

STATIONARY GE-PROCESS AND ITS APPLICATION IN ANALYZING GOLD PRICE DATA

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Abstract

In this paper we introduce a new discrete time and continuous state space stationary process $\{X_n; n = 1, 2, \dots\}$, such that X_n follows a two-parameter generalized exponential (GE) distribution. Joint distribution functions, characterization and some dependency properties of this new process have been investigated. The GE-process has three unknown parameters, two shape parameters and one scale parameter, and due to this reason it is more flexible than the existing exponential process. In presence of the scale parameter, if the two shape parameters are equal, then the maximum likelihood estimators of the unknown parameters can be obtained by solving one non-linear equation and if the two shape parameters are arbitrary, then the maximum likelihood estimators can be obtained by solving a two dimensional optimization problem. Two **synthetic** data sets, and one real gold-price data set have been analyzed to see the performance of the proposed model in practice. Finally some generalizations have been indicated.

KEY WORDS AND PHRASES: Generalized exponential distribution; maximum likelihood estimators; minification process; maxification process.

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1 INTRODUCTION

Gaussian assumptions are quite common in the theoretical development of any Markovian process. Very few Markovian models have been developed with out the Gaussian assump-

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tions. If the data indicate any non-Gaussian behavior, the usual method is to attempt to remove the skewness of the data by taking suitable transformation, and then use the Gaussian process to the transformed data. Although, it has been criticized severely in the literature. Nelson [17] as well as Granger and Andersen [6] correctly pointed out that quite often the transformed economic data are no where near Gaussian. It may not be very surprising, because Weiss [21] showed that if $\{X_t\}$ is a stationary process, and $f(\cdot)$ is a one-to-one function, then $Y_t = f(X_t)$ is time reversible if and only if $\{X_t\}$ is time reversible. Therefore, it is immediate that a process cannot be transformed to a time reversible Gaussian process unless the process itself is time reversible.

Due to this reason, several non-Gaussian processes have been introduced and studied quite extensively in the literature. For example, stationary exponential process by Tavares [20], Weibull and gamma processes by Sim [19] Logistic process by Arnold [2], Pareto process by Arnold and Hallet [4], see also Arnold [3], semi-Pareto process by Pillai [18], Marshall-Olkin bivariate Weibull processes by Jose, Ristić and Joseph [11], generalized Weibull process by Jayakumar and Girish Babu [10] and see the references cited therein. In all these cases the emphasis is to develop a stationary process which has specific marginals. In most of the cases they have been developed using minification process of autoregressive sequences.

Recently, generalized exponential (GE) distribution has received a considerable amount of attention in the statistical literature. It is a positively skewed distribution, and it can be used quite effectively to analyze lifetime data as an alternative to the popular Weibull or gamma distributions. The aim of this paper is to develop a stationary process whose marginals are identically distributed GE distributions. The GE process has been obtained using the maxification process of moving average (MA) sequences. If we use the q -th order MA process, then a q -dependent sequence with GE marginals can be generated. It has been obtained quite naturally by using the property that the GE distribution is closed under

maximization.

We study different properties of the GE process $\{X_n; n = 1, 2, \dots\}$, when $q = 1$. The joint, marginal and conditional distributions have been obtained. Some characterizations and a mixture representation have been provided. The generation of the GE process is quite straight forward, hence simulation experiments can be performed quite conveniently. The distributions of the maximum and minimum of the GE process and also the probability mass function of the stopping time have been presented. The GE process has two shape parameters and one scale parameter. When the two shape parameters are equal, then the joint distribution of X_n and X_{n+1} has a very convenient copula structure, hence several dependency properties and also dependency measures can be easily obtained. We have provided some dependency properties of the proposed GE process, and also provided different dependency measures of the process.

The estimation of the unknown parameters is an important issue in a real data analysis problem. The GE process has three unknown parameters. The most natural estimators will be the maximum likelihood estimators. If the two shape parameters are same, the maximum likelihood estimators (MLEs) can be obtained by solving one non-linear equation. Although, we could not prove it theoretically, it is observed from the profile likelihood function plot, that the MLEs exist and they are unique. If the two shape parameters are not equal, the maximum likelihood estimators can be obtained by solving a two dimensional optimization problem. In this case also, from the contour plot, it is observed that the MLEs exist and they are unique. We have analyzed **two synthetic data sets, and one gold-price data set** to show how the proposed model behaves in practice. Finally we propose some generalizations and open problems.

The main contribution of this paper is to introduce stationary GE process and derive several of its properties. Although, Weibull and gamma processes have been discussed in

the literature quite extensively, the same is not true in case of GE process, although, GE distribution becomes very popular in the last two decades. Another important contribution is the estimation of the unknown parameters of the proposed GE process. Although, Weibull and GE processes have been discussed quite extensively, no where the estimation procedures have been proposed. Similar estimation procedures what we have used here, can be used for Weibull and **gamma** processes also.

The rest of the paper is organized as follows. In Section 2, we briefly describe the GE distribution. The stationary GE process is proposed in Section 3 and its several properties have been presented. The maximum likelihood estimators are described in Section 4. The analyses of **two synthetic data sets and one gold price data set are presented in Section 5 and Section 6, respectively**. Some generalizations and open problems are indicated in Section 7.

2 GE DISTRIBUTION: A BRIEF REVIEW

The generalized exponential distribution was originally introduced by Gupta and Kundu [7] as a special case of the exponentiated Weibull distribution of Mudholkar and Srivastava [14]. The two-parameter GE distribution has the following cumulative distribution function (CDF);

$$F_{GE}(t; \alpha, \lambda) = (1 - e^{-\lambda t})^\alpha; \quad t > 0, \quad (1)$$

and 0, otherwise. Here, α and λ are the shape and scale parameters, respectively. The corresponding probability density function (PDF) becomes;

$$f_{GE}(t; \alpha, \lambda) = \alpha \lambda e^{-\lambda t} (1 - e^{-\lambda t})^{\alpha-1}; \quad t > 0, \quad (2)$$

and 0, otherwise. A GE random variable with the CDF (1) and PDF (2) will be denoted by $GE(\alpha, \lambda)$, and if $\lambda = 1$, it will be denoted by $GE(\alpha)$. For a $GE(\alpha)$ random variable the corresponding PDF and CDF will be denoted by $f_{GE}(t; \alpha)$ and $F_{GE}(t; \alpha)$, respectively.

It is immediate that when $\alpha = 1$, the GE distribution becomes an exponential distribution. Hence, the GE distribution is an extension of the exponential distribution, similar to the Weibull and gamma distributions but in different ways. It has been observed that the shapes of the PDF and hazard functions of a GE distributions are quite similar to the Weibull and gamma distributions. The hazard function of a GE distribution can be an increasing, decreasing or constant depending on the shape parameter. Since the CDF of a GE distribution is in compact form, hence, the generation of a random sample from a GE distribution is quite straight forward. The GE distribution is closed under maximum and it can be used quite effectively in place of gamma or Weibull distribution for data analysis purposes.

Different moments, order statistics, record values, various estimation procedures, closeness with other distributions like Weibull, gamma, log-normal, have been investigated by several authors. It is observed that the GE distribution is close to a gamma distribution than to a Weibull distribution. Interested readers are referred to the review articles by Gupta and Kundu [8], Nadarajah [15], Al-Hussaini and Ahsanullah [1] and see the references cited therein.

3 GE PROCESS AND ITS PROPERTIES

In this section first we define a stationary Markov process $\{X_n\}$, so that X_n follows a GE distribution and will investigate its several properties.

DEFINITION: Let U_0, U_1, \dots be a sequence of independent and identically distributed (i.i.d.) Uniform $(0, 1)$ random variables. For $\alpha_0 > 0$ and $\alpha_1 > 0$, let us define a new sequence of random variables

$$X_n = \max\{-\ln(1 - U_n^{\frac{1}{\alpha_0}}), -\ln(1 - U_{n-1}^{\frac{1}{\alpha_1}})\}. \quad (3)$$

Then the sequence of random variables $\{X_n\}$ is called a GE process.

From the definition of the GE process, it is very easy to generate random samples from a stationary GE process with a given α_0 and α_1 . We first generate random samples from $U(0,1)$, and then by the required transformation, we can generate $\{X_n\}$. The following Theorem provides the justification of the name GE process. **It shows that the marginals follow GE distribution, and it is a stationary process.**

THEOREM 1: If the sequence of random variables $\{X_n\}$ is as defined in (3), then

- (a) $\{X_n\}$ is a stationary Markov process.
- (b) $\{X_n\}$ follows $\text{GE}(\alpha_0 + \alpha_1)$.

PROOF: Part (a) is trivial.

To prove part (b), note that

$$\begin{aligned} P(X_n \leq x) &= P \left[-\ln(1 - U_n^{\frac{1}{\alpha_0}}) \leq x, -\ln(1 - U_{n-1}^{\frac{1}{\alpha_1}}) \leq x \right] \\ &= P \left[U_n \leq (1 - e^{-x})^{\alpha_0}, U_{n-1} \leq (1 - e^{-x})^{\alpha_1} \right] \\ &= (1 - e^{-x})^{\alpha_0 + \alpha_1}. \end{aligned}$$

■

The following result characterizes the GE process.

THEOREM 2: Suppose $X_1 \sim \text{GE}(\alpha_0 + \alpha_1)$ and U_i s are i.i.d. random variables with an absolute continuous distribution function $F(x)$ on $(0,1)$. Then the process as defined in (3) is a strictly stationary Markov process if and only if U_i s are i.i.d. $U(0,1)$ random variables.

PROOF: ‘If’ part is trivial. To prove the ‘only if’ part, let us assume that $F'(x) = f(x)$, for $x > 0$. Then from the definition of (3), we have for all $x \in (0, \infty)$,

$$(1 - e^{-x})^{\alpha_0 + \alpha_1} = F((1 - e^{-x})^{\alpha_0})F((1 - e^{-x})^{\alpha_1}). \quad (4)$$

Since, (4) is true for all $x \in (0, \infty)$, therefore, it can be written as

$$y^{\alpha_0+\alpha_1} = F(y^{\alpha_0})F(y^{\alpha_1}) \Leftrightarrow \frac{F(y^{\alpha_0})}{y^{\alpha_0}} \times \frac{F(y^{\alpha_1})}{y^{\alpha_1}} = 1, \quad (5)$$

for all $y \in (0, 1)$. Therefore, for all $\alpha > 0$ and for all $y \in (0, 1)$,

$$\frac{F(y^\alpha)}{y^\alpha} = 1, \quad \Rightarrow \quad F(y) = y.$$

■

The following result provides the joint distribution of X_n and X_{n+m} , for $m \geq 1$.

THEOREM 3: If the sequence of random variables $\{X_n\}$ is defined as in (3), then the joint distribution of X_n and X_{n+m} , $F_{X_n, X_{n+m}} = P(X_n \leq x, X_{n+m} \leq y)$ is

$$F_{X_n, X_{n+m}}(x, y) = \begin{cases} (1 - e^{-x})^{\alpha_0+\alpha_1} (1 - e^{-y})^{\alpha_0+\alpha_1} & \text{if } m \geq 2 \\ (1 - e^{-x})^{\alpha_1} (1 - e^{-y})^{\alpha_0} g(x, y) & \text{if } m = 1, \end{cases} \quad (6)$$

where $g(x, y) = \min\{(1 - e^{-x})^{\alpha_0}, (1 - e^{-y})^{\alpha_0}\}$.

PROOF: It mainly follows from the definition and considering the two cases $m = 1$ and $m \geq 2$, separately. ■

It immediately follows from Theorem 3 that X_n and X_{n+m} are independently distributed if $m > 1$, otherwise they are dependent. Now first we would like to study some dependency properties of X_n and X_{n+1} . The joint distribution function of X_n and X_{n+1} can be written as

$$F_{X_n, X_{n+1}}(x, y) = \begin{cases} (1 - e^{-x})^{\alpha_0+\alpha_1} (1 - e^{-y})^{\alpha_0} & \text{if } (1 - e^{-x})^{\alpha_0} \leq (1 - e^{-y})^{\alpha_1} \\ (1 - e^{-x})^{\alpha_1} (1 - e^{-y})^{\alpha_0+\alpha_1} & \text{if } (1 - e^{-x})^{\alpha_0} \geq (1 - e^{-y})^{\alpha_1}. \end{cases} \quad (7)$$

Since $F_{X_n}(x) = (1 - e^{-x})^{\alpha_0+\alpha_1}$ and $F_{X_{n+1}}(y) = (1 - e^{-y})^{\alpha_0+\alpha_1}$, it is immediate that for all $x > 0, y > 0$,

$$F_{X_n, X_{n+1}}(x, y) \geq F_{X_n}(x)F_{X_{n+1}}(y). \quad (8)$$

Hence, X_n and X_{n+1} are positive quadrant dependent (PQD), therefore $Cov(X_n, X_{n+1}) > 0$.

It can be easily verified from the definition that X_n and X_{n+1} has the total positivity of

order two (TP_2) property. Hence, (X_n, X_{n+1}) has left tail decreasing (LTD) as well as left corner set decreasing (LCSD) properties, see for example Nelsen [16].

It can be easily seen that the joint distribution function $F_{X_n, X_{n+1}}(x, y)$ has the following copula function

$$C(u, v) = \begin{cases} uv^{\frac{\alpha_0}{\alpha_0+\alpha_1}} & \text{if } u^{\frac{\alpha_0}{\alpha_0+\alpha_1}} \leq v^{\frac{\alpha_1}{\alpha_0+\alpha_1}} \\ u^{\frac{\alpha_1}{\alpha_0+\alpha_1}}v & \text{if } u^{\frac{\alpha_0}{\alpha_0+\alpha_1}} \geq v^{\frac{\alpha_1}{\alpha_0+\alpha_1}}. \end{cases} \quad (9)$$

Therefore, if we use $\delta = \frac{\alpha_0}{\alpha_0 + \alpha_1}$, then

$$C(u, v) = \begin{cases} uv^\delta & \text{if } u^\delta \leq v^{1-\delta} \\ u^{1-\delta}v & \text{if } u^\delta \geq v^{1-\delta}. \end{cases} \quad (10)$$

Based on the copula function, the following dependence measures can be easily obtained.

For example, the Kendall's τ becomes

$$\tau = \frac{\delta(1-\delta)(1-\delta(1-\delta))}{\delta^3 + \delta(1-\delta) + \delta^2(1-\delta)^2 + (1-\delta)^3}.$$

It can be easily seen that the minimum value of Kendall's τ is zero, and it becomes maximum at $\delta = \frac{1}{2}$, and the maximum value is $\frac{1}{3}$. The Spearman's ρ becomes

$$\rho = \frac{3\delta(1-\delta)}{\delta^2 - \delta + 2}.$$

In this case the minimum value of Spearman's ρ is zero, and it becomes maximum at $\delta = \frac{1}{2}$, and the maximum value is $\frac{3}{7}$. Therefore, both Kendall's τ and Spearman's ρ become maximum when $\alpha_0 = \alpha_1$.

We need the following notations for further development. The sets S_1 , S_2 and the curve C will be defined as follows.

$$S_1 = \{(x, y); x > 0, y > 0, (1 - e^{-x})^{\alpha_0} < (1 - e^{-y})^{\alpha_1}\} \quad (11)$$

$$S_2 = \{(x, y); x > 0, y > 0, (1 - e^{-x})^{\alpha_0} > (1 - e^{-y})^{\alpha_1}\} \quad (12)$$

$$C = \{(x, y); x > 0, y > 0, (1 - e^{-x})^{\alpha_0} = (1 - e^{-y})^{\alpha_1}\}. \quad (13)$$

Note that the curve C has the parametric form $(t, \gamma(t))$, where $\gamma(t) = -\ln\left(1 - (1 - e^{-t})^{\frac{\alpha_0}{\alpha_1}}\right)$, for $0 < t < \infty$.

The following theorem shows that the joint distribution of X_n and X_{n+1} is a singular distributions. It means, although both X_n and X_{n+1} are absolutely continuous, there is a positive probability that $X_n = X_{n+1}$. The joint distribution function of X_n and X_{n+1} can be decomposed uniquely as an absolute continuous part and a singular part.

THEOREM 4: If the sequence of random variables $\{X_n\}$ is same as defined in (3), then the joint CDF of X_n and X_{n+1} can be written as

$$F_{X_n, X_{n+1}}(x, y) = pF_a(x, y) + (1 - p)F_s(x, y), \quad (14)$$

here

$$p = \frac{\alpha_0^2 + \alpha_1^2}{\alpha_0^2 + \alpha_1^2 + \alpha_0\alpha_1}, \quad F_s(x, y) = (g(x, y))^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0\alpha_1}{\alpha_0\alpha_1}} \quad \text{and}$$

$$F_a(x, y) = \frac{\alpha_0^2 + \alpha_1^2 + \alpha_0\alpha_1}{\alpha_0^2 + \alpha_1^2} \left\{ (1 - e^{-x})^{\alpha_1} (1 - e^{-y})^{\alpha_0} g(x, y) \right\} - \frac{\alpha_0\alpha_1}{\alpha_0^2 + \alpha_1^2} (g(x, y))^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0\alpha_1}{\alpha_0\alpha_1}}$$

PROOF: Note that p and $F_a(x, y)$ can be obtained from $F_{X_n, X_{n+1}}(x, y)$ as follows

$$p = \int_0^\infty \int_0^\infty \frac{\partial^2}{\partial x \partial y} F_{X_n, X_{n+1}}(x, y)$$

and

$$pF_a(x, y) = \int_0^x \int_0^y \frac{\partial^2}{\partial u \partial v} F_{X_n, X_{n+1}}(u, v).$$

From

$$\frac{\partial^2}{\partial x \partial y} F_{X_n, X_{n+1}}(x, y) = \begin{cases} f_1(x, y) & \text{if } (x, y) \in S_1 \\ f_2(x, y) & \text{if } (x, y) \in S_2, \end{cases}$$

where

$$f_1(x, y) = \alpha_0(\alpha_0 + \alpha_1)e^{-(x+y)}(1 - e^{-x})^{\alpha_0 + \alpha_1 - 1}(1 - e^{-y})^{\alpha_0 - 1}$$

$$f_2(x, y) = \alpha_1(\alpha_0 + \alpha_1)e^{-(x+y)}(1 - e^{-x})^{\alpha_0 - 1}(1 - e^{-y})^{\alpha_0 + \alpha_1 - 1},$$

the expressions for p and $F_a(x, y)$ can be obtained. Once, we obtain p and $F_a(x, y)$, $F_s(x, y)$ can be obtained by subtraction.

Alternatively, the probabilistic arguments also can be given. Suppose A is the following event:

$$A = \left\{ -\ln \left(1 - U_n^{\frac{1}{\alpha_0}} \right) > -\ln \left(1 - U_{n-1}^{\frac{1}{\alpha_1}} \right) \right\} \cap \left\{ -\ln \left(1 - U_n^{\frac{1}{\alpha_1}} \right) > -\ln \left(1 - U_{n+1}^{\frac{1}{\alpha_0}} \right) \right\}$$

then

$$P(X_n \leq x, X_{n+1} \leq y) = P(X_n \leq x, X_{n+1} \leq y|A)P(A) + P(X_n \leq x, X_{n+1} \leq y|A^c)P(A^c).$$

Now consider

$$P(X_n \leq x, X_{n+1} \leq y|A) = (g(x, y))^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0 \alpha_1}{\alpha_0 \alpha_1}}$$

and

$$P(A) = P \left[U_{n-1}^{\frac{\alpha_0}{\alpha_1}} < U_n, U_{n+1}^{\frac{\alpha_1}{\alpha_0}} < U_n \right] = \int_0^1 u^{\frac{\alpha_1}{\alpha_0} + \frac{\alpha_0}{\alpha_1}} du = \frac{\alpha_0 \alpha_1}{\alpha_0^2 + \alpha_1^2 + \alpha_0 \alpha_1} = 1 - p.$$

Moreover, $P(X_n \leq x, X_{n+1} \leq y|A^c)$ can be obtained by subtraction. Clearly, $F_s(x, y)$ is the singular part, as its mixed partial derivative is 0 in $S_1 \cup S_2$, and $P(X_n \leq x, X_{n+1} \leq y|A^c)$ is the absolute continuous part, as its mixed partial derivative is a proper bivariate density function. ■

Now we would like to obtain the joint probability density function with respect to a proper dominating measure. It will be needed to compute the maximum likelihood estimators of the unknown parameters and other associated statistical inferences based on density functions. We consider the following dominating measure, similarly as in Bemis, Bain and Higgins [5]. The dominating measure is the two dimensional usual Lebesgue measure on $S_1 \cup S_2$, and one dimensional Lebesgue measure defined on the curve C . Here a length is defined as the arc length on the curve C . One natural question is whether we can get different results using

different dominating measures. Fortunately, the answer is negative due to the application of the elementary results by Halmos [9], see also Bemis, Bain and Higgins [5] in this connection.

The following theorem provides the explicit form of the joint PDF of X_n and X_{n+1} based on the above dominating measure.

THEOREM 5: If $\{X_n\}$ is same as defined in Theorem 3, then the joint PDF of X_n and X_{n+1} for $x > 0$ and $y > 0$ is

$$f_{X_n, X_{n+1}}(x, y) = \begin{cases} f_1(x, y) & \text{if } (x, y) \in S_1 \\ f_2(x, y) & \text{if } (x, y) \in S_2 \\ f_0(x) & \text{if } y = \gamma(x), \end{cases} \quad (15)$$

where $f_1(x, y)$ and $f_2(x, y)$ are same as defined before, and

$$f_0(x) = \alpha_1 \times (1 - e^{-x})^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0 \alpha_1 - \alpha_0}{\alpha_1}} \times \left(1 - (1 - e^{-x})^{\frac{\alpha_0}{\alpha_1}}\right).$$

PROOF: To prove Theorem 4, we need to show that for all $0 < x, y < \infty$,

$$F_{X_n, X_{n+1}}(x, y) = \int \int_{B_1} f_1(u, v) dudv + \int \int_{B_2} f_2(u, v) dudv + \int_0^{h(x, y)} f_0(u) |\gamma'(u)| du,$$

here for $A(x, y) = \{(u, v); 0 < u \leq x, 0 < v \leq y\}$, $B_1 = A(x, y) \cup S_1$, $B_2 = A(x, y) \cup S_2$, and $h(x, y) = \min \left\{ x, -\ln \left[1 - (1 - e^{-y})^{\frac{\alpha_1}{\alpha_0}} \right] \right\}$. The first part, namely

$$\int \int_{B_1} f_1(u, v) dudv + \int \int_{B_2} f_2(u, v) dudv = pF_a(x, y),$$

has already shown in Theorem 3. Therefore, the result is proved if we can show that

$$\int_0^{h(x, y)} f_0(u) |\gamma'(u)| du = (1 - p)F_s(x, y).$$

Since,

$$\begin{aligned} |\gamma'(u)| &= \frac{\alpha_0}{\alpha_1} \times \frac{1}{1 - (1 - e^{-u})^{\frac{\alpha_0}{\alpha_1}}} \times (1 - e^{-u})^{\frac{\alpha_0}{\alpha_1} - 1} \times e^{-u}, \\ \int_0^{h(x, y)} f_0(u) |\gamma'(u)| du &= \alpha_0 \int_0^{h(x, y)} e^{-u} (1 - e^{-u})^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0 \alpha_1}{\alpha_1} - 1} du \\ &= (1 - p) (1 - e^{-u})^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0 \alpha_1}{\alpha_1}} \Big|_0^{h(x, y)} = (1 - p)F_s(x, y). \end{aligned}$$

■

Observe that if $\alpha_0 = \alpha_1 = \alpha$, then (15) can be written as

$$f_{X_n, X_{n+1}}(x, y) = \begin{cases} f_1(x, y) & \text{if } x < y \\ f_2(x, y) & \text{if } x > y \\ f_0(x) & \text{if } x = y, \end{cases}$$

here

$$\begin{aligned} f_1(x, y) &= 2\alpha^2 e^{-(x+y)} (1 - e^{-x})^{2\alpha-1} (1 - e^{-y})^{\alpha-1} \\ f_2(x, y) &= 2\alpha^2 e^{-(x+y)} (1 - e^{-x})^{\alpha-1} (1 - e^{-y})^{2\alpha-1} \\ f_0(x) &= \alpha e^{-x} (1 - e^{-x})^{3\alpha-1}. \end{aligned}$$

It can be easily seen that when $\alpha_0 = \alpha_1$, then (X_n, X_{n+1}) follows a bivariate generalized exponential distribution as proposed by Kundu and Gupta [12]. Based on the Markovian property of $\{X_n\}$, the joint PDF of X_1, \dots, X_n can be written as

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{\prod_{i=1}^{n-1} f_{X_i, X_{i+1}}(x_i, x_{i+1})}{\prod_{i=2}^n f_{X_i}(x_i)}, \quad (16)$$

and this will be useful to develop likelihood inference. Now we will study the behavior of the maximum and minimum of a GE process. Let

$$Y_n = \max\{X_1, \dots, X_n\} \quad \text{and} \quad Z_n = \min\{X_1, \dots, X_n\}.$$

Then it can be easily seen that for $\alpha = \min\{\alpha_0, \alpha_1\}$,

$$P(Y_n \leq x) = P(X_1 \leq x, \dots, X_n \leq x) = (1 - e^{-x})^{\alpha_0 + \alpha_1 + \alpha}$$

and

$$P(Z_n \geq x) = P(X_1 \geq x, \dots, X_n \geq x) = (P(X_2 \geq x | X_1 \geq x))^{n-1} P(X_1 \geq x).$$

Moreover, by simple calculation, it follows that

$$P(X_2 \geq x | X_1 \geq x) = 1 - \frac{(1 - e^{-x})^{\alpha_0 + \alpha_1} (1 - (1 - e^{-x})^\alpha)}{1 - (1 - e^{-x})^{\alpha_0 + \alpha_1}}.$$

Now we will discuss about the stopping time. We define the stopping time as the minimum time so that the process exceeds a certain level say L . Let us define a new discrete random variable N , which denotes the stopping time, i.e.

$$\{N = k\} \Leftrightarrow \{X_1 \leq L, X_2 \leq L, \dots, X_{k-1} \leq L, X_k > L\}.$$

Here k can take values $1, 2, \dots$. Therefore, if $p = (1 - e^{-L})$, then

$$\begin{aligned} P(N = 1) &= P(X_1 > L) = 1 - P(X_1 \leq L) = 1 - p^{\alpha_0 + \alpha_1} \\ P(N = 2) &= P(X_1 \leq L, X_2 > L) = P(X_1 \leq L) - P(X_1 \leq L, X_2 \leq L) = p^{\alpha_0 + \alpha_1} - p^{\alpha_0 + \alpha_1 + \alpha} \\ &\vdots \\ P(N = k) &= p^{\alpha_0 + \alpha_1 + (k-2)\alpha} - p^{\alpha_0 + \alpha_1 + (k-1)\alpha}. \end{aligned}$$

The probability generating function $G_N(s)$ for N becomes

$$G_N(s) = E(s^N) = \frac{(1 - p^{\alpha_0 + \alpha_1})z + z^2(p^{\alpha_0 + \alpha_1} - p^\alpha)}{1 - p^\alpha z}.$$

Different properties of the stopping time can be obtained from the probability generating function of N .

4 MAXIMUM LIKELIHOOD ESTIMATION

In this section we consider the maximum likelihood estimation of the unknown parameters based on a random sample of size n , say $\mathbf{x} = \{x_1, \dots, x_n\}$, from a GE process. It is further assumed that the common scale parameter is also present, i.e. X_n can be written as

$$X_n = \frac{1}{\lambda} \max \left\{ -\ln \left(1 - U_n^{\frac{1}{\alpha_0}} \right), -\ln \left(1 - U_{n-1}^{\frac{1}{\alpha_1}} \right) \right\}. \quad (17)$$

It is clear that the $\{X_n\}$ process has two shape parameters α_0 and α_1 , and one scale parameter λ . We consider two cases separately (i) $\alpha_0 = \alpha_1$ and (ii) $\alpha_0 \neq \alpha_1$. It is observed that when

$\alpha_0 = \alpha_1 = \alpha$, the MLEs can be obtained by solving only one non-linear equation. On the other hand when $\alpha_0 \neq \alpha_1$, the MLEs are obtained by solving a two-dimensional optimization problem.

4.1 CASE I: $\alpha_0 = \alpha_1 = \alpha$

Note that in presence of the scale parameter, the joint density function of X_n and X_{n+1} is (15), where

$$\begin{aligned} f_1(x, y) &= 2\alpha^2\lambda^2 e^{-\lambda(x+y)} (1 - e^{-\lambda x})^{2\alpha-1} (1 - e^{-\lambda y})^{\alpha-1} \\ f_2(x, y) &= 2\alpha^2\lambda^2 e^{-\lambda(x+y)} (1 - e^{-\lambda x})^{\alpha-1} (1 - e^{-\lambda y})^{2\alpha-1} \\ f_0(x) &= \alpha\lambda e^{-\lambda x} (1 - e^{-\lambda x})^{3\alpha-1}. \end{aligned} \quad (18)$$

Let us use the following notations:

$$\begin{aligned} A_1 &= \{i; 1 \leq i \leq n-1, x_i < x_{i+1}\}, \quad A_2 = \{i; 1 \leq i \leq n-1, x_i > x_{i+1}\}, \\ A_0 &= \{i; 1 \leq i \leq n-1, x_i = x_{i+1}\}, \end{aligned}$$

and

$$|A_0| = n_0, |A_1| = n_1, |A_2| = n_2.$$

Clearly, $n_0 + n_1 + n_2 = n - 1$. Moreover, we also define $f_{X_i}(x; \alpha, \lambda)$ and $f_{X_i|X_{i-1}=y}(x; \alpha, \lambda)$ as the density function of X_i and the conditional density function of X_i given $X_{i-1} = y$, respectively. Now based on the observations, the log-likelihood function of the observed data becomes, see (16),

$$\begin{aligned} l(\alpha, \lambda | Data) &= \ln f_{X_1}(x_1; \alpha, \lambda) + \sum_{i=2}^n \ln f_{X_i|X_{i-1}=x_{i-1}}(x_i; \alpha, \lambda) \\ &= \sum_{i=1}^{n-1} \ln f_{X_i, X_{i+1}}(x_i, x_{i+1}; \alpha, \lambda) - \sum_{i=2}^{n-1} \ln f_{X_i}(x_i; \alpha, \lambda) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in I_0 \cup I_1 \cup I_2} \ln f_{X_i, X_{i+1}}(x_i, x_{i+1}; \alpha, \lambda) - \sum_{i=2}^{n-1} \ln f_{X_i}(x_i; \alpha, \lambda) \\
&= c + (n_1 + n_2 + 2) \ln \alpha + \alpha g_1(\lambda, \mathbf{x}) + (n_1 + n_2 + 2) \ln \lambda - \lambda g_2(\mathbf{x}) + g_3(\lambda, \mathbf{x}).
\end{aligned}$$

Here c is a constant, and

$$\begin{aligned}
g_1(\lambda, \mathbf{x}) &= 2 \sum_{i \in A_1} \ln(1 - e^{-\lambda x_i}) + \sum_{i \in A_1} \ln(1 - e^{-\lambda x_{i+1}}) + \sum_{i \in A_2} \ln(1 - e^{-\lambda x_i}) + 2 \sum_{i \in A_2} \ln(1 - e^{-\lambda x_{i+1}}) \\
&\quad - \sum_{i=2}^{n-1} \ln(1 - e^{-\lambda x_i}) \\
g_2(\mathbf{x}) &= \left[\sum_{i \in A_1 \cup A_2} (x_i + x_{i+1}) + \sum_{A_0} x_i - \sum_{i=2}^{n-1} x_i \right] \\
g_3(\lambda, \mathbf{x}) &= - \sum_{i \in A_1 \cup A_2} \ln(1 - e^{-\lambda x_i}) - \sum_{i \in A_1 \cup A_2} \ln(1 - e^{-\lambda x_{i+1}}) - \sum_{i \in A_0} \ln(1 - e^{-\lambda x_i}) + \sum_{i=2}^{n-1} \ln(1 - e^{-\lambda x_i}).
\end{aligned}$$

Hence, for a given λ , the MLE of α , say, $\hat{\alpha}(\lambda)$ can be obtained as

$$\hat{\alpha}(\lambda) = -\frac{n_1 + n_2 + 2}{g_1(\lambda, \mathbf{x})}, \tag{19}$$

and the MLE of λ can be obtained by maximizing

$$h(\lambda) = -(n_1 + n_2 + 2) \ln(g_1(\lambda, \mathbf{x})) + (n_1 + n_2 + 2) \ln \lambda - \lambda g_2(\mathbf{x}) + g_3(\lambda, \mathbf{x}).$$

The maximization of $h(\lambda)$ involves solving a one-dimensional optimization problem. We can use bisection method to compute $\hat{\lambda}$, the maximum of $h(\lambda)$. Once $\hat{\lambda}$ is obtained then $\hat{\alpha}$ can be obtained as $\hat{\alpha}(\hat{\lambda})$ from (19). Due to complicated nature of $h(\lambda)$, it is not possible to prove that it is an unimodal function. We propose to plot $h(\lambda)$ as a function of λ , to get an idea about $\hat{\lambda}$. The details have been illustrated in the Synthetic Experiments and Data Analysis sections.

4.2 CASE II: $\alpha_0 \neq \alpha_1$

Now we will consider the MLEs of the unknown parameters, when $\alpha_0 \neq \alpha_1$. Therefore, in this case we have three unknown parameters. In this case the joint PDF of X_n and X_{n+1} in

presence of the scale parameter λ is (15), where

$$\begin{aligned} f_1(x, y) &= \alpha_0(\alpha_0 + \alpha_1)\lambda^2 e^{-\lambda(x+y)} (1 - e^{-\lambda x})^{\alpha_0 + \alpha_1 - 1} (1 - e^{-\lambda y})^{\alpha_0 - 1} \\ f_2(x, y) &= \alpha_1(\alpha_0 + \alpha_1)\lambda^2 e^{-\lambda(x+y)} (1 - e^{-\lambda x})^{\alpha_1 - 1} (1 - e^{-\lambda y})^{\alpha_0 + \alpha_1 - 1} \\ f_0(x) &= \alpha_1 \lambda (1 - e^{-\lambda x})^{\frac{\alpha_0^2 + \alpha_1^2 + \alpha_0 \alpha_1 - \alpha_0}{\alpha_1}} \left(1 - (1 - e^{-\lambda x})^{\frac{\alpha_0}{\alpha_1}}\right). \end{aligned}$$

We reparametrize as $(\alpha_0, \alpha_1, \lambda)$ to $(\gamma, \alpha_1, \lambda)$, where $\gamma = \frac{\alpha_0}{\alpha_1}$. We use the following notations:

$$\begin{aligned} A_1(\gamma) &= \{i; 1 \leq i \leq n-1, (1 - e^{-\lambda x_i}) < (1 - e^{-\lambda x_{i+1}})^\gamma\}, \\ A_2(\gamma) &= \{i; 1 \leq i \leq n-1, (1 - e^{-\lambda x_i}) > (1 - e^{-\lambda x_{i+1}})^\gamma\}, \\ A_0(\gamma) &= \{i; 1 \leq i \leq n-1, (1 - e^{-\lambda x_i}) = (1 - e^{-\lambda x_{i+1}})^\gamma\}, \end{aligned}$$

and $n_1(\gamma) = |A_1(\gamma)|$, $n_2(\gamma) = |A_2(\gamma)|$, $n_0(\gamma) = |A_0(\gamma)|$. Therefore, based on the above notations, the log-likelihood function becomes

$$\begin{aligned} l(\gamma, \alpha_1, \lambda | \mathbf{x}) &= \ln f_{X_1}(x_1; \gamma, \alpha_1, \lambda) + \sum_{i=2}^n \ln f_{X_i | X_{i-1}=x_{i-1}}(x_i; \gamma, \alpha_1, \lambda) \\ &= \sum_{i=1}^{n-1} \ln f_{X_i, X_{i+1}}(x_i, x_{i+1}; \gamma, \alpha_1, \lambda) - \sum_{i=2}^{n-1} \ln f_{X_i}(x_i; \gamma, \alpha_1, \lambda) \\ &= (2n_1(\gamma) + 2n_2(\gamma) + n_0(\gamma)) \ln \alpha_1 + \alpha_1 h_1(\lambda, \gamma, \mathbf{x}) + \\ &\quad (2n_1(\gamma) + 2n_2(\gamma) + n_0(\gamma)) \ln \lambda - \lambda \sum_{i \in A_1(\gamma) \cup A_2(\gamma)} (x_i + x_{i+1}) + \\ &\quad n_1(\gamma) (\ln \gamma + \ln(1 + \gamma)) + n_2(\gamma) \ln(1 + \gamma) + \\ &\quad \sum_{i \in A_0(\gamma)} \ln(1 - (1 - e^{-\lambda x_i})^\gamma) - \sum_{i \in A_1(\gamma) \cup A_2(\gamma)} \ln(1 - e^{-\lambda x_i}) - \\ &\quad \sum_{i \in A_1(\gamma) \cup A_2(\gamma)} \ln(1 - e^{-\lambda x_{i+1}}) - \sum_{i \in A_0(\gamma)} \ln(1 - e^{-\lambda x_i}) \end{aligned}$$

where

$$\begin{aligned} h_1(\lambda, \gamma, \mathbf{x}) &= (1 + \gamma) \sum_{i \in A_1(\gamma)} \ln(1 - e^{-\lambda x_i}) + \gamma \sum_{i \in A_1(\gamma)} \ln(1 - e^{-\lambda x_{i+1}}) + \\ &\quad (1 + \gamma) \sum_{i \in A_2(\gamma)} \ln(1 - e^{-\lambda x_{i+1}}) + \gamma \sum_{i \in A_2(\gamma)} \ln(1 - e^{-\lambda x_i}) + \\ &\quad (1 + \gamma + \gamma^2) \sum_{i \in A_0(\gamma)} \ln(1 - e^{-\lambda x_i}). \end{aligned}$$

For fixed γ and λ , the MLE of α_1 , say $\hat{\alpha}_1(\gamma, \lambda)$ can be obtained as

$$\hat{\alpha}_1(\gamma, \lambda) = -\frac{2n_1(\gamma) + 2n_2(\gamma) + n_0(\gamma)}{h_1(\lambda, \gamma, \mathbf{x})}. \quad (20)$$

Hence, the MLEs of γ and λ can be obtained by maximizing numerically $l(\gamma, \hat{\alpha}_1(\gamma, \lambda), \lambda)$ with respect to γ and λ . Note that it is a two-dimensional optimization problem. Newton-Raphson or some iterative methods may be used to compute $\hat{\gamma}$ and $\hat{\lambda}$, the MLEs of γ and λ , respectively. Once $\hat{\gamma}$ and $\hat{\lambda}$ are obtained, the MLE of α can be obtained as $\hat{\alpha}(\hat{\gamma}, \hat{\lambda})$ from (20). One needs a starting values to start any iterative process. In this respect, we suggest to use the contour plot of the profile log-likelihood function $l(\gamma, \hat{\alpha}_1(\gamma, \lambda), \lambda)$. The details will be illustrated in the Synthetic Experiments and Data Analysis sections.

5 SYNTHETIC EXPERIMENTS

In this section we provide the analyses of two synthetic data sets. Two data sets have been simulated: (i) $\alpha_0 = \alpha_1 = \alpha$, (ii) $\alpha_0 \neq \alpha_1$.

5.1 SYNTHETIC DATA SET 1:

In this case we have generated the data set of size $n = 100$ with the following parameters:

$$\alpha_0 = \alpha_1 = \alpha = 2.0 \quad \text{and} \quad \lambda = 1.0.$$

The generated $\{x_1, \dots, x_{100}\}$ has been plotted in Figure 1. We first computed the MLE of λ by maximizing the profile log-likelihood function $h(\lambda)$ as mentioned in the previous section. The profile log-likelihood function $h(\lambda)$ has been plotted in Figure 2. It is an unimodal function. Therefore, MLEs are unique in this case. It clearly gives an idea that the MLE of λ lies between 0.5 and 1.5. We start our bisection method with these two boundaries, and the MLE of λ has been obtained as $\hat{\lambda} = 0.9058$. Based on $\hat{\lambda}$, the MLE of α can be obtained

as $\hat{\alpha} = 1.5164$. We have used parametric bootstrap method to compute the 95% confidence intervals of α and λ and they are $(0.7132, 1.1015)$ and $(0.9054, 2.1141)$, respectively.

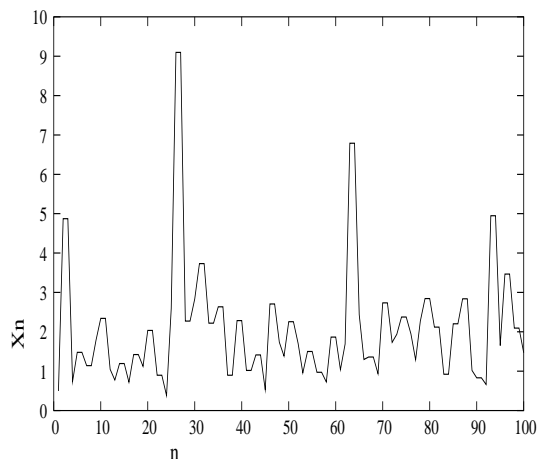


Figure 1: Generated X_n , when $\alpha_0 = \alpha_1 = \alpha = 2.0$ and $\lambda = 1$.

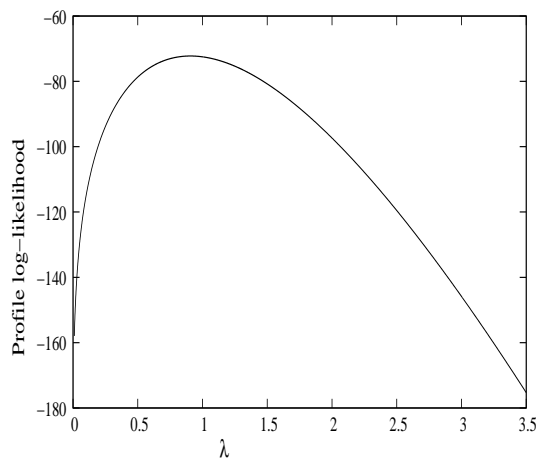


Figure 2: The profile log-likelihood of λ .

5.2 SYNTHETIC DATA SET 2:

In this case a data set of size $n = 100$, has been generated using the following parameters:

$$\alpha_0 = 2.0, \quad \alpha_1 = 3.0, \quad \text{and} \quad \lambda = 1.0.$$

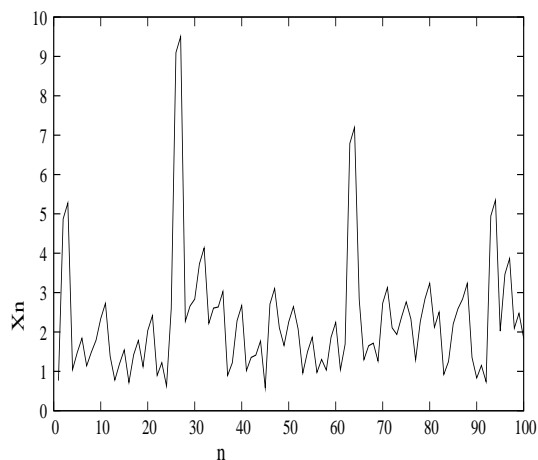


Figure 3: Generated X_n , when $\alpha_0 = 2.0$, $\alpha_1 = 3.0$ and $\lambda = 1$.

The data set has $\{x_1, \dots, x_{100}\}$ has been plotted in Figure 3.

Now we would like to compute the MLEs of the α_1 , γ and λ , and they can be obtained by maximizing first the profile log-likelihood function $h_1(\lambda, \gamma, \mathbf{x})$ as defined in Section 4.2. We provide the contour lot of $h_1(\lambda, \gamma, \mathbf{x})$ in Figure 4. It indicates that the profile log-likelihood

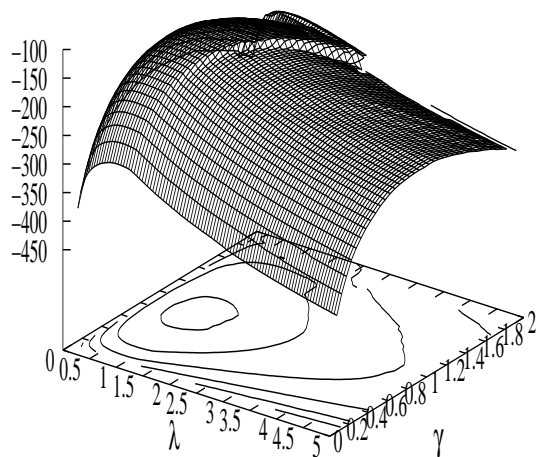


Figure 4: The contour plot of the profile log-likelihood of λ and γ .

function of λ and γ is an unimodal function, hence the MLEs are unique. **The contour plot provides a good idea about the initial guesses of λ and γ . We have started the iterative**

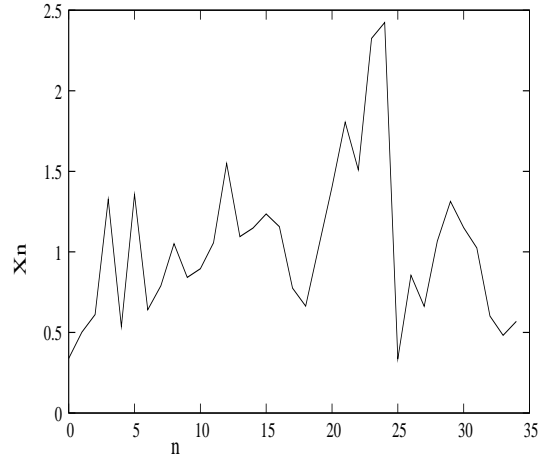


Figure 5: Gold price in Indian market for 45 days starting from October 06, 2020.

process with $\lambda = 1$ and $\gamma = 0.6$. The iteration stops at $\hat{\lambda} = 0.8699$ and $\hat{\gamma} = 0.8500$. Based on these, the MLEs of α_0 , α_1 and λ become $\hat{\alpha}_0 = 2.1338$, $\hat{\alpha}_1 = 2.5103$ and $\hat{\lambda} = 0.8699$. In this case based on the parametric bootstrap the 95% confidence intervals for α_0 , α_1 and λ are (1.5431,2.8342), (1.8775,3.2312) and (0.6754,1.1231), respectively.

6 GOLD PRICE DATA ANALYSIS

In this section we present the analysis of gold-price data based on the proposed GE process to see how the proposed model and methods can be used in practice. This is a real data set of gold price per gram in Indian Rupees in Indian market of 45 days starting from October 06, 2020, and it has been obtained from the website as follows: <https://www.bullion-rates.com/gold/INR-history.htm>. There is no trading during the weekends and holidays, hence we have data for 35 days. The minimum and maximum values were Rs. 4230.02 and Rs. 4642.32, respectively. We have scaled the data set by subtracting 4200 and divided by 100, to each data points. The scaled data set has been plotted in Figure 5. First we compute the MLEs of α and λ based on the assumption $\alpha_0 = \alpha_1 = \alpha$. The profile log-likelihood function of λ has been plotted in Figure 6. It is an unimodal function, hence the

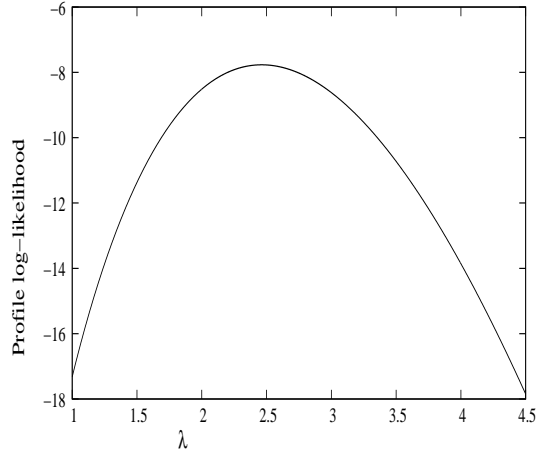


Figure 6: The profile log-likelihood function of λ for Gold price data.

MLEs are unique. The MLEs of α and λ are $\hat{\lambda} = 2.4620$ and $\hat{\alpha} = 3.3498$. The associated log-likelihood value is -11.4732, 95% confidence intervals of λ and α are (1.9854,2.6831) and (2.9552,3.7454), respectively.

Based on the assumption $\alpha_0 \neq \alpha_1$, the MLEs of λ , α_0 and α_1 are $\hat{\lambda} = 2.3449$, $\hat{\alpha}_0 = 3.5312$, $\hat{\alpha}_1 = 4.2684$. The associated log-likelihood value is -9.0123 and 95% confidence intervals of λ , α_0 and α_1 are (1.9756,2.7016), (2.9625,3.7523), (3.9598,4.6734), respectively. The contour plot of λ and γ is provided in Figure 7. From the contour plot of γ and λ , it is clear that the MLEs exist and they are unique.

Now one natural question is how to show that GE process fits the gold price data. We still do not have a proper goodness of fit test, but we have tried the following measures which ensures at least that it does not violate some of the sufficient conditions. If $\{X_n\}$ is a GE process, then $\{X_{2n-1}; n = 1, 2, \dots\}$, will be i.i.d. GE random variables, and similarity, $\{X_{2n}; n = 1, 2, \dots\}$ will be also i.i.d. GE random variables. **Now we would like to test the following: first we would like to test whether $\{x_1, x_3, \dots, x_{35}\}$ are independently distributed or not, and then test whether they follow GE distribution or not. The same we want to do for $\{x_2, x_4, \dots, x_{34}\}$ also. To test whether they are independent or not we have used run test,**

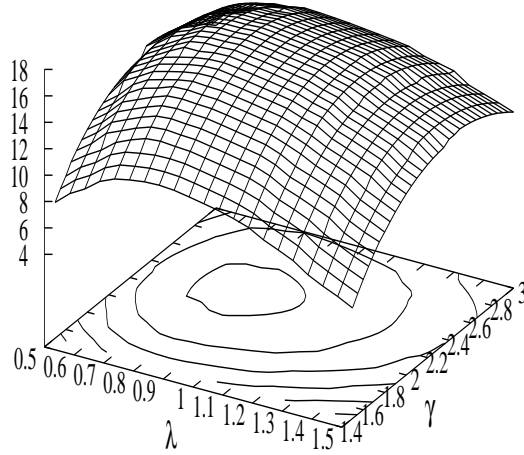


Figure 7: The contour plot of γ and λ for the gold-price log-likelihood function.

and for testing whether they follow GE distributions or not we have used Kolmogorv-Smirnov (KS) test.

We fit $GE(\beta, \theta)$ to $\{x_1, x_3, \dots, x_{35}\}$. The MLEs of β and θ are $\hat{\beta} = 7.0863$ and $\hat{\theta} = 2.5866$, respectively. The Kolmogorov-Smirnov (KS) distance between the empirical cumulative distribution function (CDF) and the fitted CDF is 0.1437 and the corresponding p value is 0.8513. We further perform the test of independence of $\{x_1, x_3, \dots, x_{35}\}$ based on run test, and the corresponding p value becomes 0.15. We have done the same procedure for $\{x_2, x_4, \dots, x_{34}\}$ also. In this case $\hat{\beta} = 6.6324$ and $\hat{\theta} = 2.3979$. The KS distance and the corresponding p values are 0.1326 and 0.9261, respectively. The p value based on run test is 0.45. Hence, we cannot reject the null hypothesis that $\{x_1, x_3, \dots, x_{35}\}$ is a random sample from a GE distribution, and the same for $\{x_2, x_4, \dots, x_{34}\}$.

We have further computed the first and second order autocorrelations of the data set and they are 0.0915 and 0.0402, respectively. We have obtained the distribution of the first order and second order autocorrelations for GE process based on simulations. The upper 90% percentile points of the first and second order autocorrelations are 0.1621 and 0.0504, respectively. Therefore, based on the observed first and second order autocorrelations, we

cannot reject the hypothesis that the data are coming from a GE process and based on the log-likelihood values, we cannot reject the hypothesis that $\alpha_0 = \alpha_1$. Hence, we conclude that GE process with two equal shape parameters, fits the gold-price data well.

7 CONCLUSIONS

In this paper we propose a new discrete time and continuous state space stationary process, and we named it as a GE process. It is called a stationary GE process as the marginals are GE distributions and it is a stationary process. The distinct feature of this proposed process is that the joint distribution of X_n and X_{n+1} is a singular distribution, due to this it can be used if there are some ties in the consecutive data points with positive probability. The existing Weibull or gamma processes do not have this feature.

The proposed GE process can be easily extended to a more general class of proportional reversed hazard (PRH) process, i.e. for a class of lifetime distribution functions which can be represented as follows:

$$F(t; \alpha) = (F_0(t))^\alpha.$$

Here $F_0(t)$ is an absolutely continuous distribution, and $F_0(t) = 0$, for $t \leq 0$, see for example Kundu and Gupta [13]. Now we can define PRH process as follows.

$$X_n = \max\{F_0^{-1}(U_n^{\frac{1}{\alpha_0}}), F_0^{-1}(U_{n-1}^{\frac{1}{\alpha_1}})\},$$

where $\{U_0, U_1, \dots, \}$ is a sequence of i.i.d. Uniform (0,1) random variables, $\alpha_0 > 0$ and $\alpha_1 > 0$. Most of the results what we have developed for the GE process, can be extended for the PRH process also. It will be interesting to develop proper inferential procedure and some model selection criteria for data analysis purposes. More work is needed along that direction.

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