Test manuscript No.
(will be inserted by the editor)

Generalized mixtures of Weibull components

Manuel Franco · Narayanaswamy Balakrishnan · Debasis Kundu · Juana-María Vivo

Received: date / Accepted: date

Abstract Weibull mixtures have been used extensively in reliability and survival analysis, and they have also been generalized by allowing negative mixing weights, which arise naturally under the formation of some structures of reliability systems. These models provide flexible distributions for modelling dependent lifetimes from heterogeneous populations. In this paper, we study conditions on the mixing weights and the parameters of the Weibull components under which the considered generalized mixture is a well-defined distribution. Specially, we characterize the generalized mixture of two Weibull components. In addition, some reliability properties are established for these generalized two-component Weibull mixture models. One real data set is also analyzed for illustrating the usefulness of the studied model.

Keywords Weibull distribution · Mixture model · Generalized mixture · Two-component mixture · Reliability · Failure rate

Mathematics Subject Classification (2000) 62E10 · 62N05

Manuel Franco (Corresponding author)
Department of Statistics and Operations Research, University of Murcia, 30100 Murcia, Spain
E-mail: mfranco@um.es

Narayanaswamy Balakrishnan
Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

Debasis Kundu
Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Pin 208016, India

Juana-María Vivo
Regional Campus of International Excellence "Campus Mare Nostrum", University of Murcia, 30100 Murcia, Spain

1 Introduction

The Weibull distribution plays an important role in the analysis of reliability and survival data (see, Johnson et al. 1994, for a detailed review), and mixtures of Weibull distributions have also been used widely to model lifetime data, since they exhibit a wide range of shapes for the failure rate function. Their great flexibility makes them suitable in numerous applications (see, for example, Sinha 1987; Xie and Lai 1995; Patra and Dey 1999; Kundu and Basu 2000; Tsionas 2002; Wondmagegnehu 2004; Marín et al. 2005; Attardi et al. 2005; Carta and Ramírez 2007; Mosler and Scheicher 2008; Farcomeni and Nardi 2010; Ebden et al. 2010; Qin et al. 2012 and Elmahdy and Aboutahoun 2013). These mixture forms have been further generalized by allowing negative mixing weights, as such models arise naturally under the formation of some structures of reliability systems (see Jiang et al. 1999; Bučar et al. 2004 and Navarro et al. 2009). They also provide suitable distributions for modelling dependent lifetimes from heterogeneous populations, as mixtures of defective devices with shorter lifetimes and standard devices with longer lifetimes. In this setting, Bučar et al. (2004) pointed out that it is crucial to be able to guarantee that these mixtures are valid probability models. Thus, it seems reasonable to use generalized mixtures of Weibull components as underlying distributions to model heterogeneous survival data and the reliability of a system when its structure is unknown or only partially known. In both these cases, the sub-populations and components may often have the same distributional type, but it is important to know what parametric constraints are necessary to have a legitimate probability distribution.

Many authors have studied generalized mixture distributions and their distributional and structural properties. For instance, conditions in terms of the mixing weights and the parameters to guarantee that generalized mixtures of exponential components are valid models have been analyzed by Steutel (1967), Bartholomew (1969), Baggs and Nagaraja (1996), and Franco and Vivo (2006). Franco and Vivo (2007) further studied generalized mixtures of a gamma and one or two exponential components. Recently, Franco and Vivo (2009a) discussed constraints for generalized mixtures of Weibull distributions having a common shape parameter.

It is well-known that log-concavity and log-convexity are useful aging properties in many applied areas (see Bagnoli and Bergstrom 2005 and Kokonendji et al. 2008). Since it is reasonable to use generalized mixtures of Weibull as underlying distributions in many situations as described above, it will naturally be of interest to study aging properties of such generalized mixtures of Weibull components.

In this regard, aging classifications of generalized mixtures of exponential distributions have been discussed earlier by Baggs and Nagaraja (1996) and Franco and Vivo (2002, 2006), while Franco and Vivo (2007, 2009b) have examined generalized mixtures of gamma and exponential components in a similar vein. Related results for ordinary mixtures of exponential and Weibull components were given by Wondmagegnehu et al. (2005), and Block et al. (2010,

2012) for exponential and gamma components. Recently, Franco et al. (2011) studied log-concavity properties of generalized mixtures of Weibull components with a common shape parameter, i.e., the monotonicity of its failure rate function.

Analytical and graphical methods have been used to fit Weibull mixture models. For instance, moments, maximum likelihood, least-squares and WPP plot methods, and different numerical algorithms have been used to fit the model and also to examine the probabilistic behavior. Inferential methods based on quasi-Newton, Newton-Raphson, conjugate gradient and EM algorithms have been developed in this regards. A brief review of these estimation methods and their applications to Weibull mixtures can be found in Bučar et al. (2004) and Carta and Ramírez (2007). Some of them may also be used to fit Weibull mixture models with negative mixing weights for getting a more flexible model; for example, Franco and Vivo (2009a) revisited the method of moments, the least-squares method and the maximum likelihood method in the context of generalized mixtures of two Weibull components with a common shape parameter. Negative estimated weights of generalized mixtures of Weibull components have also been obtained by Jiang et al. (1999) and Bučar et al. (2004).

The main aim of this paper is to establish conditions for the mixing weights and the shape and scale parameters of the Weibull components to guarantee that their generalized mixtures are valid probability models. This would answer the question posed by Bučar et al. (2004) and also extend the previously known results for the generalized mixtures of exponential or Rayleigh components, as well as mixtures of Weibull components with a common shape parameter. The work also has the motivation for providing more flexible Weibull mixture distributions for modelling dependent lifetimes from heterogeneous populations, as mentioned earlier.

The rest of this paper is organized as follows. In Section 2, we recall some concepts of aging and define the generalized mixtures of Weibull distributions, which are studied in the sequel. Some necessary conditions for these mixtures are also obtained in this section. In Section 3, some sufficient conditions are established for a Weibull mixture with negative mixing weights to be a valid probability model. In Section 4, we discuss the characterization of a generalized mixture of two Weibull components. In Section 5, we present some reliability results for the generalized two-component Weibull mixture model. Finally, in Section 6, we analyze one real data set illustrating the usefulness and flexibility of Weibull mixtures, allowing negative mixing weights.

2 Generalized finite mixtures of Weibull distributions

First, we recall some reliability properties or aging concepts that are pertinent to the developments in this paper.

Let X be a non-negative random variable with reliability or survival function $S(x) = P(X > x)$. Then, the conditional survival function of a unit of age x is defined by $S(t|x) = S(t+x)/S(x)$ for all $x, t \geq 0$, and it represents the survival probability for an additional period of duration t of a unit of age x.

Note that the log-concavity of the survival function is determined by the monotonicity of its conditional survival function. X is said to have a log-concave (log-convex) survival function if $\log S(x)$ is concave (convex), i.e., $S(t|x)$ is decreasing (increasing) in x for all $t \geq 0$. In the absolutely continuous case, this log-concavity property can be determined by the monotonicity of its failure rate function defined by $r(x) = -\frac{d}{dx} \log S(x) = -S'(x)/S(x)$, which represents the probability of instantaneous failure or death at a time x. So, the log-concave (log-convex) survival function also corresponds to increasing (decreasing) failure rate, i.e., IFR (DFR) class (see Barlow and Proschan 1981 and Shaked and Shanthikumar 2006, for further details.

It is well-known that the DFR class of distributions is closed under mixtures (Barlow and Proschan (1981)), but that closure is not preserved under generalized mixtures (see for example Navarro et al. 2009 and Franco et al. 2011). In the case of two components, Navarro et al. (2009) obtain that generalized mixtures of an IFR distribution with a positive coefficient and a DFR distribution with a negative coefficient are IFR.

We now introduce the concept of generalized mixture of Weibull distributions, and present some necessary conditions on the mixing weights and shape and scale parameters of the Weibull components in order for the mixture to be a valid probability model.

Definition 2.1 Let $(X_1, X_2, ..., X_n)$ be a random vector formed by Weibull components with survival functions $S_i(x) = \exp(-b_i x^{c_i})$ for all $x > 0$, and $b_i > 0$ and $c_i > 0$, $i = 1, ..., n$. Then, $S(x)$ is said to be a generalized mixture of Weibull survival functions, if it is given by

$$S(x) = \sum_{i=1}^{n} a_i \exp(-b_i x^{c_i}), \quad \text{for } x > 0, \tag{2.1}$$

where $a_i \in \mathbb{R}$, $i = 1, ..., n$, such that $\sum_{i=1}^{n} a_i = 1$.

Evidently, this generalized mixture may be defined by its distribution and density functions (cdf and pdf)

$$F(x) = 1 - \sum_{i=1}^{n} a_i \exp(-b_i x^{c_i}) \quad \text{and} \quad f(x) = \sum_{i=1}^{n} a_i f_i(x), \quad \text{for } x > 0. \tag{2.2}$$

Note that $a_i \neq 0$ can be assumed, $i = 1, ..., n$. Likewise, we can assume that (b_i, c_i) are different, $i = 1, ..., n$, i.e., $b_i = b_j$ and $c_i = c_j$ cannot be simultaneously satisfied for $i \neq j$. Otherwise, the two terms i and j can be merged to form a single term. Therefore, without loss of generality, we may assume that $0 < c_1 \leq c_2 \leq ... \leq c_n$ and $b_i < b_j$ for each $c_i = c_j$ with $i < j$.

Moreover, taking into account that $f_i(x) = c_i b_i x^{c_i - 1} \exp(-b_i x^{c_i})$ is the pdf of each component of the generalized mixture, it is obvious that $\int_{\mathbb{R}} f(x)dx =$

$\sum_{i=1}^{n} a_i = 1$, and consequently, $f(x)$ is a pdf if and only if $f(x) \geq 0$ for all $x > 0$, i.e., a generalized mixture of Weibull distributions is a valid probability model if and only if $f(x)$ is everywhere positive. Besides, if some of the a_i's are negative, $f(x)$ could be negative for some values of x and so may not be a valid pdf.

Let us now consider necessary conditions for $S(x)$ in (2.1) to be a valid survival function.

Theorem 2.1 *Let $S(x) = \sum_{i=1}^{n} a_i S_i(x)$ be a generalized mixture of Weibull survival functions. If $S(x)$ is a survival function, then $a_1 > 0$ and*

$$\sum_{i=1}^{n} a_i b_i I(c_i = c_1) \geq 0, \tag{2.3}$$

where $I(c_i = c_1)$ denotes the indicator function indicating whether or not the equality holds.

Proof First, in order to check $f(x) \geq 0$ for all $x > 0$, where $f(x) = -\frac{d}{dx}S(x)$ is as given in (2.2), the function $f(x)$ can be written as

$$f(x) = b_1 c_1 x^{c_1 - 1} \exp(-b_1 x^{c_1}) g(x),$$

where

$$g(x) = \sum_{i=1}^{n} a_i \frac{b_i c_i}{b_1 c_1} x^{c_i - c_1} \exp(-(b_i x^{c_i} - b_1 x^{c_1}));$$

thus, $f(x)$ and $g(x)$ have the same sign, and $\lim_{x\to\infty} g(x) = a_1$.

If we suppose that $a_1 < 0$, then there exits a x_0 such that $g(x) < 0$ for all $x > x_0$, which contradicts that $f(x)$ is a pdf. Therefore, $a_1 > 0$ becomes a necessary condition for $f(x)$ to be positive.

Moreover, it is required that $f(0) \geq 0$, and according to the lowest shape parameter, we have

$$\lim_{x\to 0} g(x) = \sum_{i=1}^{n} a_i \frac{b_i}{b_1} I(c_i = c_1)$$

and now using a similar argument, its positivity becomes a necessary condition, which is equivalent to (2.3). Hence, the theorem.

Remark 2.1 From Theorem 2.1, when $c_1 = c_2 = ... = c_n = c$, we obtain the necessary conditions for a generalized finite mixture of Weibull distributions with a common shape parameter given by Franco and Vivo (2009a), which includes as special cases the generalized mixtures of exponential components (for $c = 1$) given by Steutel (1967), and of Rayleigh components (for $c = 2$).

3 Sufficient conditions for a finite Weibull mixture

The conditions of Theorem 2.1 are not sufficient for a generalized finite mixture of Weibull distributions. For example, $S(x) = 2.5\exp(-x^{0.5}) - 1.5\exp(-2x^2)$ is not a valid survival function but (2.3) holds. The following result presents some sufficient conditions for $f(x)$ to be a valid pdf.

Theorem 3.1 *Let $f(x)$ in (2.2) be a generalized mixture of Weibull density functions. If*

$$\sum_{k=1}^{i} a_k b_k I(c_k = c_i) \geq 0, \quad \text{for } i = 1, ..., n, \tag{3.1}$$

then $f(x)$ is a pdf.

Proof Under the stated conditions, $f(x)$ can be rewritten as

$$f(x) = c_n x^{c_n - 1} \exp(-b_n x^{c_n}) \sum_{i=1}^{n} a_i b_i I(c_i = c_n) + \sum_{i=1}^{n-1} A_i(x),$$

where

$$\begin{aligned} A_i(x) = & c_i x^{c_i - 1} \exp(-b_i x^{c_i}) \sum_{k=1}^{i} a_k b_k I(c_k = c_i) \\ & - c_{i+1} x^{c_{i+1} - 1} \exp(-b_{i+1} x^{c_{i+1}}) \sum_{k=1}^{i} a_k b_k I(c_k = c_{i+1}). \end{aligned}$$

Clearly, if $\sum_{k=1}^{i} a_k b_k I(c_k = c_i) \geq 0$, then $A_i(x) \geq 0$ when $c_i < c_{i+1}$ and also when $c_i = c_{i+1}$, since $b_i < b_{i+1}$. Consequently, $f(x)$ is positive if (3.1) is satisfied, i.e., $f(x)$ is a pdf. Hence, the theorem.

Remark 3.1 The sufficient condition for a generalized finite mixture of exponential distributions given by Bartholomew (1969) are deduced from Theorem 3.1 when $c_1 = c_2 = ... = c_n = 1$.

4 Characterization of a generalized mixture of two Weibull components

In this section, we give the necessary and sufficient conditions on the mixing weights and parameters for a generalized mixture of two Weibull distributions to be a valid probability model.

Theorem 4.1 *Let $f(x) = a_1 b_1 c_1 x^{c_1 - 1} \exp(-b_1 x^{c_1}) + a_2 b_2 c_2 x^{c_2 - 1} \exp(-b_2 x^{c_2})$ be a generalized mixture of two Weibull density functions with $b_1 > 0$, $b_2 > 0$, $c_1 > 0$, $c_2 > 0$, $a_1 > 0$ and $a_2 \in \mathbb{R}$, such that $a_1 + a_2 = 1$, $0 < c_1 \leq c_2$ and $b_1 < b_2$ when $c_1 = c_2$. Then, the following statements hold:*

(i) If $c_1 = c_2$, then $f(x)$ is a pdf if and only if

$$a_1 \leq \frac{b_2}{b_2 - b_1}; \tag{4.1}$$

(ii) If $c_1 < c_2$, then $f(x)$ is a pdf if and only if

$$a_1 \leq \frac{b_1 c_1 x_0^{c_1} + c_2 - c_1}{b_1 c_1 x_0^{c_1} \left(1 - \exp((1 - b_1 x_0^{c_1})(c_2 - c_1)/c_2)\right) + c_2 - c_1} \tag{4.2}$$

where $x_0 \in (0, \infty)$ is the unique positive solution of the equation

$$b_2 c_2 x^{c_2} - b_1 c_1 x^{c_1} = c_2 - c_1. \tag{4.3}$$

Proof Firstly, Part (i) is given by Theorem 3.1 of Franco and Vivo (2009a). Also note that (4.1) is easily obtained from Equation (7) of Navarro et al. (2009).

On the other hand, to prove Part (ii), when $c_1 < c_2$, the sufficient condition in (3.1) is too restrictive. In this case, using Equation (7) of Navarro et al. (2009), $f(x)$ is pdf if and only if $a_1 \leq 1/(1 - g(x_0))$ where $g(x_0) = \inf f_1(x)/f_2(x)$ is the infimum of the ratio between the density functions of both Weibull components. Here, we have

$$g(x) = \frac{b_1 c_1}{b_2 c_2 x^{c_2 - c_1}} \exp(b_2 x^{c_2} - b_1 x^{c_1})$$

which is a decreasing function for $x \in (0, x_0)$ and an increasing function for $x \in (x_0, \infty)$ where $x_0 > 0$ is the unique positive solution of (4.3).

Therefore, $f(x)$ is pdf if and only if

$$a_1 \leq \frac{b_2 c_2 x_0^{c_2 - c_1}}{b_2 c_2 x_0^{c_2 - c_1} - b_1 c_1 \exp(b_2 x_0^{c_2} - b_1 x_0^{c_1})}$$

which is equivalent to (4.2). Hence, the theorem.

Remark 4.1 Part (i) of Theorem 4.1, when $c_1 = c_2 = c = 1$, establishes the necessary and sufficient conditions for a generalized mixture of two exponential components to be a valid pdf as discussed by Bartholomew (1969) and Baggs and Nagaraja (1996). Likewise, when $c = 2$, a generalized mixture of two Rayleigh components is a true probability model if and only if (4.1) holds.

Remark 4.2 An unresolved case of generalized mixture of two Weibull components with different shape parameters is $c_1 = 1 < c_2 = 2$, i.e., a generalized mixture of exponential and Rayleigh models, with scale parameters $b_1 > 0$ and $b_2 > 0$, respectively, and mixing weights $a_1 > 0$ and $a_2 \in \mathbb{R}$, such that $a_1 + a_2 = 1$. From Part (ii) of Theorem 4.1, the generalized mixture of their pdf's

$$f(x) = a_1 b_1 \exp(-b_1 x) + 2 a_2 b_2 x \exp(-b_2 x^2)$$

is a valid pdf if and only if

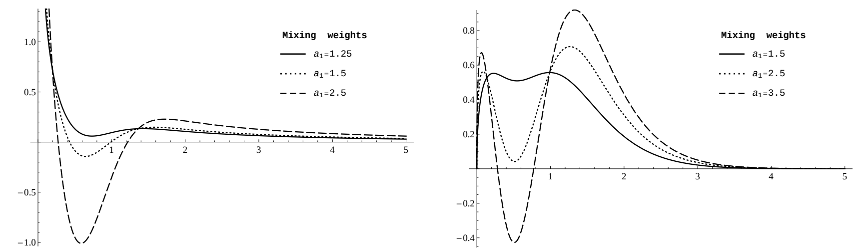
$$a_1 \leq \frac{b_1 x_0 + 1}{b_1 x_0 \left(1 - \exp((1 - b_1 x_0)/2)\right) + 1},$$

where $x_0 > 0$ is the unique positive solution of the equation $2b_2x^2 - b_1x = 1$, i.e.,

$$a_1 \leq \frac{b_1 + \sqrt{b_1^2 + 8b_2}}{b_1 + \sqrt{b_1^2 + 8b_2} - 2b_1 \exp\left[0.5 - \left(b_1 + \sqrt{b_1^2 + 8b_2}\right)\frac{b_1}{8b_2}\right]}.$$

Furthermore, we note here that the necessary and sufficient conditions for a generalized mixture of two Weibull components with common shape parameter do not ensure a valid probability model when different shape parameters are considered. Some graphs of generalized mixtures of two Weibull distributions for different mixing weights and specific values of their parameters can be used to check that $a_1b_1 + a_2b_2 \geq 0$ is neither a necessary nor a sufficient condition.

For example, Figure 1a depicts $f(x)$ for two Weibull components with $b_1 = 1$, $b_2 = 2$ and $c_1 = 0.5 < c_2 = 2$ and for varying values of the mixing weights, with the dark solid line for $a_1 = 1.25$, the dark dotted line for $a_1 = 1.5$, and the dashed line for $a_1 = 2.5$, respectively. In the three cases, $x_0 = 0.692091042972595$ is the required point in (4.2), but this inequality holds only for the first, because $a_1 = 1.25 \leq 1.326735827671239$, and so $f(x)$ is a valid pdf, with (4.1) also holding, i.e., $a_1 \leq 2$. However, $f(x)$ is not a pdf in the other two cases, i.e., (4.2) is not satisfied, since $a_1 = 1.5$ and $a_1 = 2.5$ are greater than 1.326735827671239, but $a_1 \leq 2$ and $a_1 > 2$, respectively, and consequently (4.1) is not a sufficient condition.



(a) Plots for shape parameters $c_1 = 0.5 < c_2 = 2$.

(b) Plots for shape parameters $c_1 = 1.5 < c_2 = 2$.

Fig. 1: Graphs of $f(x)$ of generalized mixture of two Weibull components with $b_1 = 1$, $b_2 = 2$ and for different mixing weights.

Similarly, Figure 1b displays $f(x)$ for two Weibull components with $b_1 = 1$, $b_2 = 2$ and $c_1 = 1.5 < c_2 = 2$ and for varying values of the mixing weights, with the dark solid line for $a_1 = 1.5$, the dark dotted line for $a_1 = 2.5$, and the dashed line for $a_1 = 3.5$, respectively. Now, $x_0 = 0.5123859916786938$ is the required point in (4.2) for these three cases. Nevertheless, the inequality (4.2) is not verified for the last case since $a_1 = 3.5 > 2.588930674148028$, and so $f(x)$ is not a valid pdf, with (4.1) also not holding, i.e., $a_1 > 2$. Moreover, $f(x)$ is a valid pdf in the two first cases, since $a_1 = 1.5$ and $a_1 = 2.5$ are lower than 2.588930674148028; but $a_1 \leq 2$ and $a_1 > 2$, respectively, and consequently (4.1) is not a necessary condition.

5 Reliability properties of a generalized mixture of two Weibull components

In this section, we present some reliability results for a generalized mixture of two Weibull distributions. For this purpose, we consider different cases depending on the shape parameters of the two components.

Assuming a common shape parameter, the log-concavity of the survival function of a generalized mixture of two Weibull components has been established in Theorem 3.1 of Franco et al. (2011), i.e., the classification in the IFR and DFR classes when $c_1 = c_2$, which of course includes the special cases of exponential and Rayleigh components.

In order to investigate the reliability properties of a generalized mixture of two Weibull distributions with different shape parameters, we first start with generalized mixtures with one component being exponential component.

Theorem 5.1 *Let X be a generalized mixture of an exponential and a Weibull component with shape parameter $0 < c < 1$, and scale parameters $b_2 > 0$ and $b_1 > 0$, respectively. Then, X cannot be IFR, and X is DFR iff either $0 < a_1 < 1$ or*

$$1 < a_1 \leq \frac{1}{1-w}, \tag{5.1}$$

where

$$w = \min_{\tau \in \{\tau_1, \tau_2\}} \left\{ \frac{cb_1(1-c)e^{-b_1\tau^c + b_2\tau}}{cb_1(1-c) + (cb_1\tau^{c-1} - b_2)^2\tau^{2-c}} \right\}, \tag{5.2}$$

with τ_1 and τ_2 being the two unique solutions of

$$cb_1 - b_2(2-c)x^{1-c} + x^c(cb_1 - b_2x^{1-c})^2 = 0 \tag{5.3}$$

such that $\tau_1 \in \left(\left(\frac{cb_1}{b_2(2-c)}\right)^{1/(1-c)}, \left(\frac{cb_1}{b_2}\right)^{1/(1-c)} \right)$ *and* $\tau_2 > \left(\frac{cb_1}{b_2}\right)^{1/(1-c)}$.

Proof It is well known that a Weibull distribution with shape parameter $c > 0$ is DFR (IFR) iff $c \leq 1$ ($c \geq 1$). Here, $0 < c < 1$, and so X is an ordinary mixture of two DFR components for $a_1 < 1$, i.e., $a_2 = 1 - a_1 > 0$, and consequently X is DFR since the DFR class is preserved by ordinary mixtures; see Barlow and Proschan (1981).

Otherwise, for $a_1 > 1$, the classification of X depends on the sign of $g(x) = S''(x)S(x) - S'(x)^2$, e.g., see Lemma 2.1 of Franco et al. (2011). Thus, $g(x)$ can be expressed as

$$g(x) = a_1 x^{c-2} e^{-b_1 x^c - b_2 x} h(x),$$

where

$$h(x) = a_2\left((b_1 c x^{c-1} - b_2)^2 x^{2-c} - b_1 c(c-1)\right) - a_1 b_1 c(c-1) e^{-b_1 x^c + b_2 x},$$

and so both functions $g(x)$ and $h(x)$ have the same sign. Note that $h(0) = b_1 c(1-c) > 0$, and so it cannot be negative for all $x > 0$, and consequently, X cannot be IFR.

Now, let us see under what conditions the sign of $h(x)$ is non-negative, and from Lemma 2.1 of Franco et al. (2011), X is DFR. For this purpose, $h(x)$ can be rewritten as

$$h(x) = (1-a_1)h_2(x) + a_1 h_1(x) = h_2(x) + a_1(h_1(x) - h_2(x)),$$

where

$$h_1(x) = b_1 c(1-c) e^{-b_1 x^c + b_2 x} \quad \text{and} \quad h_2(x) = b_1 c(1-c) + (b_1 c x^{c-1} - b_2)^2 x^{2-c},$$

both functions being positive. Obviously, $h(x) \geq 0$ for all x such that $h_1(x) \geq h_2(x)$. So, X is DFR if and only if $h(x) \geq 0$ for all x such that $h_1(x) < h_2(x)$, which is equivalent to

$$a_1 \leq \frac{h_2(x)}{h_2(x) - h_1(x)} \Leftrightarrow \frac{a_1 - 1}{a_1} \leq \frac{h_1(x)}{h_2(x)} < 1. \tag{5.4}$$

However, if (5.4) does not hold, i.e., there exist a point x such that $\frac{h_1(x)}{h_2(x)} < \frac{a_1 - 1}{a_1}$, then X cannot be DFR.

In order to analyze the ratio $h_1(x)/h_2(x)$ for x such that $h_1(x) < h_2(x)$, from the first derivatives of $h_1(x)$ and $h_2(x)$, we have that $h_1(x)$ changes its monotonicity from being decreasing to increasing for all $x > 0$, attaining its minimum at $x_1 = \left(\frac{b_1 c}{b_2}\right)^{1/(1-c)}$. Analogously, $h_2(x)$ changes its monotonicity only twice, it increases for $x < x_2 = \left(\frac{b_1 c^2}{b_2(2-c)}\right)^{1/(1-c)}$, then decreases for $x \in (x_2, x_1)$, and ultimately increases for $x > x_1$. Thus, it is straightforward to check that the sign of the first derivative of $h_1(x)/h_2(x)$ is determined by the sign of

$$h_1'(x)h_2(x) - h_1(x)h_2'(x) = -b_1 c(1-c) x^{c-1} e^{-b_1 x^c + b_2 x}(b_1 c - b_2 x^{1-c}) k(x),$$

where

$$k(x) = b_1 c - b_2(2-c)x^{1-c} + x^c(b_1 c - b_2 x^{1-c})^2.$$

Note that $k(0) = b_1 c > 0$ and $\lim_{x\to\infty} k(x) = \infty$. Furthermore, $b_1 c - b_2(2-c)x^{1-c} > 0$ for all $x < x_3$, where $x_3 = \left(\frac{b_1 c}{b_2(2-c)}\right)^{1/(1-c)} \in (x_2, x_1)$, and so the first term of $k(x)$ is negative for all $x > x_3$. Likewise, the other term of $k(x)$ is equal to $h_2(x) - b_1c(1-c) \geq 0$ for all x, and so it increases for $x < x_2$, then decreases for $x \in (x_2, x_1)$, and ultimately increases for $x > x_1$, being null at x_1. Therefore, $k(x)$ changes its sign only twice from being positive when $x < \tau_1$, then is negative for $x \in (\tau_1, \tau_2)$, and ultimately is positive for $x > \tau_2$, where $\tau_1 \in (x_3, x_1)$ and $\tau_2 > x_1$ are the two unique solutions of (5.3).

Finally, taking into account the sign of $-(b_1c - b_2x^{1-c})$ and $k(x)$, the quotient $h_1(x)/h_2(x)$ decreases in $(0, \tau_1)$, then increases in (τ_1, x_1), again decreases in (x_1, τ_2), and ultimately increases in (τ_2, ∞). Thus, $h_1(x)/h_2(x)$ attains its two local minimums at the two unique solutions τ_1 and τ_2 of (5.3), being $w_1 = h_1(\tau_1)/h_2(\tau_1) < 1$ and $w_2 = h_1(\tau_2)/h_2(\tau_2) < 1$ since $w_3 = h_1(x_1)/h_2(x_1) = e^{-b_1(1-c)x_1^c} < 1$. Therefore, (5.4) holds if and only if

$$\frac{a_1 - 1}{a_1} \leq w = \min\{w_1, w_2\},$$

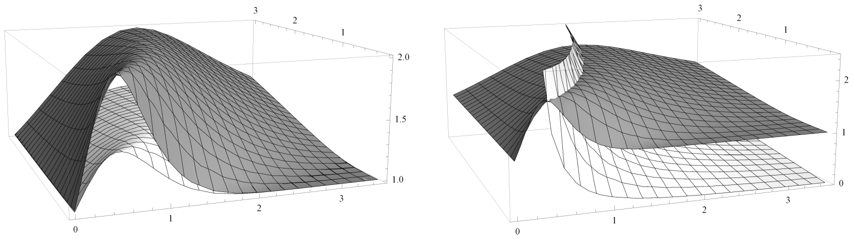
and consequently, X is DFR if and only if (5.1) holds.

Remark 5.1 From Theorem 5.1, an exponential and Weibull generalized mixture X with shape parameter $0 < c < 1$ is not always DFR, it can have neither bathtub shaped failure rate nor upside-down bathtub shaped failure rate function, denoted in short by $\cup$-shaped and $\cap$-shaped, respectively. In detail, from the proof of Theorem 5.1 and assuming (4.2), if $a_1 > 1/(1-w)$ then the failure rate of X decreases at the beginning and at the end, and has one or two piecewise increasing in the middle, i.e., X has a single or double $\setminus\!\!\wedge\!\!\setminus$-shaped failure rate. Here, X has a $\setminus\!\!\wedge\!\!\setminus$-shaped failure rate function iff either $a_1 > 1/(1-w_i)$ holds for only one of $i = 1, 2$, or $a_1 \geq 1/(1-w_3)$. Moreover, X has a double $\setminus\!\!\wedge\!\!\setminus$-shaped failure rate iff $a_1 > 1/(1-w_i)$ for $i = 1, 2$, and $a_1 < 1/(1-w_3)$.

For example, if $c = 0.5$, $b_1 = 1$ and $b_2 = 2$, X is DFR when its mixing weight $a_1 \leq 1.369955901927713$, and X has a $\setminus\!\!\wedge\!\!\setminus$-shaped failure rate when $1.369955901927713 < a_1 \leq 1.877189829645477$, since $1/(1-w_1) = 8.234894564961930$. Figure 2a displays regions for (4.2) and (5.1) according to different (b_1, b_2) when $c = 0.5$. Here, a_1 must be always below the gray surface. Moreover, X is DFR when a_1 is below of the white surface, and it has a $\setminus\!\!\wedge\!\!\setminus$-shaped failure rate when a_1 is between the white and gray surfaces.

Theorem 5.2 *Let X be a generalized mixture of an exponential and a Weibull component with shape parameter $c > 1$, and scale parameters $b_1 > 0$ and $b_2 > 0$, respectively. Then, X can be neither DFR nor IFR. Moreover, the following items hold:*

(i) If $1 < c < 2$, then X has a $\cup$-shaped ($\cap$-shaped) failure rate for $a_1 > 1$ ($a_1 < 1$), or it has a double $\cup$-shaped ($\cap$-shaped) failure rate iff $a_1 > 1$



(a) Boundary surfaces for a_1 from (4.2) and (5.1) with $c = 0.5$.

(b) Boundary surfaces for a_1 from (4.2) and Part *(ii)* of Theorem 5.2.

Fig. 2: Regions for a_1 according to $(b_1, b_2) \in (0, 3.5) \times (0, 3)$.

$(a_1 < 1)$ and

$$\begin{aligned}\frac{1-a_1}{a_1} &\in (v_1, v_2)\\ &= \left(-\exp\left(-b_1 \tfrac{c-1}{c} \left(\tfrac{b_1}{b_2 c} \right)^{\frac{1}{c-1}} \right), \left(\tfrac{(b_1 - b_2 c x_2^{c-1})^2}{b_2 c (c-1) x_2^{c-2}} - 1 \right) e^{b_2 x_2^c - b_1 x_2} \right)\end{aligned} \tag{5.5}$$

where $x_2 < x_1 = \left(\frac{b_1}{b_2 c} \right)^{1/(c-1)}$ *is the unique positive solution of*

$$(b_1 - b_2 c x^{c-1})^2 x + b_2 c x^{c-1} = (2 - c) b_1. \tag{5.6}$$

(ii) If $c = 2$, then X has a $\cup$-shaped ($\cap$-shaped) failure rate for $a_1 \in \left(1, 2b_2/b_1^2\right]$ ($a_1 < \min\{1, 2b_2/b_1^2\}$), or it has a $\cap\!\cup$-shaped ($\cup\!\cap$-shaped) failure rate for $a_1 > \max\{1, 2b_2/b_1^2\}$ ($a_1 \in \left(2b_2/b_1^2, 1\right)$).

(iii) If $c > 2$, then X has a $\cup\!\cap$-shaped failure rate for $a_1 < 1$, and X has a $\cap\!\cup$-shaped failure rate $1 < a_1 < 1/(1 + v_1)$ where v_1 is given in (5.5).

Proof From the survival and density functions of X, its failure rate function can be expressed as

$$r(x) = b_1 w(x) + b_2 c x^{c-1}(1 - w(x)) = b_2 c x^{c-1} + \left(b_1 - b_2 c x^{c-1}\right) w(x)$$

where $w(x) = a_1 e^{-b_1 x}/(a_1 e^{-b_1 x} + (1 - a_1) e^{-b_2 x^c})$, e.g., see Equation (3) in Navarro et al. (2009). Note that $w(x)$ is always positive, since it is the ratio of two survival functions and $a_1 > 0$. Also, it is easy to check that $\lim_{x\to\infty} w(x) = 1$. Moreover, the term $b_1 - b_2 c x^{c-1}$ with $c > 1$ changes its sign at $x_1 = \left(\frac{b_1}{b_2 c}\right)^{1/(c-1)}$ and converges to $-\infty$ as $x \to \infty$. Thus, we have that

$$r(x_1) = b_1 \quad \text{and} \quad \lim_{x\to\infty} r(x) = b_1,$$

i.e., the failure rate function of X crosses at x_1 or is equal to its horizontal asymptote $y = b_1$, and consequently, it must change its monotonicity at least once in (x_1, ∞). So, X can be neither DFR nor IFR for all $c > 1$.

Now, let us see the monotonicity intervals of $r(x)$. For this purpose, we have to study intervals in which the sign of its first derivative holds according to the shape parameter $c > 1$, and it is equivalent to discuss the sign of $g(x) = S''(x)S(x) - S'(x)^2$ defined in the proof of Theorem 5.1. In this case, $g(x)$ can be rewritten as $g(x) = (1 - a_1)x^{c-2}e^{-2b_2x^c}h(x)$, where

$$h(x) = a_1\left((b_1 - b_2cx^{c-1})^2x^{2-c} - b_2c(c-1)\right)e^{b_2x^c - b_1x} + (a_1 - 1)b_2c(c-1),$$

and so $g(x)$ and $h(x)$ have the same sign when $a_1 < 1$ and opposite signs when $a_1 > 1$.

First, when $1 < c < 2$, it is easy to check that $h(0) = -b_2c(c-1) < 0$ and $\lim_{x\to\infty} h(x) = \infty$, and its first derivative

$$h'(x) = -a_1(b_1 - b_2cx^{c-1})x^{1-c}e^{b_2x^c - b_1x}k(x),$$

where $k(x) = (b_1 - b_2cx^{c-1})^2x + b_2cx^{c-1} - (2-c)b_1$ is an increasing function with $k(0) = -(2-c)b_1 < 0$, $k(x_1) = (c-1)b_1 > 0$ and $\lim_{x\to\infty} k(x) = \infty$. Thus, we have that $h(x)$ changes its monotonicity only twice from being increasing in $(0, x_2)$, then decreasing in (x_2, x_1), and ultimately increasing in (x_1, ∞), where $x_2 \in (0, x_1)$ is the unique solution of (5.6).

Therefore, $h(x)$ changes its sign only once from being negative to positive when either $h(x_2) \leq 0$ or $h(x_1) \geq 0$. Otherwise, $h(x)$ changes its sign three times whenever $h(x_2) > 0$ and $h(x_1) < 0$, which is equivalent to (5.5). Consequently, the failure rate of X is decreasing (increasing) at the beginning, and then increasing (decreasing) for $a_1 > 1$ ($a_1 < 1$) when either $h(x_2) \leq 0$ or $h(x_1) \geq 0$. Moreover, if (5.5) holds, then the failure rate changes its monotonicity three times from being decreasing (increasing), then increasing (decreasing), again decreasing (increasing), and ultimately increasing (decreasing) for $a_1 > 1$ ($a_1 < 1$).

Next, when $c = 2$, $\lim_{x\to\infty} h(x) = \infty$, but $h(0) = a_1b_1^2 - 2b_2$ can be positive or negative depending on a_1. In addition, it is immediate to prove that $h(x)$ changes its monotonicity from being decreasing to increasing at $x_1 = \frac{b_1}{2b_2}$, attaining its minimum at this point, with $h(x_1) = -2b_2\left(1 - a_1\left(1 - e^{-\frac{b_1^2}{4b_2}}\right)\right) < 0$. Note that $h(x_1) \geq 0$ is equivalent to $a_1 \geq \left(1 - e^{-b_1^2/(4b_2)}\right)^{-1}$, but from (4.2) it is not satisfied for a valid pdf. Thus, $h(x)$ changes its sign only once from being negative to positive when $a_1 \leq 2b_2/b_1^2$, and it changes its sign only twice from being positive at the beginning and at the end, and negative in the middle when $a_1 > 2b_2/b_1^2$. Therefore, X has a $\cup$-shaped ($\cap$-shaped) failure rate whenever $a_1 \in \left(1, 2b_2/b_1^2\right]$ ($a_1 < \min\{1, 2b_2/b_1^2\}$), i.e., it is decreasing (increasing) at the beginning, and then increasing (decreasing). Otherwise, its failure rate function is $\wedge\!\!\vee$-shaped ($\vee\!\!\wedge$-shaped) when $a_1 > \max\{1, 2b_2/b_1^2\}$ ($a_1 \in \left(2b_2/b_1^2, 1\right)$), i.e., it increases (decreases), then decreases (increases), and ultimately increases (decreases).

Finally, when $c > 2$, the function $g(x)$ can be expressed as

$$g(x) = (1 - a_1)b_2c(c-1)x^{c-2}e^{-2b_2x^c}h(x),$$

where $h(x) = a_1\left(\frac{h_1(x)}{h_2(x)} + 1\right) - 1$ has the same sign as $g(x)$ for $a_1 < 1$ and the opposite sign for $a_1 > 1$, with

$$h_1(x) = (b_1 - b_2cx^{c-1})^2 - b_2c(c-1)x^{c-2} \quad \text{and} \quad h_2(x) = b_2c(c-1)x^{c-2}e^{b_1x-b_2x^c}.$$

Note that $\lim_{x\to 0} h(x) = \infty$ and $\lim_{x\to\infty} h(x) = \infty$. So, taking into account that $h_2(x) > 0$ for all $x > 0$, we have $h(x) > 0$ $(<)$ if and only if $\frac{h_1(x)}{h_2(x)} > \frac{1-a_1}{a_1}$ $(<)$. Thus, we have to analyze the ratio $h_1(x)/h_2(x)$ for all $x > 0$. For this purpose, it is straightforward to check that $h_1(x)/h_2(x)$ decreases and then increases, attaining its minimum at $x_1 = \left(\frac{b_1}{b_2c}\right)^{1/(c-1)}$, with $\frac{h_1(x_1)}{h_2(x_1)} = v_1 \in (-1, 0)$ as given in (5.5).

Clearly, when $a_1 < 1$, $(1 - a_1)/a_1 > 0$, and hence, there exist x_2 and x_3, $0 < x_2 < x_1 < x_3$ such that $h_1(x)/h_2(x) > (1 - a_1)/a_1$ for $x < x_2$ or $x > x_3$, and $h_1(x)/h_2(x) < (1 - a_1)/a_1$ when $x \in (x_2, x_3)$, i.e., $h(x)$ changes its sign only twice from being positive at the beginning, negative in the middle, and positive at the end. Therefore, X has a $\vee\!\!\setminus$-shaped failure rate when $a_1 < 1$, i.e., it decreases, then increases, and ultimately decreases.

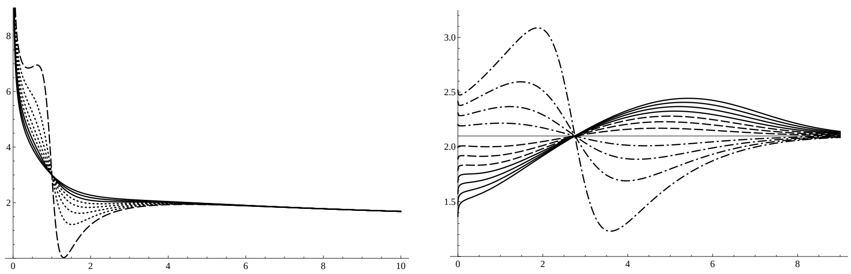
In addition, when $a_1 > 1$, $(1 - a_1)/a_1 < 0$, and so, it might be lower or upper than v_1. Note that, $(1 - a_1)/a_1 \leq v_1$ is equivalent to $a_1 \geq 1/(1 + v_1)$, which is not satisfied for a_1 bounded by (4.2) to be a valid pdf. Furthermore, for $(1 - a_1)/a_1 > v_1$, $h(x)$ changes its sign only twice, as in the case $a_1 < 1$; however now $g(x)$ has opposite sign to $h(x)$, and consequently, X has a $/\!\!\wedge$-shaped failure rate when $1 < a_1 < 1/(1 + v_1)$, i.e., it increases in $x < x_2$, then decreases in $x \in (x_2, x_3)$, and ultimately decreases in $x > x_3$.

Remark 5.2 From Theorem 5.2, an ordinary mixture of an exponential and a Weibull component with shape parameter $c > 1$ can be neither DFR nor IFR; and the shape of its failure rate function was studied by Wondmagegnehu et al. (2005).

Further, the particular case of the generalized mixture of exponential and Rayleigh components can be neither DFR nor IFR. Figure 2b shows regions for (4.2) and Part (ii) of Theorem 5.2 according to (b_1, b_2). Here, a_1 must be always below the gray surface, and the failure rate of X has a $\cap$-shaped when a_1 is below the white surface and $a_1 < 1$, $\vee\!\!\setminus$-shaped when a_1 is between the white and gray surfaces and $a_1 < 1$, $\cup$-shaped when a_1 is below the white surface and $a_1 > 1$, and $/\!\!\wedge$-shaped when a_1 is between the white and gray surfaces and $a_1 > 1$.

In this context, from Theorems 5.1 and 5.2, we show some different shapes of the failure rates of generalized mixtures of an exponential and a Weibull component with shape parameter $c = 0.75$ (Figure 3a) and $c = 1.4$ (Figure 3b).

Finally, we obtain some reliability results for generalized mixtures of two Weibull components with different shape parameters, neither of them being exponential. For this purpose, we will use the following auxiliary lemma, which proof is straightforward.



(a) Plots of DFR (solid lines), $\vee\wedge$-shaped (dotted lines) and double $\vee\wedge$-shaped (dashed lines) for $a_1 = 1.2, 1.25, ..., 1.55$, with $b_1 = 4$, $b_2 = 3$, $c = 0.75$.

(b) Plots of $\cap$-shaped (solid lines), double $\cap$-shaped (dashed lines) and double $\cup$-shaped (dotted and dashed lines) for $a_1 = 0.65, 0.7, ..., 1.2$, with $b_1 = 2.1$, $b_2 = 1$, $c = 1.4$.

Fig. 3: Failure rates of generalized mixtures of an exponential and a Weibull component.

Lemma 5.1 *The following items hold:*

(i) Let $g_1(t)$ and $g_2(t)$ be two arbitrary real functions such that its composite function is well defined, $g_1 \circ g_2(t)$. If $g_1(t)$ is concave (convex) and either $\{g_1(t)$ is non-decreasing and $g_2(t)$ is concave (convex)$\}$ or $\{g_1(t)$ is non-increasing and $g_2(t)$ is convex (concave)$\}$, then $g_1 \circ g_2(t)$ is concave (convex).
(ii) If $S(t)$ is a log-concave (log-convex) survival function and $c \geq 1$ $(0 < c \leq 1)$, then $S(t^c)$ is a log-concave (log-convex) survival function.

Theorem 5.3 *Let X be a generalized mixture of two Weibull components with different shape parameters, $c_1 < c_2$, and scale parameters $b_1 > 0$ and $b_2 > 0$, respectively.*

(i) For $0 < c_1 < c_2 < 1$, X is DFR when $a_1 < 1$ or (5.1) holds with $c = c_1/c_2$ in (5.2). Otherwise, X has a single or multiple $\vee\wedge$-shaped failure rate.
(ii) For $1 < c_1 < c_2$, X cannot be DFR. Moreover, X is either IFR or single or multiple $\wedge\vee$-shaped failure rate.
(iii) For $0 < c_1 < 1 < c_2$, X cannot be IFR. Moreover, X is either DFR or single or multiple $\vee\wedge$-shaped failure rate.

Proof For $a_1 < 1$, X is an ordinary mixture of two Weibull components with different shape parameters, $c_1 < c_2$. Thus, if $c_2 < 1$, it is a mixture of two DFR components, and consequently, X is DFR; see Barlow and Proschan (1981).

In the remaining cases, the monotonicity of the failure rate function of X with $c_1 \neq 1$ is established by the sign of $g(x) = S''(x)S(x) - S'(x)^2$ as defined in the proof of Theorem 5.1, which has the same sign as $h(x) = x^{2-c_1}e^{2b_1x^{c_1}}g(x)$

given by

$$\begin{aligned}h(x) =& - a_1^2 b_1 c_1 (c_1 - 1) - (1 - a_1)^2 b_2 c_2 (c_2 - 1) x^{c_2 - c_1} e^{2b_1 x^{c_1} - 2b_2 x^{c_2}} \\ &+ a_1(1 - a_1)\left((b_1 c_1 + b_2 c_2 x^{c_2 - c_1})^2 x^{c_1} - b_1 c_1 (c_1 - 1)\right. \\ &\left. - b_2 c_2 (c_2 - 1) x^{c_2 - c_1}\right) e^{b_1 x^{c_1} - b_2 x^{c_2}}.\end{aligned}$$

Hence, $\lim_{x\to 0} h(x) = -a_1 b_1 c_1 (c_1 - 1)$ and $\lim_{x\to\infty} h(x) = -a_1^2 b_1 c_1 (c_1 - 1)$, and so they are negative for $c_1 > 1$ and positive for $c_1 < 1$. Therefore, $g(x)$ is negative (positive) at the beginning and at the end when $c_1 > 1$ ($c_1 < 1$), and consequently, the failure rate of X is increasing (decreasing) at the beginning and at the end when $c_1 > 1$ ($c_1 < 1$).

Note that the survival function of X can be expressed as

$$S(x) = S_T(x^{c_2}), \quad \text{where} \quad S_T(t) = a_1 e^{-b_1 t^c} + a_2 e^{-b_2 t} \tag{5.7}$$

where $S_T(t)$ is the survival function of a generalized mixture of an exponential and a Weibull component with shape parameter $c = c_1/c_2 < 1$. Analogously, $S(x)$ can be rewritten as

$$S(x) = S_Y(x^{c_1}), \quad \text{where} \quad S_Y(t) = a_1 e^{-b_1 t} + a_2 e^{-b_2 t^c} \tag{5.8}$$

where $S_Y(t)$ is the survival function of a generalized mixture of an exponential and a Weibull component with $c = c_2/c_1 > 1$.

Now, let us see the monotonicity of the failure rate $r(x)$ in the middle according to the different shape parameters $c_1 < c_2$.

First, when $c_1 < c_2 < 1$, $r(x)$ decreases at the beginning and at the end. From (5.7) and Part (ii) of Lemma 5.1, the log-convex intervals of $S_T(t)$ are preserved to $S(x)$, since $c_2 < 1$, i.e., the decreasing intervals of $r_T(x)$ are preserved to $r(x)$. Thus, from Theorem 5.1, X is DFR when (5.1) holds, where w is as defined in (5.2) with $c = c_1/c_2 < 1$. Otherwise, the increasing intervals of $r_T(x)$ might not be preserved, and so $r(x)$ might have a single or multiple piecewise increasing in the middle. Therefore, X is DFR or it has a single or multiple $\vee\!\wedge$-shaped failure rate.

Secondly, when $1 < c_1 < c_2$, $r(x)$ increases at the beginning and at the end. From (5.7) and Part (ii) of Lemma 5.1, the log-concave intervals of $S_Y(t)$ are preserved to $S(x)$, since $c_1 > 1$, i.e., the increasing intervals of $r_Y(x)$ are preserved to $r(x)$. Thus, from Theorem 5.2, $S_Y(t)$ has at least one log-concave interval for all $c = c_2/c_1 > 1$, and so $r(x)$ has at least one increasing interval for all $c = c_2/c_1 > 1$. However, $r(x)$ also might have a single or multiple piecewise decreasing in the middle, since the log-convex intervals of $S_Y(x)$ might not be preserved. Therefore, the failure rate of X is either IFR or single or multiple $\wedge\!\vee$-shaped when $1 < c_1 < c_2$.

Finally, when $c_1 < 1 < c_2$, $r(x)$ decreases at the beginning and at the end. From (5.7) and Part (ii) of Lemma 5.1, the increasing intervals of $r_T(x)$ are preserved to $r(x)$, since $c_2 > 1$. Analogously, the decreasing intervals of $r_Y(x)$ are preserved to $r(x)$, since $c_1 < 1$. From Theorem 5.2, $S_Y(t)$ always has at least one log-convex interval for all $c = c_2/c_1 > 1$, and consequently, $r(x)$ has at least one increasing interval. Moreover, from Theorem 5.1, if (5.1) does not hold, where w is as defined in (5.2) with $c = c_1/c_2 < 1$, then $r(x)$ has at least one increasing interval. However, when $c_2 > 1$ and (5.1) holds, the log-convex intervals of $S_T(x)$ might not be preserved to $S(x)$. Consequently, $r(x)$ also might be decreasing or have a single or multiple piecewise increasing in the middle, i.e., X is either DFR or single of multiple $\vee\wedge$-shaped failure rate.

6 Illustrative example

In this section, we present a data analysis of the strength of carbon fibers reported by Badar and Priest (1982). The data represent the strength measured in GPa, for single carbon fibers and impregnated 1000-carbon fiber tows, which were tested under tension. For illustrative purpose, we will be considering together the single fibers and impregnated fibers at gauge lengths of 20mm. These data are presented in Table 1.

Table 1: Strength data of carbon fibers at gauge lengths of 20 mm.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554	2.566	2.570
2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	2.526
2.546	2.628	2.628	2.669	2.669	2.710	2.731	2.731	2.731	2.752	2.752	2.793	2.834	2.834
2.854	2.875	2.875	2.895	2.916	2.916	2.957	2.977	2.998	3.060	3.060	3.060	3.080	

Before going on to analyze the data using generalized two-Weibull mixture models, we have subtracted 1.250 from each data points. Clearly, it is a mixture two groups, and we have tried to fit four mixture models of two Weibull components, with negative and positive mixing weights, and equal and unequal shape parameters.

Assuming a generalized two-Weibull mixture distribution with pdf as in Theorem 4.1, the maximum likelihood estimators (MLEs) cannot be expressed in explicit forms. However, based on the random sample $\mathbf{x} = (x_1, ..., x_n)$, the maximization problem of the log-likelihood function

$$LL(\mathbf{x}; \boldsymbol{\theta}) = \sum_{i=1}^{n} \ln f(x_i; \boldsymbol{\theta}) \tag{6.1}$$

can be numerically resolved with respect to the unknown parameters $\boldsymbol{\theta} = (a_1, b_1, b_2, c)$ or $\boldsymbol{\theta} = (a_1, b_1, b_2, c_1, c_2)$, according to equal or unequal shape parameters. In addition, we provide the log-likelihood values (LL), Kolmogorov-Smirnov (KS) distances with their associated p-values, Akaike information criterion (AIC) and Bayesian information criterion (BIC) values for each fitted generalized mixture model, where $AIC(k) = 2k - 2\ln(L)$, $BIC(k) = -2\ln(L) + k\ln(n)$, and k denotes the number of parameters in each model.

Model 1. Generalized mixture of two Weibull components with unequal shape parameters.

We have the following underlying model

$$f(x_i;\boldsymbol{\theta}) = a_1 b_1 c_1 x_i^{c_1-1} e^{-b_1 x_i^{c_1}} + (1-a_1) b_2 c_2 x_i^{c_2-1} e^{-b_2 x_i^{c_2}}$$

where $\boldsymbol{\theta} = (a_1, b_1, b_2, c_1, c_2)$. The estimated scale and shape parameters of both components are $\widehat{b}_1 = 0.3161$, $\widehat{b}_2 = 0.3877$, $\widehat{c}_1 = 3.6841$ and $\widehat{c}_2 = 5.6094$, and the estimated mixing weight is $\widehat{a}_1 = 1.6251$. The standard errors of estimates based on Bootstrapping become: 0.0581, 0.0820, 0.3412, 0.6599, 0.2534, respectively. Therefore, the fitted model is a proper generalized mixture, since $\widehat{a}_2 = 1-\widehat{a}_1 = -0.6251$. For this Model 1, we have $LL = -65.5767$, $AIC = 141.1534$ and $BIC = 154.0269$. The KS distance between the empirical survival function and the fitted survival function and the associated p-value are 0.0365 and 0.9995, respectively.

Model 2. Generalized mixture of two Weibull components with equal shape parameters.

We consider the following underlying model

$$f(x_i;\boldsymbol{\theta}) = a_1 b_1 c x_i^{c-1} e^{-b_1 x_i^{c}} + (1-a_1) b_2 c x_i^{c-1} e^{-b_2 x_i^{c}}$$

where now $\boldsymbol{\theta} = (a_1, b_1, b_2, c)$. The estimated parameters for this generalized two-Weibull mixture are $\widehat{a}_1 = 1.9682$, $\widehat{b}_1 = 0.3477$, $\widehat{b}_2 = 0.4963$ and $\widehat{c} = 3.3490$. The standard errors of estimates based on Bootstrapping become: 0.2059, 0.0327, 0.0901, 0.3239, respectively. Hence, $\widehat{a}_2 = 1 - \widehat{a}_1 = -0.9682$, and then this fitted model is also a proper generalized mixture. In this case, $LL = -66.2712$, $AIC = 140.5424$ and $BIC = 150.8412$. The KS statistic and its associated p-value are 0.0738 and 0.6660, respectively.

Model 3. Mixture of two Weibull components with unequal shape parameters.

Now, we assume the following underlying model

$$f(x_i;\boldsymbol{\theta}) = a_1 b_1 c_1 x_i^{c_1-1} e^{-b_1 x_i^{c_1}} + (1-a_1) b_2 c_2 x_i^{c_2-1} e^{-b_2 x_i^{c_2}}$$

where $\boldsymbol{\theta} = (a_1, b_1, b_2, c_1, c_2)$. The estimated parameters of both Weibull components are $\widehat{b}_1 = 0.0715$, $\widehat{b}_2 = 3.8617$, $\widehat{c}_1 = 5.6294$ and $\widehat{c}_2 = 7.6858$, and the estimated mixing weight is $\widehat{a}_1 = 0.7927$. The standard errors of estimates based on Bootstrapping become: 0.0101, 0.3847, 0.6972, 0.5883, 0.1077, respectively. Thus, this fitted model is a ordinary mixture since $\widehat{a}_2 = 1 - \widehat{a}_1 = 0.2073$. Moreover, $LL = -84.2229$, $AIC = 178.4458$ and $BIC = 191.3194$. The KS statistics and its associated p-value are 0.0556 and 0.9250, respectively.

Model 4. Mixture of two Weibull components with equal shape parameters.

In this case, we have the following underlying model

$$f(x_i;\boldsymbol{\theta}) = a_1 b_1 c x_i^{c-1} e^{-b_1 x_i^c} + (1-a_1) b_2 c x_i^{c-1} e^{-b_2 x_i^c}$$

where $\boldsymbol{\theta} = (a_1, b_1, b_2, c)$. The obtained estimates of the four parameters are $\widehat{a}_1 = 0.7490$, $\widehat{b}_1 = 0.0442$, $\widehat{b}_2 = 3.0602$ and $\widehat{c} = 6.4622$, and then this fitted model is a ordinary mixture since $\widehat{a}_2 = 1 - \widehat{a}_1 = 0.2510$. The standard errors of estimates based on Bootstrapping become: 0.0963, 0.0061, 0.2765, 0.4340, respectively. Furthermore, $LL = -86.3763$, $AIC = 180.7526$ and $BIC = 191.0514$, and the KS statistic and its associated p-value are 0.0750 and 0.6462, respectively.

Table 2 summarizes the MLEs of the unknown parameters for the above four mixture models, and their log-likelihood values (LL), Kolmogorov-Smirnov (KS) distances with their associated p-values, Akaike information criterion (AIC) and Bayesian information criterion (BIC) values.

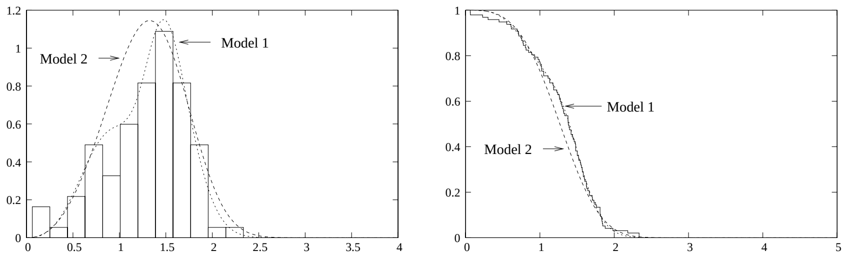
Table 2: Estimated parameters and the corresponding goodness-of-fit measures for the four fitted models.

Model	Estimates (standard errors)	LL	AIC	BIC	KS	p-value
Model 1	$\widehat{a}_1$ =1.6251 (0.2534)	-65.5767	141.1534	154.0269	0.0365	0.9995
	$\widehat{b}_1$ =0.3161 (0.0581)					
	$\widehat{b}_2$ =0.3877 (0.0820)					
	$\widehat{c}_1$ =3.6841 (0.3412)					
	$\widehat{c}_2$ =5.6094 (0.6599)					
Model 2	$\widehat{a}_1$ =1.9682 (0.2059)	-66.2712	140.5424	150.8412	0.0738	0.6660
	$\widehat{b}_1$ =0.3477 (0.0327)					
	$\widehat{b}_2$ =0.4963 (0.0901)					
	$\widehat{c}$ =3.3490 (0.3239)					
Model 3	$\widehat{a}_1$ =0.7927 (0.1077)	-84.2229	178.4458	191.3194	0.0556	0.9250
	$\widehat{b}_1$ =0.0715 (0.0101)					
	$\widehat{b}_2$ =3.8617 (0.3847)					
	$\widehat{c}_1$ =5.6294 (0.6972)					
	$\widehat{c}_2$ =7.6858 (0.5883)					
Model 4	$\widehat{a}_1$ =0.7490 (0.0963)	-86.3763	180.7526	191.0514	0.0750	0.6462
	$\widehat{b}_1$ =0.0442 (0.0061)					
	$\widehat{b}_2$ =3.0602 (0.2765)					
	$\widehat{c}$ =6.4622 (0.4340)					

From the AIC and BIC values the generalized mixture of two Weibull components with equal shape parameters (Model 2) provides the best fit among these four models. Model 1 also provides a very good fit to the above data set based on the Kolmogorov-Smirnov statistics and the associated p-values. Moreover, the probability density function based on Model 1 provides a very good match with the histogram (Figure 4). In order to test the hypothesis

$$H_0 : \text{Model 2} \quad vs. \quad H_1 : \text{Model 1},$$

based on the likelihood ratio test, we cannot reject the null hypothesis as the value of the test statistic is 1.3890 and its associated p-value is 0.2386. Based on all these, we propose to choose Model 1 for this data set.



(a) Histogram and the fitted pdf's. (b) Empirical and fitted survival functions.

Fig. 4: Graphs of fitted Model 1 and fitted Model 2 for the strength data.

7 Conclusions

In this paper, we have considered generalized mixtures of Weibull components. The generalized mixture is a very versatile model as allowing negative mixing weights, providing different probability density functions and failure rate functions for different parameter values.

We have discussed necessary and sufficient conditions for a generalized mixture of Weibull components, establishing its characterization in the case of two components. We have also studied the monotonicity of the failure rate function for this generalized two-Weibull mixture model. In addition to developing general results for the generalized two-Weibull mixture model, we have provided results when only one component is exponential, and for the special case of one exponential and the other Rayleigh.

In addition, one real data example has been analyzed for the purpose of illustration, and it has been shown that two-Weibull mixture models allowing negative mixing weights work very well. Of course, the development of different inferential issues for these mixture models and their applications are topics for further study.

Acknowledgements The authors would like to thank two referees, the Associated Editor and the Editor-in-Chief for many constructive suggestions of an earlier version of this manuscript, which have improved the presentation of this paper. The authors are also thankful to Prof. J.G. Surles for providing the real data set. This work was partially supported by Fundación Séneca of the Regional Government of Murcia (Spain) under Grant 11886/PHCS/09.

References

1. Attardi L, Guida M, Pulcini C (2005) A mixed-Weibull regression model for the analysis of automotive warranty data. Reliab Eng Syst Saf 87:265–273
2. Badar MG, Priest AM (1982) Statistical aspects of fiber and bundle strength in hybrid composites. In: Hayashi T, Kawata K, Umekawa S (eds) Progress in Science and Engineering Composites. ICCM-IV, Tokyo, pp 1129–1136
3. Baggs GE, Nagaraja HN (1996) Reliability properties of order statistics from bivariate exponential distributions. Commun Stat Stoch Models 12:611–631
4. Bagnoli M, Bergstrom T (2005) Log-concave probability and its applications. Econ Theory 26:445–469
5. Barlow RE, Proschan F (1981) Statistical Theory of Reliability and Life Testing: Probability Models. To Begin With, Silver Spring, Maryland
6. Bartholomew DJ (1969) Sufficient conditions for a mixture of exponentials to be a probability density function. Ann Math Stat 40:2183–2188
7. Block HW, Langberg NA, Savits TH (2012) A mixture of exponential and IFR gamma distributions having an upsidedown bathtub-shaped failure rate. Probab Eng Inf Sci 26:573–580
8. Block HW, Langberg NA, Savits TH, Wang J (2010) Continuous mixtures of exponentials and IFR gammas having bathtub-shaped failure rates. J Appl Probab 47:899–907
9. Bučar T, Nagode M, Fajdiga M (2004) Reliability approximation using finite Weibull mixture distributions. Reliab Eng Syst Saf 84:241–251
10. Carta JA, Ramírez P (2007) Analysis of two-component mixture Weibull statistics for estimation of wind speed distributions. Renew Energy 32:518–531
11. Ebden M, Stranjak A, Roberts S (2010) Visualizing uncertainty in reliability functions with application to aero engine overhaul. *J R Stat Soc, Ser C* 59:163–173
12. Elmahdy EE, Aboutahoun AW (2013) A new approach for parameter estimation of finite Weibull mixture distributions for reliability modeling. Appl Math Model 37:1800–1810
13. Farcomeni A, Nardi A (2010) A two-component Weibull mixture to model early and late mortality in a Bayesian framework. Comput Stat Data Anal 54:416–428
14. Franco M, Vivo JM (2002) Reliability properties of series and parallel systems from bivariate exponential models. Commun Stat Theory Methods 31:2349–2360
15. Franco M, Vivo JM (2006) On log-concavity of the extremes from Gumbel bivariate exponential distributions. Statistics 40:415–433
16. Franco M, Vivo JM (2007) Generalized mixtures of gamma and exponentials and reliability properties of the maximum from Friday and Patil bivariate exponential model. Commun Stat Theory Methods 36:2011–2025
17. Franco M, Vivo JM (2009a) Constraints for generalized mixtures of Weibull distributions with a common shape parameter. Stat Probab Lett 79:1724–1730
18. Franco M, Vivo JM (2009b) Stochastic aging classes for the maximum statistic from Friday and Patil bivariate exponential distribution family. Commun Stat Theory Methods 38:902–911
19. Franco M, Vivo JM, Balakrishnan N (2011) Reliability properties of generalized mixtures of Weibull distributions with a common shape parameter. J Stat Plan Inference 141:2600–2613
20. Jiang R, Zuo MJ, Li HX (1999) Weibull and inverse Weibull mixture models allowing negative weights. Reliab Eng Syst Saf 66:227–234
21. Johnson NL, Kotz S, Balakrishnan N (1994) Continuous Univariate Distributions - vol. 1, 2nd ed. Wiley, New York
22. Kokonendji C, Mizere D, Balakrishnan N (2008) Connections of the Poisson weight function to overdispersion and underdispersion. J Stat Plan Inference 138:1287–1296
23. Kundu D, Basu S (2000) Analysis of incomplete data in presence of competing risks. J Stat Plan Inference 87:231–239
24. Marín JM, Rodríguez MT, Wiper MP (2005) Using Weibull mixture distributions to model heterogeneous survival data. Commun Stat Simul Comput 34:673–684
25. Mosler K, Scheicher C (2008). Homogeneity testing in a Weibull mixture model. Stat Pap 49:315–332

26. Navarro J, Guillamon A, Ruiz MC (2009) Generalized mixtures in reliability modelling: Applications to the construction of bathtub shaped hazard models and the study of systems. Appl Stoch Models Bus Ind 25:323–337
27. Patra K, Dey DK (1999) A multivariate mixture of Weibull distributions in reliability modeling. Stat Probab Lett 45:225–235
28. Qin X, Zhang JS, Yan XD (2012) Two improved mixture Weibull models for the analysis of wind speed data. J Appl Meteorol Climatol 51:1321–1332
29. Shaked M, Shanthikumar JG (2006) Stochastic Orders, 2nd ed. Springer, New York
30. Sinha SK (1987) Bayesian estimation of the parameters and reliability function of mixture of Weibull life distributions. J Stat Plan Inference 16:377–387
31. Steutel FW (1967) Note on the infinite divisibility of exponential mixtures. Ann Math Stat 38:1303–1305
32. Tsionas EG (2002) Bayesian analysis of finite mixtures of Weibull distributions. Commun Stat Theory Methods 31:37–48
33. Wondmagegnehu ET (2004) On the behavior and shape of mixture failure rates from a family of IFR Weibull distributions. Nav Res Logist 51:491–500
34. Wondmagegnehu ET, Navarro J, Hernandez PJ (2005) Bathtub shaped failure rates from mixtures: a practical point of view. IEEE Trans Reliab 54:270–275
35. Xie M, Lai CD (1995) Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliab Eng Syst Saf 52:87–93