta + 1)}{\delta Z(y)(Z(y)^{\delta} - 1)},$$

and

$$\eta'(y) = \frac{-Z(y)^{2\delta}(1+\delta) + Z(y)^{\delta}(1+\delta)(2 + \alpha\delta - \delta) - (\alpha\delta + 1)}{\left(\delta Z(y)(Z(y)^{\delta} - 1)\right)^2}.$$

Clearly, as y varies from 0 to ∞, $Z(y)$ varies from 1 to ∞. Since the denominator of $\eta'(y)$ does not change sign, it is enough to consider the sign of the numerator only. If we denote the numerator of $\eta'(y)$ as $u_1(y)$, then observe that $u_1(0) = (\alpha - 1)\delta^2$ and $u_1(y) < 0$ for large y. Since $u_1(y)$ can change sign at most twice, therefore for $\alpha > 1$, it will change sign only once. It implies that the HF is either unimodal or a decreasing function, see Glaser [9]. Since for $\alpha > 1$, $h(0) = 0$, it implies the HF has to be unimodal. For $\alpha < 1$, $u_1(0) < 0$, therefore, in this case there are two possibilities (i) it changes sign twice in $(0, \infty)$ or (ii) it does not change sign in $(0,\infty)$. From the quadratic equation $-x^2 + x(2+\alpha\delta-\delta) - \dfrac{\alpha\delta+1}{\delta+1}$, since $\dfrac{2+\alpha\delta-\delta}{2} < 1$, it follows that $u_1(y)$ either changes sign twice below 0 or does not change sign at all. Therefore, $u_1(y) < 0$ for all $y \geq 0$. It implies, the HF is a decreasing function, see Glaser [9]

Proof of Theorem 2: Observe that the shape of the HF of $f(y;\alpha,\beta,1)$ will be same as the shape of the HF of $g(y;\alpha,\delta)$, where $\delta = \dfrac{1}{\beta}$ and

$$g(y;\alpha,\delta) = \begin{cases} \alpha\left(1-(1-y)^{\delta}\right)^{\alpha-1}(1-y)^{\delta-1} & \text{if} \quad 0<y<1 \\ 0 & \text{otherwise,} \end{cases}$$

Let us write

$$\eta(y) = -\frac{g'(y)}{g(y)} = -\frac{(\alpha\delta-1)(1-y)^{\delta}-\delta+1}{(1-(1-y)^{\delta})(1-y)},$$

and the numerator of $\eta'(y)$ as $v_1(y)$, where

$$v_1(y) = (\alpha\delta-1)(1-y)^{2\delta} + (1-y)^{\delta}(\alpha\delta^2-\alpha\delta-\delta-\delta^2+2) + (\delta-1).$$

We also have $v_1(0) = (\alpha-1)\delta^2$ and $v_1(1) = \delta-1$. Note that $v_1(y)$ can change sign at most two times. Clearly, for (a) $\alpha > 1$ and $\delta < 1$ $(\beta > 1)$, it will change sign once in $(0,1)$ from positive to negative. Therefore, the HF will either be bathtub shaped or an increasing function, see Glaser [9]. If $\alpha > 1$, since $h(0) = 0$, the hazard function is an increasing function and for $\alpha < 1$, since $h(0) = \infty$, the HF will be a bathtub shaped.

Proof of Theorem 3 Part (a) is straight forward and therefore it is omitted. For part (b) let $U_{(n)}$ denote the largest order statistic of a sample of size n from uniform(0,1) distribution. Then from the quantile function $Q(\cdot)$ as defined in (14), we obtain

$$Y_{(n)} \stackrel{d}{=} Q\left(U_{(n)}\right) = \theta\left(1-\left(1-U_{(n)}^{\frac{1}{\alpha}}\right)^{\beta}\right).$$

Since,

$$n^{\beta}\left[\frac{Y_{(n)}}{\theta}-1\right] = -\left[n\left(1-U_{(n)}^{\frac{1}{\alpha}}\right)\right]^{\beta},$$

therefore,

$$P\left[n\left(1-U_{(n)}^{\frac{1}{\alpha}}\right)\le x\right]=P\left[U_{(n)}\ge\left(1-\frac{x}{n}\right)^{\alpha}\right]=1-\left(1-\frac{x}{n}\right)^{\alpha}\to 1-e^{-\alpha x}.$$

∎

Proof of Theorem 4: Conditional on $\tilde{\theta}$, the modified log-likelihood function (16) satisfies all the regularity assumptions required for the asymptotic normality of the maximum likelihood estimators to follows, and therefore the result follows. See Mudholkar *et al.* [21] and Smith [23] for similar developments.

Proof of Theorem 5: In this proof only, we denote the true values of α, β and θ as α_0, β_0 and θ_0. Let $\gamma_0=(\alpha_0,\beta_0)'$, $\tilde{\gamma}=(\tilde{\alpha},\tilde{\beta})'$ and $\gamma=(\alpha,\beta)'$. Also let $L=L(\alpha,\beta,\tilde{\theta})$ be the modified log-likelihood function (16). Let $G(\gamma,\tilde{\theta})=\dfrac{\partial L}{\partial\gamma}$ and $H(\gamma,\tilde{\theta})=\dfrac{\partial L}{\partial\gamma^2}$ be the derivative vector and the Hessian matrix respectively, of L. Then clearly we have,

$$G(\tilde{\gamma},\tilde{\theta})=0. \tag{19}$$

Expanding (19) around γ_0, we get

$$0=G(\gamma_0,\tilde{\theta})+H(\gamma_0,\tilde{\theta})(\tilde{\gamma}-\gamma_0)+o_p(1/\sqrt{n}). \tag{20}$$

Taking n large enough, so that $\tilde{\theta}$ can be replaced by θ_0 in $H(\gamma_0,\tilde{\theta})$, we have

$$0=G(\gamma_0,\tilde{\theta})+H(\gamma_0,\theta_0)(\tilde{\gamma}-\gamma_0)+o_p(1/\sqrt{n}). \tag{21}$$

Consider three different cases (i) $\beta>\frac{1}{2}$, (ii) $\beta<\frac{1}{2}$ (iii) $\beta=\frac{1}{2}$.
Case (i)$\beta>\frac{1}{2}$: From (21) we can write

$$\sqrt{n}(\tilde{\gamma}-\gamma_0)=-\sqrt{n}H^{-1}(\gamma_0,\theta_0)G(\gamma_0,\tilde{\theta})+o_p(1). \tag{22}$$

Now expanding $G(\gamma_0,\tilde{\theta})$ around θ_0, we have

$$\sqrt{n}(\tilde{\gamma}-\gamma_0)=-\sqrt{n}H^{-1}(\gamma_0,\theta_0)G(\gamma_0,\theta_0)-\sqrt{n}(\tilde{\theta}-\theta_0)H^{-1}(\gamma_0,\theta_0)\frac{\partial}{\partial\theta}G(\gamma_0,\theta_0)+o_p(1). \tag{23}$$

since $\beta > \frac{1}{2}$, then in view of Theorem 3, part (b), the second term of (23) is negligible compared to the first term and the asymptotic distribution of $\tilde{\gamma}$ is a bivariate normal.
Case (ii)$\beta < \frac{1}{2}$. In this case we will obtain

$$n^{\beta}(\tilde{\gamma} - \gamma_0) = -n^{\beta} H^{-1}(\gamma_0, \theta_0) G(\gamma_0, \theta_0) - n^{\beta}(\tilde{\theta} - \theta_0) H^{-1}(\gamma_0, \theta_0) \frac{\partial}{\partial \theta} G(\gamma_0, \theta_0) + o_p(1). \quad (24)$$

Since $\beta < \frac{1}{2}$, the first term is negligible compared to the the second term. Using Theorem 3, part (b) it follows that the second term converges to a bivariate Weibull distribution.
Case (ii)$\beta = \frac{1}{2}$. From (23) it is clear that for $\beta = \frac{1}{2}$, it is mixture of normal and Weibull.

References

[1] Aarset, M.V. (1987), "How to identify a bathtub hazard rate", *IEEE Transactions on Reliability*, 36, 106-108.

[2] Badar, M.G. and Priest , A.M. (1982), "Statistical aspects of fiber and bundle strength in hybrid composites", *Progress in Science and Engineering Composites*, Eds. T. Hayashi, K. Kawata and S. Umekawa, Tokyo, 1129 - 1136, ICCM-IV.

[3] Bermudez, P.Z. and Kotz, S. (2010), "Parameter estimation of the generalized Pareto distribution-Part I", *Journal of Statistical Planning and Inference*, vol. 109, 1353 - 1373.

[4] Bermudez, P.Z. and Kotz, S. (2010), "Parameter estimation of the generalized Pareto distribution-Part I", *Journal of Statistical Planning and Inference*, vol. 109, 1374 - 1388.

[5] Bjerkedal, T. (1960), "Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli", *American Journal Hygienes*, 72, 130 - 148.

[6] Castillo, E. and Hadi, A. S. (1997), "Fitting the generalized Pareto distribution to data", *Journal of the American Statistical Association*, vol. 92, 1609 - 1620.

[7] Davison, A.C. and Smith, R.L. (1990), "Models for exceedances over high thereshholds", *Journal of the Royal Statistical Society*, Ser. B, vol. 52, 393 - 442.

[8] Grimshaw, S.D. (1993), "Computing maximum likelihood estimates for the generalized Pareto distribution", *Technometrics*, vol. 35, 185 - 191.

[9] Glaser, R. A. (1982), "Bathtub and related failure rate characterization", *Journal of the American Statistical Association*, 75, 667 - 672.

[10] Gupta, R. C., Kannan, N. and RayChoudhuri, A. (1997), "Analysis of log-normal survival data", *Mathematical Biosciences*, 139, 103-115.

[11] Gupta, R.C. and Gupta, R.D. (2007), "Proportional reversed hazard rate model", *Journal of Statistical Planning and Inference*, 137, 3525 - 3536.

[12] Gupta, R. D. and Kundu, D. (1999). "Generalized exponential distributions", *Australian and New Zealand Journal of Statistics*, 41, 173 - 188.

[13] Gupta, R.D. and Kundu, D. (2007), "Generalized exponential distribution: existing methods and recent developments", *Journal of the Statistical Planning and Inference*, vol. 137, 3537 - 3547.

[14] Hosking, J.R.M. and Wallis, J.R. (1987), "Parameter and quantile estimation for the generalized Pareto ditribution", *Technometrics*, vol. 29, 339 - 349.

[15] Hosking, J.R.M., Wallis, J.R. and Wood, E.F. (1987), "Estimation of the generalized extreme value distribution", *Technometrics*, vol. 27, 251 - 261.

[16] Johnson, N.L., Kotz, S. and Balakrishnan, N. (1994), *Continuous univariate distributions*, vol. 1, 2-nd edition, John Wiley & Sons, New York.

[17] Kundu, D. and Gupta, R.D. (2006), "Estimation of $P(Y < X)$ for Weibull distribution", *IEEE Transactions on Reliability*, 55, no. 2, 270 - 280.

[18] Kundu, D., Gupta, R.D. (2009), "Bivariate generalized exponential distribution", *Journal of Multivariate Analysis*, 100, 581593.

[19] Kundu, D., Kannan, N. and Balakrishnan, N. (2008), "On the hazard function of Birnbaum-Saunders distribution and associated inference", *Computational Statistics and Data Analysis*, 52, 2692 - 2702.

[20] Mudholkar, G.S., Srivastava, D.K. and Freimer, M. (1995), "The exponentiated Weibull family: a reanalysis of the bus motor failure data", *Technometrics*, 37, 436 - 445.

[21] Mudholkar, G.S., Srivastava, D.K. and Kollia, G.D. (1996), "A generalization of the Weibull distribution with applications to the analysis of survival data", *Journal of the American Statistical Association*, 91, 1575 - 1583.

[22] Press, W. H., Teukolsky, S. A., Vellerling, W. T. and Flannery, B. P. (1992), *Numerical Recipes in FORTRAN*, 2nd. edition, Cambridge University Press, Cambridge.

[23] Smith, R.L. (1985), "Maximum likelihood estimation in a class of non-regular cases", *Biometrika*, 72, 67 - 90.

[24] Surles, J.G. and Padgett, W.J. (2001), "Inference for reliability and stress-strength for a scaled Burr type X distribution", *Lifetime and Data Analysis*, 7, 187 - 200.