Generalized Exponential Geometric Extreme Distribution

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Abstract

Recently Louzada et al. ("The exponentiated exponential-geometric distribution: a distribution with decreasing, increasing and unimodal hazard function", Statistics, 2014) proposed a new three-parameter distribution with decreasing, increasing and unimodal hazard functions. They have provided several properties of the distribution, and discussed different inferential issues. In this paper we discuss a generalized version of the model following the approach of Marshall and Olkin ("A new method for adding a parameter to a family of distributions with application to exponential and Weibull families", Biometrika, 1997). The proposed model is more flexible than the Louzada-Marchi-Roman model, although they have the same number of parameters. The model has three unknown parameters, and the hazard function can take different shapes. We propose to use the EM algorithm to compute the maximum likelihood estimators of the unknown parameters. We further consider the bivariate generalization of the proposed model and discuss its different properties. The EM algorithm can be used to estimate the unknown parameters in case of the bivariate model also. One bivariate data set has been analyzed for illustrative purposes, and the performance is quite satisfactory.

Key Words and Phrases Generalized exponential distribution; hazard function; probability density function; maximum likelihood estimator; Fisher information matrix; competing risks, EM algorithm; geometric distribution.

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1 Introduction

Exponentiated exponential or Generalized exponential (GE) distribution has received considerable attention since its introduction by Gupta and Kundu (1999). Extensive work has been done on GE and its related distribution by several authors for the last 10-12 years. Adamidis and Loukas (1998) introduced a two-parameter exponentiated-exponential-geometric (E2G) distribution using the latent competing risks scenario. The basic idea is very simple and it is as follows. Suppose, $T_1, T_2, \ldots$ is a sequence of independent identically distributed (i.i.d.) exponential random variables, and M is a geometric random variable, which is independent of T_i's. Then the distribution of Y, where

$$Y = \min\{T_1, \ldots, T_M\}, \tag{1}$$

is known as the E2G distribution.

The two-parameter exponential-geometric distribution has been recently generalized by Louzada et al. (2014), where they have replaced the exponential distribution of T_i's by the GE distribution, and named it as the exponentiated-exponential-geometric (E2G) distribution. It is observed that the three-parameter E2G distribution is quite flexible, and its probability density function (PDF) can take different shapes. It can have increasing, decreasing and unimodal hazard functions depending on the parameter values, and it has some interesting physical interpretations also. In the same paper the authors discussed some inferential issues also.

In this paper, we consider a three-parameter distribution following a similar approach as Marshall and Olkin (1997), see also Ristić and Kundu (2014) in this respect. It is observed that the E2G distribution can be obtained as a special case of the proposed model. The proposed model can be seen as a latent competing risks or complementary risks model, depending on the parameter values. The hazard function of the proposed model can take different shapes namely (i) increasing, (ii) decreasing, (iii) unimodal (iv) U-shaped. It is observed that the generation of a random sample from the proposed model is fairly simple, and we discuss several other properties also for this new distribution. We propose to use the EM algorithm to compute the maximum likelihood estimators (MLEs) of the unknown parameters. It is observed that at each 'E'-step, the corresponding 'M'-step can be performed by solving a one dimensional optimization problem, hence it is quite easy to implement it in practice. We perform the analysis of a data set for illustrative purposes. It is observed the proposed model and the EM algorithm work quite well in practice.

We propose a bivariate generalization of the proposed model. The bivariate model has singular component along the line $x = y$, similarly, as the the Marshall-Olkin bivariate exponential model. Several one dimensional results can be easily generalized to the bivariate case. We discuss various other properties also. We discuss in details one specific case when the proposed bivariate distribution has absolute continuous probability density function, which has some interesting features. The EM algorithm can be extended for the bivariate case. One bivariate data has been analyzed for illustrative purposes. Finally we conclude the paper.

Rest of the paper is organized as follows. In Section 2, we provide a brief review of the univariate and bivariate GE distributions. The model and its properties are discussed in Section 3. In Section 4, we consider the EM algorithm. In Section 5, we discuss the bivariate extension, and one specific bivariate case is discussed in details in Section 6. We provide the bivariate extension of the EM algorithm in Section 7, and one data set has been analyzed in Section 8. Finally we conclude the paper in Section 9.

2 Preliminaries

A two-parameter GE distribution has the following cumulative distribution function (CDF);

$$F_{GE}(x;\alpha,\lambda) = (1 - e^{-\lambda x})^{\alpha}; \quad x > 0,$$

and 0 otherwise. Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively. The corresponding probability density function (PDF) and hazard function become

$$f_{GE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}; \quad x > 0, \tag{2}$$

$$h_{GE}(x;\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}}{1 - (1 - e^{-\lambda x})^{\alpha}}; \quad x > 0,$$

respectively. From now on, a GE random variable with the shape parameter α and scale parameter λ will be denoted by GE(α, λ). For different developments of the GE distribution, the readers are referred to the review article by Gupta and Kundu (2007) or Nadarajah (2011).

Kundu and Gupta (2009) introduced four-parameter bivariate generalized exponential (BGE) distribution, whose marginals are GE distributions. The model has some interesting physical interpretations also. The joint CDF of the BGE model is as follows:

$$F_{BGE}(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) = \begin{cases} F_{GE}(x;\alpha_1+\alpha_3,\lambda)F_{GE}(y;\alpha_2,\lambda) & \text{if} \quad x < y \\ F_{GE}(x;\alpha_1,\lambda)F_{GE}(y;\alpha_2+\alpha_3,\lambda) & \text{if} \quad x > y \\ F_{GE}(x;\alpha_1+\alpha_2+\alpha_3,\lambda) & \text{if} \quad x = y \end{cases} \tag{3}$$

The corresponding joint PDF becomes;

$$f_{BGE}(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) = \begin{cases} f_1(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) & \text{if} \quad x < y \\ f_2(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) & \text{if} \quad x > y \\ f_0(x;\alpha_1,\alpha_2,\alpha_3,\lambda) & \text{if} \quad x = y, \end{cases}$$

where

$$\begin{aligned} f_1(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) &= f_{GE}(x;\alpha_1+\alpha_3,\lambda) \times f_{GE}(y;\alpha_2,\lambda) \\ f_2(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) &= f_{GE}(x;\alpha_1,\lambda) \times f_{GE}(y;\alpha_2+\alpha_3,\lambda) \\ f_0(x,y;\alpha_1,\alpha_2,\alpha_3,\lambda) &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{GE}(x;\alpha_1+\alpha_2+\alpha_3,\lambda). \end{aligned}$$

Kundu and Gupta (2009) provided several properties and different inferential issues of the above model in details.

3 Model Formulation and Properties

3.1 Model Formulation

Louzada et al. (2014) proposed the E2G distribution which has the following PDF;

$$f_{E2G}(y;\alpha,\lambda,\theta) = \frac{\alpha\lambda\theta e^{-\lambda y}(1-e^{-\lambda y})^{\alpha-1}}{(1-(1-\theta)(1-(1-e^{-\lambda y})^{\alpha}))^2}; \quad y>0. \tag{4}$$

Here $\alpha > 0$ and $0 < \theta < 1$ are the shape parameters and $\lambda > 0$ is the scale parameter. When $\theta = 1$, it coincides with the GE distribution. It has been shown by Louzada et al. (2014) that it can have different shapes of the PDFs, and it can have increasing, decreasing and unimodal hazard functions. The model has been derived using (1).

We use the following notation. If M is geometric random variable such that $P(M = m) = p(1-p)^{m-1}$, $m = 1, 2, \ldots$, then it will be denoted as GM(p), for $0 < p < 1$. Now we propose the new model as follows. Consider the random variable

$$Y = \begin{cases} \min\{T_1,\ldots,T_M\} & \text{if} \quad 0<\theta<1 \\ \\ \max\{T_1,\ldots,T_M\} & \text{if} \quad 1\le\theta<\infty \end{cases} \tag{5}$$

Here T_i's are i.i.d. GE(α,λ) random variables, $M \sim$ GM(θ) and $M \sim$ GM($1/\theta$), if $0 < \theta < 1$ and $1 \le \theta < \infty$ respectively. Further M and T_i's are independently distributed. We call this as the generalized exponential geometric extreme (GE2) random variable. For $0 < \theta < 1$, the CDF of GE2 random variable is same as the CDF of a E2G random variable.

It can be easily verified that for $0 < \theta < \infty$, the CDF and PDF of a GE2 random variable are

$$F_{GE2}(y;\alpha,\lambda,\theta) = \frac{(1-e^{-\lambda y})^{\alpha}}{\theta-(\theta-1)(1-e^{-\lambda y})^{\alpha}}; \quad y>0 \tag{6}$$

and

$$f_{GE2}(y;\alpha,\lambda,\theta) = \frac{\alpha\lambda\theta e^{-\lambda y}(1-e^{-\lambda y})^{\alpha-1}}{(\theta-(\theta-1)(1-e^{-\lambda y})^{\alpha})^2}; \quad y>0, \tag{7}$$

respectively. From now on a GE2 random variable with parameters α, λ and θ will be denoted by GE2(α, λ, θ). Note that GE2 can be obtained from the Marshall and Olkin (1997) model also, by replacing the exponential or Weibull distribution with the GE distribution.

The PDF of the GE2 distribution can take different shapes. It can be increasing decreasing or unimodal. When $\theta = 1$, it coincides with the GE distribution. Therefore, clearly, it is a generalized version of the GE distribution. It is observed that when $\alpha \leq 1$, then the PDF of GE2 distribution is a decreasing function for all values of θ. When $\alpha > 1$, the PDF is an unimodal function. For large values of θ, the PDF becomes almost symmetric.

Further, the shape of the hazard function depends only on α and θ, and do not depend on λ. It is observed that the hazard function is either (a) increasing, (b) decreasing, (c) unimodal or (d) U-shaped depending on α and θ. The hazard functions of a GE2 distribution for different parameters values of α and θ, for $\lambda = 1$ are presented in Figure 1. It may be noted that this is one of the very few three-parameter distributions, which can have all four different shapes of the hazard function.

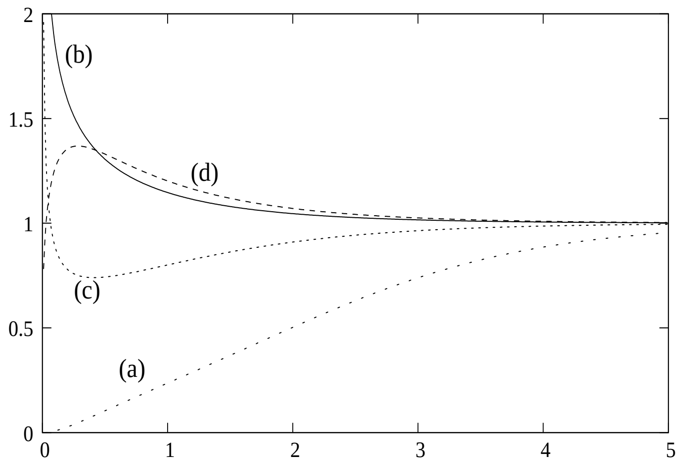

Figure 1: The hazard function of a GE2 distribution for different parameter values of α and θ; (a) $\alpha = 3.0$, $\theta = 3.0$, (b) $\alpha = 0.75$, $\theta = 0.75$, (c) $\alpha = 0.75$, $\theta = 3.0$, (d) $\alpha = 3.0$, $\theta = 0.75$

3.2 Miscellaneous Properties

In this section we provide different miscellaneous properties of the GE2 distribution.

Result 1: If $Y_1 \sim \text{GE2}(\alpha, \lambda, \theta_1)$ and $Y_2 \sim \text{GE2}(\alpha, \lambda, \theta_2)$ and $\theta_2 > \theta_1$, then Y_2 is stochastically larger than Y_1

Result 2: If $Y_1 \sim \text{GE2}(\alpha_1, \lambda, \theta)$ and $Y_2 \sim \text{GE2}(\alpha_2, \lambda, \theta)$ and $\alpha_2 > \alpha_1$, then Y_2 is stochastically larger than Y_1

Proof: Results 1 and 2 can be verified by comparing the two distribution functions. ∎

The PDF of GE2 distribution can be written as

$$f_{GE2}(y; \alpha, \lambda, \theta) = f_{GE}(y; \alpha, \lambda) w(y; \alpha, \lambda, \theta), \tag{8}$$

where $f_{GE}(\cdot; \alpha, \lambda)$ denotes the PDF of a $\text{GE}(\alpha, \lambda)$ distribution as defined in (2), and $w(\cdot; \alpha, \lambda, \theta)$ is the weight function as follows;

$$w(y; \alpha, \lambda, \theta) = \frac{\theta}{(\theta - (\theta - 1)(1 - e^{-\lambda y})^{\alpha})^2}.$$

Therefore, the GE2 distribution can be written as a weighted GE distribution. If $0 < \theta < 1$, then the weight function is a decreasing function, and it decreases from 1 to θ, and for $1 < \theta < \infty$, it is an increasing function, and it increases from 1 to θ. Since GE2 is a weighted GE distribution, many properties of the weighted distribution can be used here.

The generation from a GE2 distribution becomes quite simple using the structure of (8). If $0 < \theta < 1$, then

$$f_{GE2}(y; \alpha, \lambda, \theta) \le f_{GE}(y, \alpha, \lambda), \quad \text{for} \quad y \ge 0, \tag{9}$$

and if $1 < \theta < \infty$, then

$$f_{GE2}(y; \alpha, \lambda, \theta) \le \theta f_{GE}(y, \alpha, \lambda), \quad \text{for} \quad y \ge 0. \tag{10}$$

Therefore, acceptance-rejection method can be easily used to generate samples from a GE2 distribution, since the generation from a GE distribution is very simple, see for example Gupta and Kundu (1999). In case of GE2 model, it is observed that if θ is close to 1, the method works very well, but if θ is very close to 0 or very large, the method may not be very efficient. Alternatively, the definition of a GE2 model can be used directly to generate samples from a GE2 distribution.

RESULT 3: Suppose $Y_1, Y_2, \ldots$ is a sequence of i.i.d. GE2$(\alpha, \lambda, \theta)$ random variables, and $M \sim$ GM(p), for $0 < p < 1$. Then (a) $Z = \min\{Y_1, \ldots, Y_M\} \sim$ GE2$(\alpha, \lambda, p\theta)$ and (b) $W = \max\{Y_1, \ldots, Y_M\} \sim$ GE2$(\alpha, \lambda, \theta/p)$.

PROOF: Consider

$$\begin{aligned}
P(Z \geq z) &= \sum_{m=1}^{\infty} P(Z \geq z, M = m) = \sum_{m=1}^{\infty} P(Z \geq z | M = m) P(M = m) \\
&= p \sum_{m=1}^{\infty} (P(Y_1 \geq z))^m (1-p)^{m-1} = \frac{P(Y_1 \geq z)}{1 - (1-p) P(Y_1 \geq z)} \\
&= \frac{p\theta(1 - (1 - e^{-\lambda y})^{\alpha})}{(p\theta - (p\theta - 1)(1 - e^{-\lambda y})^{\alpha})}.
\end{aligned}$$

Similarly, (b) can be proved. ∎

The following development will be useful in deriving the EM algorithm.

Suppose $T_1, T_2, \ldots$ is a sequence of i.i.d. GE(α, λ) random variables. For $0 < \theta < 1$, suppose $M \sim$ GM(θ), and it is independent of T_i's. Let us consider the joint PDF, $f_{Y,M}$, of (Y, M), where

$$Y = \min\{T_1, \ldots, T_M\},$$

and M is same as defined above. Now

$$P(Y \geq y, M = m) = (1 - (1 - e^{-\lambda y})^{\alpha})^m \theta (1 - \theta)^{m-1}; \quad m = 1, 2, \ldots, y \geq 0.$$

Therefore,

$$
\begin{aligned}
f_{Y,M}(y,m) &= \frac{d}{dy}P(Y \geq y, M = m) \\
&= m(1-(1-e^{-\lambda y})^{\alpha})^{m-1}\alpha\lambda e^{-\lambda y}(1-e^{-\lambda y})^{\alpha-1}\theta(1-\theta)^{m-1}, \quad m = 1, 2, \ldots
\end{aligned}
$$

Hence the conditional probability mass function of M conditioning on $Y = y$ is

$$
f_{M|Y=y}(m) = m(1-(1-e^{-\lambda y})^{\alpha})^{m-1}(1-\theta)^{m-1}(\theta + (1-\theta)(1-e^{-\lambda y})^{\alpha})^2,
$$

and

$$
\begin{aligned}
E(M|Y=y) &= (\theta + (1-\theta)(1-e^{-\lambda y})^{\alpha})^2 \times \sum_{m=1}^{\infty} m^2[(1-(1-e^{-\lambda y})^{\alpha})(1-\theta)]^{m-1} \\
&= 1 + (1-(1-e^{-\lambda y})^{\alpha})(1-\theta). \qquad (11)
\end{aligned}
$$

Similarly, for $1 < \theta < \infty$, consider

$$
Y = \max\{T_1, \ldots, T_M\},
$$

where $M \sim \text{GM}(1/\theta)$, T_i's are same as defined before and they are independent of M. Therefore,

$$
P(Y \leq y, M = m) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^{m-1}(1-e^{-\lambda y})^{m\alpha}; \quad m = 1, 2, \ldots, y \geq 0.
$$

The joint PDF of Y and M is

$$
f_{Y,M}(y,m) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^{m-1} m\alpha\lambda e^{-\lambda y}(1-e^{-\lambda y})^{m\alpha-1},
$$

and the conditional mass function of M given $Y = y$ is

$$
f_{M|Y=y}(m) = \frac{m}{\theta^2}\left(1 - \frac{1}{\theta}\right)^{m-1}(1-e^{-\lambda y})^{\alpha(m-1)} \times (\theta - (\theta-1)(1-e^{-\lambda y})^{\alpha})^2; \quad m = 1, 2, \ldots, y \geq 0.
$$

Then

$$
\begin{aligned}
E(M|Y=y) &= \frac{1}{\theta^2}(\theta - (\theta-1)(1-e^{-\lambda y})^{\alpha})^2 \times \sum_{m=1}^{\infty} m^2\left(1 - \frac{1}{\theta}\right)^{m-1}(1-e^{-\lambda y})^{\alpha(m-1)} \\
&= \frac{\theta + (\theta-1)(1-e^{-\lambda y})^{\alpha}}{\theta - (\theta-1)(1-e^{-\lambda y})^{\alpha}} \qquad (12)
\end{aligned}
$$

4 Maximum Likelihood Estimation: EM Algorithm

In this section we consider the maximum likelihood estimation procedures of the unknown parameters of a GE2 distribution. We mainly develop the EM algorithm, which can be used quite successfully to compute the MLEs of the unknown parameters. Based on a random sample of size n, say $\{y_1, \ldots, y_n\} = \mathcal{D}$ from GE2$(\alpha, \lambda, \theta)$, the log-likelihood function can be written as follows:

$$\begin{aligned} \ell(\alpha, \lambda, \theta|\mathcal{D}) &= n \ln \alpha + n \ln \lambda + n \ln \theta - \lambda \sum_{i=1}^{n} y_i + (\alpha - 1) \sum_{i=1}^{n} \ln(1 - e^{-\lambda y_i}) - \\ & \quad 2 \sum_{i=1}^{n} \ln(\theta - (\theta - 1)(1 - e^{-\lambda y_i})^{\alpha}). \end{aligned} \tag{13}$$

Hence, the MLEs can be obtained by maximizing (13) with respect to α, λ and θ. It is clear that the MLEs cannot be obtained in explicit forms, they can be obtained by solving a three-dimensional optimization problem, or equivalently, by solving simultaneously three non-linear equations namely

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(1 - e^{-\lambda y_i}) + \sum_{i=1}^{n} \frac{2(\theta - 1)(1 - e^{-\lambda y_i})^{\alpha} \ln(1 - e^{-\lambda y_i})}{\theta - (\theta - 1)(1 - e^{-\lambda y_i})^{\alpha}} = 0 \tag{14}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} y_i + (\alpha - 1) \sum_{i=1}^{n} \frac{y_i e^{-\lambda y_i}}{1 - e^{-\lambda y_i}} + \sum_{i=1}^{n} \frac{2 y_i e^{-\lambda y_i} \alpha(\theta - 1)(1 - e^{-\lambda y_i})^{\alpha - 1}}{\theta - (\theta - 1)(1 - e^{-\lambda y_i})^{\alpha}} = 0 \tag{15}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{2(1 - (1 - e^{-\lambda y_i})^{\alpha}}{\theta - (\theta - 1)(1 - e^{-\lambda y_i})^{\alpha}} = 0. \tag{16}$$

The standard Newton-Raphson or Gauss-Newton algorithm may be used to solve the three non-linear equations. But it is well known that finishing the initial guesses and the convergence of the algorithm are the standard problems in such a situation. To avoid that we treat this problem as a missing value problem, and use the EM algorithm. It is observed that the EM algorithm can be used quite effectively in this case.

Now we will explain how it can be used as a missing value problem. Suppose with each Y value we observe the associated M value also. Therefore, the complete data is of the following form:

$$\mathcal{CD} = \{(y_1, m_1), \ldots, (y_n, m_n)\}. \tag{17}$$

Now based on the complete observation, the log-likelihood function for $0 < \theta < 1$ becomes;

$$\begin{aligned} \ell_{complete}(\alpha, \lambda, \theta|\mathcal{CD}) &= \sum_{i=1}^{n} \ln m_i + \sum_{i=1}^{n} (m_i - 1) \ln(1 - (1 - e^{-\lambda y_i})^{\alpha}) + n \ln \alpha + n \ln \lambda + \\ & \quad n \ln \theta + \ln(1-\theta) \sum_{i=1}^{n} (m_i - 1) - \lambda \sum_{i=1}^{n} y_i + (\alpha - 1) \sum_{i=1}^{n} \ln(1 - e^{-\lambda y_i}) \end{aligned} \tag{18}$$

and for $1 < \theta < \infty$

$$\begin{aligned} \ell_{complete}(\alpha, \lambda, \theta|\mathcal{CD}) &= -n \ln \theta + \ln \left(1 - \frac{1}{\theta}\right) \sum_{i=1}^{n} (m_i - 1) + \sum_{i=1}^{n} \ln m_i + n \ln \alpha - \\ & \quad \lambda \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} (m_i \alpha - 1) \ln(1 - e^{-\lambda y_i}) \end{aligned} \tag{19}$$

For $0 < \theta < 1$, the MLEs of the unknown parameters can be obtained by maximizing (18). In this case, the MLEs of θ becomes,

$$\widehat{\theta}_{MLE} = \frac{n}{K}, \quad \text{where} \quad K = \sum_{i=1}^{n} m_i. \tag{20}$$

Since $K \geq n$, clearly, $0 < \widehat{\theta}_{MLE} \leq 1$. For a given λ, the MLE of α, say $\widehat{\alpha}_{MLE}(\lambda)$ can be obtained by solving the following non-linear equation

$$n + \sum_{i=1}^{n} \ln Z_i^{\alpha}(\lambda) - \sum_{i=1}^{n} (m_i - 1) \frac{Z_i^{\alpha}(\lambda) \ln Z_i^{\alpha}(\lambda)}{1 - Z_i^{\alpha}(\lambda)} = 0, \tag{21}$$

where $Z_i(\lambda) = (1 - e^{-\lambda y_i})$. Then the MLE of λ can be obtained by maximizing the profile log-likelihood function

$$\begin{aligned} \ell_{profile}(\lambda|\mathcal{CD}) &= \sum_{i=1}^{n} (m_i - 1) \ln(1 - (1 - e^{-\lambda y_i})^{\widehat{\alpha}_{MLE}(\lambda)} + n \ln \widehat{\alpha}_{MLE}(\lambda) - \lambda \sum_{i=1}^{n} y_i + \\ & \quad (\widehat{\alpha}_{MLE}(\lambda) - 1) \sum_{i=1}^{n} \ln(1 - e^{-\lambda y_i}) \end{aligned} \tag{22}$$

with respect to λ. Suppose $\widehat{\lambda}_{MLE}$ maximizes (22), then the MLE of α becomes $\widehat{\alpha}_{MLE} = \widehat{\alpha}_{MLE}(\widehat{\lambda}_{MLE})$.

For $1 \le \theta < \infty$, the MLEs of the unknown parameters can be obtained by maximizing (19) with respect to the unknown parameters. In this case, the MLEs of θ becomes,

$$\widehat{\theta}_{MLE} = \frac{K}{n}, \tag{23}$$

where K is same as defined before. In this case for a given λ, the MLE of α can be obtained as

$$\widehat{\alpha}(\lambda) = -\frac{n}{\sum_{i=1}^{n} m_i \ln(1 - e^{-\lambda y_i})}, \tag{24}$$

and the MLE of λ, say $\widehat{\lambda}_{MLE}$ can be obtained by maximizing the profile log-likelihood function of λ, namely

$$\ell_{profile}(\lambda|\mathcal{CD}) = \sum_{i=1}^{n}(m_i\widehat{\alpha}_{MLE}(\lambda) - 1)\ln(1 - e^{-\lambda y_i}) + n\ln\widehat{\alpha}_{MLE}(\lambda) - \lambda\sum_{i=1}^{n} y_i. \tag{25}$$

Now we are ready to provide the EM algorithm. At the 'E-Step' of the EM algorithm the 'completed' log-likelihood function can be obtained by replacing m_i and $\ln m_i$ with $E(M|Y = y_i)$ and $E(\ln M|Y = y_i)$, respectively in (18) or (19). At the 'M-step', the maximization of the 'completed' log-likelihood function can be performed similarly, as it has been mentioned above.

5 Bivariate GE2 Model

In this section we introduce the bivariate version of the GE2 model, and we will call it as the bivariate generalized exponential geometric extreme (BGE2) model. Suppose $\{(X_{1n}, X_{2n}); n = 1, 2, \ldots\}$ is a sequence of independent identically distributed (i.i.d.) bivariate random variables with common distribution function $F(\cdot, \cdot)$, $N \sim$ GM(p) random variable and it is independent of $\{(X_{1n}, X_{2n}); n = 1, 2, \ldots\}$. Consider the following bivariate random variable (Y_1, Y_2), where

$$Y_1 = \max\{X_{11}, \ldots, X_{1N}\} \quad \text{and} \quad Y_2 = \max\{X_{21}, \ldots, X_{2N}\}. \tag{26}$$

The joint CDF of (Y_1, Y_2) for $0 < y_1, y_2 < \infty$, becomes

$$P(Y_1 \le y_1, Y_2 \le y_2) = p\sum_{n=1}^{\infty} F^n(y_1, y_2)(1-p)^{n-1} = \frac{pF(y_1, y_2)}{1-(1-p)F(y_1, y_2)}.$$

First we will define BGE2 for $\theta \ge 1$. We consider $\theta = 1/p$, and $F(y_1, y_2) = F_{BGE}(y_1, y_2)$, as defined in (3). We call this new distribution as the BGE2 distribution, and it will be denoted by BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$. Therefore, if $(Y_1, Y_2) \sim$ BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$, then the joint CDF can be written as

$$P(Y_1 \le y_1, Y_2 \le y_2) = G_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{F_{GE}(y_1;\alpha_1+\alpha_3,\lambda)F_{GE}(y_2;\alpha_2,\lambda)}{\theta-(\theta-1)F_{GE}(y_1;\alpha_1+\alpha_3,\lambda)F_{GE}(y_2;\alpha_2,\lambda)} \text{if} & y_1 \le y_2 \\ \frac{F_{GE}(y_1;\alpha_1,\lambda)F_{GE}(y_2;\alpha_2+\alpha_3,\lambda)}{\theta-(\theta-1)[F_{GE}(y_1;\alpha_1,\lambda)F_{GE}(y_2;\alpha_2+\alpha_3,\lambda)} \text{if} & y_1 > y_2. \end{cases} \tag{27}$$

Note that (27) can also be written for $z = \min\{y_1, y_2\}$, and $0 < y_1, y_2 < \infty$, as

$$G_{Y_1,Y_2}(y_1, y_2) = \frac{F_{GE}(y_1;\alpha_1,\lambda)F_{GE}(y_2;\alpha_2,\lambda)F_{GE}(z;\alpha_3,\lambda)}{\theta-(\theta-1)F_{GE}(y_1;\alpha_1,\lambda)F_{GE}(y_2;\alpha_2)F_{GE}(z;\alpha_3,\lambda)}.$$

From (27), it is observed that the marginals Y_1 and Y_2 have the following CDF

$$P(Y_1 \le y_1) = G_{Y_1}(y_1) = \frac{F_{GE}(y_1;\alpha_1+\alpha_3,\lambda)}{\theta-(\theta-1)F_{GE}(y_1;\alpha_1+\alpha_3,\lambda)} \tag{28}$$

$$P(Y_2 \le y_2) = G_{Y_2}(y_2) = \frac{F_{GE}(y_2;\alpha_2+\alpha_3,\lambda)}{\theta-(\theta-1)F_{GE}(y_2;\alpha_2+\alpha_3,\lambda)}, \tag{29}$$

respectively. Therefore, $Y_1 \sim$ GE2$(\alpha_1+\alpha_3, \lambda, \theta)$ and $Y_2 \sim$ GE2$(\alpha_2+\alpha_3, \lambda, \theta)$. The following results will provide the joint PDF of the BGE2 model.

Theorem 1: Let $(Y_1, Y_2) \sim$ BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$, then the joint PDF of (Y_1, Y_2) is

$$g(y_1, y_2) = \begin{cases} g_1(y_1, y_2) & \text{if} \quad y_1 < y_2 \\ g_2(y_1, y_2) & \text{if} \quad y_1 > y_2 \\ g_0(y) & \text{if} \quad y_1 = y_2 = y, \end{cases} \tag{30}$$

where

$$
\begin{aligned}
g_1(y_1, y_2) &= \frac{\theta f_{GE}(y_1; \alpha_1+\alpha_3, \lambda) f_{GE}(y_2; \alpha_2, \lambda)(\theta + (\theta-1) F_{GE}(y_1; \alpha_1+\alpha_3, \lambda) F_{GE}(y_2; \alpha_2, \lambda))}{(\theta - (\theta-1) F_{GE}(y_1; \alpha_1+\alpha_3, \lambda) F_{GE}(y_2; \alpha_2, \lambda))^3} \\
g_2(y_1, y_2) &= \frac{\theta f_{GE}(y_1; \alpha_1, \lambda) f_{GE}(y_2; \alpha_2+\alpha_3, \lambda)(\theta + (\theta-1) F_{GE}(y_1; \alpha_1, \lambda) F_{GE}(y_2; \alpha_2+\alpha_3, \lambda))}{(\theta - (\theta-1) F_{GE}(y_1; \alpha_1, \lambda) F_{GE}(y_2; \alpha_2+\alpha_3, \lambda))^3} \\
g_0(y) &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} \times \frac{\theta f_{GE}(y; \alpha_1+\alpha_2+\alpha_3)}{(\theta - (\theta-1)(F_{GE}(y; \alpha_1+\alpha_2+\alpha_3, \lambda))^2}.
\end{aligned}
$$

PROOF: The expressions for $g_1(y_1, y_2)$ and $g_2(y_1, y_2)$ can be obtained simply by taking $\frac{\partial^2 G_{Y_1,Y_2}(y_1, y_2)}{\partial y_1 \partial y_2}$ for $y_1 < y_2$ and $y_1 > y_2$, respectively. But $g_0(y)$ cannot be obtained in the same way. First observe that $g_0(y)$, $g_1(y_1, y_2)$ and $g_2(y_1, y_2)$ must satisfy

$$
\int_0^\infty \int_0^{y_2} g_1(y_1, y_2) dy_1 dy_2 + \int_0^\infty \int_0^{y_1} g_2(y_1, y_2) dy_2 dy_1 + \int_0^\infty g_0(y) dy = 1.
$$

After some lengthy calculations (can be obtained from the author on request), it can be shown that

$$
\int_0^\infty \int_0^{y_2} g_1(y_1, y_2) dy_1 dy_2 = \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3} \quad \text{and} \quad \int_0^\infty \int_0^{y_1} g_2(y_1, y_2) dy_2 dy_1 = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}.
$$

Therefore, $g_0(y)$ must satisfy

$$
\int_0^\infty g_0(y) dy = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}. \tag{31}
$$

Note that $g_0(y)$ can be obtained from $G_{Y_1,Y_2}(y, y)$ as follows:

$$
g_0(y) = k \ \frac{d}{dy} G_{Y_1,Y_2}(y, y),
$$

where k is the constant, so that $g_0(y)$ satisfies (31). Now by observing the facts:

$$
\frac{d}{dy} G_{Y_1,Y_2}(y, y) = \frac{\theta f_{GE}(y; \alpha_1+\alpha_2+\alpha_3)}{(\theta - (\theta-1)(F_{GE}(y; \alpha_1+\alpha_2+\alpha_3, \lambda))^2} = \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_3} \times g_0(y),
$$

and

$$
\int_0^\infty \frac{\theta f_{GE}(y; \alpha_1+\alpha_2+\alpha_3, \lambda)}{(\theta - (\theta-1)(F_{GE}(y; \alpha_1+\alpha_2+\alpha_3, \lambda))^2} dy = 1,
$$

the result follows. ∎

It should be noted that the function $g(y_1, y_2)$ may be considered to be a density function of the BGE2 distribution, if it is understood that the first two terms are density functions with respect to two dimensional Lebesgue measure, where as the third term is a density function with respect to a one dimensional Lebesgue measure, similarly as the Marshall-Olkin bivariate exponential model. From (27) it is clear that when $p = 1$, BGE2 becomes a BGE model. Therefore, BGE is a member of BGE2 model. Moreover, it is immediate that BGE2 model will also have the range of correlation from 0 to 1.

The absolute continuous part of the joint PDF of BGE2 can take different shapes. The absolute continuous part of the joint PDF of BGE2 for different parameter values are presented in Figure 2.

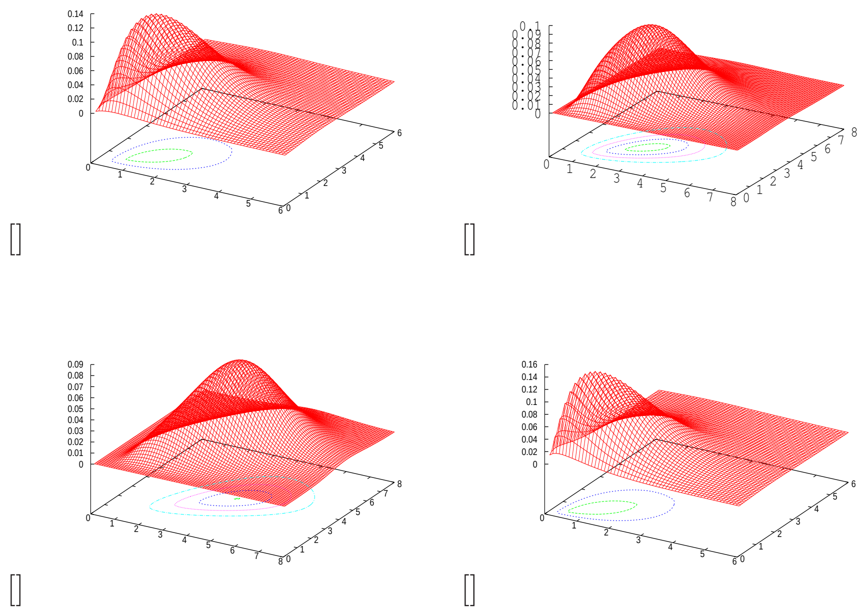

Figure 2: Contour plots of the absolute continuous part of the joint PDF of the BGE2 distribution for different parameter values: $(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$: (a) (2.0, 2.0, 2.0, 1, 2), (b) (4.0, 4.0, 1.0, 1, 4.0), (c) (8.0, 8.0, 1.0, 1, 8.0), (d) (1.5, 1.5, 1.0, 1, 2).

The generation of random sample from a BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$ is quite simple. The following algorithm can be used for that purpose.

ALGORITHM:

Step 1: Generate n from GM$(1/\theta)$.

Step 2: Generate $\{u_{11}, \ldots, u_{1n}\}$ from GE(α_1), $\{u_{21}, \ldots, u_{2n}\}$ from GE(α_2) and $\{u_{31}, \ldots, u_{3n}\}$ from GE(α_3).

Step 3: Obtain $v_{1k} = \max\{u_{1k}, u_{3k}\}$ and $v_{2k} = \max\{u_{2k}, u_{3k}\}$ for $k = 1, \ldots, n$.

Step 4: Compute the desired $y_1 = \max\{v_{11}, \ldots, v_{1n}\}$ and $y_2 = \max\{v_{21}, \ldots, v_{2n}\}$

The following result will provide explicitly the absolute continuous part and the singular part of the BGE2 model.

THEOREM 2: Let $(Y_1, Y_2) \sim$ BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$, then

$$G_{Y_1,Y_2}(y_1, y_2) = \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} G_a(y_1, y_2) + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} G_s(y_1, y_2),$$

where for $z = \min\{y_1, y_2\}$,

$$G_s(y_1, y_2) = \frac{F_{GE}(z; \alpha_1 + \alpha_2 + \alpha_3, \lambda)}{\theta - (\theta - 1) F_{GE}(z; \alpha_1 + \alpha_2 + \alpha_3, \lambda)}$$

and

$$G_a(y_1, y_2) = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_1 + \alpha_2} G_{Y_1,Y_2}(y_1, y_2) - \frac{\alpha_3}{\alpha_1 + \alpha_2} G_s(y_1, y_2).$$

Here $G_s(y_1, y_2)$ and $G_s(y_1, y_2)$ are the singular and absolute continuous parts of $G_{Y_1,Y_2}(y_1, y_2)$ respectively.

PROOF: Using Theorem 1, the proof can be obtained easily, hence it is avoided. ∎

THEOREM 3: Let $(Y_1, Y_2) \sim$ BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$ and $(Z_1, Z_2) \sim$ BGE2$(\alpha_1, \alpha_2, \alpha_3', \lambda, \theta)$, then

(a) $Y_1 \sim \text{GE2}(\alpha_1 + \alpha_3, \lambda, \theta)$ and $Y_2 \sim \text{GE2}(\alpha_2 + \alpha_3, \lambda, \theta)$.

(b) $P(Y_1 < Y_2) = \dfrac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}$.

(c) If $Z = \max\{Y_1, Y_2\}$, then $Z \sim \text{GE2}(\alpha_1 + \alpha_2 + \alpha_3, \lambda)$.

(d) If $\alpha_3 \le \alpha_3'$, then $(Y_1, Y_2) <_{st} (Z_1, Z_2)$, i.e. for any upper set $U \in \mathbb{R}^2$, $P((Y_1, Y_2) \in U) \le P((Z_1, Z_2) \in U)$.

Proof: Proof of (a) has already been provided. To prove (b), observe that

$$\begin{aligned}
P(Y_1 < Y_2) &= \sum_{n=1}^{\infty} P(Y_1 < Y_2, N = n) = \frac{1}{\theta} \sum_{n=1}^{\infty} \left(1 - \frac{1}{\theta}\right)^{n-1} \int_0^{\infty} \int_0^{y_2} f_{1n}(y_1, y_2) dy_1 dy_2 \\
&= \frac{1}{\theta} \sum_{n=1}^{\infty} \left(1 - \frac{1}{\theta}\right)^{n-1} \int_0^{\infty} n\alpha_2 e^{-y_2} (1 - e^{-y_2})^{n(\alpha_1 + \alpha_2 + \alpha_3)} \\
&= \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} \times \frac{1}{\theta} \sum_{n=1}^{\infty} \left(1 - \frac{1}{\theta}\right)^{n-1} = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}.
\end{aligned}$$

To prove (c), consider

$$\begin{aligned}
P(Z \le z) &= P(Y_1 \le z, Y_2 \le z) = \sum_{n=1}^{\infty} P(Y_1 \le z, Y_2 \le z | N = n) P(N = n) \\
&= \frac{1}{\theta} \sum_{n=1}^{\infty} \left(1 - \frac{1}{\theta}\right)^{n-1} \left(1 - e^{-\lambda z}\right)^{n(\alpha_1 + \alpha_2 + \alpha_3)} \\
&= \frac{\left(1 - e^{-\lambda z}\right)^{(\alpha_1 + \alpha_2 + \alpha_3)}}{\theta - (\theta - 1)\left(1 - e^{-\lambda z}\right)^{(\alpha_1 + \alpha_2 + \alpha_3)}}.
\end{aligned}$$

To prove (d), observe that

$$Y_1 <_{st} Z_1 \quad \text{and} \quad Y_2 <_{st} Z_2. \tag{32}$$

The proof of (32) can be easily obtained using the fact $\alpha_3 \le \alpha_3'$, and considering the two joint CDFs. The result follows since both (Y_1, Y_2) and (Z_1, Z_2) have the same copula. ∎

It may be mentioned that for every bivariate distribution function F_{X_1, X_2} with continuous marginals distribution function F_{X_1} and F_{X_2}, corresponds a unique function $C : [0, 1] \times$

$[0,1] \to [0,1]$, called a copula such that

$$F_{X_1,X_2}(x_1,x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2)); \quad \text{for} \quad (x_1,x_2) \in (-\infty,\infty)\times(-\infty,\infty).$$

Equivalently, for $0 < u_1, u_2 < 1$,

$$C(u_1,u_2) = F_{X_1,X_2}(F_{X_1}^{-1}(u_1), F_{X_2}^{-1}(u_2)).$$

It follows after a length calculation that if $(Y_1,Y_2) \sim \text{BGE2}(\alpha_1,\alpha_2,\alpha_3,\lambda,\theta)$, then for $p = 1/\theta$, it has the following copula function:

$$C(u_1,u_2) = \begin{cases} C_1(u_1,u_2) & \text{if} \quad \left[\frac{u_1}{p+u_1(1-p)}\right]^{\frac{\alpha_1}{\alpha_1+\alpha_3}} \leq \left[\frac{u_2}{p+u_2(1-p)}\right]^{\frac{\alpha_2}{\alpha_2+\alpha_3}} \\ C_2(u_1,u_2) & \text{if} \quad \left[\frac{u_1}{p+u_1(1-p)}\right]^{\frac{\alpha_1}{\alpha_1+\alpha_3}} > \left[\frac{u_2}{p+u_2(1-p)}\right]^{\frac{\alpha_2}{\alpha_2+\alpha_3}} \end{cases} \tag{33}$$

where

$$C_1(u_1,u_2) = \frac{\frac{pu_1}{p+u_1(1-p)}\left[\frac{u_2}{p+u_2(1-p)}\right]^{\frac{\alpha_2}{\alpha_2+\alpha_3}}}{1-\frac{(1-p)u_1}{p+u_1(1-p)}\left[\frac{u_2}{p+u_2(1-p)}\right]^{\frac{\alpha_2}{\alpha_2+\alpha_3}}}$$

$$C_2(u_1,u_2) = \frac{\frac{pu_2}{p+u_2(1-p)}\left[\frac{u_1}{p+u_1(1-p)}\right]^{\frac{\alpha_1}{\alpha_1+\alpha_3}}}{1-\frac{(1-p)u_2}{p+u_2(1-p)}\left[\frac{u_1}{p+u_1(1-p)}\right]^{\frac{\alpha_1}{\alpha_1+\alpha_3}}}$$

We have introduced BGE2 model for $1 \leq \theta < \infty$. Now we introduce BGE2 model when $0 < \theta < 1$. We define,

$$Y_1 = \min\{X_{11},\ldots,X_{1N}\} \quad \text{and} \quad Y_2 = \min\{X_{21},\ldots,X_{2N}\}, \tag{34}$$

where the sequence of random variables $\{(X_{1i},X_{2i}); i = 1,2,\ldots\}$ are same as defined before, and $N \sim \text{GM}(\theta)$. Hence the joint survival function of Y_1 and Y_2 for $y_1 \geq 0$ and $y_2 \geq 0$, can be written as follows

$$P(Y_1 \geq y_1, Y_2 \geq y_2) = \bar{G}_{Y_1,Y_2}(y_1,y_2) = \frac{\theta \bar{F}_{BGE}(y_1,y_2)}{1-(1-\theta)\bar{F}_{BGE}(y_1,y_2)}, \tag{35}$$

where

$$\bar{F}_{BGE}(y_1, y_2) = 1 - F_{GE}(y_1; \alpha_1 + \alpha_3, \lambda) - F_{GE}(y_2; \alpha_2 + \alpha_3, \lambda) + F_{BGE}(y_1, y_2),$$

and $F_{BGE}(y_1, y_2)$ is same as defined in (3). In this case also the random vector (Y_1, Y_2) with the joint survival function (35) will be denoted by BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$. It is immediate that from (35) that if $(Y_1, Y_2) \sim$ BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, \theta)$, then $Y_1 \sim$ GE2$(\alpha_1 + \alpha_3, \lambda, \theta)$ and $Y_2 \sim$ GE2$(\alpha_2 + \alpha_3, \lambda, \theta)$. The joint PDF of (Y_1, Y_2) can be written as follows:

$$G_{Y_1,Y_2}(y_1, y_2) = \bar{G}_{Y_1,Y_2}(y_1, y_2) + F_{GE2}(y_1; \alpha_1 + \alpha_3, \lambda, \theta) + F_{GE2}(y_2; \alpha_2 + \alpha_3, \lambda, \theta) - 1.$$

Following the similar approach as in Theorem 1, the joint PDF of (Y_1, Y_2) for $0 < \theta < 1$, can also be obtained in the same form as in (30). The exact expressions of $g(y_1, y_2)$ are different from (30) and they are quite involved. They are not presented here. Similarly, the survival copula of (Y_1, Y_2), hence the corresponding copula also can be obtained, but they are also quite involved. Hence, they are not presented here, they can be obtained from the authors on request. This is one of the major difference between GE2 and BGE2 models. In case of GE2 model the CDF has the same form for $0 < \theta < \infty$, where as in case of BGE2 model, the joint CDF or joint survival function has different forms for $0 < \theta < 1$ and $1 \leq \theta < \infty$. From now on we restrict our attention to the case $1 \leq \theta < \infty$.

We need the following results mainly to develop the EM algorithm which will be discussed in Section 7. We need the joint PDF of (Y_1, Y_2) and N, where Y_1, Y_2 and N are same as defined before. First note that

$$\begin{aligned} P(Y_1 \leq y_1, Y_2 \leq y_2, N = n) &= P(Y_1 \leq y_1, Y_2 \leq y_2 | N = n) P(N = n) \\ &= \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^{n-1} F_{BGE}(y_1, y_2; n\alpha_1, n\alpha_2, n\alpha_3, \lambda, \theta). \end{aligned}$$

Hence the joint PDF of Y_1, Y_2 and N becomes

$$f_{Y_1,Y_2,N}(y_1, y_2, n) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^{n-1} f_{BGE}(y_1, y_2; n\alpha_1, n\alpha_2, n\alpha_3, \lambda).$$

The conditional probability mass function of N given $Y_1 = y_1$ and $Y_2 = y_2$ is,

$$f_N(n|y_1,y_2) = \begin{cases} c_1(y_1,y_2)\left(1-\frac{1}{\theta}\right)^{n-1} n^2 \left(1-e^{-\lambda y_1}\right)^{(n-1)(\alpha_1+\alpha_3)} \left(1-e^{-\lambda y_2}\right)^{(n-1)\alpha_2} & \text{if} \quad y_1 < y_2 \\ c_2(y_1,y_2)\left(1-\frac{1}{\theta}\right)^{n-1} n^2 \left(1-e^{-\lambda y_1}\right)^{(n-1)\alpha_1} \left(1-e^{-\lambda y_2}\right)^{(n-1)(\alpha_2+\alpha_3)} & \text{if} \quad y_1 > y_2 \\ c_0(y)\left(1-\frac{1}{\theta}\right)^{n-1} n \left(1-e^{-\lambda y}\right)^{(n-1)(\alpha_1+\alpha_2+\alpha_3)} & \text{if} \quad y_1 = y_2 = y, \end{cases}$$

where

$$\begin{aligned} \xi_1 &= \left(1-\frac{1}{\theta}\right)\left(1-e^{-\lambda y_1}\right)^{\alpha_1+\alpha_3}\left(1-e^{-\lambda y_2}\right)^{\alpha_2} \\ \xi_2 &= \left(1-\frac{1}{\theta}\right)\left(1-e^{-\lambda y_1}\right)^{\alpha_1}\left(1-e^{-\lambda y_2}\right)^{\alpha_2+\alpha_3} \\ \xi &= \left(1-\frac{1}{\theta}\right)\left(1-e^{-\lambda y_1}\right)^{\alpha_1+\alpha_2+\alpha_3}, \end{aligned}$$

$$c_1(y_1,y_2) = \frac{(1-\xi_1)^3}{1+\xi_1}, \quad c_2(y_1,y_2) = \frac{(1-\xi_2)^3}{1+xi_2}, \quad c_0(y) = (1-\xi)^2.$$

Hence, we obtain

$$E(N|y_1,y_2) = \begin{cases} \frac{1+4\xi_1+\xi_1^2}{(1-\xi_1^2)} & \text{if} \quad y_1 < y_2 \\ \frac{1+4\xi_2+\xi_2^2}{(1-\xi_2^2)} & \text{if} \quad y_2 < y_1 \\ \frac{1+\xi}{1-\xi} & \text{if} \quad y_1 = y_2 \end{cases} \tag{36}$$

6 A Special Case

In this section we consider a special case namely when $\alpha_3 = 0$. It is further assumed that $p = 1/\theta$ and $1 \le \theta < \infty$. In this special case the BGE2 model does not have any singular component, hence it has an absolute continuous CDF. This model can be used to analyze bivariate data when there are no ties. In this case, (Y_1, Y_2) has the following joint CDF

$$G_{Y_1,Y_2}(y_1,y_2) = \frac{p(1-e^{-\lambda y_1})^{\alpha_1}(1-e^{-\lambda y_2})^{\alpha_2}}{1-(1-p)(1-e^{-\lambda y_1})^{\alpha_1}(1-e^{-\lambda y_2})^{\alpha_2}},$$

and it will be denoted by BGE2$(\alpha_1, \alpha_2, \lambda, p)$. The corresponding joint PDF is

$$g(y_1,y_2) = \frac{p\theta f_{GE}(y_1;\alpha_1,\lambda) f_{GE}(y_2;\alpha_2,\lambda)(1+(1-p)F_{GE}(y_1;\alpha_1,\lambda)F_{GE}(y_2;\alpha_2,\lambda))}{(1-(1-p)F_{GE}(y_1;\alpha_1,\lambda)f_{GE}(y_2;\alpha_2,\lambda))^3}.$$

Moreover, in this case the marginal distributions of Y_1 and Y_2 become

$$G_{Y_1}(y_1) = \frac{p(1-e^{-\lambda y_1})^{\alpha_1}}{1-(1-p)(1-e^{-\lambda y_1})^{\alpha_1}} \quad \text{and} \quad G_{Y_2}(y_2) = \frac{p(1-e^{-\lambda y_2})^{\alpha_2}}{1-(1-p)(1-e^{-\lambda y_2})^{\alpha_2}},$$

respectively.

It immediately follows that if $(Y_1, Y_2) \sim \text{BGE2}(\alpha_1, \alpha_2, \lambda, p)$, then it has the following copula function

$$C(u_1, u_2) = \frac{u_1 u_2}{1-(1-p)(1-u_1)(1-u_2)}.$$

It is known as the Ali-Mikhail-Haq copula, see for example Ali et al. (1978). Therefore, using the copula structure several properties of $\text{BGE2}(\alpha_1, \alpha_2, \lambda, p)$ can be easily obtained. We immediately have the following results.

RESULT 4: Let $(Y_1, Y_2) \sim \text{BGE2}(\alpha_1, \alpha_2, \lambda, p)$, then Y_2 is stochastically increasing in Y_1, and vice versa.

PROOF: Using the copula function, it follows that X_2 is stochastically increasing in X_1, if and only if for any v, $C(u, v)$ is a concave function of u, see Nelsen (2006). It is equivalent to saying that

$$\frac{\partial^2}{\partial u_1^2} C(u_1, u_2) \le 0.$$

Since

$$\frac{\partial^2}{\partial u_1^2} C(u_1, u_2) = -\frac{2u_2(1-u_2)(1-(1-p)(1-u_2))(1-p)}{(1-(1-p)(1-u_1)(1-u_2))^3} \le 0,$$

the result follows. ∎

A non-negative function g defined on $\mathbb{R}^2$ is totally positive of order 2, abbreviated by TP_2 if for all $u_{11} < u_{12}$ and $u_{21} < u_{22}$,

$$g(u_{11}, u_{21}) g(u_{12}, u_{22}) \ge g(u_{12}, u_{21}) g(u_{11}, u_{22}).$$

It is also known that, see Nelsen (2006), TP_2 property is a copula property. It can be easily verified that Ali-Mikhail-Haq copula satisfies TP_2 property. Therefore, we have the following result.

RESULT 5: Let $(Y_1, Y_2) \sim \text{BGE2}(\alpha_1, \alpha_2, \lambda, p)$, then, (Y_1, Y_2) has the TP_2 property.

Now we discuss some dependency measures of $\text{BGE2}(\alpha_1, \alpha_2, \lambda, p)$. In case of Ali-Mikhail-Haq copula, the Kendall's τ is

$$\tau = \frac{3p-2}{3p} - \frac{2(1-p)^2 \ln(1-p)}{3p^2}, \tag{37}$$

see Kumar (2010). Since Kendall's τ is a copula property, $\text{BGE2}(\alpha_1, \alpha_2, \lambda, \theta)$ has the Kendall's τ as given in (37). It follows from Kumar (2010) that $\tau \in [0, 1/3]$.

Spearman's correlation coefficient ρ is also a copula property. Hence we have the following result that for $\text{BGE2}(\alpha_1, \alpha_2, \lambda, p)$, the Spearman's correlation coefficient is

$$\rho = \frac{12(1+p) di\log(1-p) - 24(1-p)\ln(1-p)}{p^2} - \frac{3(p+12)}{p},$$

where the di-logarithm function $di\log(x)$ is

$$di\log(x) = \int_1^x \frac{\ln t}{1-t} dt,$$

see Kumar (2010). Moreover, $\rho \in [0, 4\pi^2 - 39]$.

The population version of medial correlation coefficient for a pair (Y_1, Y_2) of continuous random variables was defined by Blomqvist (1950). The median correlation coefficient is a copula property, and it can be obtained as $M_{Y_1,Y_2} = 4C\left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore, for BGE2, the medial correlation coefficient is $\frac{1}{3+p}$.

The bivariate tail dependence measures the amount of dependence in the upper or lower quadrant tail of a bivariate distribution. For details the reader is referred to Joe (1997). In terms of copula, the tail dependence χ can be written as

$$\chi = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1} \left\{ 2 - \frac{\ln C(u,u)}{\ln u} \right\},$$

see Coles et al. (1999). In case of Ali-Mikhail-Haq copula, using the L'Hospital's rule it immediately follows that $\chi = 0$. Tail dependence is a copula property. Hence, BGE2$(\alpha_1, \alpha_2, p, \lambda)$ has the tail dependence $\chi = 0$. Therefore, in case of BGE2$(\alpha_1, \alpha_2, p, \lambda)$ model, two marginals are asymptotically independent.

7 Estimation: BGE2 Model

In this section we consider the maximum likelihood estimation of the unknown parameters of a BGE2 model. It is assumed $\theta \geq 1$, and $p = 1/\theta$. In this section only, we will denote it as BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, p)$. We have a random sample $\{(y_{11}, y_{21}), \ldots, (y_{1m}, y_{2m})\}$ of size m from BGE2$(\alpha_1, \alpha_2, \alpha_3, \lambda, p)$, and based on the random sample we need to estimate the unknown parameters. We use the following notations:

$$I_0 = \{i; y_{1i} = y_{2i} = y_i\}, \qquad I_1 = \{i; y_{1i} < y_{2i}\}, \qquad I_2 = \{i; y_{1i} > y_{2i}\},$$

and m_0, m_1 and m_2, denote the number of observations in I_0, I_1 and I_2, respectively. Now, based on the joint PDF of (Y_1, Y_2), as provided in Theorem 1, the log-likelihood function of the observations, as a function of the parameters, can be written as

$$\ell(\alpha_1, \alpha_2, \alpha_3, \lambda, p) = \sum_{i \in I_1} \ln g_0(y_i) + \sum_{i \in I_1} \ln g_1(y_{1i}, y_{2i}) + \sum_{i \in I_3} \ln g_2(y_{1i}, y_{2i}). \tag{38}$$

Therefore, the MLEs of the unknown parameters can be obtained by maximizing (38) with respect to the unknown parameters. It involves solving a five dimensional optimization problem. To avoid that we propose to use EM algorithm similar to the method proposed by Kundu and Gupta (2014). The main ideas are as follows.

We treat this problem as a missing value problem, and assuming p to be known, we obtain the MLEs of α_1, α_2, α_3 and λ, and then maximizing the profile log-likelihood function of p, we obtain the MLE of p. This problem is being treated as a missing value problem as follows. It is assumed that the complete data are obtained from (Y_1, Y_2, N), out of which N is missing. The complete data would be of the form $\{(y_{1i}, y_{2i}, n_i); \imath = 1, \ldots, m\}$. First observe that

$$(Y_1, Y_2 | N = n) \sim \text{BGE}(n\alpha_1, n\alpha_2, n\alpha_3, \lambda). \tag{39}$$

Moreover, since p is assumed to be known, the MLEs of α_1, α_2, α_3 and λ can be obtained by maximizing the conditional log-likelihood function based on the complete observations. The conditional log-likelihood function of the complete observations can be written as

$$\ell_1(\alpha_1, \alpha_2, \alpha_3, \lambda) = \sum_{i=1}^{m} \ln f_{BGE}(y_{1i}, y_{2i}; n_i\alpha_1, n_i\alpha_2, n_i\alpha_3, \lambda).$$

Kundu and Gupta (2009) developed an efficient EM algorithm to estimate the parameters of a BGE model. Using the method developed by Kundu and Gupta (2009), and using similar approach as of Kundu and Gupta (2014), we develop the EM algorithm for the BGE2 model. Explicit details are avoided, it can be obtained on request from the corresponding author. We just provide the final algorithm.

We need to define the following notations for further development. We try to use the same notations as in Kundu and Gupta (2009) and Kundu and Gupta (2014). We denote the unknown parameter vector $\gamma = (\alpha_1, \alpha_2, \alpha_3, \lambda)$. At the k-th step of the EM algorithm, the parameter vector $\gamma^{(k)} = (\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)}, \lambda^{(k)})$.

$$u_1^{(k)} = \frac{\alpha_1^{(k)}}{\alpha_1^{(k)} + \alpha_3^{(k)}}, \quad u_2^{(k)} = \frac{\alpha_3^{(k)}}{\alpha_1^{(k)} + \alpha_3^{(k)}}, w_1^{(k)} = \frac{\alpha_2^{(k)}}{\alpha_2^{(k)} + \alpha_3^{(k)}}, \quad w_2^{(k)} = \frac{\alpha_3^{(k)}}{\alpha_2^{(k)} + \alpha_3^{(k)}}.$$

$$E(N | y_{1i}, y_{2i}, p, \gamma^{(k)}) = a_i^{(k)}.$$

Note that $a_i^{(k)}$ can be obtained from (36). Following Kundu and Gupta (2009) and Kundu and Gupta (2014), it follows that after k-th iterations, $(k+1)-$th step can be obtained using the following algorithm.

Algorithm

- Obtain $\lambda^{(k+1)}$ by solving a fixed point type equation $g(\lambda|\gamma^{(k)}) = \lambda$. The explicit expression of $g(\lambda|\gamma^{(k)})$ is provided in the Appendix. Very simple iterative procedure can be used to solve the above fixed point type equation.

- Once $\lambda^{(k+1)}$ is obtained, $\alpha_1^{(k+1)}$, $\alpha_2^{(k+1)}$, $\alpha_3^{(k+1)}$, can be obtained as $\alpha_1^{(k+1)}(\lambda^{(k+1)})$, $\alpha_2^{(k+1)}(\lambda^{(k+1)})$, $\alpha_3^{(k+1)}(\lambda^{(k+1)})$, respectively, where;

$$\begin{aligned}
\alpha_1^{(k+1)}(\lambda) &= \frac{m_1 u_1^{(k)} + m_2}{\sum_{i\in I_0} a_i^{(k)} \ln(1-e^{-\lambda y_i}) + \sum_{i \in I_1 \cup I_2} a_i^{(k)} \ln(1-e^{-\lambda y_{1i}})} \\
\alpha_2^{(k+1)}(\lambda) &= \frac{m_2 w_1^{(k)} + m_1}{\sum_{i\in I_0} a_i^{(k)} \ln(1-e^{-\lambda y_i}) + \sum_{i \in I_1 \cup I_2} a_i^{(k)} \ln(1-e^{-\lambda y_{2i}})} \\
\alpha_3^{(k+1)}(\lambda) &= \frac{m_0 + m_1 u_2^{(k)} + m_2 w_2^{(k)}}{\sum_{i\in I_0} a_i^{(k)} \ln(1-e^{-\lambda y_i}) + \sum_{i \in I_1} a_i^{(k)} \ln(1-e^{-\lambda y_{1i}}) + \sum_{i \in I_2} a_i^{(k)} \ln(1-e^{-\lambda y_{2i}})}
\end{aligned}$$

The choice of the initial estimates is an important issue. From the marginals y_{1i} and y_{2i}, we can obtain estimates of $\alpha_1+\alpha_3$, $\alpha_2+\alpha_3$, λ and θ. Similarly, using (c) of Theorem 3, from $z_i = \max\{y_{1i}, y_{2i}\}$, we can obtain estimates of $\alpha_1+\alpha_2+\alpha_3$. From the above estimates, it is possible to obtain some initial estimates of α_1, α_2, α_3, λ and θ.

8 Data Analysis

In this section we present the analysis of a data set for illustrative purposes. The data set represents the total cholesterol contents of 23 subjects measured after 4th week and 20th week of the start of the medicine. The data set has been obtained from Davis (2002), and it is represented in Table 1.

The (first quartile, median, third quartile) of the first and second marginals are (230,274,367) and (230,262,357), respectively. The above quantiles, and also the associated histograms (not presented here) indicate that both the marginals are coming from skewed distributions.

S.N.	4-th	20-th	S.N.	4-th	20-th	S.N.	4-th	20-th
1.	317	274	2.	186	197	3.	377	338
4.	229	264	5.	276	300	6.	272	228
7.	219	242	8.	260	317	9.	284	243
10.	365	311	11.	298	357	12.	274	235
13.	232	218	14.	367	338	15.	253	237
16.	230	230	17.	190	169	18.	290	299
19.	337	361	20.	283	269	21.	325	293
22.	266	245	23.	338	262			

Table 1: Total cholesterol levels of 23 subjects after 4th week and 20th week of the start of the medicine.

Moreover, from the scale TTT plots (not presented here) it is observed that the empirical hazard function of the first marginal is an increasing function, whereas the second one has an U-shaped empirical hazard function. Therefore, GE2 model may be used to analyze the marginals. Moreover, the correlation between the two marginals is positive, hence BGE2 model has been used to analyze the bivariate data. Before progressing further we make the following transformation $(X - 165)/100$ to all the observations mainly for model fitting and computational purposes. Note that the normalization is necessary, particularly the subtraction of 165, as it indicates the presence of a location parameter in the model. In this case it is assumed to be known. The division of 100 is done mainly for re-scaling the scale parameter λ.

First we would like to fit GE2 model to both the marginals. We want to use the proposed EM algorithm as suggested in Section 4, to compute the MLEs of the unknown parameters. To get an idea about the initial estimates of $(\alpha, \lambda, \theta)$ for both the marginals, we use the grid search method with a grid length of 0.25,0.5,1.0, for α, λ and θ, respectively. In case of the first marginal, the initial estimates are $(1.75, 4.5, 11.0)$. We use these initial estimates in our EM algorithm, and obtain the final estimates as $\widehat{\alpha} = 2.0277$, $\widehat{\lambda} = 2.8758$ and $\widehat{\theta} = 12.2464$, respectively after 12 iterations. The associated 95% confidence intervals of α, λ and

θ become 2.0277 ∓ 0.7519, 2.8758 ∓ 0.8874, 12.2464 ∓ 1.5629, respectively. Similarly, for the second marginal, the initial estimates are $(0.60, 4.0, 48.0)$, and we obtain the final estimates as $\widehat{\alpha} = 0.5477$, $\widehat{\lambda} = 3.1318$ and $\widehat{\theta} = 45.2060$, respectively after 10 iterations. The associated 95% confidence intervals of α, λ and θ become 0.5477 ∓ 0.2718, 3.1318 ∓ 0.9876, 45.2060 ∓ 3.1781, respectively. We have tried the EM algorithm with some other initial estimates also, and it converges to the same set of estimates, although number of iterations is different. Moreover, we have used Downhill Simplex method to compute the MLEs by maximizing directly the log-likelihood function, and it provides the same set of estimates.

To check the model validity, we compute the Kolmogorov-Smirnov distances between the empirical and the fitted distribution functions for both the marginals. The Kolmogorov-Smirnov statistic and the associated p value for the first marginal become 0.0824 and 0.9976, respectively. Similarly, for the second marginal, they are 0.1182 and 0.9049, respectively. Therefore, GE2 model provide a good fit to both the marginals.

Now we fit the BGE2 model to the bivariate data set. We use the initial estimates of α_1, α_2, α_3, λ_2 and p as 1.6, 0.2, 0.3, 3.0 and 0.2, respectively, to start the EM algorithm as proposed in Section 7, for computing the MLEs of the unknown parameters. The final estimates are obtained after 19 iterations, and they are as follows:

$$\widehat{\alpha}_1 = 1.0416, \quad \widehat{\alpha}_2 = 0.5242, \quad \widehat{\alpha}_3 = 0.3509, \quad \widehat{\lambda} = 2.8522, \quad \widehat{p} = 0.0496,$$

and the corresponding log-likelihood value becomes -10.2314. The associated 95% confidence intervals of α_1, α_2, α_3, λ and p are

$$1.0416 \mp 0.2134, \quad 0.5242 \mp 0.0786, \quad 0.3509 \mp 0.0654, \quad 2.8522 \mp 0.6542, \quad 0.0496 \mp 0.0014,$$

respectively. One question here also arises how good the proposed BGE2 model fits the data. Although there are several goodness of fit tests available for any arbitrary univariate distribution function, not much available for a general bivariate distribution function. Based on Theorem 3, we fit BGE2 distribution to the two marginals, and to the maximum. The Kolmogorov-Smirnov distance (KSD) and the associated p values are reported in Table 2. It indicates that BGE2 can be used to analyze this bivariate data set.

Distribution	KSD	p value
Y_1	0.1027	0.9683
Y_2	0.0996	0.9763
$\max\{Y_1, Y_2\}$	0.0973	0.9813

Table 2: Kolmogorov-Smirnov distances and the associated p values for BGE2 model.

For comparison purposes, we have fitted BGE$(\alpha_1, \alpha_2, \alpha_3, \lambda)$ model of Kundu and Gupta (2009) to the same bivariate data set. The MLEs of the unknown parameters are as follows:

$$\widehat{\alpha}_1 = 4.0347, \quad \widehat{\alpha}_2 = 2.7129, \quad \widehat{\alpha}_3 = 1.7701, \quad \widehat{\lambda} = 1.6200,$$

and the associated maximized log-likelihood value is -39.2682. To check whether BGE model fits the bivariate data set or, we compute the KSD and the associated p values for Y_1, Y_2 and $\max\{Y_1, Y_2\}$ for the BGE model, and it is reported in Table 3. It may be recalled, see Kundu and Gupta (2009), that if $(Y_1, Y_2) \sim$ BGE$(\alpha_1, \alpha_2, \alpha_3, \lambda)$, then similar to the BGE2 model, $Y_1 \sim$ GE$(\alpha_1 + \alpha_3, \lambda)$, $Y_2 \sim$ BGE$(\alpha_2 + \alpha_3, \lambda)$, $\max\{Y_1, Y_2\} \sim$ GE$(\alpha_1 + \alpha_2 + \alpha_3, \lambda)$. Based on Table 3 (since p values are not very small), it is clear that BGE model may be used

Distribution	KSD	p value
Y_1	0.2124	0.2503
Y_2	0.1962	0.3388
$\max\{Y_1, Y_2\}$	0.2415	0.1518

Table 3: Kolmogorov-Smirnov distances and the associated p values for BGE model.

to analyze this bivariate data set. But comparing between BGE and BGE2 model, based on AIC or KSD values it is immediate that BGE2 model provides a better fit than the BGE model.

9 Conclusions

In this paper we have discussed about the generalized exponential geometric extreme distribution, which is an extension of the recently proposed exponentiated exponential-geometric distribution by Louzada et al. (2014). We have suggested to use EM algorithm to compute the MLEs of the unknown parameters. We have further provided a bivariate version of the model and derive different properties. The EM algorithm has been extended to the bivariate case. One data analysis has been performed and it is observed that both the univariate and bivariate models, and the proposed algorithms work quite well.

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Appendix: The expression of $g(\lambda|\gamma^{(k)})$

Let us define:

$$\begin{aligned} h(\lambda|\gamma^{(k)}) \;=\; & \left[\sum_{i\in I_0} y_i + \sum_{i\in I_1\cup I_2} y_{1i} + \sum_{i\in I_1\cup I_2} y_{2i}\right] - \left(\widehat{\alpha}_1^{(k)}(\lambda) + \widehat{\alpha}_2^{(k)}(\lambda) + \widehat{\alpha}_3^{(k)}(\lambda)\right) \sum_{i\in I_0} a_i^{(k)} \frac{y_i e^{-\lambda y_i}}{1-e^{-\lambda y_i}} \\ & - \left(\widehat{\alpha}_1^{(k)}(\lambda) + \widehat{\alpha}_3^{(k)}(\lambda)\right) \sum_{i\in I_1} a_i^{(k)} \frac{y_{1i} e^{-\lambda y_{1i}}}{1-e^{-\lambda y_{1i}}} - \left(\widehat{\alpha}_2^{(k)}(\lambda) + \widehat{\alpha}_3^{(k)}(\lambda)\right) \sum_{i\in I_2} a_i^{(k)} \frac{y_{2i} e^{-\lambda y_{2i}}}{1-e^{-\lambda y_{2i}}} \\ & - \widehat{\alpha}_2^{(k)}(\lambda) \sum_{i\in I_1} a_i^{(k)} \frac{y_{2i} e^{-\lambda y_{2i}}}{1-e^{-\lambda y_{2i}}} - \widehat{\alpha}_1^{(k)}(\lambda) \sum_{i\in I_2} a_i^{(k)} \frac{y_{1i} e^{-\lambda y_{1i}}}{1-e^{-\lambda y_{1i}}} + \sum_{i\in I_0} \frac{y_i e^{-\lambda y_i}}{1-e^{-\lambda y_i}} \\ & + \sum_{i\in I_1\cup I_2} \frac{y_{1i} e^{-\lambda y_{1i}}}{1-e^{-\lambda y_{1i}}} + \sum_{i\in I_1\cup I_2} \frac{y_{2i} e^{-\lambda y_{2i}}}{1-e^{-\lambda y_{2i}}}, \end{aligned}$$

then

$$g(\lambda|\gamma^{(k)}) = (m_0 + 2m_1 + 2m_2)/h(\lambda|\gamma^{(k)}).$$

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