On Two Exponential Populations Under A Joint Adaptive Type-II Progressive Censoring

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Abstract

In this paper, we introduce a new joint adaptive Type-II progressive censoring (JAPC) scheme for independent samples from two different populations. We place two independent samples simultaneously on a life testing experiment. It is assumed that the lifetime of the experimental units of the populations follow exponential distribution with mean θ_1 and θ_2, respectively. The maximum likelihood estimators of the unknown parameters and their exact distributions are derived. Based on the exact distributions of the maximum likelihood estimators, approximate confidence intervals are constructed. Further, the Bayesian inference of the model parameters is considered under a very flexible Beta-Gamma prior. We obtain Bayes estimators and associated credible intervals of the unknown parameters under squared error loss function. Extensive simulations are performed to see the effectiveness of the proposed estimation methods. A real dataset is considered for implementing the proposed model on it. Also we use the variable neighborhood search (VNS) method proposed by Bhattacharya et al. [10] to derive the optimal censoring scheme of the model in the Bayesian framework.

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1 Introduction

In recent times, the progressive censoring scheme has received a considerable amount of attention in the statistical literature, see for example Balakrishnan and Cramer [3]. But most of the related works are based on a single sample only. However, there are situations in which the experimenter wants to compare different populations at the same time. Under such a scenario, the joint censoring schemes can be used, see for example Rasouli and Balakrishnan [24], Balakrishnan and Su [4], Balakrishnan et al. [5], Mondal and Kundu [17], Ashour and Abo-Kasem[1] for details on different joint censoring schemes.

Ng et al. [19] first proposed the adaptive progressive censoring (APC) scheme to reduce the experimental time but at the same time observing the fixed number of failures. Although there has been a significant number of articles on joint Type-II and joint progressive Type-II censoring schemes, no work has been done on adaptive censoring schemes to more than one population. For the sake of reducing cost as well as the experimental time, and at the same time observing a fixed number of failures, one may be interested to adopt the joint adaptive Type-II censoring scheme proposed in this paper. In the proposed censoring scheme, it is ensured to observe as many failures as there are in the joint progressive Type-II censoring scheme available in the literature but with lesser running time. Clearly, in a life testing experiment where products are being manufactured by two or more machines simultaneously, our proposed scheme will be useful to have lesser experimental running time (compared to the same in APC scheme) as well as ensuring a certain number of failures.

The APC scheme can be described as follows. Suppose n number of units are put on a test with an aim to obtain m failures. Before the experiment starts, we fix a time point τ and choose a progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ such that all the R_i's are non-negative integers and $\sum_{i=1}^{m} R_i = n - m$. At the time of the first failure, $T_{1:n}$ (say), R_1 items from the remaining surviving items are randomly removed. At the time of the second failure $T_{2:n}$ (say), R_2 units out of $(n - 2 - R_1)$ remaining items are randomly removed and so on. The experiment stops at the failure time $T_{m:n}$ (say) with R_m items removed, if the m^{th} failure occurs before the time point τ. However, if failure of m^{th} item does not occur before τ, no more items are removed from the experiment after τ. The experiment stops at the failure of m^{th} item with all the remaining items withdrawn. This indicates that the experiment can stop before τ or even if it does not stop before τ, the process gets faster and stops at the failure of m^{th} item. If τ is indefinitely large then the usual progressive Type-II censoring occurs and if $\tau = 0$ then we have $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$ which corresponds to the Type-II censoring scheme.

Recently, Mondal and Kundu ([16], [17]) introduced a balanced joint progressive censoring (BJPC) scheme for two populations. It has several advantages over the traditional joint progressive scheme available in the literature. It is observed that with the proper choice of R_i's the BJPC scheme provides more efficient estimators than the traditional joint progressive scheme. It is also observed that one may not have much control on the expected time of the experiment in case of a BJPC scheme. The main aim of this manuscript is to provide a new sampling scheme for two populations, following the idea of the APC scheme, so that the expected experimental time can be reduced than the BJPC scheme without sacrificing the efficiency of the estimators significantly. We have introduced a joint adaptive progressive censoring (JAPC) scheme based on the BJPC scheme and provides the necessary analysis. It is observed that the expected time of the experiment for the JAPC scheme is smaller than that of the BJPC scheme, although the efficiency of the estimators are almost same. For ceratin choice of R_i's it has been observed (see Table 2) that the expected time can be reduced upto 60% without sacrificing the efficiency of the estimators significantly. Therefore, if two machines are producing say cell phone batteries and we want to estimate their average lifetimes efficiently within a limited time, then JAPC scheme can be adopted quite effectively instead of the BJPC scheme.

We have considered only two samples of the same size and it is assumed that the lifetime of the experimental units for both the populations follow exponential distributions. The proposed scheme is also useful for the different sample sizes of the different populations. We have provided both the classical and Bayesian inference of the unknown parameters. Finally, we have discussed selecting the optimal censoring plans based on some loss function depending on the time of the experiment and precision of the estimators. We have used the variable neighborhood search (VNS) method proposed by Bhattacharya et al. [10] to derive the optimal censoring scheme in the Bayesian framework.

The rest of the paper is organized as follows. In Section 2, we introduce the JAPC scheme for two samples from two independent exponential distributions having different unknown scale parameters only. The maximum likelihood estimators and their exact distributions are provided in Section 3. Approximate confidence intervals of the parameters based on the exact distributions are provided in Section 4. In Section 5, we carry out Bayesian analysis of the proposed model. Simulation results are provided in Section 6 to assess the effectiveness of the proposed model. The above mentioned classical and the Bayesian analyses of the proposed model have been carried to a real dataset are reported in Section 7. We also discuss how to select optimal censoring plans of the proposed model in Section 8. Finally, we conclude the paper in Section 9.

2 Model Description

Suppose the products of a life testing experiment belong to two independent populations, say Pop-1 and Pop-2. We draw a random sample of size n, say Sam-1 from Pop-1 and a random sample of the same size from Pop-2 say Sam-2. One can also take the different sample sizes from different populations. Suppose the lifetime of units in Pop-1 follows a distribution with cumulative distribution function (CDF) $F(\cdot;\theta_1)$ and probability density function (PDF) $f(\cdot;\theta_1)$. Lifetime of units in that in Pop-2 follows a distribution with the CDF $G(\cdot;\theta_2)$ and the PDF $g(\cdot;\theta_2)$. The proposed two sample joint adaptive Type-II progressive censoring (JAPC) scheme can be described as follows.

Let m be a prefixed integer, τ be a prefixed time point, and $(R_1, R_2, \cdots, R_m)$ is set of pre-fixed non-negative integers such that $\sum_{i=1}^{m} R_i = n-m$. We place two independent samples simultaneously on the life testing experiment. Suppose the first failure occurs from Sam-1 (Sam-2) at time $T_{1:n}$. Then R_1 units are randomly removed from the remaining $(n-1)$ surviving units of Sam-1 (Sam-2) and at the same time, (R_1+1) units are randomly chosen and removed from the n surviving units of Sam-2 (Sam-1). Suppose the next failure occurs from Sam-2 (Sam-1) at the time $T_{2:n}$. Then R_2 units are chosen randomly from the remaining $(n-R_1-2)$ surviving units of Sam-2 (Sam-1), and they are removed. At the same time (R_2+1) units are chosen randomly and removed from the surviving $(n-R_1-1)$ units of Sam-1 (Sam-2). We continue the process until the m^{th} failure occurs or the time point τ is reached. Suppose, m^{th} failure occurs at $T_{m:n}(<\tau)$, then all the remaining surviving units are removed from both the samples and the experiment stops. On the other hand, if only $D(<m)$ many failures take place before τ, we follow the same procedure as before until $T_{D:n}$. But when the $(D+1)^{th}$ failure occurs at $T_{D+1:n}$, no more units are removed from the experiment until the m^{th} failure takes place from one of the samples. The experiment thus gets terminated with all the remaining items removed from it. In this scheme we observe m failure times and at any time of the experiment we always have the same number of surviving units in both the samples. In Figures 2.1 and 2.2, we provide a schematic diagram of our proposed JAPC model. Figure 2.1 represents the case when the m-th failure occurs after time τ, while Figure 2.2 represents the case when the m-th failure occurs before time τ. Also, notice that, as in single sample case, if τ is indefinitely large then the usual joint progressive Type-II censoring occurs and if $\tau = 0$ then we have $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n-m$, which corresponds to the joint Type-II censoring scheme.

Let us define a new set of random variables $Z_1, Z_2, \cdots, Z_m$, where $Z_i = 1$ if the i^{th} failure

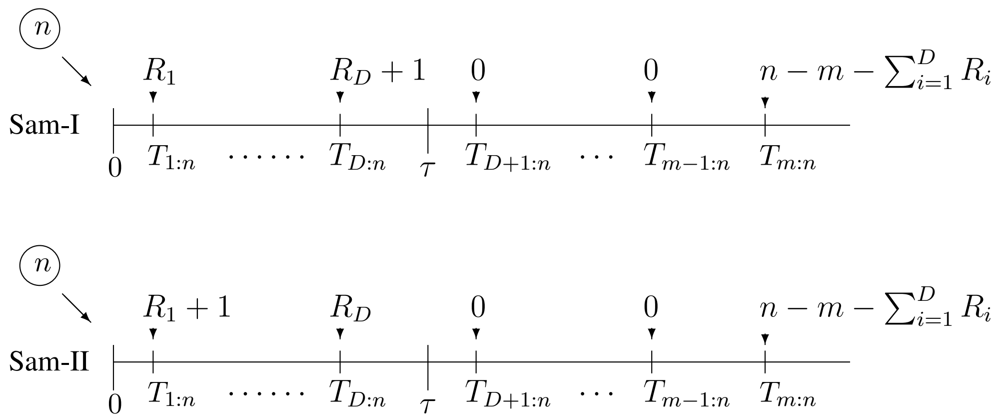
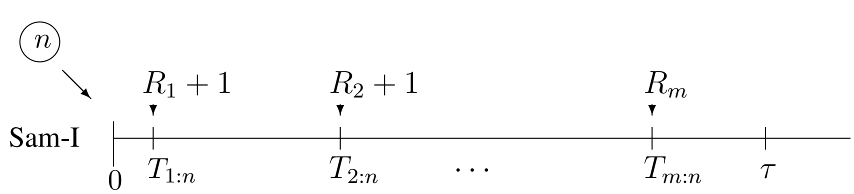


Figure (2.1) Illustration of failures and items removed when m-th failure occurs after time τ.

occurs from Sam-1 and $Z_i = 0$ if the i^{th} failure occurs from Sam-2. Hence, the random variables associated are of the form $(\mathbf{T}, \mathbf{Z})$ where, $\mathbf{T} = (T_{1:n}, T_{2:n}, \cdots, T_{m:n})$ and $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_m)$. Here, $D_1 = \sum_{i=1}^{m} Z_i$ denotes the number of failures from Sam-1 and $D_2 = \sum_{i=1}^{m}(1 - Z_i)$ denotes the number of failures from Sam-2 in the experiment. We denote the realization of the random variables $T_{i:m}$ and Z_i by t_i and z_i, respectively for $i = 1, 2, \cdots, m$. Also, we denote the realizations of the random variables D, D_1, and D_2 by d, d_1, and d_2, respectively. Let us define the following,

$$R_i^* = \begin{cases} R_i, & \text{if } i = 1, \cdots, m \text{ and } t_m \leq \tau \text{ or } i = 1, \cdots, d \text{ and } \tau < t_m, \\ 0, & \text{if } i = d+1, \cdots, m-1 \text{ and } \tau < t_m, \\ n - d - \sum_{i=1}^{d} R_i^*, & \text{if } i = m \text{ and } \tau < t_m. \end{cases}$$



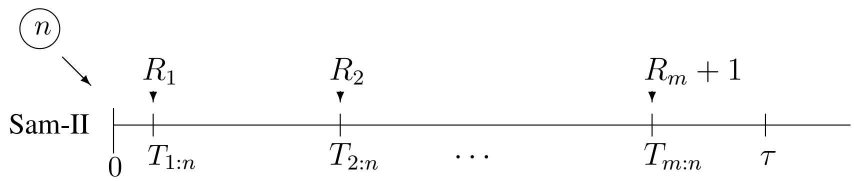


Figure (2.2) Illustration of failures and items removed when m-th failure occurs before time τ.

It is to be noted that the likelihood contribution at failure time t_1, provided that the failure has happened from Sam-1 is proportional to

$$f(t_1;\theta_1)[1-F(t_1;\theta_1)]^{R_1^*}[1-G(t_1;\theta_2)]^{R_1^*+1},$$

whereas, the likelihood contribution at failure time t_1, provided that the failure has happened from Sam-2 is proportional to

$$g(t_1;\theta_2)[1-F(t_1;\theta_1)]^{R_1^*+1}[1-G(t_1;\theta_2)]^{R_1^*}.$$

Combining the above two, we get the likelihood contribution at (t_1, z_1) as proportional to

$$f(t_1;\theta_1)^{z_1}g(t_1;\theta_2)^{1-z_1}[1-F(t_1;\theta_1)]^{R_1^*+1-z_1}[1-G(t_1;\theta_2)]^{R_1^*+z_1}.$$

Hence, for any $i = 1, 2, \cdots, m$, the likelihood contribution at (t_i, z_i) is proportional to

$$f(t_i;\theta_1)^{z_i}g(t_i;\theta_2)^{1-z_i}[1-F(t_i;\theta_1)]^{R_i^*+1-z_i}[1-G(t_i;\theta_2)]^{R_i^*+z_i}.$$

Clearly, the likelihood function is

$$L(\theta_1, \theta_2|Data) \propto \prod_{i=1}^{m} f(t_i;\theta_1)^{z_i} g(t_i;\theta_2)^{1-z_i}[1-F(t_i;\theta_1)]^{R_i^*+1-z_i}[1-G(t_i;\theta_2)]^{R_i^*+z_i}.$$

3 Maximum Likelihood Estimation

In this Section, we will obtain the maximum likelihood estimators (MLEs) of the unknown model parameters θ_1 and θ_2. We assume that lifetime distribution of the units from Pop-1 (or Sam-1), is exponential distribution with scale parameter θ_1 and lifetime distribution of the units from Pop-2 (or Sam-2), is exponential distribution with scale parameter θ_2. Following the discussion in the previous section, we can rewrite the likelihood function of the parameters as

$$L(\theta_1, \theta_2|\mathbf{T}, \mathbf{Z}) = C\frac{1}{\theta_1^{d_1}}\frac{1}{\theta_2^{d_2}}e^{-\frac{1}{\theta}\sum_{i=1}^{m} t_i(1+R_i^*)}, \tag{1}$$

where,

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}, \quad C = \prod_{i=1}^{m}\left[n - \sum_{j=1}^{i-1}(R_j+1)\right], d_1 + d_2 = m.$$

Note that when $d_1 = 0$, the likelihood function becomes

$$L(\theta_1, \theta_2|\mathbf{T}, \mathbf{Z}) \propto \frac{1}{\theta_2^{d_2}}e^{-(\frac{1}{\theta_1}+\frac{1}{\theta_2})\sum_{i=1}^{m} t_i(1+R_i^*)},$$

and is a strictly increasing function of θ_1 on $(0, \infty)$. Similarly, when $d_2 = 0$, the likelihood function is a strictly increasing function of θ_2 on $(0, \infty)$. Therefore, it is immediate that the MLEs of both θ_1 and θ_2 exist only when $1 \leq D_1 \leq m-1$, and they are as follows:

$$\hat{\theta}_1 = \frac{W}{D_1}, \;\; \hat{\theta}_2 = \frac{W}{D_2},$$

where, $W = \sum_{i=1}^{m} T_{i:n}(1+R_i^*)$.

4 Conditional Distribution of MLEs

In this section, we derive the conditional distribution of the MLEs of the parameters θ_1 and θ_2, respectively. Using them, we are able to construct approximate confidence intervals of the parameters which we report in the next section. It is to be noted that the MLEs of the parameters exist only if $D_1 > 0$ and $D_2 > 0$. Hence conditioning on the event $E = \{D_1 > 0, D_2 > 0\}$, we derive distributions of $\hat{\theta}_1$ and $\hat{\theta}_2$, by inverting moment generating function (MGF) (see, Bartholomew [7]) of them separately and report below in Theorem 1 and Theorem 2, respectively. Proofs of these theorems are supplied in the Appendix section. In Theorem 1 and Theorem 2, we have used $F_{Gamma}(x; a, b)$ to denote the value of CDF of Gamma distribution with shape and scale parameters a and b, respectively evaluated at x and $f_{Gamma}(x; a, b)$ is the associated PDF at x.

Theorem 1. *The CDF and PDF of $\hat{\theta}_1$, conditioning on the event $E = \{D_1 > 0, D_2 > 0\}$ are, respectively, given by*

$$F_{\hat{\theta}_1|E}(x) = \frac{C}{P[D_1 > 0, D_2 > 0]} \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \sum_{i=0}^{d} \sum_{s=0}^{m-d-1} \Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)} \Big(\frac{\theta_1}{\theta_1+\theta_2}\Big)^{m-d_1} \Big(\frac{\theta_2}{\theta_1+\theta_2}\Big)^{d_1}$$
$$\frac{(-1)^{i+s}\, e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i} R_j)}}{\big[\prod_{j=1}^{i} \sum_{k=d-i+1}^{d-i+j} \big(1+R_k\big)\big]\ \big[\prod_{j=1}^{d-i} \sum_{k=j}^{d-i} \big(1+R_k\big)\big] \big[n-m+s-\sum_{i=1}^{d} R_i\big]}$$
$$F_{gamma}\Big(x - \frac{\tau}{d_1}\big(n-d-1+i-\sum_{j=1}^{d-i} R_j\big); \frac{d_1}{\theta}, m\Big)\Bigg],$$

and

$$f_{\hat{\theta}_1|E}(x) = \frac{1_x(0,\infty)\, C}{P[D_1 > 0, D_2 > 0]} \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \sum_{i=0}^{d} \sum_{s=0}^{m-d-1} \Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)} \Big(\frac{\theta_1}{\theta_1+\theta_2}\Big)^{m-d_1} \Big(\frac{\theta_2}{\theta_1+\theta_2}\Big)^{d_1}$$
$$\frac{(-1)^{i+s}\, e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i} R_j)}}{\big[\prod_{j=1}^{i} \sum_{k=d-i+1}^{d-i+j} \big(1+R_k\big)\big]\ \big[\prod_{j=1}^{d-i} \sum_{k=j}^{d-i} \big(1+R_k\big)\big] \big[n-m+s-\sum_{i=1}^{d} R_i\big]}$$

$$f_{gamma}\Big(x - \frac{\tau}{d_1}(n-d-1+i-\sum_{j=1}^{d-i} R_j); \frac{d_1}{\theta}, m\Big)\Big],$$

where

$$(i): \ P(D_1>0, D_2>0) = C\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\sum_{i=0}^{d}\sum_{s=0}^{m-d-1}\Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)}\big(\frac{\theta_1}{\theta_1+\theta_2}\big)^{m-d_1}\big(\frac{\theta_2}{\theta_1+\theta_2}\big)^{d_1}$$
$$\frac{(-1)^{i+s}\ e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i} R_j)}}{\big[\prod\limits_{j=1}^{i}\sum\limits_{k=d-i+1}^{d-i+j}(1+R_k)\big]\ \big[\prod\limits_{j=1}^{d-i}\sum\limits_{k=j}^{d-i}(1+R_k)\big]\big[n-m+s-\sum_{i=1}^{d} R_i\big]}\Bigg],$$

$(ii): \ 1_x(0,\infty) = 1(\textit{or } 0),$ *according to* $x > 0$ (*or* ≤ 0).

Proof. See Appendix. □

Theorem 2. *The CDF and PDF of* $\hat{\theta}_2$*, conditioning on the event* $E = \{D_1 > 0, D_2 > 0\}$ *are, respectively, given by*

$$F_{\hat{\theta}_2|E}(x) = \frac{C}{P[D_1>0, D_2>0]}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\sum_{i=0}^{d}\sum_{s=0}^{m-d-1}\Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)}\big(\frac{\theta_2}{\theta_1+\theta_2}\big)^{m-d_1}\big(\frac{\theta_1}{\theta_1+\theta_2}\big)^{d_1}$$
$$\frac{(-1)^{i+s}\ e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i} R_j)}}{\big[\prod\limits_{j=1}^{i}\sum\limits_{k=d-i+1}^{d-i+j}(1+R_k)\big]\ \big[\prod\limits_{j=1}^{d-i}\sum\limits_{k=j}^{d-i}(1+R_k)\big]\big[n-m+s-\sum_{i=1}^{d} R_i\big]}$$
$$F_{gamma}\Big(x - \frac{\tau}{m-d_1}(n-d-1+i-\sum_{j=1}^{d-i} R_j); \frac{m-d_1}{\theta}, m\Big)\Bigg],$$

and

$$f_{\hat{\theta}_2|E}(x) = \frac{1_x(0,\infty)\ C}{P[D_1>0, D_2>0]}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\sum_{i=0}^{d}\sum_{s=0}^{m-d-1}\Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)}\big(\frac{\theta_2}{\theta_1+\theta_2}\big)^{m-d_1}\big(\frac{\theta_1}{\theta_1+\theta_2}\big)^{d_1}$$
$$\frac{(-1)^{i+s}\ e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i} R_j)}}{\big[\prod\limits_{j=1}^{i}\sum\limits_{k=d-i+1}^{d-i+j}(1+R_k)\big]\ \big[\prod\limits_{j=1}^{d-i}\sum\limits_{k=j}^{d-i}(1+R_k)\big]\big[n-m+s-\sum_{i=1}^{d} R_i\big]}$$

$$f_{gamma}\Big(x - \frac{\tau}{m-d_1}\big(n-d-1+i-\sum_{j=1}^{d-i} R_j\big); \frac{m-d_1}{\theta}, m\Big)\Big],$$

where

$$(i): \; P(D_1 > 0, D_2 > 0) = C \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \sum_{i=0}^{d} \sum_{s=0}^{m-d-1} \Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)} \big(\frac{\theta_1}{\theta_1+\theta_2}\big)^{m-d_1} \big(\frac{\theta_2}{\theta_1+\theta_2}\big)^{d_1}$$

$$\frac{(-1)^{i+s}\; e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i} R_j)}}{\big[\prod_{j=1}^{i} \sum_{k=d-i+1}^{d-i+j} \big(1+R_k\big)\big] \; \big[\prod_{j=1}^{d-i} \sum_{k=j}^{d-i} \big(1+R_k\big)\big] \big[n-m+s-\sum_{i=1}^{d} R_i\big]} \Bigg],$$

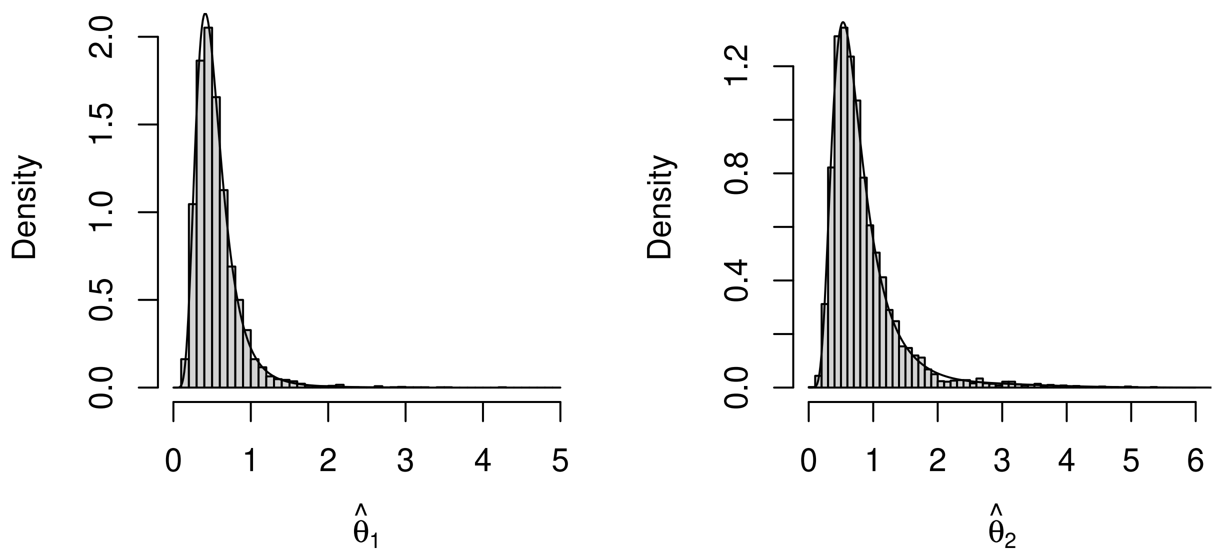
$$(ii): \; 1_x(0,\infty) = 1(\textit{or } 0), \; \textit{according to } x > 0 \; (\textit{or } \le 0).$$

Proof. See Appendix. □

It is clear that the conditional PDFs of the MLEs of the parameters are complicated in nature. To get some idea about the shapes of the PDFs, we plot the conditional PDFs of each parameter in Figures 4.1-4.3 for different choices of $n, m, R, \tau, \theta_1, \theta_2$. We also plot the histograms of MLEs, based on 5000 replications, of the parameters on the same plots and found that they are matching quite well. In the description of the plots, for brevity, we have used specific notations for censoring schemes. For example, $(2, 3, 0^{*5}, 1, 4, 0)$ means that the censoring scheme employed is $(2, 3, 0, 0, 0, 0, 0, 1, 4, 0)$.

5 Approximate Confidence Interval

In this Section, we construct a symmetric approximate confidence interval of each of the parameters θ_i, $i = 1, 2$ using conditional distribution of $\hat{\theta}_i$ as derived in Section 4. To do this, it is required that for $i = 1, 2$, $F_{\hat{\theta}_i|E}(x)$, the conditional distribution of $\hat{\theta}_i$ at any arbitrary value x (say) monotonically decreases as θ_i increases. Several authors including Balakrishnan et al. [6], Childs et al.



(a) Histogram of $\hat{\theta}_1$ along with its PDF

(b) Histogram of $\hat{\theta}_2$ along with its PDF

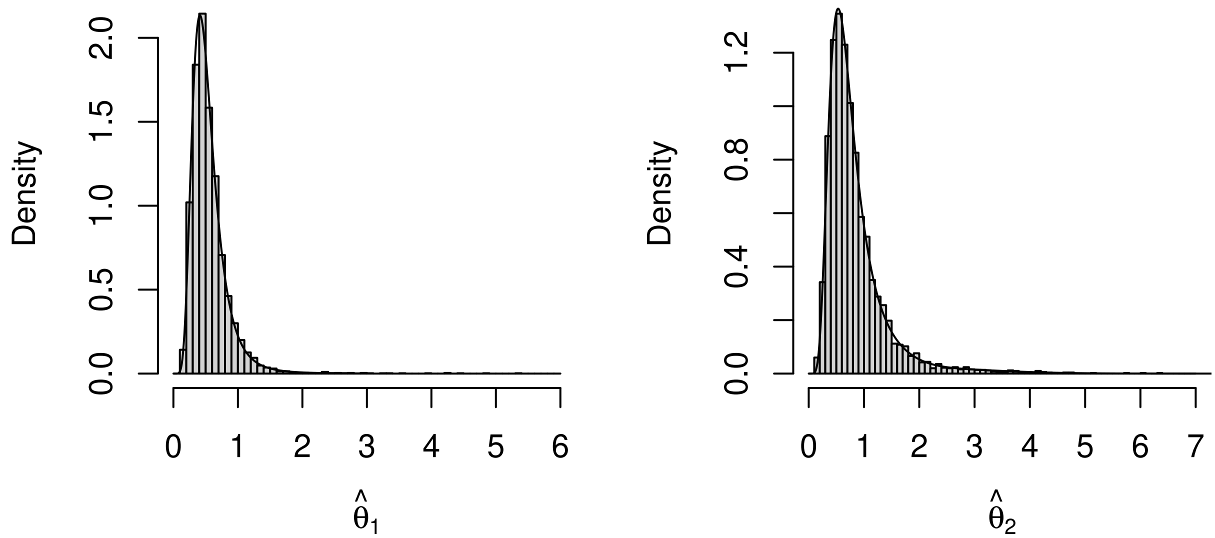
Figure (4.1) Histogram of $\hat{\theta}_1$ and $\hat{\theta}_2$ along with their PDFs, taking $(\theta_1, \theta_2) = (0.5, 0.7), (n, m) = (30, 10), \tau = 0.25, R = (2^{*10})$.

[11] used this technique to construct confidence intervals for the parameters. Although we cannot prove the monotonicity property of $F_{\hat{\theta}_i|E}(x)$ analytically, a graphical plot supports this property. In Figures 5.1, 5.2, and 5.3 we provide some graphs for illustrative purposes.

For $0 < \alpha < 1$, the $100(1-\alpha)\%$ symmetric approximate confidence interval of θ_i can be constructed using the conditional distribution of $\hat{\theta}_i$ for $i = 1, 2$. Following the approach of Koley and Kundu [12], for $i = 1, 2$, the lower (L_i) and upper (U_i) confidence limits of θ_i are obtained by solving the following two equations in θ_i:

$$F_{\hat{\theta}_i|E}(\hat{\theta}_{i\ observed}) = 1 - \frac{\alpha}{2} \quad \text{and} \quad F_{\hat{\theta}_i|E}(\hat{\theta}_{i\ observed}) = \frac{\alpha}{2}.$$

The unknown parameter θ_j, $j \neq i$ in the above equation is replaced by its estimate. Hence the



(a) Histogram of $\hat{\theta}_1$ along with its PDF

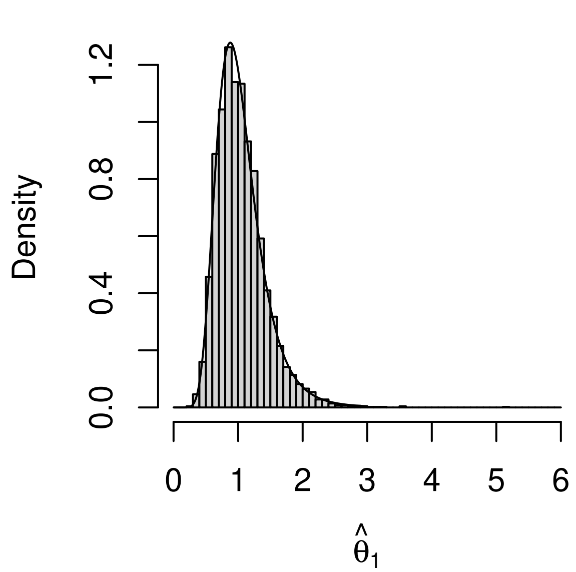
(b) Histogram of $\hat{\theta}_2$ along with its PDF

Figure (4.2) Histogram of $\hat{\theta}_1$ and $\hat{\theta}_2$ along with their PDFs, taking $(\theta_1, \theta_2) = (0.5, 0.7), (n, m) = (20, 10), \tau = 0.50, R = (2, 3, 0^{*5}, 1, 4, 0)$.

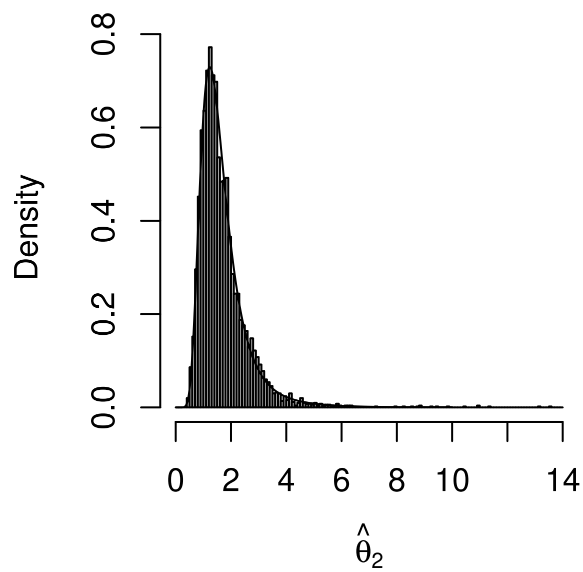
confidence interval obtained by solving the above equations is called approximate confidence interval. Clearly, due to the complicated nature of the conditional CDF of $\hat{\theta}_i$, for $i = 1, 2$, the above two equations turn out to be non-linear equations. One needs to solve them using some numerical methods such as Newton-Rapshon method or Bisection method.

6 Bayesian Analysis

Since computationally it is quite costly to obtain the confidence intervals based on the classical approach, it is quite natural to use Bayesian methodology as an alternative. In this Section, we discuss Bayes estimators of unknown parameters θ_1 and θ_2 with respect to the squared error loss function.



(a) Histogram of $\hat{\theta}_1$ along with its PDF



(b) Histogram of $\hat{\theta}_2$ along with its PDF

Figure (4.3) Histogram of $\hat{\theta}_1$ and $\hat{\theta}_2$ along with their PDFs, taking $(\theta_1, \theta_2) = (1.0, 1.5), (n, m) = (35, 15), \tau = 1.0, R = (0^{*14}, 20)$.

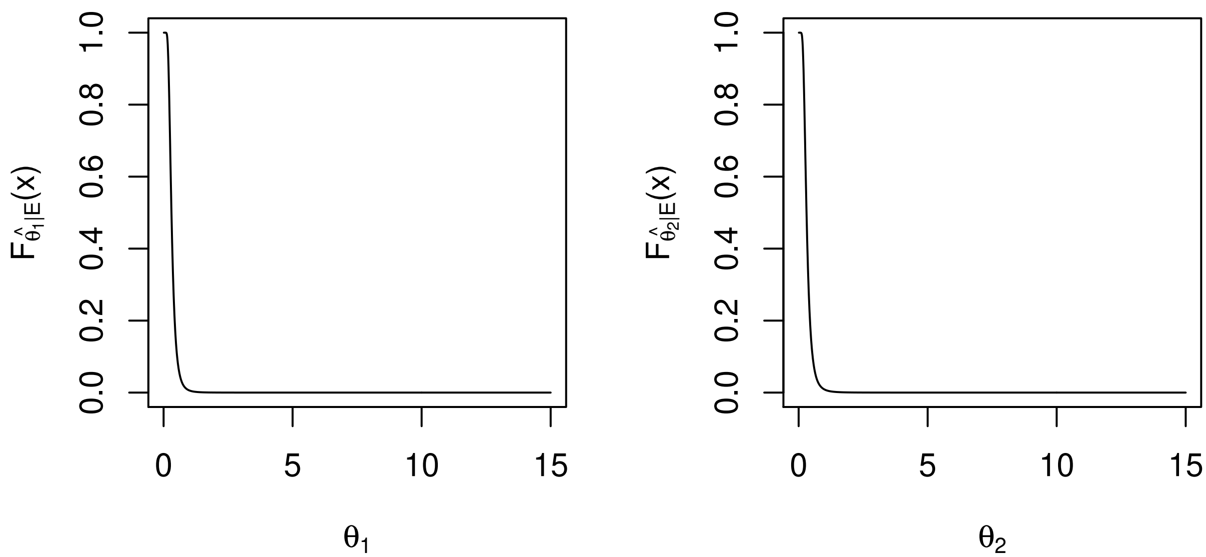
6.1 Prior Assumption

To choose a prior distribution of the unknown parameters, we make the reparameterization of the parameters as $Y_1 = \frac{1}{\theta_1}$ and $Y_2 = \frac{1}{\theta_2}$. We assume that Y_1 and Y_2 jointly follow Beta-Gamma distribution with joint PDF as,

$$\pi(y_1, y_2|a_0, b_0, a_1, a_2) = \begin{cases} C^*(y_1 + y_2)^{a_0 - a_1 - a_2} y_1^{a_1 - 1} y_2^{a_2 - 1} e^{-b_0(y_1 + y_2)}, & \text{if } y_1, y_2 > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $C^* = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)} b_0^{a_0}$ and $a_0 > 0, b_0 > 0, a_1 > 0, a_2 > 0$.

The PDF of (Y_1, Y_2) will be denoted by $BG(a_0, b_0, a_1, a_2)$. The PDF (2) can take a variety

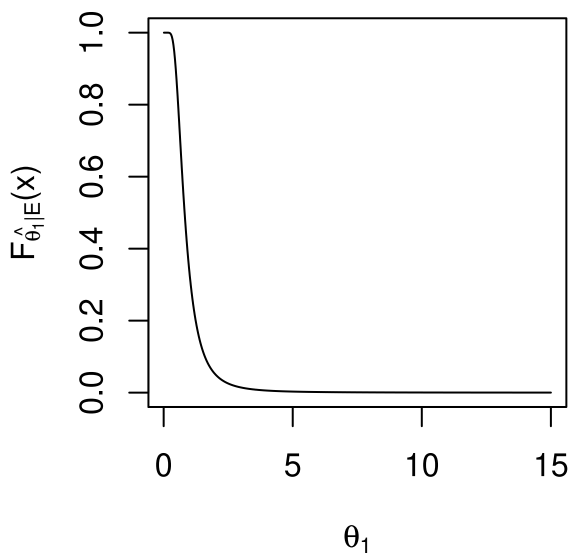


(a) Plot of $F_{\hat{\theta}_1|E}(x)$ (b) Plot of $F_{\hat{\theta}_2|E}(x)$

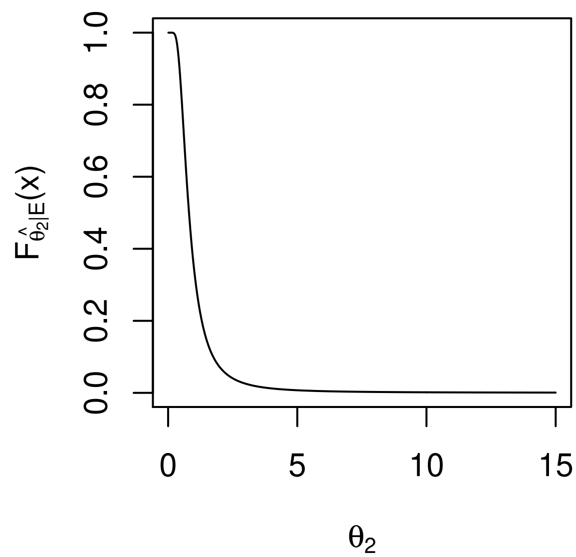
Figure (5.1) Plot of $F_{\hat{\theta}_1|E}(x)$ and $F_{\hat{\theta}_2|E}(x)$ for $(\theta_1, \theta_2) = (0.5, 0.7)$, $(n, m) = (30, 10)$, $\tau = 0.25$, $R = (2^{*10})$ at $x = 0.3$.

of shapes and is often used as a joint conjugate prior (see Pena et al. [20]). The correlation between Y_1 and Y_2 can be both positive and negative, depending on the values of a_0, a_2, and a_2. If $a_0 = a_1 + a_2$, the distributions of Y_1 and Y_2 become independent. The following three results will be useful for further development. One can see Koley and Kundu [13] for the proof of Result 1. Proof of the remaining two results are quite straightforward and hence are omitted.

RESULT 1: $(Y_1, Y_2) \sim BG(a_0, b_0, a_1, a_2)$ if and only if, $Y_1 + Y_2 \sim Gamma(a_0, b_0)$, $\frac{Y_1}{Y_1+Y_2} \sim Beta(a_1, a_2)$ and they are independent to each other.



(a) Plot of $F_{\hat{\theta}_1|E}(x)$



(b) Plot of $F_{\hat{\theta}_2|E}(x)$

Figure (5.2) Plot of $F_{\hat{\theta}_1|E}(x)$ and $F_{\hat{\theta}_2|E}(x)$ for $(\theta_1, \theta_2) = (0.5, 0.7)$, $(n, m) = (20, 10)$, $\tau = 0.50$, $R = (2, 3, 0^{*5}, 1, 4, 0)$ at $x = 0.8$.

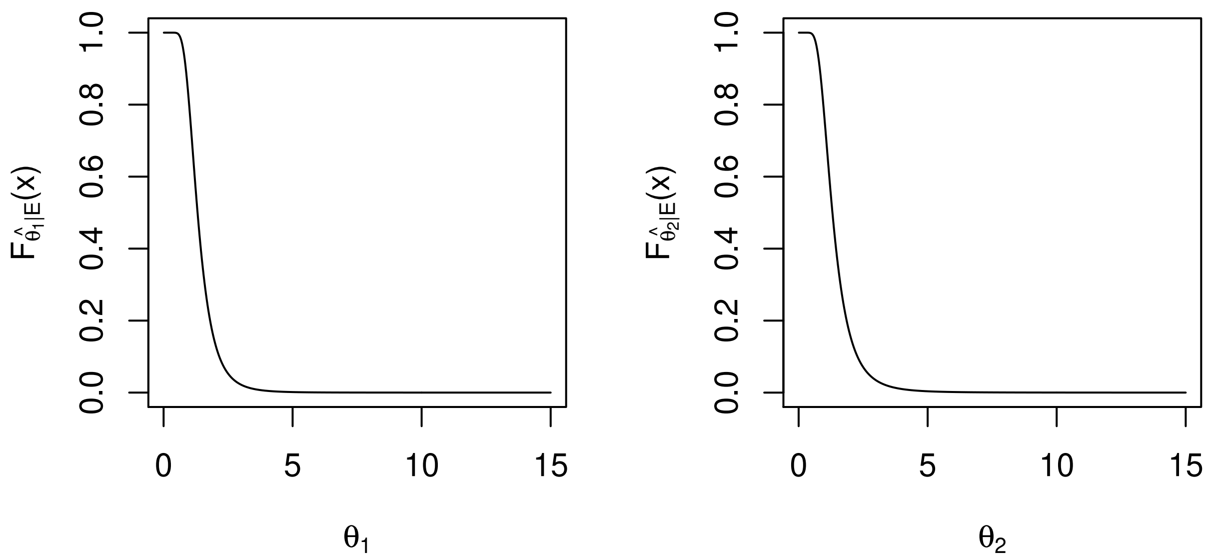
RESULT 2: If $(Y_1, Y_2) \sim BG(a_0, b_0, a_1, a_2)$, then for $i = 1, 2$,

$$E(Y_i) = \frac{a_0 a_i}{b_0(a_1 + a_2)} \text{ and } Var(Y_i) = \frac{a_0 a_i}{b_0^2(a_1 + a_2)} \times \left[\frac{(a_i + 1)(a_0 + 1)}{(a_1 + a_2 + 1)} - \frac{a_0 a_i}{(a_1 + a_2)}\right].$$

RESULT 3: If $(Y_1, Y_2) \sim BG(a_0, b_0, a_1, a_2)$, then for $i = 1, 2$,

$$E\Big(\frac{1}{Y_i}\Big) = \frac{(a_1 + a_2)b_0}{a_0 a_i} \text{ and } Var\Big(\frac{1}{Y_i}\Big) = \frac{b_0^2(a_1 + a_2)}{a_0 a_i} \times \left[\frac{(a_1 + a_2 - 1)}{(a_0 - 1)(a_i - 1)} - \frac{(a_1 + a_2)}{a_0 a_i}\right].$$

Using the same algorithm as suggested by Kundu and Pradhan [15], following steps are useful to generate samples (y_1, y_2) from $BG(a_0, b_0, a_1, a_2)$ distribution.



(a) Plot of $F_{\hat{\theta}_1|E}(x)$ (b) Plot of $F_{\hat{\theta}_2|E}(x)$

Figure (5.3) Plot of $F_{\hat{\theta}_1|E}(x)$ and $F_{\hat{\theta}_2|E}(x)$ for $(\theta_1, \theta_2) = (1.0, 1.5)$, $(n, m) = (35, 15)$, $\tau = 1.0$, $R = (0^{*14}, 20)$ at $x = 1.3$.

6.2 Posterior Distribution

The joint posterior distribution of (Y_1, Y_2) turns out to be of the following form,

$$\pi(Y_1, Y_2|Data) \propto (y_1 + y_2)^{a_0 - a_1 - a_2} y_1^{d_1 + a_1 - 1} y_2^{d_2 + a_2 - 1} e^{-(b_0 + W)(y_1 + y_2)}, \quad y_1 > 0, y_2 > 0. \tag{3}$$

Hence,

$$\pi(Y_1, Y_2|Data) \sim BG(a_0 + d_1 + d_2, b_0 + W, a_1 + d_1, a_2 + d_2).$$

Therefore, under the squared error loss function, the Bayes estimates of θ_1 and θ_2 are, respectively, obtained as

Algorithm 1: Generation of samples from $BG(a_0, b_0, a_1, a_2)$

1 Generate u from $Gamma(a_0, b_0)$.
2 Generate v from $Beta(a_1, a_2)$.
3 Obtain $y_1 = uv$ and $y_2 = u(1 - v)$.

$$\hat{\theta}_{1B} = E\left(\frac{1}{Y_1}|Data\right) = \frac{(a_1 + a_2 + d_1 + d_2)(b_0 + W)}{(a_0 + d_1 + d_2)(a_1 + d_1)}$$

and

$$\hat{\theta}_{2B} = E\left(\frac{1}{Y_2}|Data\right) = \frac{(a_1 + a_2 + d_1 + d_2)(b_0 + W)}{(a_0 + d_1 + d_2)(a_2 + d_2)}.$$

Similarly, we can obtain the corresponding posterior variances as follows:

$$V(\theta_1|Data) = V\left(\frac{1}{Y_1}|Data\right) = \frac{(a_1 + a_2 + d_1 + d_2)(b_0 + W)^2}{(a_0 + d_1 + d_2)(a_1 + d_1)} M_1 \tag{4}$$

and

$$V(\theta_2|Data) = V\left(\frac{1}{Y_2}|Data\right) = \frac{(a_1 + a_2 + d_1 + d_2)(b_0 + W)^2}{(a_0 + d_1 + d_2)(a_2 + d_2)} M_2, \tag{5}$$

where

$$M_1 = \frac{(a_1 + a_2 + d_1 + d_2 - 1)}{(a_0 + d_1 + d_2 - 1)(a_1 + d_1 - 1)} - \frac{(a_1 + a_2 + d_1 + d_2)}{(a_0 + d_1 + d_2)(a_1 + d_1)}$$

and

$$M_2 = \frac{(a_1 + a_2 + d_1 + d_2 - 1)}{(a_0 + d_1 + d_2 - 1)(a_2 + d_2 - 1)} - \frac{(a_1 + a_2 + d_1 + d_2)}{(a_0 + d_1 + d_2)(a_2 + d_2)}.$$

6.3 Credible Intervals

It is possible to construct $100(1 - \alpha)\%$ credible intervals (CRI) and Highest Posterior Density (HPD) credible intervals of the unknown parameters. Following algorithm can be considered to construct them.

Algorithm 2: Constructions of Credible and HPD intervals

1. For an observed dataset $(\mathbf{T}, \mathbf{Z})$, generate y_1 and y_2 as stated in **Algorithm-1** in subsection 6.1.
2. Obtain $\theta_1 = \frac{1}{y_1}$ and $\theta_2 = \frac{1}{y_2}$.
3. Repeat the above two steps a large number of times, say, M to obtain M values of θ_1 and θ_2. Arrange them in an increasing order and denote them as $\theta_{1(1)} \leq \theta_{1(2)} \leq \cdots \leq \theta_{1(M)}$, for the parameter θ_1 and $\theta_{2(1)} \leq \theta_{2(2)} \leq \cdots \leq \theta_{2(M)}$, for the parameter θ_2.
4. For $0 < \alpha < 1$ and $i = 1, 2;$ a $100(1-\alpha)\%$ CRI of θ_i is obtained as $\big(\theta_{i(j)}, \theta_{i(j+M(1-\alpha))}\big)$, for $j = 1, 2, \cdots, [M\alpha]$. Here $[x]$ denotes the integer part of x.
5. $100(1-\alpha)\%$, HPD credible interval of θ_i, i=1,2, is $\big(\theta_{i(j^*)}, \theta_{i(j^*+M(1-\alpha))}\big)$, such that $\theta_{i(j^*+M(1-\alpha))} - \theta_{i(j^*)} \leq \theta_{i(j+M(1-\alpha))} - \theta_{i(j)}$, for any $j = 1, 2, \cdots, [M\alpha]$.

7 Simulation Study

In this Section, a simulation study is conducted in order to evaluate the performance of the proposed estimation methodology for both classical as well as the Bayesian framework, discussed in the preceding sections. We compare the performances of the proposed estimators and the Bayes estimators using non-informative priors. For comparison purposes, we also compute 95% confidence and credible intervals of the unknown parameters in terms of their coverage percentages and HPD credible lengths. We consider different sample sizes $n = 30, 40$ and different choices for R_i, m with varying values of τ. The parameters (θ_1, θ_2) are chosen to be $(0.5, 0.7)$ and $(1, 1.5)$. In case of the Bayesian estimation we have taken the non-informative prior and the hyperparameter values are taken as $a_0 = b_0 = 0, a_1 = a_2 = 1$.

In each case, we draw a random sample with the pre-specified model parameters, the given censoring scheme and compute $\hat{\theta}_1$ and $\hat{\theta}_2$. We repeat this process 2,000 times and compute the average estimates (AE) of the MLEs and the associated mean squared errors (MSEs). We also compute 95% confidence intervals based on the conditional distributions of $\hat{\theta}_1$ and $\hat{\theta}_2$, and report the corresponding average lengths (ALs), the coverage percentages (CPs). Under the Bayesian framework, the AEs of the Bayes estimators (BEs) are calculated. We also report the average lengths of credible intervals (AL CRI), average lengths of HPD (AL HPD) credible intervals of both the unknown parameters. All the results for the classical framework are reported in Table 3-6. Results obtained by using the Bayesian method are reported in Table 7-10.

Some of the points are quite clear from the above Tables. It is clear that the performance of the estimators is quite satisfactory for both classical and Bayesian methods. In most cases, it is observed that the MSEs of both the estimators are smaller in the case of the Bayesian method than the classical one. Regarding the confidence intervals, it is observed that the confidence intervals obtained using the exact distribution and also using the HPD credible intervals provide satisfactory results. In all the cases, the coverage percentages are very close to the nominal level. Regarding the length of the confidence intervals, the HPD confidence intervals perform better than the exact confidence intervals.

8 Real Life Data Example

In this section we carry out a real data analysis both in classical and Bayesian approach. We take a dataset from Proschan [23]. The original data represents the intervals between failures (in hours) of the air conditioning system of a fleet of 13 Boeing 720 jet air planes. After analysing the data, he found that the failure distribution of the air-conditioning system for each of the planes is well approximated by exponential distribution. For our illustrative purpose, we take observations from Plane 7913 and Plane 7914. There are 27 observations from Plane 7913 and 24 observations from Plane 7914 in the original dataset. For our illustrative purpose, we take 24 observations from both the planes and they are listed below:

Plane 7913: 1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163.

Plane 7914: 3, 5, 5, 13, 14, 15, 22, 22, 23, 30, 36, 39, 44, 46, 50, 72, 79, 88, 97, 102, 139, 188, 197, 210.

For our analytical purpose, we take $\tau = 30, m = 14, R_{\frac{m}{2}} = n - m, R_i = 0$ for $i \neq \frac{m}{2}$ with $n = 24$. Based on this, we generate the following dataset. We further assume observations from Plane 7913 and Plane 7914 are coming from exponential distribution with mean θ_1 and θ_2, respectively.

The following generated dataset is obtained:

$\mathbf{T} = (1, 3, 4, 5, 5, 11, 13, 15, 16, 22, 22, 31, 44, 163)$ and

$\mathbf{Z} = (0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0)$.

Based on the obtained data and censoring scheme taken, we compute maximum likelihood estimates and Bayes estimates (under square error loss function) of the parameters. Also the 95% confidence intervals and 95% HPD credible intervals of the parameters are obtained. All the outputs are recorded in the following table.

Table (1) Real data analysis

	θ_1	θ_2
Maximum likelihood estimate	60.625	80.833
Lower confidence Limit	32.597	39.552
Upper confidence Limit	137.070	211.121
Bayes estimate	61.545	79.129
HPD Lower credible Limit	28.658	33.772
HPD Upper credible Limit	124.504	176.151

From the analysis, it is to be noted that the maximum likelihood estimates and the Bayes estimates are quite close to each other for every parameter. Also the length of HPD credible interval for each parameter is smaller than the length of confidence interval for the corresponding parameter.

9 Optimal Censoring Scheme

Before the start of any life testing experiment, it is very important to choose an "optimal" censoring scheme in a class of all possible available censoring schemes. One natural question is whether they should be chosen arbitrarily or they should be chosen based on some criterion. In any life testing experiment often there is a constraint on time and at the same time, it is also important to design the test such that the unknown quantities of interest can be estimated efficiently based on the data obtained under such a test. Thus the determination of optimal life testing plans is an important aspect of such studies and is critical for efficient designing of a life test.

In this section, we establish optimal censoring schemes under the Bayesian framework, for the proposed JAPC scheme. Determination of optimal life testing plans for Type-II progressive censoring scheme is discussed by various authors. For instance, Ng et al. [18] discussed A- and D-optimality criteria for constructing optimal plans for Weibull distribution; Pradhan and Kundu [21] obtained optimal censoring schemes for a generalized exponential distribution. Recently, Pradhan and Kundu [22] discussed Birnbaum-Saunders distribution and obtained optimal plans based on different criteria including minimizing $Var(\ln \hat{X}_p)$, where $\hat{X}_p$ represents the ML estimate of X_p, the p^{th} quantile of some random variable X. Notice that the criterion based on p^{th} quantile depends upon the value of p such that $0 < p < 1$.

In the above mentioned works, either sub-optimal censoring plans are obtained or optimal censoring plans are obtained by using complete search method for small values of n and m. Since for fixed n and m, the total number of feasible censoring schemes is $\binom{n-1}{m-1}$, even for a moderate higher values of n and m, establishing an optimal censoring scheme among all possible schemes turns out to be computationally inconvenient. Recently, Bhattacharya et al. [10] proposed a meta-heuristic based variable neighborhood search (VNS) algorithm to find an optimum or near optimum solution within a reasonable computation time for moderate to large values of n and m. Bhattacharya and Pradhan [8] proposed a modified version of the VNS algorithm and applied to compute optimum schemes under Type-I progressive hybrid censoring scheme.

Our aim here is to obtain the optimal progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$ such that $\sum_{i=1}^{m} R_i = n - m$, for some pre-fixed τ and different choices of n and m based on a variance minimization criterion as considered in Bhattacharya et al. [9]. In this paper, we consider a weighted posterior variance with the expected total time of the experiment. The expected running time of the experiment, denoted by $E(T_{m:m})$, for our proposed JAPC censoring scheme, can be calculated as follows

$$
\begin{aligned}
E(T_{m:n}) = C \sum_{d_1=1}^{m-1} \binom{m}{d_1} \Big(\frac{\theta_2}{\theta_1+\theta_2}\Big)^{d_1} \Big(\frac{\theta_1}{\theta_1+\theta_2}\Big)^{m-d_1} \\
\Bigg[\Bigg\{ \sum_{d=0}^{m-1} \sum_{i=0}^{d} \sum_{s=0}^{m-d-1} \frac{(-1)^{i+s}\binom{m-d-1}{s} e^{-\frac{\tau}{\theta}(i+s+1+\sum_{j=d-i+1}^{d} R_j)}}{\{\prod_{j=1}^{i}(j+\sum_{k=d-i+1}^{d-i+j} R_k)\}\{\prod_{j=1}^{d-i}(d-i-j+1+\sum_{k=j}^{d-i} R_k)\}} \Big(\frac{\theta}{s+1}\Big) \\
\frac{1}{(m-d-1)\,!}\Bigg\} + \Bigg\{ \sum_{i=0}^{m-1} \frac{(-1)^{i}}{\{\prod_{j=i}^{i}(j+\sum_{k=m-i}^{m-1-i+j} R_k)\}\{\prod_{j=1}^{m-1-i}(m-i-j+\sum_{k=j}^{m-1-j} R_k)\}} \\
\Big(\frac{1}{\theta} - \big(\frac{1}{\theta} + \frac{\tau}{(i+1+\sum_{j=m-i}^{m} R_j)}\big) e^{-\frac{\tau}{\theta}(i+1+\sum_{j=m-i}^{m} R_j)}\Big)\Bigg\}\Bigg]. \qquad (6)
\end{aligned}
$$

The above expression is obtained from the distribution of $T_{m:n}$, derived by the moment generating function technique. Details of the derivation are omitted and can be obtained from the authors upon request.

The constrained optimization problem is now formally formulated as follows:

$$
\text{Minimize } \ \phi(R) = w_1 E(\,V(\theta_1|Data)) + w_2 E(\,V(\theta_2|Data)) + w_3\, E(T_{m:n}), \qquad (7)
$$

such that $\sum_{i=1}^{m} R_i = n-m, 0 \leq w_i \leq 1, i = 1, 2, 3$, and $\sum_{i=1}^{3} w_i = 1$. Furthermore, $E(V(\theta_1|Data))$ and $E(V(\theta_2|Data))$ are the expected posterior variances of θ_1 and θ_2 respectively which can be obtained using the equations (4) and (5) and $E(T_{m:n})$ can be calculated from equation (6). To obtain the average values of posterior variances, one can also use the method given by Kundu [14]. Note that the above mentioned optimization problem (7) is a mixed integer non-linear programming problem in R, where R is a vector of non-negative integers. To find the optimum solution, we use a VNS based algorithm given by Bhattacharya et al. [10]. An initial guess $R_0 = (R_{01}, R_{02}, \cdots, R_{0m})$, is taken from the set $CS(n, m) = \Big\{R = (R_1, R_2, \cdots, R_m) :$

$\sum_{i=1}^{m} R_i = n - m \Big\}$ and denote $l_{max} = maxR_{0j}$, $1 \leq j \leq m$. For $l = 1, \cdots, l_{max}$, construct the neighborhood $N_l(R_0)$ of R_0 given by $N_l(R_0) = \Big\{ R : R \in CS(n,m)$ and $\|R - R_0\| < l + 1 \Big\}$ where $\|\cdot\|$ denotes standard Euclidean norm. Finally for fixed n, m, and τ with initial guess R_0 we obtain the optimum censoring scheme by solving the constrained optimization problem (7). Note that the desired calculation can be performed using *gtools* package in R software.

The computational algorithm for the optimization problem (7) is presented in Algorithm 3 in the Appendix section. For the purpose of illustration, we have considered various values of input parameters, and the computed optimum plans are reported in Table 11. In this table, we can see that when we put the whole weight on w_3, say, $w_3 = 1$, i.e., our main objective is to reduce the experimental time, then clearly the optimum solution should be typically Type-II censoring scheme, which one can see from the result of the table. Similarly, when we put the weight on the variance part, i.e., $w_1 + w_2 = 1$, then early removal is preferable, and it is likely reflected in Table 11. If we put positive weights on all w_i's then an optimum censoring scheme is somewhere in between, which also can be seen in Table 11. It is interesting to check that whether our proposed JAPC scheme takes lesser time to run the experiment than that of in BJPC scheme. With different choices of the parameters, n, m, R, and τ we compare $E(T_{m:n})$ obtained from equation (6) with the time to run the experiment in BJPC scheme numerically. It is to be noted that when $\tau \to \infty$, JAPC scheme coincides with the BJPC scheme. We report our findings in Table 2. In the table, we have used the term ERT to denote the experimental running time. Clearly, from Table 2, it is observed that the experimental time of the JAPC scheme is lesser than the BJPC scheme. Also in the same table we report MSE of MLE of each parameter both under JAPC scheme and BJPC scheme for comparison purpose. However we can not compare about the MSE of MLE of a parameter under JAPC scheme with BJPC scheme. For example, when $n = 20, m = 10, \tau = 0.25$ and $R = (5, 2, 0^{*4}, 2, 0, 1, 0)$, MSE of $\hat{\theta}_1$ under JAPC scheme is greater than MSE of $\hat{\theta}_1$ under BJPC model. The same is true for $\hat{\theta}_2$. But this is not true for some other cases.

Table (2) Comparison of experimental running time between JAPC and BJPC model with different choice of (n, m) and τ.

					ERT		MSE (JAPC)		MSE (BJPC)	
n	m	τ	R	(θ_1, θ_2)	JAPC	BJPC	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_1$	$\hat{\theta}_2$
10	5	0.25	$(0^{*3}, 2, 3)$	(0.5,0.7)	0.209	0.211	0.195	0.315	0.206	0.334
	5	0.50	$(2, 1, 1, 1, 0)$		0.513	0.530	0.194	0.332	0.199	0.337
20	10	0.25	$(5, 2, 0^{*4}, 2, 0, 1, 0)$		0.524	0.652	0.083	0.350	0.085	0.372
	10	0.50	$(0, 2, 0^{*2}, 2, 3, 2, 0, 1, 0)$		0.598	0.615	0.094	0.334	0.094	0.350
30	15	0.25	$(2^{*5}, 1, 1, 3, 0{*}7)$		0.867	0.871	0.049	0.185	0.045	0.180
	15	0.50	$(0^{*5}, 1, 0^{*2}, 2^{*7})$		0.335	0.338	0.039	0.190	0.041	0.234
10	5	0.25	$(0^{*3}, 2, 3)$	(1.0,2.0)	0.444	0.491	0.671	2.095	0.782	2.043
	5	0.50	$(2, 1, 1, 1, 0)$		1.005	1.193	0.764	2.127	0.733	1.994
20	10	0.25	$(5, 2, 0^{*4}, 2, 0, 1, 0)$		0.891	1.527	0.228	3.409	0.226	3.138
	10	0.50	$(0, 2, 0^{*2}, 2, 3, 2, 0, 1, 0)$		1.098	1.399	0.246	3.177	0.241	3.179
30	15	0.25	$(2^{*5}, 1, 1, 3, 0^{*7})$		1.335	1.999	0.128	2.290	0.124	2.383
	15	0.50	$(0^{*5}, 1, 0^{*2}, 2^{*7})$		0.691	0.779	0.124	2.386	0.119	2.664

10 Conclusion

In this work, we have proposed the JAPC scheme for two exponential populations. We provide both classical and the Bayesian inferences of the unknown parameters and the performances of both the estimators are quite satisfactory. Also, we have shown that our proposed scheme takes less experimental time compared to the existing BJPC scheme. Further, we have formulated an optimization problem, in which we minimize the weighted posterior variance and expected total time on test and we have computed the optimum schemes by VNS algorithm for different n and m. In this paper, we consider only one parameter exponential distribution. It will be interesting to develop inference procedures for other distributions like two-parameter Weibull or generalized exponential distributions. It may be mentioned that although in the case of exponential distribution it is possible to derive the exact distribution of the MLEs, it may not be the case for other distributions. It is more challenging in those cases. Also, in this paper we consider the problem on two populations with the same sample size only but one can also generalize this idea for more than two populations as well as different sample sizes. More work is needed along that direction.

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Appendix

1. Proof of Theorem 1

To prove Theorem 1, we need the following theorem.

Theorem A:

For $d \in \{0, 1, \cdots, m\}, d_1 \in \{1, \cdots, m-1\}$, the conditional distribution of $\{T_{1:n}, \cdots, T_{m:n}\}$ conditioning on the event $\{D = d, D_1 = d_1, D_2 = m - d_1\} = G(d, d_1)$ (say) is,

$$f_{T_{1:n},\ldots,T_{m:n}|G(d,d_1)}(t_1 \ldots t_m) = \frac{1}{P(G(d,d_1))} C \left(\frac{1}{\theta_1}\right)^{d_1} \left(\frac{1}{\theta_2}\right)^{m-d_1} e^{-\frac{1}{\theta}\sum_{i=1}^{m} t_i(1+R_i^*)},$$
$$\text{where, } 0 < t_1 < \ldots < t_d < \tau < t_{d+1} < \ldots < t_m.$$

Proof of **Theorem A** can be obtained from the given likelihood function.

$E\left[e^{t\hat{\theta}_1}|D_1 > 0, D_2 > 0\right]$, the conditional MGF of $\hat{\theta}$, conditioning on $E = \{D_1 > 0, D_2 > 0\}$ is obtained below.

$$E\left[e^{t\hat{\theta}_1}|E\right]$$
$$= \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} E\left[e^{t\hat{\theta}_1}|G(d, d_1)\right] P[G(d, d_1)|E]$$

$$=\frac{1}{P(E)}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}E\big[e^{t\hat{\theta}_1}|G(d,d_1)\big]P(G(d,d_1))$$

$$=\frac{1}{P(E)}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\int\underset{V_d}{\cdots}\int e^{\frac{t}{d_1}\sum_{i=1}^{m}x_i(1+R_i^*)}f_{T_{1:m},\ldots,T_{m:m}|G(d,d_1)}(t_1,\ldots,t_m)\,dt_1\ldots dt_m\;P(G(d,d_1))$$

{where, V_d is the region $\{0<t_1<\ldots<t_d<\tau<t_{d+1}<\ldots<t_m\}$}

$$=\frac{C}{P[E]}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\int\underset{V_d}{\cdots}\int\big(\frac{1}{\theta_1}\big)^{d_1}\big(\frac{1}{\theta_2}\big)^{m-d_1}\;e^{-(\frac{1}{\theta}-\frac{t}{d_1})\sum_{i=1}^{m}t_i(1+R_i^*)}\,dt_1\ldots dt_m$$

$$=\frac{C\,\left(\frac{1}{\theta}-\frac{t}{d_1}\right)^{-m}}{P[E]}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\big(\frac{1}{\theta_1}\big)^{d_1}\big(\frac{1}{\theta_2}\big)^{m-d_1}\int\underset{V_d}{\cdots}\int\big(\frac{1}{\theta}-\frac{t}{d_1}\big)^{m}\;e^{-(\frac{1}{\theta}-\frac{t}{d_1})\sum_{i=1}^{m}t_i(1+R_i^*)}\,dt_1\ldots dt_m$$

$$=\frac{C\,\left(1-\frac{t\theta}{d_1}\right)^{-m}}{P[E]}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\sum_{i=0}^{d}\sum_{s=0}^{m-d-1}\Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)}\big(\frac{\theta_1}{\theta_1+\theta_2}\big)^{m-d_1}\big(\frac{\theta_2}{\theta_1+\theta_2}\big)^{d_1}$$

$$\times\frac{(-1)^{i+s}\;e^{-\left(\frac{1}{\theta}-\frac{t}{d_1}\right)\tau\left(n-d-1+i-\sum_{j=1}^{d-i}R_j\right)}}{\big[\prod_{j=1}^{i}\sum_{k=d-i+1}^{d-i+j}\big(1+R_k\big)\big]\;\big[\prod_{j=1}^{d-i}\sum_{k=j}^{d-i}\big(1+R_k\big)\big]\big[n-m+s-\sum_{i=1}^{d}R_i\big]}\Bigg]$$

{Part of the above integration is carried out by using the theorem from the paper Balakrishnan et al. [2]}

Hence, by the inversion property of MGF, the conditional distribution of $\hat{\theta}_1$, conditioning on the event $E=\{D_1>0,D_2>0\}$ is obtained as,

$$f_{\hat{\theta}_1|D_1>0,D_2>0}(x)=\frac{1_x(0,\infty)\;C}{P[D_1>0,D_2>0]}\sum_{d=0}^{m}\sum_{d_1=1}^{m-1}\sum_{i=0}^{d}\sum_{s=0}^{m-d-1}\Bigg[\frac{\binom{(m-d-1)}{s}}{(m-d-1)}\big(\frac{\theta_1}{\theta_1+\theta_2}\big)^{m-d_1}\big(\frac{\theta_2}{\theta_1+\theta_2}\big)^{d_1}$$

$$\times\frac{(-1)^{i+s}\;e^{-\frac{1}{\theta}\tau(n-d-1+i-\sum_{j=1}^{d-i}R_j)}}{\big[\prod_{j=1}^{i}\sum_{k=d-i+1}^{d-i+j}\big(1+R_k\big)\big]\;\big[\prod_{j=1}^{d-i}\sum_{k=j}^{d-i}\big(1+R_k\big)\big]\big[n-m+s-\sum_{i=1}^{d}R_i\big]}$$

$$\times f_{gamma}\Big(x-\frac{\tau}{d_1}\big(n-d-1+i-\sum_{j=1}^{d-i}R_j\big);\frac{d_1}{\theta},m\Big)\Bigg],$$

where, $1_x(0,\infty)=1$(or 0), according to $x>0$ (or ≤ 0).

Also,

$$
\begin{aligned}
&P(D_1 > 0, D_2 > 0) \\
&= \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} P(D = d, D_1 = d_1, D_2 = m - d_1) \\
&= \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \int \underset{V_d}{\cdots} \int f_{T_{1:m},\ldots,T_{m:m}|D=m,D_1=d_1,D_2=m-d_1}(t_1, \ldots, t_m)\, dt_1 \ldots dt_m
\end{aligned}
$$

$\{$where, V_d is the region $\{0 < t_1 < \ldots < t_d < \tau < t_{d+1} < \ldots < t_m\}\}$

$$
\begin{aligned}
&= \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \int \underset{V_d}{\cdots} \int C \left(\frac{1}{\theta_1}\right)^{d_1} \left(\frac{1}{\theta_2}\right)^{m-d_1} e^{-\left(\frac{1}{\theta} - \frac{t}{d_1}\right) \sum_{i=1}^{m} t_i (1+R_i^*)}\, dt_1 \ldots dt_m \\
&= C \left(\frac{1}{\theta}\right)^{-m} \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \left(\frac{1}{\theta_1}\right)^{d_1} \left(\frac{1}{\theta_2}\right)^{m-d_1} \int \underset{V_d}{\cdots} \int \left(\frac{1}{\theta}\right)^{m} e^{-\frac{1}{\theta} \sum_{i=1}^{m} t_i (1+R_i^*)}\, dt_1 \ldots dt_m \\
&= C \sum_{d=0}^{m} \sum_{d_1=1}^{m-1} \sum_{i=0}^{d} \sum_{s=0}^{m-d-1} \Bigg[\frac{\binom{m-d-1}{s}}{(m-d-1)} \left(\frac{\theta_1}{\theta_1 + \theta_2}\right)^{m-d_1} \left(\frac{\theta_2}{\theta_1 + \theta_2}\right)^{d_1} \\
&\times \frac{(-1)^{i+s}\, e^{-\frac{1}{\theta}\tau\left(n-d-1+i-\sum_{j=1}^{d-i} R_j\right)}}{\left[\prod_{j=1}^{i} \sum_{k=d-i+1}^{d-i+j} \left(1 + R_k\right)\right] \left[\prod_{j=1}^{d-i} \sum_{k=j}^{d-i} \left(1 + R_k\right)\right] \left[n - m + s - \sum_{i=1}^{d} R_i\right]} \Bigg]
\end{aligned}
$$

$\{$Part of the above integration is carried out by using the theorem from the paper Balakrishnan et al. [2]. $\}$

The PDF of $\hat{\theta}_1$ is easily obtained from its CDF.

2. Proof of Theorem 2

Theorem 2 can be proved by the similar way as above and hence details are omitted.

3. Description of Algorithm 3

Algorithm 3: Constrained optimization method using VNS based algorithm

Input:
Fix n, m, and τ
Output:
$R^{opt} = R^{new}$
Choose a random point R_0 from $CS(n, m)$;
Find $E(T_m)$ by solving (6) for given n, m, and τ;
Set $R^{int} = R_0$ as initial solution;
Suppose, $l_{max} = maxR_{0j}, 1 \leq j \leq m$;
Set $l = 1$
while $l \leq l_{max}$ **do**
K_l denotes the number of points in each neighborhood.
for $i = 1, \cdots, K_l$ **do**
Perform local search on $N_l(R_0)$.
Suppose R^{new} is a neighbor of R_0 in $N_l(R_0)$
if $(\phi(R^{new}) < \phi(R^{int}))$ **then**
$R^{new} = R^{int}$
$l = 1$
end
else
$l = l + 1$
end
end
end
$R^{opt} = R^{new}$.

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Table (3) AE, MSEs of the MLES and interval estimates of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (0.5, 0.7)$ under classical framework.

			θ_1				θ_2			
(n,m)	Scheme	τ	AE	MSE	AL	CP	AE	MSE	AL	CP
$(30,10)$		0.25	0.551	0.078	1.937	0.959	0.859	0.409	7.085	0.940
			0.537	0.076	1.692	0.955	0.853	0.342	7.360	0.951
	$(20, 0^{*9})$		0.545	0.095	1.956	0.956	0.843	0.363	6.838	0.944
	$(0^{*9}, 20)$	0.5	0.559	0.097	1.942	0.974	0.859	0.389	7.289	0.944
			0.544	0.076	1.859	0.955	0.858	0.383	7.308	0.946
	(2^{*10})		0.543	0.078	1.935	0.946	0.846	0.340	7.208	0.954
		1	0.555	0.083	1.874	0.966	0.859	0.389	7.289	0.944
			0.544	0.076	1.859	0.955	0.858	0.383	7.308	0.946
			0.553	0.111	2.165	0.942	0.848	0.367	7.145	0.955
$(30,15)$		0.25	0.533	0.045	0.982	0.951	0.788	0.217	2.673	0.951
			0.527	0.039	0.983	0.960	0.767	0.159	2.549	0.952
	$(15, 0^{*14})$		0.533	0.045	0.967	0.960	0.774	0.160	2.459	0.950
	$(0^{*14}, 15)$	0.5	0.533	0.045	0.983	0.951	0.788	0.217	2.673	0.951
			0.529	0.040	0.941	0.952	0.789	0.169	2.523	0.963
	(1^{*15})		0.521	0.039	0.919	0.951	0.806	0.233	2.728	0.944
		1	0.533	0.045	0.982	0.951	0.788	0.217	2.673	0.951
			0.529	0.040	0.941	0.952	0.789	0.169	2.523	0.963
			0.524	0.042	0.930	0.954	0.797	0.186	2.693	0.942

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Table (4) AE, MSEs of the MLES and interval estimates of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (1, 1.5)$ under classical framework.

			θ_1				θ_2			
(n, m)	Scheme	τ	AE	MSE	AL	CP	AE	MSE	AL	CP
$(30, 10)$		0.5	1.112	0.356	3.498	0.958	1.833	1.683	14.075	0.950
			1.065	0.260	3.123	0.964	1.866	1.958	14.412	0.950
	$(20, 0^{*9})$		1.089	0.416	3.272	0.950	1.866	2.119	13.649	0.939
	$(0^{*9}, 20)$	1	1.112	0.356	3.498	0.958	1.833	1.683	14.075	0.950
	(2^{*10})		1.077	0.273	3.216	0.963	1.860	1.949	14.009	0.947
			1.062	0.261	3.240	0.952	1.856	1.790	14.142	0.951
		1.5	1.112	0.356	3.498	0.958	1.833	1.683	14.075	0.950
			1.077	0.273	3.216	0.963	1.860	1.949	14.009	0.947
			1.078	0.348	3.450	0.951	1.847	1.840	13.812	0.948
$(30, 15)$		0.5	1.059	0.250	1.947	0.945	1.673	0.817	5.277	0.957
			1.056	0.151	1.896	0.963	1.614	0.642	5.506	0.960
	$(15, 0^{*14})$		1.066	0.197	1.965	0.948	1.704	0.974	5.638	0.957
	$(0^{*14}, 15)$	1	1.059	0.250	1.947	0.945	1.673	0.817	5.277	0.957
	(1^{*15})		1.050	0.145	1.799	0.953	1.732	1.084	5.817	0.955
			1.069	0.165	1.858	0.959	1.748	1.316	6.532	0.944
		1.5	1.059	0.250	1.947	0.945	1.673	0.817	5.277	0.957
			1.050	0.145	1.799	0.953	1.732	1.084	5.817	0.955
			1.045	0.177	1.863	0.952	1.739	1.079	6.618	0.953

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Table (5) AE, MSEs of the MLES and interval estimates of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (0.5, 0.7)$ under classical framework.

			θ_1				θ_2			
(n, m)	Scheme	τ	AE	MSE	AL	CP	AE	MSE	AL	CP
$(40, 15)$		0.25	0.494	0.022	0.829	0.980	0.813	0.279	2.902	0.940
			0.539	0.045	0.979	0.960	0.812	0.267	2.811	0.949
	$(25, 0^{*14})$		0.518	0.032	0.915	0.976	0.843	0.350	3.294	0.948
	$(0^{*14}, 25)$	0.5	0.498	0.038	0.979	0.972	0.811	0.275	2.903	0.954
			0.519	0.034	0.923	0.964	0.785	0.148	2.259	0.964
	$(2^{*10}, 1^{*5})$		0.510	0.037	0.886	0.956	0.784	0.154	2.599	0.968
		1	0.498	0.038	0.979	0.971	0.813	0.279	2.902	0.952
			0.519	0.034	0.923	0.964	0.781	0.144	2.260	0.962
			0.531	0.038	0.954	0.960	0.745	0.117	2.007	0.980
$(40, 25)$		0.25	0.512	0.020	0.619	0.970	0.769	0.067	1.740	0.967
			0.509	0.018	0.613	0.960	0.725	0.065	1.118	0.973
	$(15, 0^{*24})$		0.518	0.021	0.623	0.947	0.749	0.088	1.191	0.966
	$(0^{*24}, 15)$	0.5	0.512	0.020	0.619	0.972	0.723	0.065	1.250	0.972
			0.531	0.021	0.648	0.964	0.735	0.093	1.144	0.968
	$(0^{*7}, 2^{*5}, 0^{*8}, 1^{*5})$		0.508	0.019	0.605	0.954	0.731	0.062	1.129	0.961
		1	0.512	0.020	0.619	0.953	0.733	0.054	1.148	0.971
			0.531	0.021	0.646	0.962	0.726	0.061	1.117	0.969
			0.518	0.022	0.624	0.946	0.719	0.051	1.107	0.973

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Table (6) AE, MSEs of the MLES and interval estimates of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (1, 1.5)$ under classical framework.

			θ_1				θ_2			
(n,m)	Scheme	τ	AE	MSE	AL	CP	AE	MSE	AL	CP
$(40,15)$		0.5	1.059	0.145	1.815	0.958	1.657	0.625	5.507	0.961
			1.040	0.162	1.815	0.950	1.646	0.566	5.109	0.967
	$(25, 0^{*14})$		1.058	0.162	1.826	0.950	1.677	0.600	5.419	0.973
	$(0^{*14}, 25)$	1	1.054	0.151	1.823	0.955	1.657	0.625	5.507	0.961
			1.060	0.169	1.843	0.950	1.640	0.585	5.092	0.963
	$(2^{*10}, 1^{*5})$		1.053	0.156	1.826	0.954	1.647	0.615	5.254	0.951
		1.5	1.061	0.164	1.883	0.955	1.657	0.625	6.268	0.961
			1.047	0.157	1.814	0.957	1.640	0.585	5.092	0.963
			1.060	0.160	1.837	0.950	1.647	0.620	5.366	0.959
$(40,25)$		0.5	1.029	0.080	1.212	0.957	1.573	0.324	2.491	0.952
			1.026	0.077	1.209	0.966	1.592	0.325	2.511	0.962
	$(15, 0^{*24})$		1.031	0.079	1.213	0.950	1.590	0.337	2.561	0.953
	$(0^{*24}, 15)$	1	1.033	0.075	1.211	0.951	1.573	0.324	2.491	0.952
			1.027	0.078	1.205	0.952	1.602	0.343	2.600	0.950
	$(0^{*7}, 2^{*5}, 0^{*8}, 1^{*5})$		1.026	0.068	1.209	0.973	1.582	0.335	2.548	0.962
		1.5	1.034	0.079	1.220	0.960	1.573	0.324	2.491	0.952
			1.027	0.078	1.205	0.952	1.602	0.343	2.600	0.950
			1.025	0.069	1.125	0.955	1.619	0.374	2.629	0.956

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Table (7) AE, MSEs of the Bayes estimates and symmetric and HPD credible interval of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (0.5, 0.7)$ under Bayesian framework.

			θ_1						θ_2					
(n,m)	Scheme	τ	AE	MSE	AL CRI	CP CRI	AL HPD	CP HPD	AE	MSE	AL CRI	CP CRI	AL HPD	CP HPD
$(30,10)$		0.25	0.586	0.042	1.105	0.988	0.960	0.966	0.797	0.087	1.761	0.981	1.476	0.957
			0.586	0.044	1.105	0.983	0.960	0.964	0.789	0.085	1.737	0.979	1.457	0.952
	$(20, 0^{*9})$		0.584	0.042	1.104	0.983	0.959	0.962	0.771	0.073	1.6833	0.978	1.415	0.954
	$(0^{*9}, 20)$	0.5	0.573	0.041	1.078	0.979	0.938	0.961	0.786	0.084	1.743	0.979	1.459	0.958
	(2^{*10})		0.583	0.043	1.100	0.983	0.955	0.967	0.794	0.084	1.757	0.982	1.472	0.958
			0.581	0.043	1.099	0.980	0.954	0.963	0.776	0.074	1.707	0.985	1.432	0.958
		1	0.582	0.041	1.096	0.990	0.953	0.976	0.787	0.084	1.733	0.974	1.454	0.955
			0.576	0.043	1.085	0.982	0.943	0.964	0.795	0.087	1.767	0.978	1.478	0.956
			0.579	0.043	1.095	0.989	0.951	0.964	0.771	0.071	1.689	0.977	1.418	0.950
$(30,15)$		0.25	0.569	0.032	0.842	0.984	0.763	0.970	0.788	0.072	1.386	0.976	1.218	0.960
			0.573	0.035	0.852	0.983	0.771	0.969	0.783	0.070	1.374	0.976	1.208	0.955
	$(15, 0^{*9})$		0.560	0.029	0.834	0.986	0.755	0.967	0.762	0.064	1.337	0.975	1.176	0.956
	$(0^{*9}, 15)$	0.5	0.573	0.034	0.847	0.978	0.767	0.963	0.793	0.075	1.394	0.977	1.225	0.956
	(1^{*15})		0.572	0.033	0.846	0.984	0.767	0.971	0.791	0.072	1.388	0.983	1.220	0.962
			0.563	0.032	0.832	0.983	0.754	0.965	0.786	0.073	1.388	0.981	1.219	0.957
		1	0.573	0.036	0.850	0.982	0.770	0.964	0.791	0.076	1.394	0.970	1.224	0.958
			0.574	0.033	0.850	0.985	0.770	0.972	0.788	0.073	1.381	0.976	1.214	0.953
			0.573	0.034	0.849	0.991	0.769	0.974	0.784	0.070	1.371	0.980	1.206	0.960

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Table (8) AE, MSEs of the Bayes estimates and symmetric and HPD credible interval of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (1, 1.5)$ under Bayesian framework.

(n, m)	Scheme	τ	θ_1 AE	θ_1 MSE	θ_1 AL CRI	θ_1 CP CRI	θ_1 AL HPD	θ_1 CP HPD	θ_2 AE	θ_2 MSE	θ_2 AL CRI	θ_2 CP CRI	θ_2 AL HPD	θ_2 CP HPD
(30, 10)		0.5	1.028	0.073	1.841	0.986	1.616	0.969	1.775	0.586	4.206	0.976	3.461	0.955
			1.293	0.361	2.455	0.967	2.127	0.972	1.860	0.719	4.284	0.980	3.544	0.965
	$(20, 0^{*9})$		1.285	0.331	2.4417	0.978	2.115	0.972	1.841	0.736	4.237	0.974	3.507	0.952
	$(0^{*9}, 20)$	1	1.275	0.355	2.416	0.965	2.095	0.971	1.853	0.722	4.289	0.977	3.543	0.961
	(2^{*10})		1.244	0.262	2.363	0.986	2.048	0.972	1.683	0.402	3.742	0.975	3.127	0.955
			1.249	0.271	2.379	0.982	2.062	0.967	1.667	0.377	3.690	0.978	3.089	0.955
		1.5	1.242	0.261	2.353	0.989	2.040	0.982	1.688	0.377	3.753	0.975	3.138	0.954
			1.240	0.264	2.350	0.983	2.039	0.969	1.693	0.382	3.766	0.981	3.148	0.964
			1.236	0.263	2.343	0.985	2.033	0.971	1.691	0.389	3.773	0.979	3.151	0.961
(30, 15)		0.5	1.203	0.195	1.782	0.967	1.613	0.975	1.694	0.335	3.011	0.977	2.637	0.958
			1.182	0.184	1.7425	0.967	1.578	0.972	1.686	0.337	2.999	0.976	2.628	0.961
	$(15, 0^{*14})$		1.151	0.151	1.702	0.976	1.541	0.972	1.647	0.309	2.940	0.977	2.574	0.954
	$(0^{*9}, 15)$	1	1.192	0.190	1.770	0.969	1.601	0.966	1.680	0.346	2.994	0.976	2.621	0.957
	(1^{*15})		1.193	0.188	1.768	0.962	1.601	0.972	1.671	0.340	2.968	0.975	2.601	0.949
			1.198	0.200	1.776	0.964	1.607	0.973	1.687	0.335	2.999	0.983	2.628	0.964
		1.5	1.188	0.192	1.757	0.967	1.591	0.973	1.673	0.341	2.965	0.970	2.601	0.948
			1.194	0.199	1.769	0.960	1.601	0.969	1.688	0.341	3.008	0.970	2.634	0.949
			1.187	0.186	1.753	0.969	1.588	0.970	1.681	0.327	2.983	0.977	2.616	0.958

Table (9) AE, MSEs of the Bayes estimates and symmetric and HPD credible interval of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (1, 1.5)$ under Bayesian framework.

(n, m)	Scheme	τ	θ_1 AE	θ_1 MSE	θ_1 AL CRI	θ_1 CP CRI	θ_1 AL HPD	θ_1 CP HPD	θ_2 AE	θ_2 MSE	θ_2 AL CRI	θ_2 CP CRI	θ_2 AL HPD	θ_2 CP HPD
(40, 15)		0.5	1.148	0.138	1.693	0.987	1.535	0.973	1.577	0.213	2.748	0.976	2.420	0.951
			1.154	0.138	1.705	0.983	1.545	0.968	1.572	0.206	2.731	0.972	2.406	0.947
	$(25, 0^{*14})$		1.136	0.128	1.685	0.984	1.526	0.972	1.545	0.201	2.690	0.974	2.369	0.942
	$(0^{*14}, 25)$	1	1.137	0.133	1.676	0.979	1.519	0.965	1.570	0.201	2.742	0.979	2.414	0.954
	$(2^{*10}, 1^{*5})$		1.144	0.139	1.686	0.982	1.528	0.967	1.583	0.209	2.766	0.980	2.435	0.955
			1.147	0.136	1.696	0.983	1.537	0.966	1.558	0.208	2.710	0.975	2.387	0.953
		1.5	1.146	0.140	1.693	0.979	1.534	0.963	1.571	0.211	2.743	0.975	2.414	0.951
			1.146	0.136	1.687	0.980	1.529	0.963	1.591	0.213	2.783	0.979	2.449	0.961
			1.142	0.139	1.690	0.975	1.531	0.964	1.561	0.208	2.722	0.974	2.397	0.954
(40, 25)		0.5	1.107	0.090	1.207	0.963	1.135	0.967	1.613	0.173	2.140	0.975	1.968	0.964
			1.119	0.103	1.220	0.949	1.148	0.964	1.661	0.253	2.234	0.973	2.048	0.962
	$(15, 0^{*24})$		1.065	0.085	1.167	0.959	1.097	0.953	1.575	0.219	2.125	0.970	1.947	0.949
	$(0^{*24}, 15)$	1	1.117	0.099	1.216	0.958	1.144	0.960	1.680	0.272	2.270	0.980	2.080	0.967
	$(0^{*7}, 2^{*5}, 0^{*8}, 1^{*5})$		1.116	0.106	1.221	0.952	1.148	0.959	1.644	0.245	2.208	0.979	2.024	0.959
			1.118	0.105	1.225	0.952	1.151	0.959	1.651	0.244	2.220	0.971	2.035	0.954
		1.5	1.120	0.104	1.223	0.947	1.150	0.962	1.663	0.257	2.238	0.978	2.051	0.965
			1.125	0.109	1.233	0.949	1.159	0.964	1.654	0.264	2.227	0.977	2.041	0.958
			1.121	0.108	1.226	0.944	1.153	0.956	1.668	0.277	2.253	0.979	2.063	0.961

Table (10) AE, MSEs of the Bayes estimates and symmetric and HPD credible interval of θ_1 and θ_2 for different choices of τ when $(\theta_1, \theta_2) = (0.5, 0.7)$ under Bayesian framework.

			θ_1						θ_2					
(n, m)	Scheme	τ	AE	MSE	AL CRI	CP CRI	AL HPD	CP HPD	AE	MSE	AL CRI	CP CRI	AL HPD	CP HPD
$(40, 15)$		0.25	0.567	0.033	0.846	0.981	0.765	0.965	0.747	0.050	1.286	0.977	1.135	0.960
			0.569	0.033	0.851	0.982	0.770	0.965	0.744	0.049	1.281	0.972	1.130	0.949
	$(25, 0^{*14})$		0.566	0.033	0.848	0.981	0.767	0.961	0.734	0.045	1.263	0.976	1.114	0.954
	$(0^{*14}, 25)$	0.5	0.572	0.034	0.853	0.979	0.772	0.963	0.751	0.050	1.294	0.979	1.142	0.962
	$(2^{*10}, 1^{*5})$		0.575	0.036	0.863	0.985	0.780	0.967	0.740	0.049	1.267	0.973	1.119	0.953
			0.570	0.034	0.852	0.980	0.770	0.961	0.744	0.048	1.278	0.975	1.128	0.954
		1	0.569	0.034	0.848	0.982	0.767	0.964	0.748	0.047	1.285	0.982	1.135	0.962
			0.571	0.033	0.854	0.984	0.772	0.970	0.741	0.048	1.270	0.981	1.121	0.965
			0.573	0.034	0.857	0.985	0.775	0.972	0.747	0.049	1.284	0.977	1.134	0.958
$(40, 25)$		0.25	0.558	0.028	0.618	0.944	0.580	0.960	0.788	0.068	1.050	0.972	0.963	0.961
			0.561	0.029	0.622	0.947	0.584	0.953	0.802	0.074	1.076	0.960	0.986	0.964
	$(15, 0^{*24})$		0.532	0.021	0.589	0.956	0.553	0.950	0.758	0.062	1.015	0.963	0.931	0.954
	$(0^{*24}, 15)$	0.5	0.564	0.029	0.629	0.948	0.590	0.956	0.784	0.070	1.043	0.959	0.957	0.957
	$(0^{*7}, 2^{*5}, 0^{*8}, 1^{*5})$		0.554	0.025	0.612	0.953	0.574	0.956	0.793	0.072	1.063	0.963	0.974	0.961
			0.558	0.027	0.619	0.955	0.581	0.956	0.792	0.073	1.062	0.966	0.973	0.958
		1	0.551	0.024	0.608	0.958	0.571	0.965	0.792	0.070	1.059	0.966	0.9714	0.961
			0.558	0.026	0.619	0.955	0.581	0.962	0.785	0.070	1.046	0.964	0.959	0.959
			0.553	0.024	0.609	0.962	0.572	0.962	0.793	0.070	1.059	0.960	0.972	0.963

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Table (11) Optimal progressive censoring schemes under the Bayesian framework with different choice of (n, m) and τ.

n	m	τ	(θ_1, θ_2)	(w_1, w_2, w_3)	Optimum Scheme	Optimum value
10	5	0.25	(0.5,0.7)	$(0.35, 0.35, 0.3)$	$(0^{*3}, 2, 3)$	0.3514
	5			$(0.45, 0.45, 0.1)$	$(3, 0, 1, 1, 0)$	0.3829
	5			$(0.5, 0.5, 0)$	$(3, 0, 1, 1, 0)$	0.3742
	5			$(0, 0, 1)$	$(0^{*4}, 5)$	0.1877
	7	0.5	$(0.5, 0.7)$	$(0.45, 0.45, 0.1)$	$(0^{*6}, 3)$	0.3127
	7			$(0.1, 0.1, 0.8)$	$(0^{*6}, 3)$	0.3206
	7			$(0, 0, 1)$	$(0^{*6}, 3)$	0.3182
	7			$(1, 0, 0)$	$(2, 0^{*4}, 1, 0)$	0.1712
	7			$(0, 1, 0)$	$(1, 0^{*2}, 1, 0, 1, 0)$	0.4495
	7			$(0.5, 0.5, 0)$	$(0^{*2}, 2, 0^{*2}, 1, 0)$	0.2980
20	10	0.25	(0.5,0.7)	$(0, 0, 1)$	$(0^{*9}, 10)$	0.1950
	10			$(0.5, 0.5, 0)$	$(5, 2, 0^{*4}, 2, 0, 1, 0)$	0.1785
	10			$(0.1, 0.1, 0.8)$	$(0^{*9}, 10)$	0.1966
	10			$(1, 0, 0)$	$(10, 0^{*9})$	0.1922
	15	0.5	$(0.5, 0.7)$	$(0, 0, 1)$	$(0^{*14}, 5)$	0.3825
	15			$(0.1, 0.1, 0.8)$	$(0^{*14}, 5)$	0.3282
	15			$(0.2, 0.2, 0.6)$	$(0^{*14}, 5)$	0.2742
	15			$(1, 0, 0)$	$(2^{*2}, 0^{*4}, 0^{*6}, 1, 0^{*2})$	0.0457