Order Restricted Inference of a Multiple Step-Stress Model

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Abstract

In this manuscript both the classical and Bayesian analyses of a multiple step-stress model have been considered. The lifetime distributions of the experimental units at each stress level follow two-parameter generalized exponential distribution and they are related through the cumulative exposure model assumptions. Recently Abdel-Hamid and Al-Hussaini (Computational Statistics and Data Analysis, 53:1328–1338, 2009) provided the classical inference of the model parameters of a simple step-stress model, under the same set of assumptions. In a typical step-stress experiment, it is expected that the lifetime of the experimental units will be shorter at the higher stress level. The main aim of this paper is to develop the order restricted inference of the model parameters of a multiple step-stress model based on both the classical and Bayesian approaches. An extensive simulation study has been performed and one data set has been analyzed for illustrative purposes.

Key Words Step-stress life tests; Cumulative exposure model; Maximum likelihood estimator; Credible interval; Bootstrap confidence interval.

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1 Introduction

In today's competitive world the industrial products become highly reliable. Therefore, for any analysis purposes, it becomes very difficult to get sufficient failure time data during the normal experimental time. The accelerated life testing (ALT) experiment is frequently being used to overcome this problem. The ALT experiments are introduced to conduct the experiment under one or more extreme operating conditions and thus increasing the number of failures within an affordable experimental time. The factors which directly affect the lifetime of the products are called stress factors, for example, voltage, temperature, humidity could be some of the stress factors for testing an electronic equipment. Some of the key references on different ALT models are Nelson [14], Bagdonavicius and Nikulin [4] and the references cited therein.

A special case of the ALT experiment is known as the step-stress life testing (SSLT) experiment, where the stress changes at a given time or after a specified number of failures. In a SSLT experiment if we consider only two stress levels, then it is known as a simple step-stress experiment. In a review article Balakrishnan [5] extensively discussed different inferential issues of a step-stress model when the lifetime distributions of the experimental units follow exponential distribution. In a recent monograph, Kundu and Ganguly [12] provided an extensive review of the different step-stress models.

Let us assume that the cumulative distribution function (CDF) of the lifetime at the stress level S_{i-1} is $F_i(\cdot)$. To analyze a data obtained from a SSLT experiment, one needs a model which relates the CDFs of lifetime under different stress levels to the CDF of the lifetime of the product under the SSLT experiment. Several models are available in the literature to describe this relationship. The most popular one is known as the cumulative exposure model (CEM) originally proposed by Sedyakin [16] and later quite extensively studied by Bagdonavicius [3] and Nelson [14]. This model assumes that the remaining lifetime of an experimental unit depends only on the cumulative exposure accumulated at the current stress level, irrespective of how the exposure has actually been accumulated. An extensive amount of work has been done in developing the statistical inference of the model parameters in a SSLT set up under the CEM assumptions, see for example the recent Ph.D. thesis by Ganguly [7] or the monograph by Kundu and Ganguly [12] for an extensive list of references on different step-stress models.

The main objective of a SSLT experiment is to reduce the lifetime of the experimental units by increasing the stress level. Therefore, it is quite natural to assume that the expected lifetime of the experimental units is lower at the higher stress level. Balakrishnan et al. [6] first incorporated this information and considered the estimation of the model parameters based on the assumption that the lifetime distribution of the experimental units follow exponential distribution. They obtained the order restricted maximum likelihood estimators (MLEs) and also discussed the hypothesis testing problems under order restrictions in case of Type-I and Type-II censored data. Recently, Samanta et al. [15] developed the order restricted Bayesian inference of the model parameters under the same set of assumptions.

A two-parameter generalized exponential distribution (GE) has received a considerable amount of attention since its introduction by Gupta and Kundu [9]. A two-parameter GE distribution with the shape parameter $\alpha > 0$ and scale parameter $\theta > 0$ has the following CDF, probability density function (PDF) and hazard function (HF), respectively,

$$F(t;\alpha,\theta) = (1-e^{-\theta t})^{\alpha}, \qquad t>0, \tag{1}$$

$$f(t;\alpha,\theta) = \theta\alpha e^{-\theta t}(1-e^{-\theta t})^{\alpha-1}, \qquad t>0, \tag{2}$$

$$H(t;\alpha,\theta) = \frac{\alpha\theta(1-e^{-\theta t})^{\alpha-1}e^{-\theta t}}{1-(1-e^{-\theta t})^{\alpha}}, \qquad t>0. \tag{3}$$

From now on a GE distribution with the shape parameter α and the scale parameter θ will be denoted by GE(α,θ). Due to presence of the shape parameter the GE distribution is a very flexible model. The PDF of a GE distribution can be a decreasing or an unimodal function. Moreover, the hazard function of a GE distribution can be an increasing, decreasing or a constant function depending on the shape parameter. Conventional exponential distribution is a special case of the GE distribution. Therefore, Weibull, gamma and the GE distributions are all extensions of the exponential distribution but in different ways. It has been shown by Gupta and Kundu [10] that the GE distribution can be a good alternative to a gamma or a Weibull distribution. In fact in many cases it may provide a better fit to a given data set, than the gamma or the Weibull distribution. Interested readers are referred to Gupta and Kundu [11], Al-Hussaini and Ahsnullah [2], Nadarajah [13] and the references cited therein for different developments associated with the GE distribution.

In this paper we consider the analysis of a given data set, obtained from a multiple SSLT experiment. It is assumed that the lifetime distribution of the experimental unit under each stress level follows a two parameter GE distribution with the same shape parameter but different scale parameters, and it satisfies the CEM assumptions. It is further assumed that the expected lifetime of the experimental units at the higher stress level is smaller compared to a lower stress level. We provide the order restricted inference of the model parameters both under the classical and Bayesian set up. We provide both the point and interval estimators of the unknown parameters associated with the model. An extensive simulation experiment has been performed to see the effectiveness of the order restricted inference under both classical and Bayesian methods. It is observed that the performances of the Bayes estimators even with non-informative priors are significantly better than the classical estimators in terms of biases and mean squared errors (MSEs). We provide the analysis of one data set for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we provide the model assumptions and the likelihood function based on the available data. In Section 3, we obtain the MLEs and the associated Fisher information matrices of the unknown parameters. The Bayes estimators and their credible intervals are provided in Section 4. In Section 5 we present the simulation results and the analysis of one data set. Finally we conclude the paper in Section 6.

2 Model Assumptions and the Likelihood Function

Let us assume that there are $m+1$ stress levels, say $S_0, S_1, \ldots S_m$ and the expected lifetime of experimental units is shorter at the stress level S_k than at the stress level S_{k-1}; $k = 1, \ldots, m$. Suppose n experimental units are subjected to a life testing experiment at the time point 0, under the stress level S_0. The stress level is increased to S_1 at a pre-fixed time τ_1 and then to S_2 at a pre-fixed time τ_2 and so on. Finally the stress level is increased to S_m at the time point τ_m, and the experiment continues till all the n items fail.

Failure time data obtained from this multiple SSLT experiment is denoted by

$$\begin{aligned}\mathcal{D} &= \{t_{1:n} < \ldots < t_{n_1:n} < \tau_1 < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n} < \tau_2 < \ldots \\ &\quad < \tau_m < t_{(n_1+\ldots+n_m+1):n} < \ldots < t_{n:n}\}. \end{aligned} \tag{4}$$

Here n_i is the number of failures under the stress level S_{k-1} $(k = 1, \ldots, m+1)$. It is assumed that the lifetime distribution of the experimental units under the stress level S_{k-1} follows GE(α, θ_k). Hence, for $\alpha > 0$, $\theta_k > 0$ and $t > 0$,

$$F_k(t) = \left(1 - e^{-\theta_k t}\right)^{\alpha}, \qquad k = 1, \ldots, m+1.$$

Since it is assumed that $F_1(\cdot), \ldots, F_{m+1}(\cdot)$ satisfy the CEM assumptions, we have

$$F(t) = \begin{cases} F_1(t) & \text{if } 0 < t \leq \tau_1, \\ F_k(c_{k-1} + t - \tau_{k-1}) & \text{if } \tau_{k-1} < t < \tau_k, \\ F_{m+1}(c_m + t - \tau_m) & \text{if } \tau_m < t < \infty, \end{cases} \tag{5}$$

where

$$F_k(c_{k-1}) = F_{k-1}(c_{k-2} + \tau_{k-1} - \tau_{k-2}); \quad k = 2, 3, \ldots, m+1.$$

By solving the above recursion relations one can easily obtain

$$c_{k-1} = \frac{1}{\theta_k}\sum_{j=1}^{k-1}\theta_j(\tau_j - \tau_{j-1}); \quad k = 2, 3, \ldots, m+1,$$

with $c_0 = 0$ and $\tau_0 = 0$. Hence, the probability density function (PDF) associated with the CDF (5) is given by

$$f(t) = \begin{cases} \alpha\theta_1(1 - e^{-\theta_1 t})^{\alpha-1}e^{-\theta_1 t} & \text{if } 0 < t \leq \tau_1, \\ \alpha\theta_k(1 - e^{-\theta_k(c_{k-1}+t-\tau_{k-1})})^{\alpha-1}e^{-\theta_k(c_{k-1}+t-\tau_{k-1})} & \text{if } \tau_{k-1} < t < \tau_k, \text{ for } k = 2, 3, \ldots, m \\ \alpha\theta_{m+1}(1 - e^{-\theta_{m+1}(c_m+t-\tau_m)})^{\alpha-1}e^{-\theta_{m+1}(c_m+t-\tau_m)} & \text{if } \tau_m < t < \infty. \end{cases} \tag{6}$$

Therefore, based on the complete data $\mathcal{D}$, the likelihood function can be written as

$$\begin{aligned} l(\alpha, \theta_1, \ldots, \theta_{m+1}|\mathcal{D}) \;\propto\; & \alpha^n\theta_1^{n_1}\ldots\theta_{m+1}^{n_{m+1}}\prod_{i=1}^{n_1}(1 - e^{-\theta_1 t_{i:n}})^{\alpha-1}e^{-\theta_1\sum_{i=1}^{n_1} t_{i:n}} \times \\ & \prod_{i=n_1+1}^{\overline{n}_2}(1 - e^{-\theta_2(t_{i:n}+\frac{\theta_1}{\theta_2}\tau_1-\tau_1)})^{\alpha-1}e^{-\theta_2\sum_{i=n_1+1}^{\overline{n}_2}(t_{i:n}+\frac{\theta_1}{\theta_2}\tau_1-\tau_1)} \times \\ & \prod_{i=\overline{n}_2+1}^{\overline{n}_3}(1 - e^{-\theta_3(t_{i:n}+\frac{\theta_1}{\theta_3}\tau_1+\frac{\theta_2}{\theta_3}(\tau_2-\tau_1)-\tau_2)})^{\alpha-1}e^{-\theta_3\sum_{i=\overline{n}_2+1}^{\overline{n}_3}(t_{i:n}+\frac{\theta_1}{\theta_3}\tau_1+\frac{\theta_2}{\theta_3}(\tau_2-\tau_1)-\tau_2)} \times \\ & \vdots \\ & \prod_{i=\overline{n}_m+1}^{n}(1 - e^{-\theta_{m+1}(t_{i:n}+\frac{\theta_1}{\theta_{m+1}}\tau_1+\frac{\theta_2}{\theta_{m+1}}(\tau_2-\tau_1)+\ldots+\frac{\theta_m}{\theta_{m+1}}(\tau_m-\tau_{m-1})-\tau_m)})^{\alpha-1} \times \\ & e^{-\theta_{m+1}\sum_{i=\overline{n}_m+1}^{n}(t_{i:n}+\frac{\theta_1}{\theta_{m+1}}\tau_1+\frac{\theta_2}{\theta_{m+1}}(\tau_2-\tau_1)+\ldots+\frac{\theta_m}{\theta_{m+1}}(\tau_m-\tau_{m-1})-\tau_m)}, \end{aligned} \tag{7}$$

where $\overline{n}_k = \sum_{j=1}^{k} n_j$. In the next section we provide the MLEs of the unknown parameters and the associated confidence intervals based on the Fisher information with the order restriction on $\theta_1 \ldots \theta_{m+1}$. It should be mentioned that through out this paper it is assumed that if $\theta_1^0, \ldots, \theta_{m+1}^0$ are the true values of $\theta_1, \ldots, \theta_{m+1}$ respectively, then $\theta_1^0 < \ldots < \theta_{m+1}^0$.

3 Maximum Likelihood Estimators

In a typical step-stress experiment it is reasonable to assume that as the stress level increases the expected lifetime of the experimental units decrease. In this section we use this information to compute the MLEs of the unknown parameters. In this case due to the assumption of the lifetime distribution of the experimental units the expected lifetime at the stress level S_{i-1} $(i = 1, \ldots, m+1)$ is

$$\frac{1}{\theta_i}\left(\psi(\alpha+1) - \psi(1)\right),$$

where $\psi(x) = \dfrac{d}{dx}\ln\Gamma(x)$ is the digamma function, see Gupta and Kundu [9]. Therefore, we have the natural restriction on the scale parameters as $\theta_1 < \ldots < \theta_{m+1}$. Hence the MLEs of the unknown parameters can be obtained by maximizing (7) with respect to α, $\theta_1, \ldots, \theta_{m+1}$ based on the order restriction $\theta_1 < \ldots < \theta_{m+1}$. It is shown by Balakrishnan et al. [6] that the order restricted inference is a challenging problem both numerically and theoretically. Due to this reason Samanta et al. [15] considered a re-parameterization of the model parameters, which transforms the order restriction problem to a problem without any order restriction. In this paper we also use similar re-parameterization of the model parameters. We make the following transformation of the model parameters: $\theta_k = \beta_k\beta_{k+1}\ldots\beta_m\theta_{m+1}$, $k = 1, \ldots, m$. where $0 < \beta_1, \ldots, \beta_m < 1$. Since there is a one to one correspondence between $\{\alpha, \theta_1, \ldots, \theta_{m+1}\}$ and $\{\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1}\}$, the statistical inference based on the two sets of parameters will be equivalent. Hence, the log likelihood function of α, $\beta_1, \ldots, \beta_m$ and θ_{m+1}, without the additive constant can be written as

$$
\begin{aligned}
l_1(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1}|\mathcal{D}) \;=\; & n\ln(\alpha) + \overline{n}_1 \ln(\beta_1) + \ldots + \overline{n}_m \ln(\beta_m) + n\ln(\theta_{m+1}) \\
& -\beta_1 \ldots \beta_m \theta_{m+1} \sum_{i=1}^{n_1} t_{i:n} - \beta_2 \ldots \beta_m \theta_{m+1} \sum_{i=n_1+1}^{\overline{n}_2} (t_{i:n} - \tau_1) - \ldots \\
& -\theta_{m+1} \sum_{i=\overline{n}_m+1}^{n} (t_{i:n} - \tau_m) - \beta_1 \ldots \beta_m \theta_{m+1}(n - \overline{n}_1)\tau_1 \\
& -\beta_2 \ldots \beta_m \theta_{m+1}(n - \overline{n}_2)(\tau_2 - \tau_1) - \ldots \\
& -\beta_m \theta_{m+1}(n - \overline{n}_m)(\tau_m - \tau_{m-1}) + (\alpha - 1)\big(A_1(\beta_1, \ldots, \beta_m, \theta_{m+1}) \\
& + \ldots + A_{m+1}(\beta_1, \ldots, \beta_m, \theta_{m+1})\big). \qquad (8)
\end{aligned}
$$

where,

$$
\begin{aligned}
A_1(\beta_1, \ldots, \beta_m, \theta_{m+1}) &= \sum_{i=1}^{n_1} \ln(1 - e^{-\beta_1, \ldots \beta_m, \theta_{m+1} t_{i:n}}), \\
A_2(\beta_1, \ldots, \beta_m, \theta_{m+1}) &= \sum_{i=n_1+1}^{\overline{n}_2} \ln\left(1 - e^{-(\beta_1, \ldots \beta_m, \theta_{m+1}\tau_1 + \beta_2, \ldots \beta_m, \theta_{m+1}(t_{i:n} - \tau_1))}\right), \\
&\vdots \\
A_{m+1}(\beta_1, \ldots, \beta_m, \theta_{m+1}) &= \sum_{i=\overline{n}_m+1}^{n} \ln\left(1 - e^{-(\beta_1, \ldots \beta_m, \theta_{m+1}\tau_1 + \beta_2, \ldots \beta_m, \theta_{m+1}(\tau_2 - \tau_1) + \ldots + \theta_{m+1}(t_{i:n} - \tau_m))}\right).
\end{aligned}
$$

Therefore, the MLEs of the unknown parameters can be obtained by maximizing (8) with respect to $\alpha > 0$, $\theta_{m+1} > 0$ and $0 < \beta_1, \ldots, \beta_m < 1$. Note that without any bounded restriction on β_i's the MLEs of α, θ_{m+1} and $\beta_1, \ldots, \beta_m$ can be obtained by solving the normal equations

$$
\frac{\partial l_1}{\partial \alpha} = 0, \quad \frac{\partial l_1}{\partial \theta_{m+1}} = 0, \quad \frac{\partial l_1}{\partial \beta_k} = 0, \quad k = 1, \ldots, m.
$$

In the Appendix, we have provided the normal equations explicitly. Let α^*, θ^*_{m+1} and β^*_k for $k = 1, \ldots, m$ be the unique solutions of the above normal equations. If for all $k = 1, \ldots, m$, $0 < \beta^*_k \leq 1$, then clearly, $\alpha^*, \theta^*_{m+1}, \beta^*_k$ are the MLEs of α, θ_{m+1} and β_k, respectively, for $k = 1, \ldots, m$, otherwise not. We use Algorithm 1 to obtain the MLEs under the constraints $\alpha > 0$, $\theta_{m+1} > 0$ and $0 < \beta_1, \ldots, \beta_m < 1$.

Algorithm 1:

Step 1: For the given data set $\mathcal{D}$, obtain α^*, $\beta_1^*, \ldots, \beta_m^*$, θ_{m+1} and check whether ${\beta_k}^* \leq 1$ for all $k = 1, \ldots, m$ or not.

Step 2: If all ${\beta_k}^* \leq 1$, for $k = 1, \ldots, m$, then $\alpha^*, \theta_{m+1}^*, \beta_k^*$ are the MLEs of α, θ_{m+1} and β_k, respectively, for $k = 1, \ldots, m$. The algorithm stops.

Step 3: If one or some of the ${\beta_k}^* > 1$ then replace all of them by 1 in the log-likelihood (8) and obtain the normal equations for the remaining parameters and re-estimate them by solving those normal equations. Let us denote them also as α^*, $\beta_1^*, \ldots, \beta_m^*$, θ_{m+1}. Note that some of the β_k^*'s are 1.

Step 4: Go to Step 2.

Due to absence of the closed form expressions of the MLEs in this case, it is difficult to obtain exact confidence intervals (CIs) of the unknown parameters. Hence, we propose to use the confidence intervals based on the inverse of the observed Fisher information matrix. Therefore under the assumption of asymptotic normality of the MLEs and if $0 < \widehat{\beta}_i < 1$ $(i = 1, \ldots, m)$ then the $100(1-\gamma)\%$ asymptotic CIs of α, β_i's and θ_{m+1} are

$$[\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}} \sqrt{V_{11}}], \quad [\widehat{\beta}_i \pm z_{1-\frac{\gamma}{2}} \sqrt{V_{i+1i+1}}], \quad [\widehat{\theta}_{m+1} \pm z_{1-\frac{\gamma}{2}} \sqrt{V_{m+2m+2}}],$$

respectively, where V_{ij} is the $(i,j)^{th}$, for $i, j = 1, \ldots, m+2$, element of the inverse of the observed Fisher information matrix and $z_{1-\frac{\gamma}{2}}$ is the upper $\left(1 - \dfrac{\gamma}{2}\right)$-th point of standard normal distribution. The elements of the observed Fisher information matrix for three-stress level are given in Appendix 7.2. Similarly, it can be easily obtained for more than three stress levels. Note that, the boundary points of the confidence interval of β_i's may fall outside $(0,1)$ and the lower bound of α and θ_{m+1} may be less than zero. In all those cases when lower bound is less than zero then it will be replaced by 0 and for β_i's if the upper bound exceeds 1, then it will be replaced by 1. Now when $\widehat{\beta}_i = 1$ then a left sided CI of β_i will be considered as $[1 - z_\gamma \sqrt{V_{i+1i+1}}, 1]$. By using delta method it is quite straight forward to compute the confidence intervals of $(\alpha, \theta_1, \ldots, \theta_{m+1})$, the details are avoided.

4 Bayesian Inference

In this section we consider the Bayesian inference of the unknown parameters. We have made very general prior assumptions and obtained the Bayes estimates and the associated credible intervals. Although, we have mainly considered the squared error loss function, any other loss functions also can be easily incorporated. Before progressing further we would like to introduce the following notations which will be used in this section. A gamma random variable with the shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ has the PDF

$$f_{GA}(x; \alpha, \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}; \quad x > 0,$$

and it will be denoted by GA(α, λ). A beta random variable with the parameters $a > 0$ and $b > 0$ has the PDF

$$f_{BE}(x; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}; \quad 0 < x < 1,$$

and it will be denoted by BE(a, b). Similar to the frequentist method here also we have used the parameterization. Therefore, under order restriction we have $\alpha > 0$, $\theta_{m+1} > 0$ and $0 < \beta_i < 1$.

In case of prior selection we use the same notations, it should be clear from the context. It is assumed that α, β_i, and θ_{m+1} have priors $\pi_0(\cdot)$, $\pi_i(\cdot)$ $(i = 1, \ldots m)$ and $\pi_{m+1}(\cdot)$, respectively. For $a_0 > 0, b_0 > 0, a_i > 0, b_i > 0, a_{m+1} > 0, b_{m+1} > 0$, the priors for the order restricted case are

$$\begin{aligned} \pi_0(\alpha; a_0, b_0) &\sim \text{GA}(a_0, b_0), \\ \pi_i(\beta_i; a_i, b_i) &\sim \text{BE}(a_i, b_i), \\ \pi_{m+1}(\theta_{m+1}; a_{m+1}, b_{m+1}) &\sim \text{GA}(a_{m+1}, b_{m+1}). \end{aligned}$$

4.1 Posterior Analysis

Based on the data (4) and the prior assumptions mentioned as above, the posterior distribution of $\alpha, \beta_1, \ldots, \beta_m$ and θ_{m+1} is given by

$$
\begin{aligned}
l_2(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha|\mathcal{D}) \;\propto\;& \prod_{i=1}^{m} \beta_i^{\overline{n}_i + a_i - 1}(1-\beta_i)^{b_i-1}\theta_{m+1}^{n+b_{m+1}-1} e^{-h_1(\beta_1,\ldots,\beta_m)\theta_{m+1}} \\
& \alpha^{n+b_0-1} e^{-h_2(\beta_1,\ldots,\beta_m,\,\theta_{m+1})\alpha} \prod_{i=1}^{n_1}(1-e^{-\beta_1,\ldots,\beta_m\theta_{m+1}t_{i:n}})^{-1} \times \\
& \prod_{i=n_1+1}^{\overline{n}_2} (1-e^{-(\beta_1\ldots\beta_m\theta_{m+1}\tau_1+\beta_2\ldots\beta_m\theta_{m+1}(t_{i:n}-\tau_1))})^{-1} \times \ldots \times \\
& \prod_{i=\overline{n}_m+1}^{n} (1-e^{-(\beta_1\ldots\beta_m\theta_{m+1}\tau_1+\beta_2\ldots\beta_m\theta_{m+1}(\tau_2-\tau_1)+\ldots+\theta_{m+1}(t_{i:n}-\tau_m))})^{-1},
\end{aligned}
$$

where,

$$
\begin{aligned}
h_1(\beta_1, \ldots, \beta_m) &= a_{m+1} + \beta_1\ldots\beta_m \sum_{i=1}^{n_1} t_i + \beta_2\ldots\beta_m \sum_{i=n_1+1}^{\overline{n}_2}(t_i-\tau_1) + \ldots + \\
& \sum_{i=\overline{n}_m+1}^{n}(t_{i:n}-\tau_m) + (n-\overline{n}_1)\beta_1\ldots\beta_m\tau_1 + (n-\overline{n}_2)\beta_2\ldots\beta_m(\tau_2-\tau_1) \\
& + \ldots + (n-\overline{n}_m)\beta_m(\tau_m-\tau_{m-1}), \\
h_2(\beta_1, \ldots, \beta_m, \theta_{m+1}) &= a_0 - A_1(\beta_1, \ldots, \beta_m, \theta_{m+1}) - \ldots - A_{m+1}(\beta_1, \ldots, \beta_m, \theta_{m+1}).
\end{aligned}
$$

The Bayes estimate of some function of α, $\beta_1, \ldots, \beta_m$ and θ_{m+1}, say $g(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1})$, under the squared error loss function is the posterior expectation of $g(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1})$ and it is given by

$$
\begin{aligned}
& \widehat{g}_B(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1}) \\
&= E_{\beta,\theta_2,\alpha|Data}\left(g(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1})\right) \\
&= \frac{\int_0^1 \ldots \int_0^1 \int_0^\infty \int_0^\infty g(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1}) l_2(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha|\mathcal{D}) d\alpha d\theta_{m+1} d\beta_1 \ldots d\beta_m}{\int_0^1 \ldots \int_0^1 \int_0^\infty \int_0^\infty l_2(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha|\mathcal{D}) d\alpha d\theta_{m+1} d\beta_1 \ldots d\beta_m},
\end{aligned}
\tag{9}
$$

provided the expectation exists. It is clear that (9) cannot be obtained in explicit form in general, hence, we propose to use importance sampling technique to compute the Bayes estimate and associated credible interval. Note that posterior density of $(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha)$ can be written as

$$l_2(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha \,|\, \mathcal{D}) \propto h(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha) \prod_{i=1}^{m} l_i(\beta_i) l_{m+1}(\theta_{m+1} \,|\, \beta_1, \ldots, \beta_m) \times l_0(\alpha \,|\, \theta_{m+1}, \beta_1, \ldots, \beta_m), \tag{10}$$

where

$$\begin{aligned}
h(\beta_1, \ldots, \beta_m, \theta_{m+1}, \alpha) &= \textstyle\prod_{i=1}^{m} \beta_i^{\overline{n}_i + a_i - 1} (1-\beta_i)^{b_i - 1} \prod_{i=1}^{n_1} (1 - e^{-\beta_1, \ldots, \beta_m \theta_{m+1} t_{i:n}})^{-1} \times \\
&\quad \textstyle\prod_{i=n_1+1}^{\overline{n}_2} (1 - e^{-(\beta_1 \ldots \beta_m \theta_{m+1} \tau_1 + \beta_2 \ldots \beta_m \theta_{m+1} (t_{i:n} - \tau_1)})^{-1} \times \ldots \times \\
&\quad \textstyle\prod_{i=\overline{n}_m+1}^{n} (1 - e^{-(\beta_1 \ldots \beta_m \theta_{m+1} \tau_1 + \beta_2 \ldots \beta_m \theta_{m+1} (\tau_2 - \tau_1) + \ldots + \theta_{m+1} (t_{i:n} - \tau_m))})^{-1} \\
&\quad \times [h_1(\beta_1, \ldots, \beta_m)]^{-(n + b_{m+1})} [h_2(\beta_1, \ldots, \beta_m, \theta_{m+1})]^{-(n+b_0)}, \\
l_i(\beta_i) &= 1 \quad \text{for } 0 < \beta_i < 1, \quad i = 1, \ldots, m, \\
l_{m+1}(\theta_{m+1} | \beta_1, \ldots, \beta_m) &= \tfrac{[h_1(\beta_1, \ldots, \beta_m)]^{n + b_{m+1}}}{\Gamma(n + b_{m+1})} \theta_{m+1}^{n + b_{m+1} - 1} e^{-h_1(\beta_1, \ldots, \beta_m)\, \theta_{m+1}} \quad \text{for } \theta_{m+1} > 0, \\
l_0(\alpha | \theta_{m+1}, \beta_1, \ldots, \beta_m) &= \tfrac{[h_2(\beta_1, \ldots, \beta_m, \theta_{m+1})]^{n + b_0}}{\Gamma(n + b_0)} \alpha^{n + b_0 - 1} e^{-h_2(\beta_1, \ldots, \beta_m, \theta_{m+1})\, \alpha} \quad \text{for } \alpha > 0.
\end{aligned}$$

Algorithm 2 can be used to compute Bayes estimate and the associated credible interval of $g(\alpha, \beta_1, \ldots, \beta_m, \theta_{m+1})$.

Algorithm 2:

Step 1: Generate β_{i1} $(i = 1, \ldots, m)$ from Uniform$(0, 1)$, θ_{m+11} from GA$(n + b_{m+1}, h_1(\beta_{11}, \ldots, \beta_{m1}))$, and α_1 from GA$(n + b_0, h_2(\beta_{11}, \ldots, \beta_{m1}, \theta_{m+11}))$ distribution.

Step 2: Repeat Step 1, N times to obtain $(\beta_{11}, \ldots, \beta_{m1}, \theta_{m+11}, \alpha_1), \ldots, (\beta_{1N}, \ldots, \beta_{mN}, \theta_{m+1N}, \alpha_N)$.

Step 3: Calculate $g_i = g(\alpha_i, \beta_{1i}, \ldots, \beta_{mi}, \theta_{m+1i})$ and $w_i = \dfrac{h(\beta_{1i}, \ldots, \beta_{mi}, \theta_{m+1i}, \alpha_i)}{\sum_{i=1}^{N} h(\beta_{1i}, \ldots, \beta_{mi}, \theta_{m+1i}, \alpha_i)}$.

Step 4: The approximate value of (9) can be obtained as $\sum_{i=1}^{N} w_i g_i$.

Step 5: Rearrange $(g_1, w_1), (g_2, w_2), \ldots, (g_N, w_N)$ as $(g_{(1)}, w_{(1)}), (g_{(2)}, w_{(2)}), \ldots, (g_{(N)}, w_{(N)})$

where $g_{(1)} \leq g_{(2)} \leq \ldots \leq g_{(N)}$. Note that $w_{(i)}$'s are not ordered, they are just associated with $g_{(i)}$'s.

Step 6: A $100(1-\gamma)\%$ credible interval for $g(\alpha,\, \beta_1, \ldots, \beta_m,\, \theta_{m+1})$ can be obtain as $(g_{j_1},\, g_{j_2})$, where j_1 and j_2 satisfy

$$j_1,\, j_2 \in \{1,\, 2,\, \ldots,\, N\}\,, \quad j_1 < j_2, \quad \sum_{i=j_1}^{j_2} w_{(i)} \leq 1 - \gamma < \sum_{i=j_1}^{j_2+1} w_{(i)}. \tag{11}$$

The $100(1-\gamma)\%$ HPD CRI of $g(\alpha,\, \beta_1, \ldots, \beta_m,\, \theta_{m+1})$ becomes $\left(g_{(j_1^*)},\, g_{(j_2^*)}\right)$, where $1 \leq j_1^* < j_2^* \leq N$ satisfy

$$\sum_{i=j_1^*}^{j_2^*} w_{(i)} \leq 1 - \gamma < \sum_{i=j_1^*}^{j_2^*+1} w_{(i)}, \quad \text{and} \quad g_{(j_2^*)} - g_{(j_1^*)} \leq g_{(j_2)} - g_{(j_1)},$$

for all j_1 and j_2 satisfying (11).

5 Simulation and Data Analysis

5.1 Simulation

Extensive simulation experiments have been performed to observe the performances of the different methods. In our simulation experiment we have considered three stress levels. We have mainly observed average biases and the associated MSEs for both the MLEs and Bayes estimators. We have also reported the percentages of cases when $\theta_1 = \theta_2$ and $\theta_2 = \theta_3$. Different confidence and credible intervals are compared based on their average lengths and their coverage percentages. We have taken different sample sizes, $n = 20,\ 30,\ 40$ and 50, different (τ_1, τ_2) values, $(\tau_1, \tau_2) = (4{,}8),\ (6{,}8)$ and $(6{,}10)$, and different sets of parameters. The parameter values are $\alpha = 0.8$ and 1.5, and $\theta_1 = 0.1$, $\theta_2 = 0.2$ and $\theta_3 = 0.3$. All the results are based on 1000 replications. In case of Bayes estimates we consider the following hyper parameters: $a_0 = b_0 = a_3 = b_3 = 0.0001$ and $a_1 = b_1 = a_2 = b_2 = 1$, and $N = 10{,}000$. The above hyper parameters have been chosen so that they behave almost like non-informative priors.

In each case we have computed the average values of the MLEs and the associated MSEs based on 1000 replications. In each case we also compute the 95% confidence intervals of the unknown parameters based on the observed Fisher information matrix, and report the average lengths and the associated coverage percentages based on 1000 replications. In each case we also compute the Bayes estimates and report the average values of the Bayes estimates and the associated MSEs. Similarly 95% symmetric and HPD credible intervals of all the parameters are computed and we report their average lengths and the associated coverage percentages. All the results are reported in Tables 1 - 4.

Some of the points are very clear from these extensive simulation experiments. In all the cases for fixed τ_1 and τ_2 as n increases the average biases, MSEs, percentage of equality of scale parameters, average lengths of the confidence and credible intervals decrease. It verifies the consistency properties of the different estimators. Now comparing the performances of the MLEs (Table 1), it is clear that the performances of the order restricted MLEs are quite satisfactory. Similarly, observing the performances of the confidence intervals based on the MLEs it is clear that in all the cases the coverage percentages of the confidence intervals are very close to the nominal values.

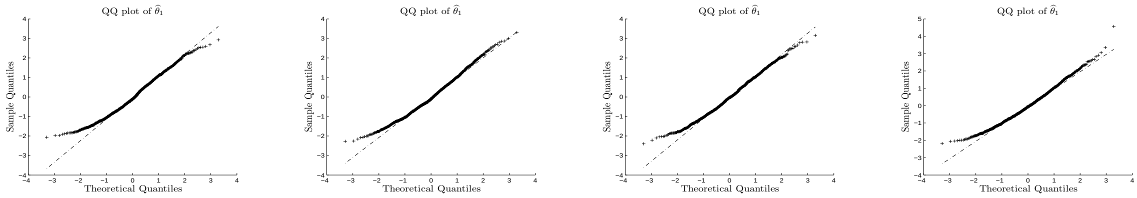
Now from the Table 3 and Table 4 it is clear that the performances of the Bayes estimators based on non-informative priors are quite satisfactory. The 95% symmetric and HPD credible intervals perform quite well and in all the cases the coverage percentages are very close to the nominal level. Between symmetric and HPD credible intervals clearly HPD credible intervals provide shorter average lengths.

Now comparing the performances between the MLEs and the Bayes estimators, the average biases and the MSEs of the MLEs are significantly larger than those of the Bayes estimators in both the cases. Moreover, the average lengths of the confidence intervals are significantly larger than the average lengths of the corresponding credible intervals. Comparing all the points we recommend to use the Bayes estimators with order restricted non-informative priors for step-stress models.

It is observed that the asymptotic confidence intervals perform quite well even for small sample sizes. We have provided the quantile-quantile (q-q) plots of the MLEs to verify their normality assumptions. The q-q plots of the MLEs of unknown parameters for three stress levels are given from Figure (1) to Figure (4) for a particular choice of parameter values and for different n. It is observed as expected that as the sample size increases, the observed and the theoretical quantiles becomes closer for all the parameters. Even for small sample sizes the matches are quite well for most of the parameters.

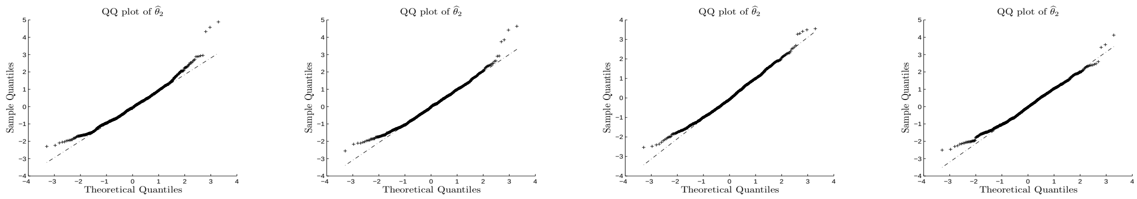
Table 1: MLEs and MSEs of model parameters (Actual value of $\theta_1 = 0.1$ $\theta_2 = 0.2$ and $\theta_3 = 0.3$).

				α		θ_1		θ_2		θ_3		% of cases	
α	n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE	$\widehat{\theta}_1 = \widehat{\theta}_2$	$\widehat{\theta}_2 = \widehat{\theta}_3$
0.8	20	4	8	0.8613	0.2472	0.1058	0.0157	0.2121	0.0183	0.4058	0.0895	0.40	22.00
	20	6	8	0.9017	0.2898	0.1096	0.0122	0.2242	0.0201	0.3841	0.0594	0.60	26.80
	20	6	10	0.8738	0.2198	0.1039	0.0078	0.2154	0.0117	0.4429	0.1463	0.80	27.00
	30	4	8	0.8534	0.1095	0.1050	0.0043	0.2090	0.0060	0.3578	0.0287	0.10	19.60
	30	6	8	0.8571	0.1095	0.1066	0.0036	0.2152	0.0090	0.3594	0.0272	0.20	21.40
	30	6	10	0.8502	0.0996	0.1046	0.0045	0.2105	0.0082	0.3848	0.0650	0.20	19.80
	40	4	8	0.8492	0.0738	0.1061	0.0023	0.2047	0.0036	0.3464	0.0189	0.10	16.10
	40	6	8	0.8381	0.0572	0.1059	0.0023	0.2157	0.0059	0.3387	0.0150	0.20	17.80
	40	6	10	0.8369	0.0746	0.1021	0.0032	0.2090	0.0055	0.3588	0.0367	0.20	16.60
	50	4	8	0.8383	0.0594	0.1053	0.0049	0.2080	0.0057	0.3333	0.0163	0.00	14.80
	50	6	8	0.8445	0.0575	0.1049	0.0020	0.2096	0.0049	0.3329	0.0105	0.00	15.50
	50	6	10	0.8423	0.0438	0.1051	0.0013	0.2038	0.0030	0.3470	0.0224	0.00	13.50
1.5	20	4	8	1.4796	0.9012	0.0940	0.0057	0.1924	0.0079	0.3590	0.0381	0.00	22.80
	20	6	8	1.5734	1.0332	0.0994	0.0046	0.1908	0.0112	0.3413	0.0206	0.00	28.90
	20	6	10	1.5423	1.0018	0.0974	0.0044	0.1871	0.0076	0.3859	0.0451	0.00	26.10
	30	4	8	1.5282	0.7621	0.0988	0.0048	0.1927	0.0055	0.3393	0.0153	0.00	20.20
	30	6	8	1.5863	0.7830	0.1005	0.0034	0.1875	0.0071	0.3295	0.0117	0.00	20.70
	30	6	10	1.5268	0.7421	0.0982	0.0032	0.1857	0.0054	0.3577	0.0216	0.00	18.30
	40	4	8	1.5008	0.5866	0.0990	0.0041	0.1857	0.0039	0.3232	0.0091	0.00	14.40
	40	6	8	1.5893	0.5658	0.1025	0.0026	0.1876	0.0052	0.3190	0.0067	0.00	18.50
	40	6	10	1.5798	0.6273	0.1027	0.0028	0.1889	0.0038	0.3411	0.0137	0.00	17.30
	50	4	8	1.5188	0.5773	0.1005	0.0039	0.1899	0.0034	0.3220	0.0073	0.00	11.70
	50	6	8	1.5614	0.3964	0.1016	0.0021	0.1913	0.0042	0.3172	0.0056	0.00	13.90
	50	6	10	1.5424	0.4479	0.1012	0.0023	0.1847	0.0029	0.3317	0.0091	0.00	13.90



(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 1: QQ Plots of $\widehat{\theta}_1$ with parameter values $\alpha = 1.5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$, and for $\tau_1 = 6$, $\tau_2 = 8$.



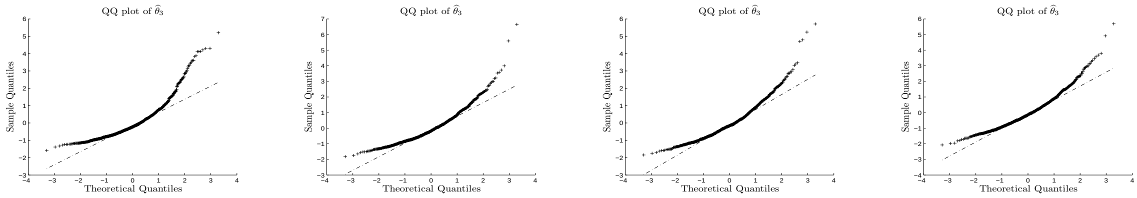
(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 2: QQ Plots of $\widehat{\theta}_2$ with parameter values $\alpha = 1.5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$, and for $\tau_1 = 6$, $\tau_2 = 8$.

5.2 Data Analysis

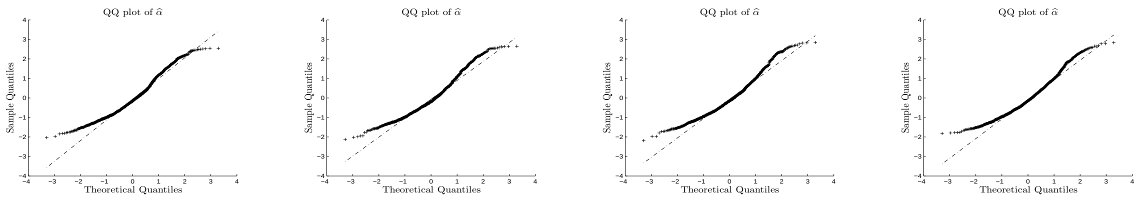
In this section we provide the analysis of a multiple step-stress data set obtained from Greven et al. [8]. A sample of 15 fishes were swum at initial flow rate 15 cm/sec. The time at which a fish could not maintain its position is recorded as the failure time. To ensure the early failure, the stress level was increased (flow rate by 5 cm/sec) at time 110, 130, 150 and 170. The observed failure time data is 91.00, 93.00, 94.00, 98.20, 115.81, 116.00, 116.50, 117.25, 126.75, 127.50, 154.33, 159.50, 164.00, 184.14, 188.33. Note that here we have five stress levels and number of failure at each stress level is 4, 6, 0, 3 and 2 respectively.

We assume that the above failure data follow GE distribution at each stress level with same shape parameter but different scale parameter. We have subtracted 80 for each data points and then analyze the data. Under the above model assumptions the MLEs of $\alpha = 1.6117$, $\theta_1 = 0.0206$, $\theta_2 = 0.0268$, $\theta_3 = 0.0268$, $\theta_4 = 0.0462$ and $\theta_5 = 0.0626$. Asymptotic CIs of model parameters are given in Table 5. Next to check the performance of the fitted model we calculate the Kolmogorov-Smirnov (KS) distance between the empirical



(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 3: QQ Plots of $\widehat{\theta}_3$ with parameter values $\alpha = 1.5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$, and for $\tau_1 = 6$, $\tau_2 = 8$.



(a) $n = 20$ (b) $n = 30$ (c) $n = 40$ (d) $n = 50$

Figure 4: QQ Plots of $\widehat{\alpha}$ with parameter values $\alpha = 1.5$, $\theta_1 = 0.1$, $\theta_2 = 0.2$, $\theta_3 = 0.3$, and for $\tau_1 = 6$, $\tau_2 = 8$.

distribution function (EDF) and the fitted distribution function (FDF) and also obtain the associate p-value. The KS distance and the associated p-value based on MLEs are 0.1513 and 0.8331 respectively which indicates very good fit of the given data. It may be mentioned that since there is no failure in one stress level, the MLEs of all the parameters do not exist without the order restriction.

Next we want to test whether there exists any linear trend on θ_i's or not, i.e., we want to test $H_0 : \theta_i = a + ib$ $(i = 1, 2, 3, 4)$, where a and b are constant. We have performed the likelihood ratio test for testing H_0 against there is no linear trend. The maximum likelihood estimates of a and b under H_0 are 0.0055 and 0.0120, respectively. Subsequently, under H_0 the MLEs of α, θ_1, θ_2, θ_3 and θ_4 are 1.4686, 0.0175, 0.0295, 0.0415 and 0.0535, respectively. The likelihood ratio test statistics is $\Lambda = 0.9083$ and hence $-2\log(\Lambda) = 0.1923$ with p-value $= 0.09$. Hence, based on the likelihood ratio test with 10% level of significance we reject H_0.

We have also obtained the order restricted Bayes estimates of the model parameters based on the same data. Order restricted Bayes estimates of $\alpha = 1.1229$, $\theta_1 = 0.0120$, $\theta_2 = 0.0202$, $\theta_3 = 0.0255$, $\theta_4 = 0.0427$ and $\theta_5 = 0.0736$. 90%, 95% and 99% symmetric and HPD CRIs

Table 2: 95% asymptotic confidence interval of model parameters (Actual value of $\theta_1 = 0.1$ $\theta_2 = 0.2$ and $\theta_3 = 0.3$).

				α		θ_1		θ_2		θ_3	
α	n	τ_1	τ_2	AL	CP	AL	CP	AL	CP	AL	CP
0.8	20	4	8	1.4390	95.60	0.2796	93.70	0.3978	96.50	0.8031	97.40
	20	6	8	1.3754	96.00	0.2304	95.50	0.5566	99.00	0.6949	98.20
	20	6	10	1.3263	97.20	0.2237	94.60	0.4518	98.70	0.9625	97.80
	30	4	8	1.1517	97.00	0.2294	95.50	0.3167	97.40	0.5494	98.30
	30	6	8	1.0280	95.60	0.1846	93.20	0.4577	98.50	0.5092	97.40
	30	6	10	1.0057	97.60	0.1821	96.10	0.3396	97.30	0.7060	96.80
	40	4	8	0.9729	95.60	0.1984	93.60	0.2639	97.50	0.4498	96.70
	40	6	8	0.8521	97.00	0.1591	95.90	0.3823	98.20	0.4032	96.30
	40	6	10	0.8572	94.40	0.1561	94.10	0.2921	97.70	0.5294	96.60
	50	4	8	0.8362	96.80	0.1754	94.70	0.2344	97.60	0.3793	95.80
	50	6	8	0.7702	94.70	0.1420	95.20	0.3357	98.10	0.3445	97.30
	50	6	10	0.7635	96.40	0.1419	95.90	0.2536	97.40	0.4306	96.00
1.5	20	4	8	4.6408	92.10	0.4204	89.00	0.3514	93.60	0.5195	92.90
	20	6	8	3.8845	94.70	0.2836	92.60	0.4006	94.80	0.4907	97.00
	20	6	10	3.9667	93.40	0.2891	90.90	0.3168	94.20	0.6060	93.40
	30	4	8	4.3262	93.00	0.3923	90.80	0.2982	94.60	0.4194	92.90
	30	6	8	3.2175	93.60	0.2376	92.70	0.3272	94.60	0.3939	94.70
	30	6	10	2.9732	92.40	0.2310	92.10	0.2500	94.70	0.4837	94.50
	40	4	8	3.3848	91.40	0.3241	90.80	0.2479	94.60	0.3456	94.40
	40	6	8	2.6100	93.80	0.2021	92.50	0.2743	95.20	0.3273	96.30
	40	6	10	2.7526	95.20	0.2134	94.90	0.2172	96.10	0.4179	94.60
	50	4	8	3.6045	92.30	0.3453	91.90	0.2544	93.30	0.3837	93.60
	50	6	8	2.2091	94.50	0.1779	93.50	0.2446	96.20	0.2873	96.30
	50	6	10	2.2728	93.80	0.1841	94.20	0.1905	94.50	0.3587	95.40

are given in Table 6. The KS distance and the associated p-value between the EDF and the FDF based on the Bayes estimates are 0.1971 and 0.5404, respectively. Comparing the KS distance and associated p-values it can be said that MLEs fit the data better than the Bayes estimates though both the methods fit the data quite well.

Table 3: Bayes estimates and MSEs of model parameters (Actual value of $\theta_1 = 0.1$ $\theta_2 = 0.2$ and $\theta_3 = 0.3$).

				α		θ_1		θ_2		θ_3	
α	n	τ_1	τ_2	AE	MSE	AE	MSE	AE	MSE	AE	MSE
0.8	20	4	8	0.8808	0.1182	0.1089	0.0023	0.2020	0.0055	0.4208	0.1692
	20	6	8	0.8879	0.1263	0.1079	0.0021	0.2019	0.0063	0.3812	0.0522
	20	6	10	0.8844	0.0985	0.1084	0.0018	0.2046	0.0053	0.4276	0.1478
	30	4	8	0.8650	0.0666	0.1098	0.0018	0.2021	0.0035	0.3607	0.0242
	30	6	8	0.8876	0.0762	0.1095	0.0016	0.1991	0.0034	0.3446	0.0157
	30	6	10	0.8604	0.0652	0.1081	0.0013	0.2010	0.0034	0.3856	0.0405
	40	4	8	0.8488	0.0509	0.1085	0.0015	0.1990	0.0025	0.3466	0.0196
	40	6	8	0.8508	0.0485	0.1061	0.0011	0.1948	0.0027	0.3276	0.0101
	40	6	10	0.8547	0.0495	0.1070	0.0011	0.1971	0.0026	0.3479	0.0153
	50	4	8	0.8590	0.0426	0.1094	0.0013	0.1973	0.0019	0.3352	0.0108
	50	6	8	0.8441	0.0384	0.1067	0.0010	0.1931	0.0022	0.3225	0.0065
	50	6	10	0.8324	0.0333	0.1060	0.0010	0.1945	0.0021	0.3432	0.0139
1.5	20	4	8	1.5955	0.3221	0.0991	0.0014	0.1879	0.0033	0.3357	0.0146
	20	6	8	1.5961	0.3889	0.0979	0.0015	0.1850	0.0042	0.3303	0.0120
	20	6	10	1.6158	0.3931	0.0998	0.0016	0.1895	0.0036	0.3548	0.0265
	30	4	8	1.5340	0.2026	0.0968	0.0012	0.1855	0.0024	0.3183	0.0079
	30	6	8	1.5566	0.2387	0.0973	0.0012	0.1835	0.0027	0.3168	0.0072
	30	6	10	1.5726	0.2289	0.0995	0.0011	0.1899	0.0025	0.3334	0.0115
	40	4	8	1.4953	0.1458	0.0975	0.0012	0.1862	0.0018	0.3140	0.0058
	40	6	8	1.5188	0.1521	0.0974	0.0010	0.1863	0.0023	0.3120	0.0049
	40	6	10	1.5400	0.1538	0.1000	0.0009	0.1884	0.0017	0.3181	0.0066
	50	4	8	1.4847	0.1103	0.0957	0.0010	0.1868	0.0015	0.3093	0.0041
	50	6	8	1.5093	0.1209	0.0973	0.0008	0.1843	0.0020	0.3054	0.0034
	50	6	10	1.5010	0.1232	0.0978	0.0008	0.1886	0.0016	0.3105	0.0046

Table 4: 95% symmetric and HPD CRIs of model parameters (Actual value of $\theta_1 = 0.1$ $\theta_2 = 0.2$ and $\theta_3 = 0.3$).

					α		θ_1		θ_2		θ_3	
CRI	α	n	τ_1	τ_2	AL	CP	AL	CP	AL	CP	AL	CP
Symmetric	0.8	20	4	8	1.2218	96.30	0.2025	97.50	0.2750	95.50	0.7279	94.80
		20	6	8	1.1908	95.20	0.1808	96.50	0.3109	96.20	0.5672	94.90
		20	6	10	1.1721	96.90	0.1790	97.90	0.2875	96.40	0.8255	96.50
		30	4	8	0.9853	97.20	0.1758	97.40	0.2257	94.90	0.4618	95.00
		30	6	8	0.9735	96.40	0.1549	95.90	0.2568	97.30	0.4034	95.20
		30	6	10	0.9285	96.80	0.1518	97.40	0.2355	96.10	0.5617	95.60
		40	4	8	0.8313	97.20	0.1557	97.10	0.1952	95.30	0.3851	95.40
		40	6	8	0.7987	95.80	0.1340	96.40	0.2246	97.10	0.3327	95.70
		40	6	10	0.7946	95.80	0.1325	96.60	0.2028	95.40	0.4188	97.30
		50	4	8	0.7598	96.40	0.1440	96.50	0.1744	95.80	0.3273	94.90
		50	6	8	0.7043	95.60	0.1217	96.50	0.2050	97.50	0.2940	95.80
		50	6	10	0.6874	94.70	0.1200	96.20	0.1826	95.50	0.3730	95.50
	1.5	20	4	8	2.5522	98.30	0.1904	99.50	0.2371	96.90	0.4059	96.30
		20	6	8	2.4501	97.20	0.1655	97.90	0.2603	96.50	0.3716	95.00
		20	6	10	2.4797	96.50	0.1655	97.60	0.2321	95.30	0.4660	94.50
		30	4	8	1.9812	98.10	0.1621	98.40	0.1916	95.60	0.3087	94.60
		30	6	8	1.9398	96.80	0.1393	96.90	0.2161	96.50	0.2886	94.40
		30	6	10	1.9407	97.20	0.1388	97.20	0.1915	95.00	0.3536	95.60
		40	4	8	1.6783	97.30	0.1465	98.10	0.1693	95.20	0.2648	94.50
		40	6	8	1.6136	96.60	0.1236	97.10	0.1933	95.20	0.2444	93.80
		40	6	10	1.6103	97.80	0.1226	97.80	0.1647	96.30	0.2894	95.60
		50	4	8	1.4806	97.90	0.1332	98.00	0.1511	94.10	0.2317	93.80
		50	6	8	1.4001	96.80	0.1098	95.60	0.1744	94.80	0.2126	94.50
		50	6	10	1.3958	96.70	0.1106	96.80	0.1489	94.00	0.2525	94.90
HPD	0.8	20	4	8	1.1420	93.90	0.1896	95.70	0.2605	93.40	0.6574	97.40
		20	6	8	1.1166	93.40	0.1699	94.70	0.2912	93.60	0.5264	95.40
		20	6	10	1.1037	95.50	0.1690	95.60	0.2706	94.40	0.7243	97.10
		30	4	8	0.9316	95.30	0.1663	95.50	0.2154	94.30	0.4313	95.80
		30	6	8	0.9214	94.70	0.1472	94.20	0.2432	96.40	0.3824	94.70
		30	6	10	0.8795	94.90	0.1440	95.70	0.2239	95.20	0.5182	96.80
		40	4	8	0.7884	95.80	0.1478	95.00	0.1869	93.50	0.3629	96.00
		40	6	8	0.7585	93.60	0.1274	94.40	0.2139	95.00	0.3175	94.60
		40	6	10	0.7577	93.70	0.1261	94.30	0.1937	93.70	0.3933	97.40
		50	4	8	0.7239	94.60	0.1369	95.20	0.1673	94.90	0.3116	95.50
		50	6	8	0.6712	93.80	0.1160	93.90	0.1958	95.70	0.2823	95.80
		50	6	10	0.6553	93.80	0.1143	94.30	0.1746	93.20	0.3535	95.90
	1.5	20	4	8	2.3601	96.90	0.1779	98.30	0.2263	94.70	0.3856	95.80
		20	6	8	2.2759	94.70	0.1548	95.10	0.2479	93.00	0.3546	94.30
		20	6	10	2.3085	93.80	0.1554	94.00	0.2218	92.50	0.4377	95.90
		30	4	8	1.8607	96.00	0.1528	96.40	0.1833	92.70	0.2945	94.20
		30	6	8	1.8167	94.80	0.1310	93.20	0.2071	94.00	0.2769	93.00
		30	6	10	1.8164	94.60	0.1306	94.20	0.1833	92.90	0.3361	95.00
		40	4	8	1.5784	95.10	0.1378	95.50	0.1616	92.80	0.2539	93.50
		40	6	8	1.5206	94.40	0.1166	94.10	0.1853	92.90	0.2351	93.40
		40	6	10	1.5188	95.40	0.1155	94.50	0.1577	94.40	0.2767	94.30
		50	4	8	1.3975	95.80	0.1255	95.30	0.1440	93.00	0.2224	92.60
		50	6	8	1.3238	94.00	0.1036	91.20	0.1670	92.30	0.2040	93.00
		50	6	10	1.3175	94.70	0.1041	93.80	0.1425	91.90	0.2410	93.60

Table 5: Asymptotic CI of parameters based on the Fish data.

	α		θ_1		θ_2		θ_3		θ_4		θ_5	
Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
90%	0	3.6162	0	0.0539	0.0059	0.0477	0.0047	0.0489	0.0029	0.0895	0	0.1346
95%	0	4.0073	0	0.0604	0.0018	0.0518	0.0004	0.0532	0	0.0979	0	0.1486
99%	0	4.7529	0	0.0728	0	0.0595	0	0.0615	0	0.1140	0	0.1754

Table 6: Symmetric and HPD CRI of parameters based on the Fish data.

		α		θ_1		θ_2		θ_3		θ_4		θ_5	
CRI	Level	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL	LL	UL
	90%	0.4255	2.3398	0.0017	0.0273	0.0069	0.0353	0.0100	0.0444	0.0191	0.0743	0.0312	0.1382
Symm.	95%	0.3601	2.5327	0.0009	0.0321	0.0052	0.0410	0.0083	0.0490	0.0163	0.0841	0.0270	0.1597
	99%	0.2715	2.8621	0.0002	0.0329	0.0032	0.0480	0.0057	0.0563	0.0122	0.1004	0.0195	0.2099
	90%	0.3048	1.9550	0.0002	0.0239	0.0069	0.0352	0.0082	0.0407	0.0161	0.0684	0.0229	0.1218
HPD	95%	0.3048	2.4176	0.0002	0.0273	0.0032	0.0356	0.0057	0.0448	0.0115	0.0752	0.0195	0.1401
	99%	0.2508	2.7843	0.0003	0.0329	0.0035	0.0480	0.0057	0.0553	0.0107	0.0946	0.0128	0.1857

6 Conclusion

In this paper we consider the analysis of a multiple step-stress model based on the assumptions that the lifetime distribution of the experimental units follow two parameter GE distribution at the different stress levels. It is further assumed that when the stress factor changes the scale parameter of the GE distribution changes where as the shape parameter remains constant. We have considered both the classical and Bayesian inference when the order restriction on the average lifetime exists. Though the article is developed based on complete data, it can also be easily incorporated for censored data. Based on our extensive simulation experiments it is observed that if it is known a priori that there exists an order restriction on the expected lifetime of the experimental units, it is better to use that information. To analyze the fish data it is assumed that the location parameter is known. It would be interesting to develop a methodology for unknown location parameter, more work is needed along that direction.

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7 Appendix

7.1 Normal Equations

$$\frac{\partial l_1}{\partial \alpha} = \frac{n}{\alpha} + A_1(\beta_1, \ldots, \beta_m, \theta_{m+1}) + \ldots + A_{m+1}(\beta_1, \ldots, \beta_m, \theta_{m+1}) = 0.$$

$$\frac{\partial l_1}{\partial \theta_{m+1}} = 0 \Rightarrow$$

$$\frac{n}{\theta_{m+1}} - \beta_1 \ldots \beta_m \sum_{i=1}^{n_1} t_{i:n} - \beta_2 \ldots \beta_m \sum_{i=n_1+1}^{\overline{n}_2} (t_{i:n} - \tau_1) - \ldots - \sum_{i=\overline{n}_m+1}^{n} (t_{i:n} - \tau_m) +$$

$$(\alpha - 1) \sum_{i=1}^{n_1} \frac{\beta_1 \ldots \beta_m t_{i:n} e^{-\beta_1 \ldots \beta_m \theta_{m+1} t_{i:n}}}{1 - e^{-\beta_1 \ldots \beta_m \theta_{m+1} t_{i:n}}} +$$

$$(\alpha - 1) \sum_{i=n_1+1}^{\overline{n}_2} \frac{[\beta_1 \ldots \beta_m \tau_1 + \beta_2 \ldots \beta_m (t_{i:n} - \tau_1)] e^{-(\beta_1 \ldots \beta_m \theta_{m+1} \tau_1 + \beta_2 \ldots \beta_m \theta_{m+1} (t_{i:n} - \tau_1))}}{1 - e^{-(\beta_1 \ldots \beta_m \theta_{m+1} \tau_1 + \beta_2 \ldots \beta_m \theta_{m+1} (t_{i:n} - \tau_1))}} +$$

$$\vdots$$

$$(\alpha - 1) \sum_{i=\overline{n}_m+1}^{n} \frac{[\beta_1 \ldots \beta_m \tau_1 + \ldots + (t_{i:n} - \tau_m)] e^{-(\beta_1 \ldots \beta_m \theta_{m+1} \tau_1 + \beta_2 \ldots \beta_m \theta_{m+1} (\tau_2 - \tau_1) + \ldots + \theta_{m+1} (t_{i:n} - \tau_m))}}{1 - e^{-\beta_1 \ldots \beta_m \theta_{m+1} \tau_1 + \beta_2 \ldots \beta_m \theta_{m+1} (\tau_2 - \tau_1) + \ldots + \theta_{m+1} (t_{i:n} - \tau_m))}}$$

$$-\beta_1 \ldots \beta_m (n - \overline{n}_1) \tau_1 - \beta_2 \ldots \beta_m (n - \overline{n}_2)(\tau_2 - \tau_1) - \ldots - \beta_m (n - \overline{n}_m)(\tau_m - \tau_{m-1}) = 0. \quad (12)$$

$$
\begin{aligned}
&\frac{\partial l_1}{\partial \beta_k} = 0 \Rightarrow \\
&\frac{\overline{n}_k}{\beta_k} - \beta_1 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1} \sum_{i=1}^{n_1} t_{i:n} - \beta_2 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1} \sum_{i=1}^{n_1} (t_{i:n} - \tau_1) \\
&- \ldots - \beta_{k+1} \ldots \beta_m \theta_{m+1} \sum_{i=\overline{n}_{k-1}+1}^{\overline{n}_k} (t_{i:n} - \tau_{k-1}) - \beta_1 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1} (n - \overline{n}_1)\tau_1 - \\
&\beta_2 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1} (n - \overline{n}_2)(\tau_2 - \tau_1) - \ldots - \beta_{k+1} \ldots \beta_m \theta_{m+1} (n - \overline{n}_k)(\tau_k - \tau_{k-1}) \\
&+(\alpha - 1)\big[D_1(\beta_k) + \ldots + D_{m+1}(\beta_k)\big] = 0; \quad \text{for} \quad k = 1, \ldots, m, \qquad (13)
\end{aligned}
$$

where

$$
\begin{aligned}
D_1(\beta_k) &= \tfrac{\partial A_1(\beta_1 \ldots \beta_{m+1}\theta_{m+1})}{\partial \beta_k} = \textstyle\sum_{i=1}^{n_1} \tfrac{\beta_1 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1} t_{i:n} e^{-\beta_1 \cdots \beta_m \theta_{m+1} t_{i:n}}}{1 - e^{-\beta_1 \cdots \beta_m \theta_{m+1} t_{i:n}}} \\
D_2(\beta_k) &= \tfrac{\partial A_2(\beta_1 \ldots \beta_{m+1}\theta_{m+1})}{\partial \beta_k} \\
&= \textstyle\sum_{i=n_1+1}^{\overline{n}_2} \tfrac{\left[\beta_1 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1}\tau_1 + \beta_2 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1}(t_{i:n} - \tau_1) I_{[k \geq 2]}\right] e^{-\theta_{m+1}(\beta_1 \cdots \beta_m \tau_1 + \beta_2 \cdots \beta_m (t_{i:n} - \tau_1))}}{1 - e^{-(\beta_1 \cdots \beta_m \theta_{m+1}\tau_1 + \beta_2 \cdots \beta_m \theta_{m+1}(t_{i:n} - \tau_1))}} \\
&\vdots \\
D_{m+1}(\beta_k) &= \tfrac{\partial A_{m+1}(\beta_1 \ldots \beta_{m+1}\theta_{m+1})}{\partial \beta_k} = \textstyle\sum_{i=\overline{n}_m+1}^{n} \tfrac{B e^{-(\beta_1 \cdots \beta_m \theta_{m+1}\tau_1 + \beta_2 \cdots \beta_m \theta_{m+1}(\tau_2 - \tau_1) + \ldots + \theta_{m+1}(t_{i:n} - \tau_m))}}{1 - e^{-(\beta_1 \cdots \beta_m \theta_{m+1}\tau_1 + \beta_2 \cdots \beta_m \theta_{m+1}(\tau_2 - \tau_1) + \ldots + \theta_{m+1}(t_{i:n} - \tau_m))}}, \\
\text{where, } B &= \beta_1 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1}\tau_1 + \beta_2 \ldots \beta_{k-1}\beta_{k+1} \ldots \beta_m \theta_{m+1}(\tau_2 - \tau_1) + \ldots + \\
&\quad \beta_{k+1} \ldots \beta_m \theta_{m+1}(\tau_k - \tau_{k-1}) + \theta_{m+1}(\tau_m - \tau_{m-1}) I_{[k=m]}.
\end{aligned}
$$

where $I_A = 1$, if A occur, otherwise zero.

7.2 Elements of Fisher Information Matrix

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} &= -\frac{n}{\alpha^2},\\
\frac{\partial^2 l}{\partial \beta_1^2} &= -\frac{n}{\beta_1^2} + (\alpha-1)[A_1''(\beta_1) + A_2''(\beta_1) + A_3''(\beta_1)],\\
\frac{\partial^2 l}{\partial \beta_2^2} &= -\frac{\overline{n}_2}{\beta_2^2} + (\alpha-1)[A_1''(\beta_2) + A_2''(\beta_2) + A_3''(\beta_2)],\\
\frac{\partial^2 l}{\partial \theta_3^2} &= -\frac{n}{\theta_3^2} + (\alpha-1)[A_1''(\theta_3) + A_2''(\theta_3) + A_3''(\theta_3)],\\
\frac{\partial^2 l}{\partial \alpha \partial \beta_1} &= A_1'(\beta_1) + A_2'(\beta_1) + A_3'(\beta_1),\\
\frac{\partial^2 l}{\partial \alpha \partial \beta_2} &= A_1'(\beta_2) + A_2'(\beta_2) + A_3'(\beta_2),\\
\frac{\partial^2 l}{\partial \alpha \partial \theta_3} &= A_1'(\theta_3) + A_2'(\theta_3) + A_3'(\theta_3),\\
\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} &= -\theta_3 \sum_{i=1}^{n_1} t_{i:n} - \theta_3(n_2+n_3)\tau_1 + (\alpha-1)[A_1''(\beta_1,\beta_2) + A_2''(\beta_1,\beta_2) + A_3''(\beta_1,\beta_2)],\\
\frac{\partial^2 l}{\partial \beta_1 \partial \theta_3} &= -\beta_2 \sum_{i=1}^{n_1} t_{i:n} - \beta_2(n_2+n_3)\tau_1 + (\alpha-1)[A_1''(\beta_1,\theta_3) + A_2''(\beta_1,\theta_3) + A_3''(\beta_1,\theta_3)],\\
\frac{\partial^2 l}{\partial \beta_2 \partial \theta_3} &= -\beta_1 \sum_{i=1}^{n_1} t_{i:n} - \sum_{i=n_1+1}^{\overline{n}_2} (t_{i:n}-\tau_1) - \beta_1(n_2+n_3)\tau_1 - n_3(\tau_2-\tau_1),\\
&\quad +(\alpha-1)[A_1''(\beta_2,\theta_3) + A_2''(\beta_2,\theta_3) + A_3''(\beta_2,\theta_3)], \qquad \text{where}\\
A_1'(\beta_1) &= \sum_{i=1}^{n_1} \frac{\beta_2\theta_3 t_{i:n} e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}},\\
A_1'(\beta_2) &= \sum_{i=1}^{n_1} \frac{\beta_1\theta_3 t_{i:n} e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}},\\
A_1'(\theta_3) &= \sum_{i=1}^{n_1} \frac{\beta_1\beta_2 t_{i:n} e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}},\\
A_2'(\beta_1) &= \sum_{i=n_1+1}^{\overline{n}_2} \frac{\beta_2\theta_3\tau_1 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}}{1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}},\\
A_2'(\beta_2) &= \sum_{i=n_1+1}^{\overline{n}_2} \frac{[\beta_1\theta_3\tau_1+\theta_3(t_{i:n}-\tau_1)] e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}}{1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}},\\
A_2'(\theta_3) &= \sum_{i=n_1+1}^{\overline{n}_2} \frac{[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)] e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}}{1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}},
\end{aligned}
$$

$$
\begin{aligned}
A_3'(\beta_1) &= \sum_{i=\overline{n}_2+1}^{n} \frac{\beta_2\theta_3\tau_1 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}}{1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}},\\
A_3'(\beta_2) &= \sum_{i=\overline{n}_2+1}^{n} \frac{[\beta_1\theta_3\tau_1+\theta_3(\tau_2-\tau_1)]e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}}{1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}},\\
A_3'(\theta_3) &= \sum_{i=\overline{n}_2+1}^{n} \frac{[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+(t_{i:n}-\tau_2)]e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}}{1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}},\\
A_1''(\beta_1) &= -\sum_{i=1}^{n_1} \frac{(\beta_2\theta_3 t_{i:n})^2 e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{[1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}]^2},\\
A_1''(\beta_2) &= -\sum_{i=1}^{n_1} \frac{(\beta_1\theta_3 t_{i:n})^2 e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{[1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}]^2},\\
A_1''(\theta_3) &= -\sum_{i=1}^{n_1} \frac{(\beta_1\beta_2 t_{i:n})^2 e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{[1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}]^2},\\
A_2''(\beta_1) &= -\sum_{i=n_1+1}^{\overline{n}_2} \frac{(\beta_2\theta_3\tau_1)^2 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}}{[1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}]^2},\\
A_2''(\beta_2) &= -\sum_{i=n_1+1}^{\overline{n}_2} \frac{[\beta_1\theta_3\tau_1+\theta_3(t_{i:n}-\tau_1)]^2 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}}{[1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}]^2},\\
A_2''(\theta_3) &= -\sum_{i=n_1+1}^{\overline{n}_2} \frac{[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]^2 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}}{[1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]}]^2},\\
A_3''(\beta_1) &= -\sum_{i=\overline{n}_2+1}^{n} \frac{(\beta_2\theta_3\tau_1)^2 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}}{[1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}]^2},\\
A_3''(\beta_2) &= -\sum_{i=\overline{n}_2+1}^{n} \frac{[\beta_1\theta_3\tau_1+\theta_3(\tau_2-\tau_1)]^2 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}}{[1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}]^2},\\
A_3''(\theta_3) &= -\sum_{i=\overline{n}_2+1}^{n} \frac{[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+(t_{i:n}-\tau_2)]^2 e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}}{[1-e^{-[\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(\tau_2-\tau_1)+\theta_3(t_{i:n}-\tau_2)]}]^2},\\
A_1''(\beta_1,\beta_2) &= \sum_{i=1}^{n_1} \theta_3 t_{i:n}\left[\frac{1-(1+\beta_1\beta_2\theta_3 t_{i:n})e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{[1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}]^2}-1\right],\\
A_1''(\beta_1,\theta_3) &= \sum_{i=1}^{n_1} \beta_2 t_{i:n}\left[\frac{1-(1+\beta_1\beta_2\theta_3 t_{i:n})e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{[1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}]^2}-1\right],\\
A_1''(\beta_2,\theta_3) &= \sum_{i=1}^{n_1} \beta_1 t_{i:n}\left[\frac{1-(1+\beta_1\beta_2\theta_3 t_{i:n})e^{-\beta_1\beta_2\theta_3 t_{i:n}}}{[1-e^{-\beta_1\beta_2\theta_3 t_{i:n}}]^2}-1\right],\\
A_2''(\beta_1,\beta_2) &= \sum_{i=n_1+1}^{\overline{n}_2} \theta_3\tau_1\left[\frac{[1-(1+\beta_1\beta_2\theta_3\tau_1)e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]}}{[1-e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]}]^2}-1\right],
\end{aligned}
$$

$$\begin{aligned}
A_2''(\beta_1,\theta_3) &= \sum_{i=n_1+1}^{\overline{n}_2} \beta_2\tau_1\left[\frac{[1-[1+\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]}}{[1-e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]}]^2}-1\right],\\
A_2''(\beta_2,\theta_3) &= \sum_{i=n_1+1}^{\overline{n}_2} (\beta_1\tau_1+t_{i:n}-\tau_1)\left[\frac{[1-[1+\beta_1\beta_2\theta_3\tau_1+\beta_2\theta_3(t_{i:n}-\tau_1)]e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]}}{[1-e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(t_{i:n}-\tau_1)]}]^2}-1\right],\\
A_3''(\beta_1,\beta_2) &= \sum_{i=\overline{n}_2+1}^{n} \theta_3\tau_1\left[\frac{1-(1+\beta_1\beta_2\theta_3\tau_1)e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]}}{[1-e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]}]^2}-1\right],\\
A_3''(\beta_1,\theta_3) &= \sum_{i=\overline{n}_2+1}^{n} \beta_2\tau_1\left[\frac{1-\big[1+\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]\big]e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]}}{[1-e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]}]^2}-1\right]\\
A_3''(\beta_2,\theta_3) &= \sum_{i=\overline{n}_2+1}^{n} (\beta_1\tau_1+\tau_2-\tau_1)\left[\frac{C_i}{[1-e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]}]^2}-1\right],\\
\text{where}\quad C_i &= 1-\big[1+\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]\big]e^{-\theta_3[\beta_1\beta_2\tau_1+\beta_2(\tau_2-\tau_1)+t_{i:n}-\tau_2]},
\end{aligned}$$