Hybrid Censoring: Models, Inferential Results and Applications

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Abstract

A hybrid censoring scheme is a mixture of Type-I and Type-II censoring schemes. In this review, we first discuss Type-I and Type-II hybrid censoring schemes and associated inferential issues. Next, we present details on developments regarding generalized hybrid censoring and unified hybrid censoring schemes that have been introduced in the literature. Hybrid censoring schemes have been adopted in competing risks set-up and in step-stress modeling and these results are outlined next. Recently, two new censoring schemes, viz., progressive hybrid censoring and adaptive progressive censoring schemes have been introduced in the literature. We discuss these censoring schemes and describe inferential methods based on them, and point out their advantages and disadvantages. Determining an optimal hybrid censoring scheme is an important design problem, and we shed some light on this issue as well. Finally, we present some examples to illustrate some of the results described here. Throughout the article, we mention some open problems and suggest some possible future work for the benefit of readers interested in this area of research.

Key Words and Phrases: Type-I and Type-II hybrid censoring schemes; Progressive censoring scheme; Adaptive progressive censoring; Competing risks; Fisher information; maximum likelihood estimators; Optimum sampling plans; Step-stress testing.

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1 Introduction

Type-I and Type-II censoring schemes are the two most common and popular censoring schemes. In Type-I censoring scheme, the experimental time is fixed, but the number of observed failures is a random variable. On the other hand, in Type-II censoring scheme, number of observed failures is fixed, but the experimental time is a random variable. The mixture of Type-I and Type-II censoring schemes is known as hybrid censoring scheme, and it can be described as follows.

Consider the following life-testing experiment in which n units are placed on test. The lifetimes of the sample units are assumed to be independent and identically distributed ($i.i.d.$) random variables. Let the ordered lifetimes of these units be denoted by $X_{1:n}, \cdots, X_{n:n}$, respectively. The test is terminated when a pre-fixed number, $r < n$, out of n items have failed, or when a pre-fixed time, T, has been reached. In other words, the life-test is terminated at a random time $T_* = \min\{X_{r:n}, T\}$. It is also usually assumed that the failed units are not replaced in the experiment.

This hybrid censoring scheme, which was originally introduced by Epstein (1954), has been used quite extensively in reliability acceptance test in MIL-STD-781-C (1977). From now on, we refer to this hybrid censoring scheme as *Type-I hybrid censoring scheme* (*Type-I HCS*). It is evident that the complete sample situation as well as Type-I and Type-II right censoring schemes are all special cases of this Type-I HCS.

Since the introduction of Type-I HCS by Epstein (1954), extensive work has been done on hybrid censoring and many different variations of it. In his pioneering work, Epstein (1954) introduced the Type-I HCS and considered the special case when the lifetime distribution is exponential with mean lifetime θ. He discussed estimation methods for θ and also proposed a two-sided confidence interval for θ, without presenting a formal proof of its construction.

Later, Fairbanks et al. (1982) modified slightly the proposition of Epstein (1954), and suggested a simple set of confidence intervals. Motivated by the works of Bartholomew (1963) and Barlow et al. (1968), Chen and Bhattacharyya (1988) derived the exact distribution of the conditional maximum likelihood estimator (MLE) of θ by using the conditional moment generating function approach, and used it to construct an exact lower confidence bound for θ. Childs et al. (2003) derived a simplified but an equivalent form of the exact distribution of the MLE of θ as derived by Chen and Bhattacharyya (1988). In constructing exact confidence intervals for θ from the exact conditional densities, these authors made a critical assumption about monotonicity of tail probabilities and this was formally proved only recently by Balakrishnan and Iliopoulos (2009). Draper and Guttman (1987) considered the Bayesian inference for θ, and obtained the Bayesian estimate and a two-sided credible interval for θ by using an inverted gamma prior. A comparison of different methods of estimation, by using extensive Monte Carlo simulations, was carried out by Gupta and Kundu (1998). While all these results were developed for the case of exponential distribution, Type-I HCS has been discussed for some other lifetime distributions such as two-parameter exponential, Weibull, log-normal and generalized exponential. All these developments pertaining to this Type-I HCS will be reviewed in Section 2.

As in the case of conventional Type-I censoring, the main disadvantage of Type-I HCS is that most of the inferential results need to be developed in this case under the condition that the number of observed failures is at least one; moreover, there may be very few failures occurring up to the pre-fixed time T which results in the estimator(s) of the model parameter(s) having low efficiency. For this reason, Childs et al. (2003) introduced an alternative hybrid censoring scheme that would terminate the experiment at the random time $T^* = \max\{X_{r,n}, T\}$. This hybrid censoring scheme is called *Type-II hybrid censoring scheme* (*Type-II HCS*), and it has the advantage of guaranteeing at least r failures to be observed by the end of the experiment. If r failures occur before time T, then the experiment would continue up to time T which may end up yielding possibly more than r failures in the data. On the other hand, if the r-th failure does not occur before time T, then the experiment would continue until the time when the r-th failure occurs in which case we would observe exactly r failures in the data. All developments pertaining to this Type-II HCS will be reviewed in Section 3.

In a direct comparison of Type-I and Type-II hybrid censoring schemes, we observe the following advantages and disadvantages in them:

TYPE-I HCS: In this case, the termination time is pre-fixed, which is clearly an advantage from an experimenter's point of view. However, if the mean lifetime of experimental units is somewhat larger than the pre-fixed time T, then with high probability, far fewer failures than the pre-fixed r would be observed before time T. This will definitely have an adverse effect on the efficiency of inferential procedures based on Type-I HCS.

TYPE-II HCS: In this case, the termination time is a random variable, which is clearly a disadvantage from the experimenter's point of view. On the other hand, at least r failures and possibly more than r failures would be observed by the termination time, and this will result in efficient inferential procedures in this case.

To overcome some of the drawbacks in these schemes, Chandrasekar et al. (2004) introduced two extensions, and called them *generalized Type-I and Type-II hybrid censoring schemes*. We will review in Section 4 all developments relating to these generalized Type-I HCS and Type-II HCS. Recently, Balakrishnan et al. (2008) combined all these censoring schemes, and introduced the so-called *unified hybrid censoring scheme*. Details on this unified hybrid censoring scheme and related inferential issues will be reviewed in Section 5.

In many life-testing experiments, quite often there may be more than one risk factor that could cause the failure of units. In such situations, an investigator would naturally be interested in the assessment of a specific risk factor in the presence of all other risk factors. Such a scenario is referred to in the literature as the *competing risks* problem. Analysis of data from hybrid life-testing experiments in the presence of competing risks was discussed by Kundu and Gupta (2007), and this will be the focus of discussion in Section 6.

Recently, *progressive censoring* has been discussed quite extensively in the literature as a very flexible censoring scheme as it allows for the removal of live experimental units at various intermittent times during the experiment in addition to removal at the termination of the experiment. For an elaborate treatment on the theory, methods and applications of progressive censoring, one may refer to the monograph of Balakrishnan and Aggarwala (2000) and the recent review article by Balakrishnan (2007). Hybrid censoring schemes have been introduced in the context of progressive censoring as well. Inferential results have been developed by Kundu and Joarder (2006) and Childs et al. (2008) based on such *progressive hybrid censoring schemes*, and these results will be reviewed in Section 7.

Ng et al. (2009) and Lin et al. (2009) have proposed *adaptive progressive hybrid censoring schemes* and then developed inferential methods for the unknown parameter(s) of exponential and Weibull distributions, respectively. Such adaptive progressive hybrid censoring schemes allow the experimenter to modify the censoring scheme adaptively during the life-testing experiment. The corresponding models and inferential results will be reviewed in Section 8.

Often, in reliability and life-testing experiments, one is interested in the effects of varying stress factors, such as temperature, voltage and load, on the lifetimes of experimental units. Since many modern products are highly reliable and hence a life-test under normal condition would tend to last a long time, experimenters often resort to an *accelerated life-testing* (ALT) experiment. Such experiments allow the experimenter to obtain adequate data for the items under accelerated stress conditions, which cause the items to fail much more quickly than under normal operating conditions. For some key references in the area of accelerated testing, one may refer to Nelson (1990), Meeker and Escobar (1998), Bagdonavicius and Nikulin (2002), and the references cited therein. A special class of the ALT is called *step-stress testing*, which allows the experimenter to change the stress level in multiple incremental steps during the experimentation. Balakrishnan and Xie (2007a,b) developed statistical inference for the model parameters in a simple step-stress model (meaning only two stress levels) with Type-I and Type-II hybrid censored data from an exponential distribution. We shall review these developments in Section 9. Some other related issues such as determination of optimal hybrid censoring plans, construction of acceptance sampling plans, testing of exponentiality based on hybrid censored data, and Bayesian prediction of lifetimes will all be discussed briefly in Section 10.

Finally, in Section 11, we present a few numerical examples in order to illustrate some of the results described here and also to highlight the usefulness of some of the forms of hybrid censoring scheme discussed in this article.

During the course of our discussion, we shall also point out several open problems and possible directions for future work. Through out the article, unless otherwise mentioned, it is assumed that n identical units are placed simultaneously on a life-testing experiment at time zero, and that the failed units are not replaced.

2 Type-I HCS

2.1 Form of Data

In this section, we assume that the data are Type-I hybrid censored, i.e., we have one of the following two forms of observations as our observed data:

Case I: $\{x_{1:n} < \cdots < x_{r:n}\}$ if $x_{r:n} \le T$;

Case II: $\{x_{1:n} < \cdots < x_{D:n}\}$ if $x_{r:n} > T$,

where D denotes the number of failures observed before time T. In the rest of this section, we consider different distributions for the lifetimes of the units and then describe pertinent inferential methods.

2.2 Exponential Distribution

2.2.1 MLE and its Exact Distribution

Suppose the lifetimes of the units under test are i.i.d. exponential random variables with probability density function (PDF)

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x \ge 0, \quad \theta > 0. \tag{1}$$

Then, based on the observed sample, it can be easily seen [see Chen and Bhattacharyya (1988)] that the MLE of θ does not exist if $d = 0$, and if $d > 0$, it is given by

$$\widehat{\theta} = \begin{cases} \frac{1}{d}\left\{\sum_{i=1}^{d} x_{i:n} + (n-d)T\right\} & \text{if} \quad x_{r:n} > T \\ \frac{1}{r}\left\{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}\right\} & \text{if} \quad x_{r:n} \le T. \end{cases} \tag{2}$$

Clearly, $\widehat{\theta}$ is the conditional MLE of θ, conditioned on having observed at least one failure, and it has a bounded support. Chen and Bhattacharyya (1988) derived the moment generating function of $\widehat{\theta}$ by using the expression

$$M_{\widehat{\theta}}(\omega) = E_\theta(e^{\omega\widehat{\theta}}) = \sum_{d=1}^{r-1} E_\theta(e^{\omega\widehat{\theta}} \mid D = d) P_\theta(D = d) + \sum_{d=r}^{n} E_\theta(e^{\omega\widehat{\theta}} \mid D = d) P_\theta(D = d), \tag{3}$$

where $P_\theta(D = d)$ is the conditional probability that $D = d$, given $D \ge 1$. The expression obtained by Chen and Bhattacharyya (1988) is quite complicated, and an equivalent but simplified expression has been obtained by Childs et al. (2003) as follows:

$$M_{\widehat{\theta}}(\omega) \;=\; (1-q^n)^{-1}\left[\sum_{d=1}^{r-1}\binom{n}{d}\frac{q^{(n-d)(1-\theta\omega/d)}}{(1-\omega\theta/d)^d}(1-q^{(1-\theta\omega/d)})^d + r\binom{n}{r}(1-\theta\omega/r)^{-r}\right.$$

$$\times \sum_{k=0}^{r-1} \frac{(-1)^k}{(n-r+k+1)} \binom{r-1}{k} (1 - q^{(1-\theta\omega/r)(n-r+k+1)}) \Bigg], \quad \omega < \frac{1}{\theta}, \tag{4}$$

where $q = e^{-T/\theta}$. Now, upon using the moment generating functions of gamma and shifted gamma distributions and by inverting (4), Childs et al. (2003) obtained the exact conditional PDF of $\widehat{\theta}$ as

$$\begin{aligned} f_{\widehat{\theta}}(x) &= (1-q^n)^{-1} \left[\sum_{d=1}^{r-1} \sum_{k=0}^{d} C_{k,d}\, g\left(x - T_{k,d}; \frac{d}{\theta}, d\right) + g\left(x; \frac{r}{\theta}, r\right) \right. \\ &\left. + r \binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k q^{n-r+k}}{n-r+k} \binom{r-1}{k-1} g\left(x - T_{k,r}; \frac{r}{\theta}, r\right) \right], \quad 0 < x < nT, \end{aligned} \tag{5}$$

where $C_{k,d} = (-1)^k \binom{n}{d}\binom{d}{k} q^{n-d+k}$, $T_{k,d} = (n-d+k)T/d$, and $g(x; \lambda, \alpha)$ is the two-parameter gamma density given by

$$g(x;\lambda,\alpha) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \le 0. \end{cases} \tag{6}$$

It is clear from (5) that the conditional PDF of $\widehat{\theta}$ is a generalized mixture of gamma and shifted gamma densities, in which the mixing coefficients may also be negative. The corresponding cumulative distribution function (CDF) and the survival function (SF) of $\widehat{\theta}$ can be obtained readily from (5) in terms of incomplete gamma function. The SF of $\widehat{\theta}$ is presented below since it is useful in constructing exact confidence intervals for θ:

$$\begin{aligned} P_\theta(\widehat{\theta} > b) &= (1-q^n)^{-1} \left[\sum_{d=1}^{r-1} \sum_{k=0}^{d} \frac{C_{k,d}}{(d-1)!} \Gamma(d, A_d(T_{k,d})) + \frac{\Gamma(r, rb/\theta)}{(r-1)!} \right. \\ &+ \left. \frac{r}{(r-1)!} \binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k q^{n-r+k}}{n-r+k} \binom{r-1}{k-1} \Gamma(r, A_r(T_{k,r})) \right], \end{aligned} \tag{7}$$

where $A_k(a) = \frac{k}{\theta} < b - a >$, $< x >= \max\{x, 0\}$, and $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the upper incomplete gamma function.

Because of the explicit expression of the PDF of the MLE $\widehat{\theta}$ in (5), moments of $\widehat{\theta}$ can be easily obtained; see Chen and Bhattacharyya (1988). It is observed that the MLE of θ is a biased estimate of θ. From the PDF in (5), it can be easily observed that when $T \to \infty$,

$f_{\widehat{\theta}}(x) = g\left(x; \frac{r}{\theta}, r\right)$, which yields the well-known result in this case that $\frac{2r\widehat{\theta}}{\theta}$ has a chi-square distribution with $2r$ degrees of freedom; see Balakrishnan and Cohen (1991). Observe in this case that the Type-I HCS simply reduces to a conventional Type-II censored sample. Also, if we set $r = n$, we simply deduce the PDF of the MLE of $\widehat{\theta}$ for the conventional Type-I censoring case as was originally derived by Bartholomew (1963).

2.2.2 Exact Confidence Intervals

In this section, we present two different exact confidence intervals for the parameter θ. Fairbanks et al. (1982) proposed a set of two-sided 100(1-α)% confidence intervals for θ in the Type-I HCS as follows:

$$\begin{array}{lll} \left[\frac{2S}{\chi^2_{2,\alpha/2}}, \infty\right] & \text{if} & d = 0, \\ \left[\frac{2S}{\chi^2_{2d+2,\alpha/2}}, \frac{2S}{\chi^2_{2d,1-\alpha/2}}\right] & \text{if} & 1 \le d \le r-1, \\ \left[\frac{2S}{\chi^2_{2r,\alpha/2}}, \frac{2S}{\chi^2_{2r,1-\alpha/2}}\right] & \text{if} & d = r, \end{array} \tag{8}$$

where S is the total time on test given by

$$S = \begin{cases} \sum_{i=1}^{d} x_{i:n} + (n-d)T & \text{if} \quad x_{r:n} > T, \\ \sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n} & \text{if} \quad x_{r:n} \le T, \end{cases} \tag{9}$$

and $\chi^2_{m,\beta}$ denotes the upper β percentage point of a chi-square distribution with m degrees of freedom. Fairbanks et al. (1982) provided a formal proof in the 'with replacement' case, and mentioned that the proof can be extended for the 'without replacement' case as well. It is not clear how the proof for the 'without replacement' case will proceed since one of the key assumptions in their proof for the 'with replacement' case is that, for $0 \le j \le r-1$,

$$P\{j \text{ items fail by the decision time}\} = \frac{e^{-nT/\theta}\,(nT/\theta)^j}{j!}. \tag{10}$$

Clearly, (10) will not be valid for the 'without replacement' case. Thus, the problem for the 'without replacement' case appears to be an open one!

Now, we describe the exact two-sided confidence interval for θ by using the exact distribution of $\widehat{\theta}$, as proposed by Chen and Bhattacharyya (1988), based on the assumption that $P_\theta(\widehat{\theta} > b)$ is an increasing function of θ for fixed b, although they did not provide a formal proof for this. A formal proof of the monotonicity of this tail probability has been given recently by Balakrishnan and Iliopoulos (2009). If (θ_L, θ_U) denotes the exact two-sided 100(1-α)% symmetric confidence interval for θ, then the limits θ_L and θ_U can be obtained by solving the following two non-linear equations:

$$\begin{aligned} \frac{\alpha}{2} &= (1-e^{-nT\theta_L})^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d}\frac{C_{k,d}}{(d-1)!}\Gamma\left(d, \frac{k}{\theta_L} < \widehat{\theta} - T_{k,d} >\right) + \frac{\Gamma(r, r\widehat{\theta}/\theta_L)}{(r-1)!}\right. \\ &\left. + \frac{r}{(r-1)!}\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k e^{-T\theta_L(n-r+k)}}{n-r+k}\binom{r-1}{k-1}\Gamma\left(r, \frac{r}{\theta_L} < \widehat{\theta} - T_{k,r} >\right)\right] \end{aligned} \quad (11)$$

and

$$\begin{aligned} 1-\frac{\alpha}{2} &= (1-e^{-nT\theta_U})^{-1}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d}\frac{C_{k,d}}{(d-1)!}\Gamma\left(d, \frac{k}{\theta_U} < \widehat{\theta} - T_{k,d} >\right) + \frac{\Gamma(r, r\widehat{\theta}/\theta_U)}{(r-1)!}\right. \\ &\left. + \frac{r}{(r-1)!}\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k e^{-T\theta_U(n-r+k)}}{n-r+k}\binom{r-1}{k-1}\Gamma\left(r, \frac{r}{\theta_U} < \widehat{\theta} - T_{k,r} >\right)\right]. \end{aligned} \quad (12)$$

Since these equations do not admit explicit solutions, the confidence limits θ_L and θ_U need to be computed numerically by solving the above two non-linear equations in (11) and (12).

2.2.3 Approximate Methods

Since the computation of the above exact confidence interval is somewhat involved and poses a problem when n is large, Gupta and Kundu (1998) proposed an approximate confidence interval by using the asymptotic normality of the MLE $\widehat{\theta}$ and the observed Fisher information. Specifically, the distribution of $\widehat{\theta}$ for large n is approximately normal with mean θ and variance $\left(\frac{S}{\widehat{\theta}^3} - \frac{d}{\widehat{\theta}^2}\right)$. Using this, an approximate 100(1-α)% confidence interval for θ can be easily constructed.

It should be mentioned that parametric and non-parametric bootstrap can also be used effectively for constructing approximate confidence intervals. Comparative comments on all these methods are made later in Section 2.2.6.

2.2.4 Fisher Information

Park et al. (2008) derived the Fisher information for Type-I HCS by following an approach used by Wang and He (2005). In the case of exponential distribution with mean θ, the Fisher information in Type-I hybrid censored data can be expressed as a sum of single integrals as

$$I_1(\theta) = \frac{1}{\theta^2} \sum_{i=1}^{r} F_i\left(\frac{T}{\theta}\right), \tag{13}$$

where

$$F_i(x) = \int_{e^{-x}}^{1} \frac{n!}{(i-1)!(n-i)!} t^{n-i}(1-t)^{i-1} dt \quad \text{for } x > 0. \tag{14}$$

These authors also used this expression to compare the Fisher information contained in different Type-I HCS. Moreover, this expression can also be utilized to construct confidence intervals for θ by using $I_1(\widehat{\theta})$.

2.2.5 Bayesian Inference

Draper and Guttman (1987) considered the Bayesian inference for the parameter θ based on a Type-I hybrid censored data. For this purpose, they assumed that θ has an inverted gamma prior with PDF

$$\pi(\theta) = \frac{\lambda^\beta}{\Gamma(\beta)} \theta^{-(\beta+1)} e^{-\lambda/\theta}, \quad \theta > 0, \tag{15}$$

where $\beta > 0$ and $\lambda > 0$ are the hyper-parameters. If $\beta = \lambda = 0$, then the prior in (15) becomes a non-informative prior. Based on the inverted gamma prior in (15), and the observed data, the posterior density function of θ becomes

$$l(\theta|Data) = \frac{(S_d + \lambda)^{d+\beta}}{\Gamma(d+\beta)} \theta^{-(d+\beta+1)} e^{-(S_d+\lambda)/\theta}, \tag{16}$$

where

$$S_d = \begin{cases} nT & \text{if} \quad d = 0 \\ \sum_{i=1}^{d} x_{i:n} + (n-d)T & \text{if} \quad 1 \le d \le r-1 \\ \sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n} & \text{if} \quad d = r. \end{cases} \tag{17}$$

Since $(S_d + \lambda)/\theta$ is distributed as $\chi^2_{2(\beta+\lambda)}/2$ a posteriori, the Bayesian estimator of θ under the squared-error loss function, being the posterior mean, is simply obtained as

$$\widehat{\theta}_{Bayes} = \frac{(S_d + \lambda)}{(d + \beta - 1)} \tag{18}$$

provided $d+\beta > 1$. It is clear that the MLE of θ presented in (2) coincides with the Bayesian estimator in (18) when $\beta = \lambda = 0$, i.e., when the prior becomes non-informative. Moreover, a 100(1-α)% credible interval for θ is easily obtained as

$$\left(\frac{2(S_d + \lambda)}{\chi^2_{2(d+\beta),\alpha/2}}, \frac{2(S_d + \lambda)}{\chi^2_{2(d+\beta),1-\alpha/2}} \right) \tag{19}$$

provided $d + \beta > 0$, where $\chi^2_{m,\alpha}$ denotes as before the upper α percentage point of the χ^2 distribution with m degrees of freedom.

2.2.6 Comparison of Different Methods

Gupta and Kundu (1998) carried out an extensive Monte Carlo simulation study to evaluate the performance of all these methods of inference. First of all, the confidence intervals obtained from the exact distribution perform very well in terms of the width of confidence intervals as well as in terms of coverage probabilities, even though it is computationally quite involved. The performance of the Bayesian credible intervals based on a non-informative prior is not very satisfactory, especially when the pre-fixed time T is small compared to θ. The Bayesian credible intervals based on informative priors work well, but selecting the proper prior (i.e., a specific choice for the hyper-parameters) is always a difficult problem. For moderate or large values of n and when T is not particularly small, confidence intervals for θ based on the asymptotic distribution of the MLE work quite well.

2.3 Two-Parameter Exponential Distribution

Recently, Childs et al. (2012) discussed likelihood inference for the parameters of a two-parameter exponential distribution when the data are Type-I hybrid censored. With the PDF of the two-parameter exponential distribution as

$$f(x;\mu,\theta) = \frac{1}{\theta}e^{-(x-\mu)/\theta}, \quad x > \mu, \tag{20}$$

where $-\infty < \mu < \infty$ and $\theta > 0$ are the threshold and scale parameters, respectively. Based on the observed sample as provided in Section 2.1, it has been observed by Childs et al. (2012) that the MLEs of θ and μ do not exist when $d = 0$. When $r = 1$, the MLEs of θ and μ are $\widehat{\theta} = 0$ and $\widehat{\mu} = x_{1:n}$, respectively. For $r > 1$,

$$\widehat{\mu} = x_{1:n}$$

and

$$\widehat{\theta} = \begin{cases} \frac{1}{d}\left[\sum_{i=1}^{d}(x_{i:n} - x_{1:n}) + (n-d)(T - x_{1:n})\right] & \text{if} \quad d = 1, 2, \cdots, r-1 \\ \frac{1}{r}\left[\sum_{i=1}^{r}(x_{i:n} - x_{1:n}) + (n-r)(x_{r:n} - x_{1:n})\right] & \text{if} \qquad d = r. \end{cases} \tag{21}$$

From the joint moment generating function, the conditional marginal PDF of $(\widehat{\mu} - \mu)/\theta$ can be obtained as

$$\begin{aligned} f_{\frac{\widehat{\mu}-\mu}{\theta}}(x) \;=\; & (1-q^n)^{-1}\left[\sum_{j=1}^{r-1}\sum_{k=0}^{j-1} C_{j,k}\, g(x; j-k, 1) + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1} B_{k,j}\, g\left(x - \frac{T-\mu}{\theta}; j-k, 1\right)\right. \\ & + g(x; n, 1) - q^n g\left(x - \frac{T-\mu}{\theta}; n, 1\right) + \sum_{k=1}^{r-1} D_k\, g\left(x - \frac{T-\mu}{\theta}; k, 1\right) \\ & \left. + \sum_{k=1}^{r-1} E_k\, g(x; k, 1)\right], \end{aligned} \tag{22}$$

where

$$\begin{aligned} B_{j,k} &= (-1)^{k+1}\binom{n}{j}\binom{j}{k}q^n, \quad j = 1, 2, \cdots r-1, \; k = 0, \cdots, j-1 \\ D_k &= r\binom{n}{r}\frac{1}{n-k}(-1)^{r+k-1}\binom{r-1}{k}q^n, \quad k = 1, 2, \cdots, r-1, \end{aligned}$$

$$E_k = r\binom{n}{r}\frac{1}{n-k}(-1)^{r+k}\binom{r-1}{k}q^{n-k}, \quad k = 1, 2, \cdots, r-1,$$

with q, $C_{j,k}$ and $g(\cdot)$ being the same as defined earlier. Moreover, the conditional marginal PDF of $\widehat{\theta}/\theta$ is

$$\begin{aligned} f_{\frac{\widehat{\theta}}{\theta}}(x) &= (1-q^n)^{-1}\left[\sum_{j=1}^{r-1}\sum_{k=0}^{j-1} C_{j,h} h_1(x - T_{j,k}; j, j-1, \alpha_j, k)\right. \\ &+ \sum_{j=1}^{r-1}\sum_{k=0}^{j-1} B_{j,k} h_1(x; j, j-1, \alpha_{j,k}) + (1-q^n) g(x; r, r-1) \\ &+ \left.\sum_{k=1}^{r-1} D_k h_1(x; r, r-1, \alpha_k) + \sum_{k=1}^{r-1} E_k h_1(x - T_k; r, r-1, \alpha_k)\right], \end{aligned} \tag{23}$$

where

$$h_1(x; \alpha, p, \beta) = \sum_{k=0}^{p-1} p_k g(x; \alpha, p-k) + \left(1 - \sum_{k=0}^{p-1} p_k\right) g(-x, \beta, 1), \quad -\infty < x < \infty,$$

and the rest of the quantities are as defined earlier. The construction of the exact confidence intervals can be done in an analogous manner as before. The computation of these exact confidence intervals is quite involved in this case, however.

Recently, Kundu et al. (2011) considered the Bayesian inference for the unknown parameters θ and μ of the two-parameter exponential model in (20). It is assumed that $\lambda = \dfrac{1}{\theta}$ has a gamma prior with the shape and scale parameters as a and b, respectively. In addition, it is assumed that μ has a non-proper uniform prior over the whole real line and that μ and λ are independently distributed. Based on the above priors, the posterior density function of λ and μ can be written as follows:

$$l(\mu, \lambda|\text{Data}) \propto \lambda^{d^*+a-1} e^{-\lambda(b + \sum_{i=1}^{d^*}(x_{i:n}-\mu) + (n-d^*)(U-\mu))}; \quad \lambda > 0, \ \mu < x_{1:n}. \tag{24}$$

Here for Case I, $d^* = r$, and $U = x_{r:n}$, and for Case II, $d^* = d$, $0 \le d \le r-1$, and $U = T$. The Bayes estimate of any function of λ and μ cannot be obtained in explicit form, in general.

For this reason, the authors proposed to generate samples from the joint posterior density function in (24) and to use them to compute the Bayes estimate of any function of λ and μ, and also to construct associated HPD credible intervals.

2.4 Weibull Distribution

2.4.1 MLEs

In this section, we assume that the lifetimes of the units are i.i.d. Weibull random variables with PDF

$$f(x;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-(x/\lambda)^{\alpha}}, \quad x>0, \tag{25}$$

where $\alpha>0$ and $\lambda>0$ are the shape and scale parameters, respectively. Then, based on a Type-I HCS, the likelihood functions for Cases I and II are

$$L(\alpha,\lambda) = \left(\frac{\alpha}{\lambda}\right)^{r} \prod_{i=1}^{r}\left(\frac{x_{i:n}}{\lambda}\right)^{\alpha-1} e^{-\left\{\sum_{i=1}^{r}(x_{i:n}/\lambda)^{\alpha}+(n-r)(x_{r:n}/\lambda)^{\alpha}\right\}} \tag{26}$$

and

$$L(\alpha,\lambda) = \begin{cases} \left(\frac{\alpha}{\lambda}\right)^{d}\prod_{i=1}^{d}\left(\frac{x_{i:n}}{\lambda}\right)^{\alpha-1} e^{-\left\{\sum_{i=1}^{d}(x_{i:n}/\lambda)^{\alpha}+(n-d)(T/\lambda)^{\alpha}\right\}} & \text{if} \quad d>0, \\ e^{-n(T/\lambda)^{\alpha}} & \text{if} \quad d=0, \end{cases} \tag{27}$$

respectively. The MLEs can be obtained by maximizing (26) or (27) with respect to the unknown parameters α and λ. Not surprisingly, the MLEs of α and λ can not be obtained explicitly. Kundu (2007) noted that the MLE of α can be obtained by finding the unique solution of a fixed-point type equation

$$h(\alpha) = \alpha, \tag{28}$$

where, for Case I

$$h(\alpha) = r\left[-\sum_{i=1}^{r}\ln x_{i:n} + \frac{1}{u(\alpha)}\left\{\sum_{i=1}^{r}x_{i:n}^{\alpha}\ln x_{i:n} + (n-r)x_{r:n}^{\alpha}\ln x_{r:n}\right\}\right]^{-1}, \tag{29}$$

and for Case II

$$h(\alpha) = d\left[-\sum_{i=1}^{d}\ln x_{i:n} + (1/u(\alpha))\left\{\sum_{i=1}^{d} x_{i:n}^{\alpha}\ln x_{i:n} + (n-d)T^{\alpha}\ln T\right\}\right]^{-1}, \tag{30}$$

with

$$u(\alpha) = \begin{cases} r\left[\sum_{i=1}^{r} x_{i:n}^{\alpha} + (n-r)x_{r:n}^{\alpha}\right]^{-1} & \text{for} \quad \text{Case I,} \\ d\left[\sum_{i=1}^{d} x_{i:n}^{\alpha} + (n-d)T^{\alpha}\right]^{-1} & \text{for} \quad \text{Case II.} \end{cases} \tag{31}$$

Kundu (2007) then suggested a very simple iterative scheme for finding $\widehat{\alpha}$, the MLE of α, from (28). Start with an initial guess of α, say $\alpha^{(0)}$, obtain $\alpha^{(1)} = h(\alpha^{(0)})$, and proceed in this manner to obtain $\alpha^{(k+1)} = h(\alpha^{(k)})$. Stop the iterative process when $|\alpha^{(k+1)} - \alpha^{(k)}| < \epsilon$, some pre-assigned tolerance level. Once $\widehat{\alpha}$ is obtained in this iterative manner, then $\widehat{\lambda}$, the MLE of λ, can be obtained as $\widehat{\lambda} = \{u(\widehat{\alpha})\}^{1/\widehat{\alpha}}$.

It should be noted here again that when $d = 0$, the MLEs do not exist. Since the MLEs are not available in an explicit form, Kundu (2007) proposed approximate maximum likelihood estimators (AMLEs) which are in explicit form, and these can therefore be used as initial values in the iterative procedure for the MLEs.

2.4.2 Approximate Maximum Likelihood Estimators

Suppose X has a Weibull distribution with PDF in (25), then $Y = \ln X$ has the extreme value distribution with PDF

$$f_Y(y;\mu,\sigma) = \frac{1}{\sigma}e^{(y-\mu)/\sigma - e^{(y-\mu)/\sigma}}, \quad -\infty < y < \infty, \tag{32}$$

where $\mu = \ln\lambda$ and $\sigma = 1/\alpha$ are the location and scale parameters. The density function in (32) is the so-called extreme value density. Thus, the Weibull density in (25) and the extreme value density in (32) are equivalent models in the sense that inferential procedures developed for one model can be easily adapted for the other model. Although they are equivalent models, it is sometimes easier to work with (32) since it belongs to the location-scale family of densities. Evidently, if estimates of μ and σ are obtained from the transformed data on $Y = \ln X$, then the corresponding estimates of the Weibull parameters α and λ can be obtained easily through the relations $\alpha = 1/\sigma$ and $\lambda = e^{\mu}$.

Let us denote $y_{i:n} = \ln x_{i:n}$ for $i = 1, \cdots, n$, and $\widetilde{T} = \ln T$. Then, the likelihood function for the observed data $y_{i:n}$ in Case I is given by

$$l(\mu, \sigma) \propto \frac{c}{\sigma^r} \prod_{i=1}^{r} g(z_{i:n}) \left\{ \bar{G}(z_{r:n}) \right\}^{n-r}, \tag{33}$$

and in Case II, it is given by

$$L(\mu, \sigma) \propto \frac{c}{\sigma^d} \prod_{i=1}^{d} g(z_{i:n}) \left\{ \bar{G}(V) \right\}^{n-d}, \tag{34}$$

where

$$g(z) = e^{z-e^z}, \quad \bar{G}(z) = e^{-e^z}, \quad z_{i:n} = \frac{y_{i:n} - \mu}{\sigma}, \quad V = \frac{\widetilde{T} - \mu}{\sigma}, \quad \mu = \ln \lambda, \quad \sigma = \frac{1}{\alpha}.$$

Let us first consider Case I. In this case, by taking the logarithm of the likelihood function in (33) and ignoring the constant, we obtain

$$l(\mu, \sigma) = \ln L(\mu, \sigma) = -r \ln \sigma + \sum_{i=1}^{r} \ln g(z_{i:n}) + (n-r) \ln \bar{G}(z_{r:n}). \tag{35}$$

By taking derivatives of (35) with respect to μ and σ, we obtain the two likelihood equations as

$$-\sum_{i=1}^{r} \frac{g'(z_{i:n})}{g(z_{i:n})} + (n-r)\frac{g(z_{r:n})}{\bar{G}(z_{r:n})} = 0, \tag{36}$$

$$-r - \sum_{i=1}^{r} \frac{g'(z_{i:n}) z_{i:n}}{g(z_{i:n})} + (n-r)\frac{g(z_{r:n}) z_{r:n}}{\bar{G}(z_{r:n})} = 0. \tag{37}$$

Since (36) and (37) do not have explicit solutions, let us expand the function $g'(z_{i:n})/g(z_{i:n})$ and $g(z_{r:n})/\bar{G}(z_{r:n})$ in Taylor series around the points $G^{-1}(p_i) = \ln(-\ln q_i) = \mu_i$ (say) and $G^{-1}(p_r) = \ln(-\ln q_r) = \mu_r$, where $p_i = i/(n+1)$, $q_i = 1 - p_i$ for $i = 1, \cdots, r$, as done earlier by Balakrishnan and Varadhan (1991), Balasooriya and Balakrishnan (2000), and

Balakrishnan et al. (2004). Now, upon using the approximations

$$\frac{g'(z_{i:n})}{g(z_{i:n})} \approx \alpha_i - \beta_i z_{i:n},$$
$$\frac{g(z_{r:n})}{\bar{G}(z_{r:n})} \approx 1 - \alpha_r + \beta_r z_{r:n},$$

Eqs. (36) and (37) can be approximated as

$$-\sum_{i=1}^{r}(\alpha_i - \beta_i z_{i:n}) + (n-r)(1-\alpha_r+\beta_r z_{r:n}) = 0, \tag{38}$$

$$-r - \sum_{i=1}^{r}(\alpha_i - \beta_i z_{i:n})z_{i:n} + (n-r)(1-\alpha_r+\beta_r z_{r:n})z_{r:n} = 0, \tag{39}$$

where $\alpha_i = 1 + \ln q_i(1 - \ln(-\ln q_i))$ and $\beta_i = -\ln q_i$, for $i = 1, \cdots, r$. From Eqs. (38) and (39), explicit solutions for μ and σ, say $\widetilde{\mu}$ and $\widetilde{\sigma}$, can be obtained as

$$\widetilde{\mu} = A - B\widetilde{\sigma} \quad \text{and} \quad \widetilde{\sigma} = \frac{-D + \sqrt{D^2 + 4rE}}{2r}, \tag{40}$$

where

$$A = \frac{\sum_{i=1}^{r}\beta_i y_{i:n} + \beta_r(n-r)y_{r:n}}{\sum_{i=1}^{r}\beta_i + \beta_r(n-r)}, \quad B = \frac{\sum_{i=1}^{r}\alpha_i - (n-r)(1-\alpha_r)}{\sum_{i=1}^{r}\beta_i + \beta_r(n-r)},$$

$$D = \sum_{i=1}^{r}\alpha_i(y_{i:n} - A) - (n-r)(1-\alpha_r)(y_{r:n} - A) - 2B\sum_{i=1}^{r}\beta_i(y_{i:n} - A) - 2(n-r)\beta_r B(y_{r:n} - A)$$

and

$$E = \sum_{i=1}^{r}\beta_i(y_{i:n} - A)^2 + (n-r)\beta_r(y_{r:n} - A)^2;$$

see Kundu (2007) for detailed derivations. The estimators $\widetilde{\mu}$ and $\widetilde{\sigma}$ in (40) are said to be the AMLEs of μ and σ, respectively. The AMLEs of α and λ, say $\widetilde{\alpha}$ and $\widetilde{\lambda}$ respectively, are then readily obtained as

$$\widetilde{\alpha} = \frac{1}{\widetilde{\sigma}} \quad \text{and} \quad \widetilde{\lambda} = e^{\widetilde{\mu}}. \tag{41}$$

Next, for Case II, the AMLEs of μ and σ are obtained to be same as in (40), but with A, B, D and E defined as follows:

$$A = \frac{\sum_{i=1}^{d}\beta_i y_{i:n} + \beta_{d^*}(n-d)\widetilde{T}}{\sum_{i=1}^{d}\beta_i + \beta_{d^*}(n-d)}, \quad B = \frac{\sum_{i=1}^{d}\alpha_i - (n-d)(1-\alpha_{d^*})}{\sum_{i=1}^{d}\beta_i + \beta_{d^*}(n-d)},$$

$$D = \sum_{i=1}^{d} \alpha_i (y_{i:n} - A) - (n-d)(1-\alpha_{d^*})(\widetilde{T} - A) - 2B \sum_{i=1}^{d} \beta_i (\widetilde{T} - A) - 2(n-d)\beta_{d^*} B(\widetilde{T} - A)$$

and

$$E = \sum_{i=1}^{d} \beta_i (\widetilde{T} - A)^2 + (n-d)\beta_{d^*} (\widetilde{T} - A)^2.$$

2.4.3 Confidence Intervals

In this case, it is not possible to construct exact confidence intervals for the scale and shape parameters. However, for known shape parameter, after transforming the data, the exact distributional result for the exponential case can be exploited to develop exact confidence intervals for the scale parameter. But when both parameters are unknown, it is possible to use the asymptotic distribution of the MLEs to construct approximate confidence intervals. Through Monte Carlo simulation experiments, Kundu (2007) has shown that the confidence intervals based on the asymptotic distribution perform well in terms of coverage probabilities when the sample size is at least moderate. Alternatively, parametric as well as nonparametric bootstrap methods can also be used for constructing confidence intervals.

2.4.4 Bayesian Inference

We now develop Bayesian estimates for the unknown parameters and also describe the construction of highest posterior density credible intervals for the parameters. We reparametrize the model by using $\theta = 1/\lambda^{\alpha}$, and with this parametrization, we consider the Bayesian estimates of α and θ.

Prior Distribution: When the shape parameter is known, the scale parameter θ has a conjugate gamma prior; see, for example, Zhang and Meeker (2005). So, it is assumed that the prior distribution of θ is Gamma(a, b) with PDF

$$\pi_1(\theta|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta} & \text{if} \quad \theta > 0, \\ 0 & \text{if} \quad \theta \le 0, \end{cases} \tag{42}$$

with hyper-parameters $a > 0$ and $b > 0$. In most practical applications, with proper information on the Weibull scale parameter, the prior variance is usually finite.

Now, let us consider the more practical case when the shape parameter is also unknown. It is known in this case that the Weibull distribution does not have a continuous conjugate prior, although there exists a continuous-discrete joint prior distribution; see Soland (1969). The continuous component of this prior distribution is related to the scale parameter, while the discrete part is related to the shape parameter. This model has been widely criticized due to its difficulty in applying to real-life situations [see Kaminskiy and Krivtosov (2005)], and so it has not received much attention in the literature.

Following the approach of Berger and Sun (1993), Kundu (2007) assumed the same prior on θ, as given in (42), but no specific form of prior, $\pi_2(\alpha)$, on α is assumed. It is assumed that the support of $\pi_2(\alpha)$ is $(0, \infty)$, it has a log-concave PDF, and that it is independent of $\pi_1(\theta)$. Based on these assumptions, the posterior density function of α and θ can be written as

$$l(\alpha, \theta|\text{data}) = \frac{l(\text{data}, \alpha, \theta)}{\int_0^\infty \int_0^\infty l(\text{data}, \alpha, \theta) d\alpha d\theta}; \tag{43}$$

here, for Case I, we have

$$l(\text{data}, \alpha, \theta) \propto \alpha^r \theta^{a+r-1} \prod_{i=1}^{r} x_{i:n}^{\alpha-1} e^{-\theta\left[\sum_{i=1}^{r} x_{i:n}^{\alpha} + (n-r)x_{r:n}^{\alpha} + b\right]} \pi_2(\alpha),$$

and for Case II, we have

$$l(\text{data}, \alpha, \theta) \propto \alpha^d \theta^{a+d-1} \prod_{i=1}^{d} x_{i:n}^{\alpha-1} e^{-\theta\left[\sum_{i=1}^{d} x_{i:n}^{\alpha} + (n-d)T^{\alpha} + b\right]} \pi_2(\alpha) \quad \text{if} \;\; d > 0,$$

$$l(\text{data}, \alpha, \theta) \propto \theta^{a-1} e^{-\theta[nT^{\alpha} + b]} \pi_2(\alpha) \quad \text{if} \;\; d = 0.$$

Hence, the Bayesian estimate of any function of α and θ, say $g(\alpha, \theta)$, with respect to the squared error loss function, can be obtained as the posterior mean given by

$$\widehat{g}_{Bayes}(\alpha, \theta) = \int_0^\infty \int_0^\infty g(\alpha, \theta) l(\alpha, \theta|\text{data}) d\alpha d\theta. \tag{44}$$

It is clear that even if an explicit form is assumed on $\pi_2(\alpha)$, it may not be possible to compute (44) in most cases. For this reason, Kundu (2007) proposed to use the Gibbs sampling procedure for computing the Bayesian estimate as well as for constructing the HPD credible interval of any function of α and θ.

The following results are useful in generating samples from the posterior distribution function given in (43); see Kundu (2007) for their proofs.

RESULT 1: The conditional density function of θ, given α and data, for Case I is

$$\pi_1(\theta|\alpha, \text{data}) = \text{Gamma}\left(a + r, \sum_{i=1}^{r} x_{i:n}^\alpha + (n - r)x_{r:n}^\alpha + b\right),$$

while for Case II

$$\pi_1(\theta|\alpha, \text{data}) = \text{Gamma}\left(a + d, \sum_{i=1}^{d} x_{i:n}^\alpha + (n - d)T^\alpha + b\right)$$

if $d > 0$, and

$$\pi_1(\theta|\alpha, \text{data}) = \text{Gamma}(a, nT^\alpha + b)$$

if $d = 0$. ∎

RESULT 2: The conditional PDF of α, given the data, is as follows: For Case I,

$$l(\alpha|\text{data}) \propto \frac{\alpha^r \pi_2(\alpha) \prod_{i=1}^r x_{i:n}^{\alpha-1}}{\left(\sum_{i=1}^r x_{i:n}^\alpha + (n - r)x_{r:n}^\alpha + b\right)^{a+r}}, \tag{45}$$

and for Case II,

$$l(\alpha|\text{data}) \propto \frac{\alpha^d \pi_2(\alpha) \prod_{i=1}^d x_{i:n}^{\alpha-1}}{\left(\sum_{i=1}^d x_{i:n}^\alpha + (n - d)T^\alpha + b\right)^{a+d}} \tag{46}$$

if $d > 0$, and

$$l(\alpha|\text{data}) \propto \frac{\pi_2(\alpha)}{(nT^\alpha + b)^a} \tag{47}$$

if $d = 0$. Moreover, (45), (46) and (47) are all log-concave. ∎

It may be mentioned that the general method proposed by Devroye (1984) can be used to generate samples from any log-concave density function. Now, by employing the idea of Geman and Geman (1984), the following algorithm can be used to generate samples from the joint posterior density function of α and θ.

ALGORITHM:

- Step 1: Generate α_1 from the log-concave density function $l(\alpha|\text{data})$, as given in (45), (46) or (47) depending on the situation;
- Step 2: Generate θ_1 from $\pi_1(\theta|\alpha, \text{data})$ as provided in Result 1;
- Step 3: Repeat Steps 1 and 2 M times to generate $(\alpha_1, \theta_1), \cdots, (\alpha_M, \theta_M)$.

Note that once we have generated samples from the posterior density function, they can be used in a simple manner to determine the Bayes estimate or the associated credible interval for any function of α and θ.

2.5 Log-Normal Distribution

2.5.1 MLEs

In this section, we assume the lifetimes of the units are to be i.i.d. log-normal random variables with parameters as μ and σ, and with PDF

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0, \quad -\infty < \mu < \infty, \quad \sigma > 0. \tag{48}$$

It is well known that if a random variable X has a log-normal distribution with PDF in (48), then $Y = \ln X$ has a normal distribution with mean μ and standard deviation σ. First, we discuss the maximum likelihood estimation of the parameters μ and σ, as developed recently by Dube et al. (2011).

Based on the observed data, the likelihood functions for Cases I and II can be expressed as follows:

$$L(\mu,\sigma) \propto \left(\frac{1}{\sigma}\right)^R \prod_{i=1}^{R} exp\left\{-\frac{(\ln x_{i:n}-\mu)^2}{2\sigma^2}\right\}\left\{1-\Phi\left(\frac{c-\mu}{\sigma}\right)\right\}^{n-R}, \tag{49}$$

where

$$R = \begin{cases} r & \text{for} \quad \text{Case I} \\ \\ d & \text{for} \quad \text{Case II} \end{cases} \quad \text{and} \quad c = \begin{cases} \ln x_{r:n} & \text{for} \quad \text{Case I} \\ \\ \ln T & \text{for} \quad \text{Case II} \end{cases},$$

and $\Phi(\cdot)$ is the CDF of the standard normal distribution.

Note that if $d = 0$, the likelihood function is simply

$$L(\mu,\sigma) = \left\{1-\Phi\left(\frac{c-\mu}{\sigma}\right)\right\}^{n}. \tag{50}$$

Clearly, from (50), we observe in this case that the MLEs of μ and σ do not exist since L increases with both μ and σ. Hence, from now on, we shall assume that $d > 0$. Taking the logarithm of the likelihood function, and setting $y_{i:n} = \ln x_{i:n}$, we obtain the log-likelihood function as

$$l(\mu,\sigma) = -R\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{R}(y_{i:n}-\mu)^2 + (n-R)\ln\left\{1-\Phi\left(\frac{c-\mu}{\sigma}\right)\right\}. \tag{51}$$

Upon taking derivatives of (51) with respect to μ and σ, the likelihood equations are obtained as

$$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{R}(y_{i:n}-\mu) + (n-R)\frac{\phi(z^*)}{\sigma\Phi(-z^*)} = 0, \tag{52}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{R}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{R}(y_{i:n}-\mu)^2 + (n-R)\frac{z^*\phi(z^*)}{\sigma\Phi(-z^*)} = 0, \tag{53}$$

where $z^* = \dfrac{c-\mu}{\sigma}$ and $\phi(\cdot)$ is the PDF of the standard normal distribution. It is readily seen that the likelihood equations are implicit in nature, and so some numerical algorithm is needed to solve Eqs. (52) and (53) for the determination of the MLEs. Though the Newton-Raphson algorithm may be used for solving these equations, it has been observed by Dube et al. (2011) that it may not always converge. For this reason, they suggested the use of the expectation maximization (EM) algorithm for the computation of the MLEs in this case.

In developing the EM algorithm, it needs to be noted first of all that the problem can be treated as a missing value problem as follows. Let $\mathbf{X} = (x_{1:n}, \cdots, x_{R:n})$ be the observed data and $\mathbf{U} = (u_1, \cdots, u_{n-R})$ be the censored data. For a given R, $u_1, \cdots, u_{n-R}$ are not observable. The censored data vector $\mathbf{U}$ can be thought of as the missing data and $\mathbf{W} = (\mathbf{X}, \mathbf{U})$ as the complete data. Let $L_c(\mathbf{w}; \mu, \sigma)$ be the log-likelihood function based on the complete data, where $\mathbf{w} = (w_1, \cdots, w_n)$. Then,

$$L_c(\mathbf{w}; \mu, \sigma) = -n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (w_i - \mu)^2 = -n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{R} (x_{i:n} - \mu)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n-R} (u_i - \mu)^2. \tag{54}$$

At the k-th iterate of the EM algorithm, let $(\mu_{(k)}, \sigma_{(k)})$ be the estimate of (μ, σ). At the k-th iterate, in the 'E-step', one then needs to compute the pseudo log-likelihood function given by

$$\begin{aligned} L_s(\mu, \sigma) &= -n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{R} (x_{i:n} - \mu)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n-R} E\left((U_i - \mu)^2 \sim U_i > c\right) \\ &= -n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{R} (x_{i:n} - \mu)^2 - \frac{n-R}{2\sigma^2} B(c; \mu_{(k)}, \sigma_{(k)}) \\ &\quad + \frac{n-R}{2\sigma^2} \left\{ A(c; \mu_{(k)}, \sigma_{(k)}) - \mu^2 \right\}, \end{aligned} \tag{55}$$

where

$$\begin{aligned} A(c; \mu, \sigma) &= E(U \sim U > c) = \sigma Q + \mu, \\ B(c; \mu, \sigma) &= E(U^2 \sim U > c) = \sigma^2 (1 + \xi Q) + 2\sigma\mu Q + \mu^2, \end{aligned}$$

$$\xi = \frac{c-\mu}{\sigma},\ Q = \frac{\phi(\xi)}{\Phi(-\xi)}.$$

Next, in the 'M-step', one needs to maximize (55) to compute $(\mu_{(k+1)}, \sigma_{(k+1)})$, and this can be obtained as

$$\mu_{(k+1)} = \frac{1}{n}\left\{\sum_{i=1}^{R} x_{i:n} + (n-R)A(c;\mu_{(k)},\sigma_{(k)})\right\},$$
$$\sigma_{(k+1)} = \left[\frac{1}{n}\left\{\sum_{i=1}^{R} x_{i:n}^2 + (n-R)B(c;\mu_{(k)},\sigma_{(k)})\right\} - \mu_{(k)}^2\right]^{1/2}. \quad (56)$$

Since at the 'M-step' the estimates are in explicit form, the implementation of the EM algorithm becomes simpler in this case. For the implementation of the EM algorithm, one needs to have some initial guess for μ and σ. Ignoring the censoring, the initial guesses of μ and σ can be obtained as the sample mean and sample standard deviation of $\{\ln x_{1:n}, \cdots, \ln x_{R:n}\}$, for example. Alternatively, the approximate maximum likelihood estimators described in the next subsection could be used as initial values for the iterative algorithm.

2.5.2 Approximate Maximum Likelihood Estimators

It may be observed that the likelihood equations in (52) and (53) do not have explicit solutions due to the presence of the non-linear term $h(z^*) = \frac{\phi(z^*)}{\Phi(-z^*)}$. Tiku et al. (1986) and Balakrishnan and Cohen (1991) have discussed some approximate solutions in cases where such non-linear terms are present in the likelihood equations.

Let us first consider Case I. If $z_r = \frac{\ln x_{r:n} - \mu}{\sigma}$, then upon expanding $h(z_r) = \frac{\phi(z_r)}{\Phi(-z_r)}$ around the value $\nu = \Phi^{-1}\left(\frac{r}{n+1}\right)$, then keeping only the first-order term and neglecting all higher-order terms, we obtain

$$h(z_r) \approx h(\nu) + (z_r - \nu)h'(\nu) = \alpha + \beta z_r \quad \text{(say)}, \quad (57)$$

where $\alpha = h(\nu) - \nu h'(\nu)$ and $\beta = h'(\nu) \geq 0$. Using the approximation in (57) into Eqs. (52) and (53) and then solving them, we obtain the approximate maximum likelihood estimates

(AMLEs) of μ and σ as

$$\widetilde{\sigma} = \frac{A_1 + \sqrt{4rA_2}}{2r} \quad \text{and} \quad \widetilde{\mu} = K + M\widetilde{\sigma}, \tag{58}$$

where

$$A_1 = \alpha(n-r)(\ln x_{r:n} - K), \quad A_2 = \sum_{i=1}^{r}(\ln x_{i:n} - K)^2 + (n-r)\beta(\ln x_{r:n} - K)^2,$$

$$K = \frac{\sum_{i=1}^{r} \ln x_{i:n} + (n-r)\beta \ln x_{r:n}}{r + (n-r)\beta} \quad \text{and} \quad M = \frac{\alpha(n-r)}{r + \beta(n-r)}.$$

For Case II, by proceeding in an analogous manner, we obtain the AMLEs of μ and σ exactly as in (58), but with

$$A_1 = \alpha(n-d)(\ln T - K), \quad A_2 = \sum_{i=1}^{r}(\ln x_{i:n} - K)^2 + (n-d)\beta(\ln T - K)^2,$$

$$K = \frac{\sum_{i=1}^{r} \ln x_{i:n} + (n-d)\beta \ln T}{d + (n-d)\beta} \quad \text{and} \quad M = \frac{\alpha(n-d)}{d + \beta(n-d)}.$$

2.6 Generalized Exponential Distribution

Generalized exponential (GE) distribution has been studied extensively since it was introduced by Gupta and Kundu (1999). The two-parameter GE distribution has its PDF as

$$f(x; \alpha, \lambda) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha - 1}, \quad x > 0. \tag{59}$$

Here, $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. The two-parameter GE distribution is a special case of an exponentiated Weibull distribution originally proposed by Mudholkar and Srivastava (1993). The density can take on different shapes and the hazard function also can be increasing, decreasing or constant depending on the value of the shape parameter. Thus, this provides a flexible model to effectively analyze lifetime data. The GE distribution is quite similar to the gamma distribution, but since it has a closed-form distribution function, it is a convenient model for fitting censored data. Considerable work has been done on various aspects of the GE distribution, and one may refer to Gupta and Kundu (2007) for an overview of all these developments. Recently, Kundu and Pradhan (2009) discussed the Bayesian and frequentist inferential procedures on the parameters of the GE distribution, and they will be detailed here.

2.6.1 MLEs

Based on the observed data from a Type-I HCS, the log-likelihood function for Cases I and II, without the additive constant, are as follows:

$$\begin{aligned} l(\alpha, \lambda|\text{data}) &= R\ln\alpha + R\ln\lambda - \lambda\sum_{i=1}^{R} x_{i:n} + (\alpha-1)\sum_{i=1}^{R}\ln\left(1-e^{-\lambda x_{i:n}}\right) \\ &\quad + (n-R)\ln\left\{1-\left(1-e^{-\lambda c}\right)^{\alpha}\right\}. \end{aligned} \tag{60}$$

Note that as before, for Case I, $R = r$ and $c = x_{r:n}$, while for Case II, $0 \leq R = d \leq r-1$ and $c = T$. When $d = 0$, $l(\alpha, \lambda|\text{data}) = n\ln\left\{1-\left(1-e^{-\lambda c}\right)^{\alpha}\right\}$, which can be seen to increase with α for any λ, which means that the MLEs do not exist when $d = 0$, and so for the determination of the MLEs, we shall assume $d > 0$.

In this case, once again, explicit solutions for the two likelihood equations can not be obtained. So, one needs to use some numerical algorithm like Newton-Raphson method for solving the two non-linear likelihood equations. It has been observed that even though the Newton-Raphson algorithm can be used to determine the MLEs, but it may not always converge. Kundu and Pradhan (2009), therefore, proposed to use the EM algorithm for the computation of the MLEs. We treat this problem exactly as we did before in the log-normal case. Upon using the same notation as before, i.e., $\mathbf{X} = (x_{1:n}, \cdots, x_{R:n})$, $\mathbf{U} = (u_1, \cdots, u_{n-R})$, and $\mathbf{W} = (\mathbf{X}, \mathbf{U})$ as the observed data, censored data and complete data, respectively, and ignoring the constant term, the log-likelihood function of the complete data can be expressed as

$$
\begin{aligned}
l_c(\mathbf{W};\alpha,\lambda) &= n\ln\alpha + n\ln\lambda - \lambda\left(\sum_{i=1}^{R} x_{i:n} + \sum_{i=1}^{n-R} z_i\right) \\
&\quad +(\alpha-1)\left\{\sum_{i=1}^{R}\ln\left(1-e^{-\lambda x_{i:n}}\right) + \sum_{i=1}^{n-R}\ln\left(1-e^{-\lambda z_i}\right)\right\}. \qquad (61)
\end{aligned}
$$

For the 'E-step' of the EM algorithm, we need to compute 'pseudo' log-likelihood function at the k-th stage as $l_s(\alpha,\lambda|\text{data},\alpha_{(k)},\lambda_{(k)}) = E\left[l_c(\mathbf{W};\alpha,\lambda)|\mathbf{X},\alpha_{(k)},\lambda_{(k)}\right]$, where $\alpha_{(k)}$ and $\lambda_{(k)}$ are the estimates of α and λ at the k-th stage. Therefore,

$$
\begin{aligned}
l_s(\alpha,\lambda|\text{data},\alpha_{(k)},\lambda_{(k)}) &= n\ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^{R} x_{i:n} + (\alpha-1)\sum_{i=1}^{R}\ln\left(1-e^{-\lambda x_{i:n}}\right) \\
&\quad +(\alpha-1)\sum_{i=1}^{n-R} E\left[\ln\left(1-e^{-\lambda Z_i}\right)|Z_i > c,\alpha_{(k)},\lambda_{(k)}\right] \\
&\quad -\sum_{i=1}^{R} E\left[Z_i|Z_i > c,\alpha_{(k)},\lambda_{(k)}\right]. \qquad (62)
\end{aligned}
$$

We need the following results to proceed further, and proofs are simple and can be found, for example, in Ng et al. (2002, 2004).

RESULT 3: Given $X_{1:n} = x_{1:n},\cdots,X_{R:n} = x_{R:n}$, the conditional PDF of Z_j, for $j = 1,\cdots,n_R$, is

$$
f_{Z|X}(z_j|X_{1:n} = x_{1:n},\cdots,X_{R:n} = x_{R:n}) = \frac{f(z_j;\alpha,\lambda)}{1-F(x_{R:n};\alpha,\lambda)}, \quad z_j > x_{R:n}, \qquad (63)
$$

and Z_j and Z_k (for $j \neq k$) are conditionally independent. Here, $f(\cdot;\alpha,\lambda)$ is the PDF of the GE distribution in (59) and $F(\cdot;\alpha,\lambda)$ is the corresponding CDF given by $F(x;\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha}$ for $x > 0$. ∎

RESULT 4: Given d and $X_{1:n} = x_{1:n},\cdots,X_{d:n} = x_{d:n} < T$, the conditional PDF of Z_j, for $j = 1,\cdots,n-d$, is

$$
f_{Z|X}(z_j|X_{1:n} = x_{1:n},\cdots,X_{d:n} = x_{d:n} < T) = \frac{f(z_j;\alpha,\lambda)}{1-F(T;\alpha,\lambda)}, \quad z_j > T, \qquad (64)
$$

and Z_j and Z_k (for $j \neq k$) are conditionally independent, where once again $f(\cdot;\alpha,\lambda)$ and

$F(\cdot;\alpha,\lambda)$ are the PDF and CDF of the GE distribution with shape parameter α and scale parameter λ. ∎

Now, we can express

$$\begin{aligned} A(c,\alpha,\lambda) = E(Z_j|Z_j > c) &= \frac{\alpha\lambda}{1-F(c;\alpha,\lambda)} \int_c^\infty x e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1} dx \\ &= -\frac{\alpha}{\lambda(1-F(c;\alpha,\lambda))} u(\lambda c,\alpha), \end{aligned} \tag{65}$$

where

$$u(a,b) = \int_0^{e^{-a}} (1-z)^{b-1} \ln z \; dz$$

and

$$\begin{aligned} B(c,\alpha,\lambda) &= E\left[\ln(1-e^{-\lambda Z_j})|Z_j > c\right] \\ &= \frac{1}{\alpha(1-F(c;\alpha,\lambda))}\left[\left(1-e^{-c\lambda}\right)^\alpha\left\{1-\alpha\ln(1-e^{-c\lambda})\right\}-1\right]. \end{aligned}$$

Now, the 'M-step' involves the maximization of the 'pseudo' log-likelihood function in (62), and $(\alpha_{(k+1)},\lambda_{(k+1)})$ can be obtained by maximizing

$$\begin{aligned} g(\alpha,\lambda) &= n\ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^R x_{i:n} + (\alpha-1)\sum_{i=1}^R \ln(1-e^{-\lambda x_{i:n}}) \\ &\quad - \lambda(n-R)A(c;\alpha_{(k)},\lambda_{(k)}) + (\alpha-1)(n-R)B(c;\alpha_{(k)},\lambda_{(k)}). \end{aligned} \tag{66}$$

The maximization of (66) can be performed by using a fixed-point type algorithm as

$$h(\lambda) = \lambda, \tag{67}$$

where the function $h(\lambda)$ is given by

$$h(\lambda) = \left\{\frac{1}{n}\sum_{i=1}^R x_{i:n} + \frac{n-R}{n}A - \frac{1}{n}(\widehat{\alpha}(\lambda)-1)\sum_{i=1}^R \frac{x_{i:n}e^{-\lambda x_{i:n}}}{1-e^{-\lambda x_{i:n}}}\right\}^{-1},$$

with

$$A = A(c,\alpha_{(k)},\lambda_{(k)}), \quad B = B(c,\alpha_{(k)},\lambda_{(k)})$$

and

$$\widehat{\alpha}(\lambda) = -\frac{n}{\sum_{i=1}^{R} \ln(1 - e^{-\lambda x_{i:n}}) + (n-R)B}.$$

Kundu and Pradhan (2009) then suggested the following simple iterative scheme. Starting with some initial guess of (α, λ), say $(\alpha_{(0)}, \lambda_{(0)})$, obtain $\lambda_{(1)} = h(\lambda_{(0)})$, then obtain $\alpha_{(1)}$ as $\alpha_{(1)} = \widehat{\alpha}(\lambda_{(1)})$, and continue this process until convergence.

2.6.2 Bayesian Inference

We shall describe here the Bayesian estimation as well as the construction of the associated HPD credible intervals for the parameters α and λ. Unfortunately, when both parameters are unknown, there do not exist any natural conjugate priors for them. Using the approach of Raqab and Madi (2005), Kundu and Pradhan (2009) assumed the following priors on the unknown parameters α and λ:

$$\pi_1(\alpha) \propto \alpha^{a_1-1} e^{-b_1\alpha}, \quad \alpha > 0, \tag{68}$$

$$\pi_2(\lambda) \propto \lambda^{a_2-1} e^{-b_2\lambda}, \quad \lambda > 0. \tag{69}$$

All the hyper-parameters a_1, b_1, a_2, b_2 are assumed to be known and non-negative. Now, based on the observed sample $\{x_{1:n}, \cdots, x_{R:n}\}$ from the hybrid censoring scheme, the likelihood function becomes

$$l(\text{data}|\alpha, \lambda) \propto \alpha^R \lambda^R e^{-\lambda \sum_{i=1}^{R} x_{i:n}} e^{(\alpha-1)\sum_{i=1}^{R} \ln(1-e^{-\lambda x_{i:n}})} e^{(n-R)\ln\{1-(1-e^{-\lambda c})^\alpha\}}. \tag{70}$$

The joint posterior density function of α and λ can be written as

$$\begin{aligned}\pi(\alpha, \lambda|\text{data}) \quad &\propto \alpha^{a_1+d-1} e^{-\alpha(b_1 - \sum_{i=1}^{R} \ln(1-e^{-\lambda x_{i:n}}))} \lambda^{a_2+d-1} e^{-\lambda(b_2 + \sum_{i=1}^{R} x_{i:n})} \\ &\times e^{-\sum_{i=1}^{R} \ln(1-e^{-\lambda x_{i:n}}) + (n-R)\ln\{1-(1-e^{-\lambda c})^\alpha\}}.\end{aligned} \tag{71}$$

From (71), the Bayesian estimate of any function of α and λ, say $\theta(\alpha, \lambda)$, under the squared error loss function is the posterior mean of $\theta(\alpha, \lambda)$ given by

$$\widehat{\theta}_B = E_{\alpha,\lambda|\text{data}}(\theta(\alpha, \lambda)) \;=\; \int_0^\infty \int_0^\infty \theta(\alpha, \lambda)\pi(\alpha, \lambda|\text{data}) d\alpha d\lambda$$

$$= \frac{\int_0^\infty \int_0^\infty \theta(\alpha,\lambda) l(\text{data}|\alpha,\lambda)\pi_1(\alpha)\pi_2(\lambda) d\alpha d\lambda}{\int_0^\infty \int_0^\infty l(\text{data}|\alpha,\lambda)\pi_1(\alpha)\pi_2(\lambda) d\alpha d\lambda}. \tag{72}$$

It is not possible to compute (72) analytically except for some forms of $\theta(\alpha,\lambda)$. Kundu and Pradhan (2009) suggested the use of importance sampling for computing the Bayesian estimate of $\theta(\alpha,\lambda)$ in (72) and also for constructing the HPD credible interval for $\theta(\alpha,\lambda)$.

The joint posterior density function of α and λ in (71) can be expressed as

$$\pi(\alpha,\lambda|\text{data}) \propto g_\lambda(a_2^*, b_2^*)\ g_{\alpha|\lambda}(a_1^*, b_1^*)\ g_3(\alpha,\lambda), \tag{73}$$

where $g_{\alpha|\lambda}(a_1^*, b_1^*)$ is a gamma density function with the shape and scale parameters as $a_1^* = a_1 + R$ and $b_1^* = b_1 - \sum_{i=1}^{R} \ln(1 - e^{-\lambda x_{i:n}})$, respectively; $g_\lambda(a_2^*, b_2^*)$ is a gamma density function with the shape and scale parameters as $a_2^* = a_2 + R$ and $b_2^* = b_2 + \sum_{i=1}^{R} x_{i:n}$, respectively; finally,

$$g_3(\alpha,\lambda) = \frac{1}{\left\{b_1 - \sum_{i=1}^{R} \ln(1 - e^{-\lambda x_{i:n}})\right\}^{a_1+R}} \times e^{(n-d)\ln\{1-(1-e^{-\lambda c})^\alpha\} - \sum_{i=1}^{R} \ln(1-e^{-\lambda x_{i:n}})}$$

is a function of α and λ.

Kundu and Pradhan (2009) proposed a simulated consistent estimator of $\widehat{\theta}_B$ by using the importance sampling as follows:

- Step 1: Generate λ_1 from $g_\lambda(a_2^*, b_2^*) \sim \text{gamma}(a_2 + R, b_2 \sum_{i=1}^{R} x_{i:n})$;
- Step 2: Generate α_1 from $g_{\alpha|\lambda}(a_1^*, b_1^*) \sim \text{gamma}(a_1 + R, b_1 - \sum_{i=1}^{R} \ln(1 - e^{-\lambda_1 x_{i:n}}))$;
- Step 3: Repeat Steps 1 and 2 N times to obtain $(\alpha_1,\lambda_1), \cdots, (\alpha_N,\lambda_N)$;
- Step 4: The Bayes estimate of θ under the squared-error loss function can then be approximated as

$$\widehat{\theta}_B \approx \frac{\sum_{i=1}^{N} \theta(\alpha_i,\lambda_i) g_3(\alpha_i,\lambda_i)}{\frac{1}{N}\sum_{i=1}^{N} g_3(\alpha_i,\lambda_i)}.$$

Next, for computing the HPD credible interval for θ, let us use $\pi(\theta|\text{data})$ and $\Pi(\theta|\text{data})$ to denote the posterior density function and distribution function of θ, respectively. Moreover, let $\theta^{(\beta)}$ be the β-th quantile of θ (for $0 < \beta < 1$), i.e.,

$$\theta^{(\beta)} = \inf\{\theta : \Pi(\theta|\text{data}) \geq \beta\}.$$

Observe that, for a given θ^*,

$$\Pi(\theta^*|\text{data}) = E[1_{\theta \leq \theta^*}|\text{data}],$$

where $1_{\theta \leq \theta^*}$ is the indicator function. Then, a simulated consistent estimator of $\Pi(\theta^*|\text{data})$ is given by

$$\Pi(\theta^*|\text{data}) = \frac{\frac{1}{N}\sum_{i=1}^{N} 1_{\theta \leq \theta^*}(\alpha_i, \lambda_i)}{\frac{1}{N}\sum_{i=1}^{N} g_3(\alpha_i, \lambda_i)}.$$

Let $\{\theta_{(i)}\}$ be the ordered values of $\{\theta_i\}$, and let

$$w_i = \frac{g_3(\alpha_{(i)}, \lambda_{(i)})}{\sum_{i=1}^{N} g_3(\alpha_i, \lambda_i)}$$

for $i = 1, \cdots, N$. Then, we have

$$\Pi(\theta^*|\text{data}) = \begin{cases} 0 & \text{if} \quad \theta_{(i)} \leq \theta^* < \theta_{(i+1)}, \\ \sum_{j=1}^{i} w_j & \text{if} \quad \theta_{(i)} \leq \theta^* < \theta_{(i+1)}, , \\ 1 & \text{if} \quad \theta^* \geq \theta_{(n)} \end{cases} \tag{74}$$

with which $\theta^{(\beta)}$ can be approximated by

$$\widehat{\theta}^{(\beta)} = \begin{cases} \theta_{(1)} & \text{if} \quad \beta = 0, \\ \theta_{(i)} & \text{if} \quad \sum_{j=1}^{i-1} w_j < \beta < \sum_{j=1}^{i} w_j. \end{cases} \tag{75}$$

For computing a 100(1 - β)% HPD credible interval for θ, let $R_j = \left(\widehat{\theta}^{\left(\frac{1}{N}\right)}, \widehat{\theta}^{\left(\frac{j+(1-\beta)N}{N}\right)}\right)$ for $j = 1, \cdots, [\beta N]$, where $[a]$ denotes the largest integer less than or equal to a. Then, among all the R_j's, choose the one with the smallest width.

3 Type-II HCS

3.1 Form of Data

In this section, we assume that the data are Type-II hybrid censored, i.e., we have one of the following three forms of observations as our data:

Case I: $\{x_{1:n} < \cdots < x_{r:n}\}$ if $x_{r:n} \geq T$;

Case II: $\{x_{1:n} < \cdots < x_{d:n}\}$ if $x_{r:n} < T$,

where $r \leq d \leq n$ denotes the number of observed failures before time T, and $x_{d+1} > T$; here, when $d = n$, we take $x_{d+1} = \infty$.

We assume, as before, that the lifetimes of the units are i.i.d. random variables.

3.2 Exponential Distribution

3.2.1 MLE and its Exact Distribution

Under the assumption that the lifetimes of the units are i.i.d. exponential random variables with mean θ, the MLE of θ always exists in this case, unlike in the Type-I hybrid censoring case. It is given by

$$\widehat{\theta} = \frac{1}{R}\left\{\sum_{i=1}^{R} x_{i:n} + (n-R)c\right\}, \tag{76}$$

where $R = r$ and $c = x_{r:n}$ for Case I, and $R = d$ and $c = T$ for Case II.

For deriving the exact distribution of $\widehat{\theta}$, Childs et al. (2003) first obtained the moment generating function of $\widehat{\theta}$ as follows:

$$\begin{aligned} M_{\widehat{\theta}}(\omega) &= \left(1-\frac{\theta\omega}{r}\right)^{-r}\sum_{d=0}^{r-1}\binom{n}{d}q^{(n-d)(1-\theta\omega/r)}\left(1-q^{(1-\theta\omega/r)}\right)^{d} \\ &\quad + \sum_{d=r}^{n}\binom{n}{d}q^{(n-d)(1-\theta\omega/d)}\left(1-q^{(1-\theta\omega/d)}\right)^{d}\left(1-\frac{\theta\omega}{d}\right)^{-d}, \quad \omega < \frac{r}{\theta}. \end{aligned} \tag{77}$$

From the moment generating function of $\widehat{\theta}$ in (77), by using the inversion formula, the PDF of $\widehat{\theta}$ can be readily obtained as

$$f_{\widehat{\theta}}(x) = q^n g\left(x - \frac{nT}{r}; \frac{r}{\theta}, r\right) + \sum_{d=1}^{r-1}\sum_{k=0}^{d} C_{k,d}\, g\left(x - a_{k,d}; \frac{r}{\theta}, r\right) + \sum_{d=r}^{r}\sum_{k=0}^{d} C_{k,d}\, g\left(x - a_{k,d}; \frac{d}{\theta}, d\right); \tag{78}$$

here, $C_{k,d}$ and g are as defined in (5) and (6), and

$$a_{k,d} = \begin{cases} (n-d+k)T/r & \text{if} \quad d = 0, 1, \cdots, r-1, \\ (n-d+k)T/d & \text{if} \quad d = r, r+1, \cdots, n, k = 0, 1, \cdots, d. \end{cases}$$

From (78), we observe that $\widehat{\theta}$ has its PDF as a generalized mixture of shifted gamma distributions. In the special case when $T \to 0$, $f_{\widehat{\theta}}(x) = g(x; r/\theta, r)$, which is the well-known result that $2r\widehat{\theta}/\theta$ has a chi-square distribution with $2r$ degrees of freedom. From the mixture form of the exact PDF of $\widehat{\theta}$ in (78), the bias and variance of $\widehat{\theta}$ can be easily obtained.

3.2.2 Exact Confidence Intervals

In this section, we describe the construction of an exact confidence interval for θ. As in the case of Type-I HCS, based on the assumption that $P_\theta(\widehat{\theta} > b)$ is an increasing function of θ, Childs et al. (2003) developed an exact two-sided confidence interval for θ. A formal proof of the monotonicity of this probability has been given recently by Balakrishnan and Iliopoulos (2009). If (θ_L, θ_U) denotes the exact two-sided 100(1-α)% symmetric confidence interval of θ, then θ_L and θ_U can be obtained as the solutions of the following two non-linear equations:

$$\frac{\alpha}{2} = \sum_{d=0}^{r-1}\sum_{k=0}^{d} \frac{C_{k,d}}{(r-1)!}\Gamma\left(r, \frac{r}{\theta_L} < \widehat{\theta} - a_{k,d} >\right) + \sum_{d=r}^{n}\sum_{k=0}^{d} \frac{C_{k,d}}{(d-1)!}\Gamma\left(d, \frac{d}{\theta_L} < \widehat{\theta} - a_{k,d} >\right), \tag{79}$$

$$1 - \frac{\alpha}{2} = \sum_{d=0}^{r-1}\sum_{k=0}^{d} \frac{C_{k,d}}{(r-1)!}\Gamma\left(r, \frac{r}{\theta_U} < \widehat{\theta} - a_{k,d} >\right) + \sum_{d=r}^{n}\sum_{k=0}^{d} \frac{C_{k,d}}{(d-1)!}\Gamma\left(d, \frac{d}{\theta_U} < \widehat{\theta} - a_{k,d} >\right). \tag{80}$$

The lower and upper limits θ_L and θ_U need to be computed from (79) and (80), respectively, by using some numerical method.

3.2.3 Approximate Confidence Intervals

Since the exact method is computationally quite demanding, approximate confidence intervals can be constructed by using the asymptotic normality of the MLE $\widehat{\theta}$ and the inverse of the Fisher information. Park et al. (2008) derived the Fisher information for Type-II HCS by using the approach of Wang and He (2005). In this case, the Fisher information of θ is given by

$$I_2(\theta) = \frac{1}{\theta^2}\left\{n\left(1 - e^{-T/\theta}\right) + r - \sum_{i=1}^{r} F_i\left(\frac{T}{\theta}\right)\right\}, \tag{81}$$

where $F_i(x)$ is as defined earlier in (14). Hence, the asymptotic variance of $\widehat{\theta}$ is $1/I_2(\theta)$, which can be used in the construction of the approximate confidence interval for θ.

3.2.4 Fisher Information

Park et al. (2008) derived an explicit expression for the Fisher information for Type-II hybrid censored data. In the case of exponential distribution with mean θ, the Fisher information in Type-II hybrid censored data can be expressed as a sum of single integrals as

$$I_2(\theta) = \frac{1}{\theta^2}\ \left\{n\left(1 - e^{-T/\theta}\right) + r - \sum_{i=1}^{r} F_i\left(\frac{T}{\theta}\right)\right\}, \tag{82}$$

where $F_i(x)$ is as defined earlier in (14). These authors also used this expression to compare the Fisher information contained in different Type-II HCS. Moreover, this expression can also be utilized to construct confidence intervals for θ by using $I_2(\widehat{\theta})$.

3.2.5 Bayesian Inference

In this case as well, the Bayesian estimate and the associated credible intervals can be constructed very much along the lines of the Type-I HCS case. Upon using the same prior as in (15), the posterior density function of θ in this case becomes

$$l(\theta|data) = \frac{(S_d^* + \lambda)^{d+\beta}}{\Gamma(d+\beta)} \theta^{-(d+\beta+1)} e^{-(S_d^*+\lambda)\theta}, \tag{83}$$

where

$$S_d^* = \begin{cases} \sum_{i=1}^{d} x_{i:n} + (n-d)T & \text{if} \quad d > r, \\ \sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n} & \text{if} \quad d = r. \end{cases} \tag{84}$$

Since $2(S_d^* + \lambda)/\theta$ is distributed as $\chi^2_{2(\beta+\lambda)}$ a posteriori, the Bayesian estimate of θ under the squared-error loss function is given by

$$\widehat{\theta}_{Bayes} = \frac{(S_d^* + \lambda)}{(d+\beta-1)} \tag{85}$$

if $(d+\beta-1) > 0$. As expected, the Bayesian estimate under the non-informative prior coincides with the MLE of θ in this case as well. Moreover, a 100(1-α)% credible interval of θ can be obtained as

$$\left(\frac{2(S_d^* + \lambda)}{\chi^2_{2(d+\beta),\alpha/2}}, \frac{2(S_d^* + \lambda)}{\chi^2_{2(d+\beta),1-\alpha/2}} \right).$$

3.3 Weibull Distribution

3.3.1 MLEs

We shall now consider the case when the lifetimes of the units are i.i.d. Weibull random variables, with α and λ as the shape and scale parameters, respectively.

Suppose the available data are Type-II HCS. Then, as presented in Section 3.1, the likelihood function in all three cases can be combined together and expressed as

$$L(\text{data}|\alpha, \lambda) = \alpha^R \lambda^R \prod_{i=1}^{R} x_{i:n}^{\alpha-1} e^{-\lambda\{x_{i:n}^{\alpha} + (n-R)U^{\alpha}\}}; \tag{86}$$

here, R denotes the number of failures as before. In this case, $U = x_{r:n}$ if $R = r$, and $U = T$ if $R > r$. The MLEs of λ and α can be obtained by solving the following two non-linear normal equations:

$$\frac{R}{\lambda} - \sum_{i=1}^{R} x_{i:n}^{\alpha} + (n-R)U^{\alpha} = 0, \tag{87}$$

$$\frac{R}{\alpha} + \sum_{i=1}^{R} \ln x_{i:n} - \lambda \left\{ \sum_{i=1}^{R} x_{i:n}^{\alpha} \ln x_{i:n} + (n-R)U^{\alpha} \ln U \right\} = 0. \tag{88}$$

For fixed α, the MLE of λ, say $\widehat{\lambda}(\alpha)$, can be readily obtained from (87) as

$$\widehat{\lambda}(\alpha) = \frac{R}{\sum_{i=1}^{R} x_{i:n}^{\alpha} + (n-R)U^{\alpha}}. \tag{89}$$

With $\widehat{\lambda}(\alpha)$ as in (89), the MLE of α can be obtained by maximizing the profile log-likelihood function of α, viz.,

$$\ln L(\alpha, \widehat{\lambda}(\alpha)) = R \ln \alpha + R \ln R - R \ln \left\{ \sum_{i=1}^{R} x_{i:n}^{\alpha} + (n-R)U^{\alpha} \right\} + (\alpha - 1) \sum_{i=1}^{R} \ln x_{i:n}. \tag{90}$$

It can be easily shown that the profile log-likelihood function in (90) is a log-concave function, and it goes to $-\infty$ as $\alpha \downarrow 0$ or $\alpha \to \infty$. Therefore, the profile log-likelihood function of α has a unique maximum. Here, one may also use the simple arguments of Balakrishnan and Kateri (2008), based on an application of the Cauchy-Schwarz inequality, to prove the existence and uniqueness of the MLEs in this case. Furthermore, Banerjee and Kundu (2008) proposed solving the fixed-point type equation

$$h(\alpha) = \alpha, \tag{91}$$

where

$$h(\alpha) = \frac{R}{\widehat{\lambda}(\alpha) \left\{ \sum_{i=1}^{R} x_{i:n}^{\alpha} \ln x_{i:n}(n-R)U^{\alpha} \ln U \right\} - \sum_{i=1}^{R} \ln x_{i:n}}$$

for finding the MLE of α. Iterative schemes similar to ones suggested before can be used to solve Eq. (91).

3.3.2 Approximate Maximum Likelihood Estimators

Following the approach as detailed in Section 2.4.2, the approximate maximum likelihood estimates of the unknown parameters α and λ can be obtained from the transformed data

$y_{i:n} = \ln x_{i:n}$. The method is exactly the same as outlined in Section 2.4.2, whereby the approximate maximum likelihood estimators of $\mu = \ln \lambda$, and $\sigma = \dfrac{1}{\alpha}$ are obtained in explicit form as follows:

$$\widehat{\sigma} = \frac{-F + \sqrt{F^2 + 4RG}}{2R} \quad \text{and} \quad \widehat{\mu} = A - B\widehat{\sigma}, \tag{92}$$

where

$$A = \frac{\sum_{i=1}^{R} \beta_i y_{i:n} + \beta_R (n-R) y_{R:n}}{\sum_{i=1}^{n} \beta_i + \beta_R (n-R)}, \quad B = \frac{\sum_{i=1}^{R} \alpha_i - (n-R)(1-\alpha_R)}{\sum_{i=1}^{n} \beta_i + \beta_R (n-R)},$$

$$F = \sum_{i=1}^{R} \alpha_i (y_{i:n} - A) - (n-R)(1-\alpha_R)(y_{R:n} - A) - 2B \sum_{i=1}^{R} \beta_i (y_{i:n} - A) - 2(n-R)\beta_R B (y_{R:n} - A),$$

$$G = \sum_{i=1}^{R} \beta_i (y_{i:n} - A)^2 + (n-R)\beta_R (y_{R:n} - A)^2$$

for Case I, and

$$A = \frac{\sum_{i=1}^{R} \beta_i y_{i:n} + \beta_{R^*} (n-R) \ln T}{\sum_{i=1}^{n} \beta_i + \beta_{D^*} (n-R)}, \quad B = \frac{\sum_{i=1}^{R} \alpha_i - (n-R)(1-\alpha_{R^*})}{\sum_{i=1}^{n} \beta_i + \beta_{R^*} (n-R)},$$

$$F = \sum_{i=1}^{R} \alpha_i (y_{i:n} - A) - (n-R)(1-\alpha_{D^*})(\ln T - A) - 2B \sum_{i=1}^{R} \beta_i (y_{i:n} - A) - 2(n-R)\beta_{D^*} B (\ln T - A),$$

$$G = \sum_{i=1}^{R} \beta_i (y_{i:n} - A)^2 + (n-R)\beta_{D^*} (\ln T - A)^2$$

for Case II. In the above, $p_i = \dfrac{i}{n+1}$, $q_i = 1 - p_i$, $i = 1, \cdots, n$, $p_{R^*} = \dfrac{p_R + p_{R+1}}{2}$, $q_{R^*} = 1 - p_{R^*}$, $\alpha_i = 1 + \ln q_i \{1 - \ln(-\ln q_i)\}$, $\beta_i = -\ln q_i$, $i = 1, \cdots, n$, $\beta_{R^*} = -\ln q_{R^*}$, and $\alpha_{R^*} = 1 + \ln q_{R^*} \{1 - \ln(-\ln q_{R^*})\}$. In an analogous manner, the approximate MLEs can be developed in explicit form for the Type-II HCS as well.

3.3.3 Confidence Intervals

In this case, also, exact confidence intervals for α and λ can not be obtained. Banerjee and Kundu (2008) suggested the use of confidence intervals based on the asymptotic bivariate normal distribution of the MLEs of α and λ. Also, parametric or non-parametric bootstrap methods can also be used for this purpose.

3.3.4 Bayesian Inference

Banerjee and Kundu (2008) discussed the Bayesian inference for the unknown parameters in the case of Type-HCS. They have assumed independent gamma priors for the shape and scale parameters as

$$\pi_1(\lambda) \sim \text{Gamma}(a, b), \quad \pi_2(\alpha) \sim \text{Gamma}(c, d). \tag{93}$$

Since, based on the priors for the parameters α and λ in (93), the Bayesian estimates of α and λ can not be obtained explicitly under the squared-error loss function, Banerjee and Kundu (2008) adopted Lindley's approximation to obtain the approximate Bayesian estimates of α and λ as

$$\begin{aligned}
\widehat{\alpha}_B &= \widehat{\alpha} + \frac{1}{2}\left[\left\{\frac{2R}{\widehat{\alpha}^3} - \widehat{\lambda}\left(\sum_{i=1}^{R} x_{i:n}^{\widehat{\alpha}}(\ln x_{i:n})^3 + (n-R)U^{\widehat{\alpha}}(\ln U)^3\right)\right\}\tau_{11}^2 + \frac{2R}{\widehat{\lambda}^3}\tau_{21}\tau_{22}\right. \\
&\quad \left. -3\tau_{11}\tau_{12} \times \left(\sum_{i=1}^{R} x_{i:n}^{\widehat{\alpha}}(\ln x_{i:n})^2 + (n-R)U^{\widehat{\alpha}}(\ln U)^2\right)\right] + \tau_{11}\left(\frac{c-1}{\widehat{\alpha}} - d\right) \\
&\quad +\tau_{12}\left(\frac{a-1}{\widehat{\lambda}} - b\right)
\end{aligned} \tag{94}$$

and

$$\begin{aligned}
\widehat{\lambda}_B &= \widehat{\lambda} + \frac{1}{2}\left[\left\{\frac{2R}{\widehat{\alpha}^3} - \widehat{\lambda}\left(\sum_{i=1}^{R} x_{i:n}^{\widehat{\alpha}}(\ln x_{i:n})^3 + (n-R)U^{\widehat{\alpha}}(\ln U)^3\right)\right\}\tau_{11}\tau_{12} + \frac{2R}{\widehat{\lambda}^3}\tau_{22}^2\right. \\
&\quad \left. -(\tau_{11}\tau_{22} + 2\tau_{12}^2) \times \left(\sum_{i=1}^{R} x_{i:n}^{\widehat{\alpha}}(\ln x_{i:n})^2 + (n-R)U^{\widehat{\alpha}}(\ln U)^2\right)\right] + \tau_{21}\left(\frac{c-1}{\widehat{\alpha}} - d\right) \\
&\quad +\tau_{22}\left(\frac{a-1}{\widehat{\lambda}} - b\right),
\end{aligned} \tag{95}$$

where

$$\tau_{11} = \frac{W}{SW - V^2}, \quad \tau_{12} = \tau_{21} = \frac{-V}{SW - V^2}, \quad \tau_{22} = \frac{S}{SW - V^2},$$

$$S = \frac{R}{\widehat{\alpha}^2} + \widehat{\lambda}\left[\sum_{i=1}^{R} x_{i:n}^{\widehat{\alpha}}(\ln x_{i:n})^2 + (n-R)U^{\widehat{\alpha}}(\ln U)^2\right],$$

$$V = \left[\sum_{i=1}^{R} x_{i:n}^{\widehat{\alpha}}\ln x_{i:n} + (n-R)U^{\widehat{\alpha}}\ln U\right] \quad \text{and} \quad W = \frac{R}{\widehat{\lambda}^2}.$$

One may refer to Banerjee and Kundu (2008) for pertinent details on the derivation of these formulas. In this case, by using the Gibbs sampling, the HPD credible interval can be obtained for any function of the parameters α and λ.

3.3.5 Comparison of Methods of Estimation

Due to the complicated nature of the different estimators, it is not easy to compare them analytically. For this reason, Banerjee and Kundu (2008) carried out a comparative study by means of Monte Carlo simulations. They compared the performance of the MLEs, AMLEs, and the Bayesian estimates with respect to the non-informative priors (i.e., $a = b = c = d = 0$) in terms of bias and MSE (mean squared error) of the estimates. They also compared the confidence intervals obtained from the asymptotic distribution of the MLEs with the corresponding credible intervals in terms of coverage probabilities and confidence/credible widths.

They observed that the performance of the MLEs and AMLEs are quite similar in terms of both bias and mean squared error. Hence, for all practical purposes, the AMLEs are preferable to the MLEs since the AMLEs are explicit and do not require any numerical iterative procedure. Moreover, the Bayesian estimates based on non-informative priors behave quite similar to the MLEs in terms of bias and MSE. In the comparison of confidence and credible intervals, they observed that the confidence intervals based on the asymptotic distribution of the MLEs are quite satisfactory as long as the sample size is at least 30. But for small r and small T, these confidence intervals are not at all satisfactory. On the other hand, the credible intervals obtained from the Gibbs sampling procedure are quite satisfactory even for small r and small T. The coverage probabilities are very close to the nominal levels, and in addition, the average widths of the credible intervals turn out to be slightly smaller than those of the corresponding confidence intervals.

4 Generalized Hybrid Censoring Schemes

So far, we have discussed the Type-I and Type-II HCS at length. Both these schemes have certain advantages (as explained earlier in Section 1) as well as some disadvantages. Even though Type-I HCS has the distinct advantage of having a bounded experimental time, it has the disadvantage of having possibly very few failures before the termination of the experiment. This will, of course, have an adverse effect on the efficiency of the estimators. For this reason, Childs et al. (2003) proposed the Type-II HCS that guarantees a specified number of failures, but the main disadvantage of the Type-II HCS is that it might require a long time to observe the required r failures and complete the life-test.

To overcome these problems and to specifically provide a guarantee in terms of the number of failures as well as a limit on the experimental time, Chandrasekar et al. (2004) proposed two *generalized hybrid censoring schemes*. These are designed to take care of the disadvantages inherent in the Type-I and Type-II HCS. Using the same notation as before, and with n units placed on a life-test at time 0, the two generalized hybrid censoring schemes can be described as follows.

4.1 Generalized Type-I HCS

Fix integers $r, k \in \{1, 2, \cdots, n\}$ such that $k < r < n$, and time $T \in (0, \infty)$. If the k-th failure occurs before time T, terminate the experiment at $\min\{X_{r:n}, T\}$. If the k-th failure occurs after time T, terminate the experiment at $X_{k:n}$. It is evident that this HCS modifies the Type-I HCS by allowing the experiment to continue beyond time T if very few failures had been observed up to that time point. It should be noted that under this censoring scheme, the experimenter would like to observe r failures, but is willing to settle for a bare minimum of k failures.

4.1.1 Form of Data

Under such a generalized Type-I HCS, we will observe one of the following forms of observations:

Case I: $\{x_{1:n} < \cdots < x_{k:n}\}$ if $x_{k:n} > T$;

Case II: $\{x_{1:n} < \cdots < x_{k:n} < \cdots < x_{r:n}\}$ if $x_{r:n} < T$;

Case III: $\{x_{1:n} < \cdots < x_{k:n} < \cdots < x_{d:n}\}$ if $T < x_{r:n}$.

4.1.2 Exponential Distribution

Let us now assume that the lifetimes of the experimental units are i.i.d. exponential random variables with mean θ. If D denotes the number of failures that occur before time point T, then based on the three forms of the sample presented above, the likelihood function is given by

$$L(\theta|\text{data}) = \begin{cases} \frac{n!}{(n-k)!}\lambda^k e^{-\lambda\left\{\sum_{i=1}^{k} x_{i:n}+(n-k)x_{k:n}\right\}} & \text{if} \quad D = 0, 1, \cdots, k-1, \\ \frac{n!}{(n-D)!}\lambda^D e^{-\lambda\left\{\sum_{i=1}^{D} x_{i:n}+(n-D)T\right\}} & \text{if} \quad D = k, k+1, \cdots, r-1, \\ \frac{n!}{(n-r)!}\lambda^r e^{-\lambda\left\{\sum_{i=1}^{r} x_{i:n}+(n-r)x_{r:n}\right\}} & \text{if} \quad D = r. \end{cases} \tag{96}$$

In this case, the MLE of θ always exists, and is given by

$$\widehat{\theta} = \begin{cases} \frac{\sum_{i=1}^{k} x_{i:n}+(n-k)x_{k:n}}{k} & \text{if} \quad D = 0, 1, \cdots, k-1, \\ \frac{\sum_{i=1}^{D} x_{i:n}+(n-D)T}{D} & \text{if} \quad D = k, k+1, \cdots, r-1, \\ \frac{\sum_{i=1}^{r} x_{i:n}+(n-r)x_{r:n}}{r} & \text{if} \quad D = r. \end{cases} \tag{97}$$

To derive the exact distribution of $\widehat{\theta}$ in (97), the moment generating function of $\widehat{\theta}$ can be obtained first as follows:

$$\begin{aligned} M_{\widehat{\theta}}(\omega) = E(e^{\omega\widehat{\theta}}) &= \left(1 - \frac{\theta\omega}{k}\right)^{-k} \sum_{l=0}^{k-1} \binom{n}{l} p_k^j q_k^{n-l} + \sum_{l=k}^{r-1} \left(1 - \frac{\theta\omega}{l}\right)^{-l} \binom{n}{l} p_l^l q_l^{n-l} \\ &\quad + \left(1 - \frac{\theta\omega}{r}\right)^{-r} \sum_{l=r}^{n} \binom{n}{l} p_l^l q_l^{n-l}, \quad \omega < \frac{k}{\theta}, \end{aligned} \tag{98}$$

where $q_j = 1 - p_j = e^{-(1-\theta\omega j)T/\theta}$, $j = k, k+1, \cdots, r$; see Chandrasekar et al. (2004).

From the moment generating function of $\widehat{\theta}$ in (98), the PDF of $\widehat{\theta}$ is obtained to be

$$f_{\widehat{\theta}}(x) = \sum_{l=0}^{n} \sum_{j=0}^{l} c_{jl} g(x - T_{jl}; \alpha_j, \beta_l), \tag{99}$$

where

$$C_{jl} = (-1)^j \binom{n}{l} \binom{l}{j} e^{-(n-l+j)T/\theta}, \quad l = 0, 1, \cdots n,$$

$$T_{jl} = \begin{cases} \frac{(n-l+j)T}{k \vee l} & \text{if} \quad l = 0, 1, \cdots, r-1, \\ \frac{(n-l+j)T}{r} & \text{if} \quad l = r, \cdots, n, \end{cases}$$

$$\alpha_l = \begin{cases} \frac{k \vee l}{\theta} & \text{if} \quad l = 0, 1, \cdots, r-1, \\ \frac{r}{\theta} & \text{if} \quad l = r, \cdots, n, \end{cases}$$

and

$$\beta_l = \begin{cases} k \vee l, & \text{if} \quad l = 0, 1, \cdots, r-1, \\ r, & \text{if} \quad l = r, \cdots, n. \end{cases}$$

In the above expressions, $a \vee b$ denotes $\max\{a, b\}$, and $g(\cdot; \alpha_j, \beta_j)$ denotes the gamma PDF defined earlier in (6). From the PDF of $\widehat{\theta}$, the mean, variance and other moments of $\widehat{\theta}$ can be easily obtained; see Chandrasekar et al. (2004) for further details. From the expression of the mean of $\widehat{\theta}$, they noted that it is a biased estimator. In this case also, under the assumption that $P_\theta(\widehat{\theta} > b)$ is an increasing function of θ, a two-sided exact confidence interval can be constructed exactly along the same lines as in Section 2.2.2.

4.1.3 Fisher Information

Park and Balakrishnan (2009) derived an explicit expression for the Fisher information contained in generalized Type-I HCS, for the case of exponential distribution with mean θ, as a sum of single integrals as

$$I_{G_1}(\theta) = \frac{1}{\theta^2} \left\{ k + \sum_{i=k+1}^{r} F_i \left(\frac{T}{\theta} \right) \right\}, \tag{100}$$

where $F_i(x)$ is as defined earlier in (14). Park and Balakrishnan (2009) also used this expression to compare the Fisher information contained in different generalized Type-I HCS. Moreover, this expression can also be utilized to construct confidence intervals for θ by using $I_{G_1}(\widehat{\theta})$.

4.2 Generalized Type-II HCS

For describing the generalized Type-II HCS introduced by Chandrasekar et al. (2004), let us fix $r \in \{1, 2, \cdots, n\}$, and time points $T_1, T_2 \in (0, \infty)$, with $T_1 < T_2$. If the r-th failure occurs before the time point T_1, terminate the experiment at T_1. If the r-th failure occurs between T_1 and T_2, terminate the experiment at $X_{r:n}$. Finally, if the r-th failure occurs after T_2, terminate the experiment at T_2. This hybrid censoring scheme modifies the Type-II HCS by guaranteeing that the experiment will be completed by time T_2, and thus T_2 serves as the absolute maximum time that the experiment would be allowed to go for.

4.2.1 Form of Data

Under such a generalized Type-II HCS, we will observe one of the following forms of observations:

Case I: $\{x_{1:n} < \cdots < x_{r:n}\}$ if $x_{r:n} < \cdots < x_{d_1:n} < T_1$;

Case II: $\{x_{1:n} < \cdots < x_{d_1:n} < \cdots < x_{r:n}\}$ if $T_1 < x_{r:n} < T_2$;

Case III: $\{x_{1:n} < \cdots < x_{d_2:n} < T_2\}$ if $x_{r:n} \geq T_2$.

4.2.2 Exponential Distribution

Let us now assume that the lifetimes of the experimental units are i.i.d. exponential random variables with mean θ. If D_1 and D_2 denote the number of failures that occur before time points T_1 and T_2, respectively, then based on the three forms of the sample presented above, the likelihood function is given by

$$L(\theta|\text{data}) = \begin{cases} \frac{n!}{(n-d_1)!}\lambda^{d_1} e^{-\lambda\left\{\sum_{i=1}^{d_1} x_{i:n}+(n-d_1)T_1\right\}} & \text{if} \quad d_1 = r, r+1, \cdots, n, \\ \frac{n!}{(n-r)!}\lambda^{r} e^{-\lambda\left\{\sum_{i=1}^{r} x_{i:n}+(n-r)x_{r:n}\right\}} & \text{if} \quad d_1 = 0, 1, \cdots, r-1,\ d_2 = r, \\ \frac{n!}{(n-d_2)!}\lambda^{d_2} e^{-\lambda\left\{\sum_{i=1}^{d_2} x_{i:n}+(n-d_2)T_2\right\}} & \text{if} \quad d_2 = 0, 1, \cdots, r-1. \end{cases} \tag{101}$$

In this case, the MLE of θ does not exist if $d_2 = 0$, and it exists if $d_2 \geq 1$, and it is given by

$$\widehat{\theta} = \begin{cases} \frac{\sum_{i=1}^{d_1} x_{i:n}+(n-d_1)T_1}{d_1} & \text{if} \quad d_1 = r, r+1, \cdots, n, \\ \frac{\sum_{i=1}^{r} x_{i:n}+(n-r)x_{r:n}}{r} & \text{if} \quad d_1 = 0, 1, \cdots, r-1, \quad d_2 = r, \\ \frac{\sum_{i=1}^{d_2} x_{i:n}+(n-d_2)T_2}{d_2} & \text{if} \quad d_2 = 1, 2, \cdots, r-1. \end{cases} \tag{102}$$

It is possible to derive the exact distribution of $\widehat{\theta}$ in this case as well from the conditional moment generating function. Conditioned on $d_2 \geq 1$, the conditional moment generating function of $\widehat{\theta}$ can be shown to be

$$\begin{aligned} M_{\widehat{\theta}}(\omega) = E(e^{\omega\widehat{\theta}}) &= \frac{1}{1-q_2^n}\left\{\sum_{j=r}^{n}\left(1-\frac{\theta\omega}{j}\right)^{-j}\binom{n}{j} p_{1,j}^{j} q_{1,j}^{n-j}\right. \\ &\quad \left(1-\frac{\theta}{\omega}\right)^{-r}\sum_{j=0}^{r-1}\sum_{l=r-j}^{n-j}\frac{n!}{l!j!(n-l-j)!}p_{1,r}^{j}(q_{1,r}-q_{2,r})^{l} q_{2,r}^{n-l-j}, \\ &\quad \left.\sum_{j=1}^{r-1}\left(1-\frac{\theta\omega}{j}\right)^{-j}\binom{n}{j}p_{2,j}^{j}q_{2,j}^{n-j}\right\}, \quad \omega < \frac{1}{\theta}, \end{aligned} \tag{103}$$

where $q_j = e^{T_j/\theta}$, $j = 1, 2$, $q_{1,j} = 1 - p_{1,j} = e^{-(1-\theta\omega/j)T_1/\theta}$, $j = r, \cdots, n$, and $q_{2,l} = 1 - p_{2,l} = e^{-(1-\theta\omega/l)T_2/\theta}$, $l = 1, 2, \cdots, r$. Along the same lines as before, the PDF of $\widehat{\theta}$ conditioning on $d_2 \geq 1$, can be obtained as

$$f_{\widehat{\theta}}(x) = (1-q_2^n)^{-1}\left\{\sum_{j=1}^{n}\sum_{l=0}^{j} B_{j,l}\, g\left(x - T_{jl}^{*}; \frac{j}{\theta}, j\right)\right.$$

$$+\sum_{j=0}^{r-1}\sum_{l=r-j}^{n-j}\sum_{i_1=0}^{j}\sum_{i_2=0}^{l} B_{jli_1i_2} g\left(x - T^*_{jli_1i_2}; \frac{r}{\theta}, r\right)\Bigg\}, \tag{104}$$

where

$$B_{j,l} = \begin{cases} (-1)^l \binom{n}{j}\binom{j}{l} q_1^{n-j+l} & \text{for} \quad j = r, \cdots, n, \; l = 0, 1 \cdots, j, \\ (-1)^j \binom{n}{j}\binom{j}{l} q_2^{n-j+l} & \text{for} \quad j = 1, 2, \cdots, (r-1), \; l = 0, 1 \cdots, j, \end{cases}$$

$$B_{jli_1i_2} = (-1)^{i_1+i_2}\binom{n}{j}\binom{n-j}{l}\binom{j}{i_1}\binom{l}{i_2} q_1^{l+i_1-i_2} q_2^{n+i_2-j-l}$$

for

$$i_1 = 0, 1, \cdots, j, \; \imath_2 = 0, 1, \cdots, l, \; j = 0, 1, \cdots, r-1, \; l = r-j, \cdots, n-j$$

$$T^*_{j,l} = \begin{cases} \frac{(n-j+l)T_1}{j} & \text{if} \quad j = r, \cdots, n, \; l = 0, 1, \cdots, j, \\ \frac{(n-j+l)T_2}{j} & \text{if} \quad j = 1, \cdots, r-1, \; l = 0, 1, \cdots, j, \end{cases}$$

and

$$T^*_{jli_1i_2} = \frac{(l+i_1-i_2)T_1 + (n+i_2-l-j)T_2}{r},$$

for $j = 0, 1, \cdots, r-1$, $l = r-j, \cdots, n-j$, $i_1 = 0, 1, \cdots, j$, $i_2 = 0, 1, \cdots, l$.

In this case also, it may be observed that the distribution of the MLE of $\widehat{\theta}$ can be written as a weighted sum of the random variables of the type $Y + A$, where Y is a gamma random variable and A is a constant. In this case also the exact confidence interval can be constructed along the same line as before.

4.2.3 Fisher Information

Park and Balakrishnan (2009) derived an explicit expression for the Fisher information contained in a generalized Type-II HCS, for the case of exponential distribution with mean θ, as a sum of single integrals in the form

$$I_{G_2}(\theta) = \frac{1}{\theta^2}\left[n\left(1 - e^{-T_1/\theta}\right) + \sum_{i=1}^{r}\left\{F_i\left(\frac{T_2}{\theta}\right) - F_i\left(\frac{T_1}{\theta}\right)\right\}\right], \tag{105}$$

where $F_i(x)$ is as defined earlier in (14). Park and Balakrishnan (2009) also used this expression to compare the Fisher information contained in different generalized Type-II HCS. Once again, this expression can also be utilized to construct confidence intervals for θ by using $I_{G_2}(\widehat{\theta})$.

5 Unified HCS

Even though generalized Type-I HCS and generalized Type-II HCS are improvements over Type-I HCS and Type-II HCS as described in the preceding Section, yet they have some drawbacks. For example, in the generalized Type-I HCS, due to the certain termination of the life-test at or before time T, we can not guarantee observing r failures. On the other hand, in the generalized Type-II HCS, there is a possibility of not observing any failure at all or observing only few failures until the pre-fixed time T_2, and it therefore has the same problem as the Type-I HCS.

To overcome these problems, Balakrishnan et al. (2009) proposed an *unified hybrid censoring scheme* (*UHCS*), which can be described as follows. Fix integers $k, r \in \{1, \cdots, n\}$ such that $k < r$, and time points $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$. If the k-th failure occurs before time T_1, terminate the experiment at $\min\{\max\{X_{r:n}, T_1\},\ T_2\}$. If the k-th failure occurs between T_1 and T_2, terminate the experiment at $\min\{X_{r:n}, T_2\}$. Finally, if the k-th failure occurs after time T_2, terminate the experiment at $X_{k:n}$. Thus, under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failures, and if this is not the case we can guarantee exactly k failures.

5.1 Form of Data

Under such a UHCS, we will observe one of the following forms of observations:

Case I: $\{0 < x_{k:n} < x_{r:n} < T_1\}$, the experiment is terminated at T_1;

Case II: $\{0 < x_{k:n} < T_1 < x_{r:n} < T_2\}$, the experiment is terminated at $x_{r:n}$;

Case III: $\{0 < x_{k:n} < T_1 < T_2 < x_{r:n}\}$, the experiment is terminated at T_2;

Case IV: $\{0 < T_1 < x_{k:n} < x_{r:n} < T_2\}$, the experiment is terminated at $x_{r:n}$;

Case V: $\{0 < T_1 < x_{k:n} < T_2 < x_{r:n}\}$, the experiment is terminated at T_2;

Case VI: $\{0 < T_1 < T_2 < x_{k:n} < x_{r:n}\}$, the experiment is terminated at $x_{k:n}$.

5.2 Exponential Distribution

Let us now assume that the lifetimes of the experimental units are i.i.d. exponential random variables with mean θ. Then, with D_1 and D_2 denoting the number of failures to occur by times T_1 and T_2, respectively, the likelihood function corresponding to the six cases listed above can be expressed as follows:

$$L(\theta|\text{data}) = \begin{cases} \frac{n!}{(n-d)!\theta^d} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{d} x_{i:n} + (n-d)T_1\right\}} & \text{if} \quad d_1 = d_2 = d = r, \cdots, n, \\ \frac{n!}{(n-r)!\theta^r} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}\right\}} & \text{if} \quad d_1 = k, \cdots, r-1,\ d_2 = r, \\ \frac{n!}{(n-d_2)!\theta^{d_2}} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{d_2} x_{i:n} + (n-d_2)T_2\right\}} & \text{if} \quad k \le d_1 \le d_2 \le r-1, \\ \frac{n!}{(n-r)!\theta^r} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}\right\}} & \text{if} \quad d_1 = 0, \cdots, k-1, d_2 = r, \\ \frac{n!}{(n-d_2)!\theta^{d_2}} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{d_2} x_{i:n} + (n-d_2)T_2\right\}} & \text{if} \quad d_1 = 0, \cdots, k-1, d_2 = k, \cdots r-1, \\ \frac{n!}{(n-k)!\theta^k} e^{-\frac{1}{\theta}\left\{\sum_{i=1}^{k} x_{i:n} + (n-k)x_{k:n}\right\}} & \text{if} \quad d_2 = 0, \cdots, k-1. \end{cases} \tag{106}$$

The MLE of θ can be easily obtained by maximizing the likelihood function in (106), and is given by

$$\widehat{\theta} = \begin{cases} \frac{\sum_{i=1}^{d} x_{i:n} + (n-d)T_1}{d} & \text{if} \quad d_1 = d_2 = d = r, \cdots, n, \\ \frac{\sum_{i=1}^{r} x_{i:n} + (n-r)x_{r:n}}{r} & \text{if} \quad d_1 = k, \cdots, r-1, \ \ d_2 = r, \text{ or } d_1 = 0, \cdots, k-1, d_2 = r, \\ \frac{\sum_{i=1}^{d_2} x_{i:n} + (n-d_2)T_2}{d_2} & \text{if} \quad k \le d_1 \le d_2 \le r-1, \text{ or } d_1 = 0, \cdots, k-1, d_2 = k, \cdots, r-1, \\ \frac{\sum_{i=1}^{k} x_{i:n} + (n-k)x_{k:n}}{k} & \text{if} \quad d_2 = 0, \cdots, k-1. \end{cases} \tag{107}$$

Using the same conditional moment generating function approach, Balakrishnan et al. (2009) derived the exact PDF of $\widehat{\theta}$ as follows:

$$\begin{aligned} f_{\widehat{\theta}}(x) &= \frac{1}{1-q_2^n}\left\{\sum_{l=r}^{n}\sum_{t=0}^{l} B_{l,t}\, g\left(x - T^*_{l.t}; \frac{l}{\theta}, l\right) + \sum_{j=0}^{r-1}\sum_{l=r-j}^{n-j}\sum_{t=0}^{j}\sum_{u=0}^{l} B_{j,l,t,u}\, g\left(x - T^*_{j,l,t,u}; \frac{r}{\theta}, r\right)\right. \\ &\quad \left. + \sum_{l=k}^{r-1}\sum_{j=0}^{l}\sum_{h=0}^{l-j}\sum_{v=0}^{j} B_{l,j,h,v}\, g\left(x - T^*_{l,j,h,v}; \frac{l}{\theta}, l\right) + \sum_{l=0}^{k-1}\sum_{g=0}^{l} B_{l,g}\, g\left(x - T^*_{l,g}; \frac{k}{\theta}, k\right)\right\}, \end{aligned} \tag{108}$$

where

$$\begin{aligned} B_{l,t} &= \binom{n}{l}\binom{l}{t}(-1)^t q_1^{n-l+t}, \quad T^*_{l,t} = \frac{T_1(n-l+t)}{l}, \\ B_{l,g} &= \binom{n}{l}\binom{l}{t}(-1)^g q_2^{n-l+g}, \quad T^*_{l,g} = \frac{T_2(n-l+g)}{k}, \\ B_{j,l,t,u} &= \binom{n}{j,l}\binom{j}{t}\binom{l}{u}(-1)^{t+u} q_1^{l-u+t} q_2^{n-j-l+u}, \\ B_{l,j,h,v} &= \binom{n}{l}\binom{l-j}{h}\binom{l}{j}\binom{j}{v}(-1)^{h+v} q_1^{l-j-h+v} q_2^{n-l+h}, \end{aligned}$$

with $q_j = e^{-T_j/\theta}$, for $j = 1, 2$.

Under the same assumption that $P_\theta(\widehat{\theta} > b)$ is an increasing function of θ as before, Balakrishnan et al. (2009) discussed the construction of two-sided confidence intervals for θ. An explicit expression for the Fisher information in UHCS has been derived by Park and Balakrishnan (2012). In fact, in doing so, they have also considered a more general censoring scheme which includes UHCS as a special case and derived the Fisher information for this case as well.

Open Problem 1: Under such a general censoring scheme considered by Park and Balakrishnan (2012), exact inferential results have not been developed yet.

6 Hybrid Life-Tests Under Competing Risks

In medical and reliability studies, it is quite common to have more than one risk factor to cause the failure of the subject/unit under study. In this case, the experimenter will often be interested in the assessment of a specific risk in the presence of other risk factors. The data in such cases usually consist of a failure time and an indicator for the specific cause of failure. Such a model is referred to as a *competing risks model* in the statistical literature. For a general introduction to the competing risks problem and associated inferential methods, the readers are referred to the monographs by Crowder (2001) and Pintilie (2006). The causes of failure may be assumed to be dependent or independent. Even though the assumption of independence may appear to be somewhat restrictive, there is a problem of identifiability in the case of dependent risks without the presence of covariates. For this reason, the latent failure time model has been discussed extensively in the literature starting from the pioneering work of Cox (1959), wherein it is assumed that the failure times due to different risks are indeed independently distributed.

Kundu and Gupta (2007) considered the analysis of hybrid life-tests in the presence of competing risks. They assumed that the lifetimes corresponding to the different causes of failures are independent exponential random variables. They discussed only two causes of failures, but as mentioned by the authors, the results can be readily extended to the case when there are more than two causes of failures. With T_i denoting the lifetime of Cause i, for $i = 1, 2$, it is assumed that T_1 and T_2 are independent exponential random variables with means θ_1 and θ_2, respectively. Moreover, the observed data will be of the form (Z, Δ), where

$$Z = \min\{T_1, T_2\}, \quad \Delta = \begin{cases} 1 & \text{if} \quad T_1 < T_2, \\ 2 & \text{if} \quad T_2 < T_1. \end{cases} \tag{109}$$

With n units placed on a life-test, and each unit being exposed to two different types of risks, the experiment would terminate when a pre-fixed number R out of n units have failed or when a pre-fixed time T has been reached.

6.1 Form of Data

Now, let $(z_1, \delta_1), \cdots, (z_n, \delta_n)$ be n independent and identically distributed variables from (Z, Δ). Further, let $z_{1:n} < \cdots < z_{n:n}$ be the ordered values of $z_1, \cdots, z_n$. Finally, let $T^* = \min\{z_{R:n}, T\}$ denote the termination time of the life-test. We then have the following forms of observations:

Case I: $\{(z_{1:n}, \delta_1), \cdots, (z_{R:n}, \delta_n)\}$ if $z_{R:n} < T$;

Case II: $\{(z_{1:n}, \delta_1), \cdots, (z_{D:n}, \delta_D)\}$ if $z_{D:n} < T < z_{D+1:n}$.

Here, D denotes the number of observed failures up to time T for Case II. Finally, let D_1 and D_2 denote the number of failures due to Causes 1 and 2, respectively.

6.2 Exponential Distribution

Now, based on the assumption that T_i has exponential distribution with mean θ_i, for $i = 1$, 2, the log-likelihood function of the observed data (without the additive constant) is

$$l(\theta_1, \theta_2) = -D_1 \ln \theta_1 - D_2 \ln \theta_2 - W \left(\frac{1}{\theta_1} + \frac{1}{\theta_2} \right), \tag{110}$$

where $D_2 = R - D_1$ for Case I and $D - D_1$ for Case II, and $W = \sum_{i=1}^{R} z_{i:n} + (n-R)z_{R:n}$ for Case I and $\sum_{i=1}^{D} z_{i:n} + (n-D)T$ for Case II representing the total time on test (TTT). It is of interest to mention here that (W, D_1) is a joint minimal sufficient statistic (θ_1, θ_2) for Case I, while (W, D_1, D) is a joint minimal sufficient statistic for (θ_1, θ_2) for Case II.

It is clear from (110), the MLEs of θ_1 and θ_2 exist only when $D_1 > 0$ and $D_2 > 0$, and are given by

$$\widehat{\theta}_1 = \frac{W}{D_1} \quad \text{and} \quad \widehat{\theta}_2 = \frac{W}{D_2}. \tag{111}$$

Kundu and Gupta (2007) then derived the conditional PDFs of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, conditioned on $D_1 > 0$ and $D_2 > 0$, once again by utilizing the conditional moment generating function approach. The exact expressions of these conditional densities can be found in Kundu and Gupta (2007), using which they have also discussed the construction of exact confidence intervals for θ_1 and θ_2.

6.3 Bayesian Inference

Kundu and Gupta (2007) also discussed the Bayesian inference for the parameters θ_1 and θ_2 by assuming as priors inverse gamma distributions with parameters (a_1, b_1) and (a_2, b_2) for θ_1 and θ_2, respectively. An inverse gamma distribution with parameters (a, b), where $a > 0, b > 0$, has the PDF

$$f_{IG}(x; a, b) = \frac{b^a}{\Gamma(a)} e^{-\frac{b}{x}} x^{-a-1}, \quad x > 0, \tag{112}$$

and henceforth we shall denote it by IG(a, b).

With the assumed priors for θ_1 and θ_2, the joint posterior density function of θ_1 and θ_2, given the data, is given by

$$l(\theta_1, \theta_2 | Data) \propto \frac{1}{\theta_1^{D_1+a_1+1}} e^{-\frac{W+b_1}{\theta_1}} \; \frac{1}{\theta_2^{D_2+a_2+1}} e^{-\frac{W+b_2}{\theta_2}}. \tag{113}$$

It is clear from (113) that the posterior densities of θ_1 and θ_2 are independent, and that the posterior density of θ_1 is IG$(D_1 + a_1, W + b_1)$, and that of θ_2 is IG$(D_2 + a_2, W + b_2)$. Hence, the Bayesian estimates of θ_1 and θ_2 under the squared-error loss functions are readily obtained as

$$\widehat{\theta}_{1,Bayes} = \frac{W + b_1}{D_1 + a_1} \quad \text{and} \quad \widehat{\theta}_{2,Bayes} = \frac{W + b_2}{D_2 + a_2}. \tag{114}$$

As expected, when $a_1 = b_1 = a_2 = b_2 = 0$, the Bayesian estimates in (114) reduce to the corresponding MLEs. Furthermore,

$$Z_1 = \frac{2(W + b_1)}{\theta_1} \quad \text{and} \quad Z_2 = \frac{2(W + b_2)}{\theta_2} \tag{115}$$

a posteriori follow $\chi^2_{2(D_1+a_1)}$ and $\chi^2_{2(D_2+a_2)}$, respectively, provided $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are positive integers. Therefore, the 100(1-α)% credible intervals for θ_1 and θ_2 are given by

$$\left[\frac{2(W + b_1)}{\chi^2_2(D_1 + a_1), 1 - \alpha/2}, \frac{2(W + b_1)}{\chi^2_2(D_1 + a_1), \alpha/2}\right] \quad \text{and} \quad \left[\frac{2(W + b_2)}{\chi^2_2(D_2 + a_2), 1 - \alpha/2}, \frac{2(W + b_2)}{\chi^2_2(D_2 + a_2), \alpha/2}\right]$$

for $D_1 + a_1 > 0$ and $D_2 + a_2 > 0$, respectively. If $2(D_1 + a_1)$ and $2(D_2 + a_2)$ are not integers, then the corresponding credible intervals can be obtained using the gamma distribution.

6.4 Comparison of Different Methods

Kundu and Gupta (2007) carried out an extensive Monte Carlo simulation study to compare the performance of different confidence/credible intervals in terms of coverage probabilities and widths of the confidence/credible intervals. They observed that the asymptotic confidence intervals perform well even when the sample size is as small as 25. The widths of the asymptotic confidence intervals are the smallest, but their coverage probabilities are slightly lower than the nominal level. The performance of the percentile bootstrap confidence intervals is quite good as well, with the average widths of the percentile bootstrap confidence intervals being close to those of the asymptotic method, and the coverage probabilities being quite close to the nominal level. The Bayesian credible intervals, while always maintaining the nominal levels, have their widths to be slightly larger than those of the asymptotic and percentile bootstrap intervals. Computationally, the asymptotic confidence intervals and the Bayesian credible intervals are simpler and more efficient than the percentile bootstrap confidence intervals. Thus, overall, the asymptotic method is the one that these authors recommend, and the percentile bootstrap method if the computational burden is not a major concern.

7 Progressive Hybrid Censoring Schemes

7.1 Type-I PHCS

Type-I progressive hybrid censoring scheme (*Type-I PHCS*), discussed by Kundu and Joarder (2006) and Childs et al. (2008), can be described as follows. Suppose n units are placed on a life-test and $(R_1, \cdots, R_m)$ is the pre-fixed progressive censoring scheme. Under the Type-II progressive censoring scheme, at the time of the first failure, R_1 of the $n-1$ surviving units are randomly withdrawn from the life-test, then at the time of the second failure R_2 of the $n - R_1 - 2$ surviving units are withdrawn, and so on, and finally at the time of the m-th failure all $R_m = n - R_1 - \cdots - R_{m-1} - m$ surviving units are withdrawn from the life-test. Since R_i's are pre-fixed, let us denote these failure times by $x_{1:n} \leq \cdots \leq x_{m:n}$, though their distributions depend on R_i's. Then, for a Type-I PHCS, we will have the termination time of the experiment to be $T^* = \min\{x_{m:n}, T\}$, where T is a pre-fixed time point. In this case, the experiment will stop at $x_{m:n}$ if $x_{m:n}$ occurs before time T, and otherwise it will stop at time T. Clearly, the experimental time can not exceed T, and due to the presence of progressive censoring, this sampling scheme will likely provide more information about the tail of the lifetime distribution.

Under such a Type-I PHCS, the available data will be one of the following two forms:

Case I: $\{x_{1:n} < \cdots < x_{m:n}\}$ if $x_{m:n} \le T$;

Case II: $\{x_{1:n} < \cdots < x_{D:n}\}$ if $x_{m:n} > T$,

where D denotes the number of failures that occur before time T.

Under the assumption that the lifetime distribution of the n units is exponential with mean θ, the MLE of θ exists when $D > 0$, and is given by

$$\widehat{\theta} = \begin{cases} \frac{1}{D}\left\{\sum_{i=1}^{D}(R_i+1)x_{i:n} + R_{D+1}^{*}T\right\} & \text{if} \quad D = 1, 2, \cdots, m-1, \\ \frac{1}{m}\sum_{i=1}^{m}(R_i+1)x_{i:n} & \text{if} \quad D = m, \end{cases} \tag{116}$$

where $R_{D+1}^{*} = \sum_{k=D+1}^{m}(R_k+1)$.

Childs et al. (2008) derived the exact conditional density of $\widehat{\theta}$, conditioned on $D \ge 1$, by using the conditional moment generating function approach, as

$$f_{\widehat{\theta}}(x) = (1-q^n)^{-1}\sum_{d=1}^{m} c'(n,d) \times \sum_{i=0}^{d} c_{i,d}(R_1+1, \cdots, R_d+1) q^{R_d^{*}-i+1} g\left(x - R_{d-i+1}^{*}T/d; \frac{1}{\theta}, d\right), \tag{117}$$

where $q = e^{-T/\theta}$, $g(x;\cdot)$ is the gamma density defined earlier in (6), and for the vector $(a_1, \cdots, a_r)$,

$$c_{i,r}(a_1, \cdots, a_r) = \frac{(-1)^i}{\left\{\prod_{j=1}^{i}\sum_{k=r-i+1}^{r-i+j} a_k\right\}\left\{\prod_{j=1}^{r-i} a_k\right\}}, \quad c'(n,d) = \prod_{j=1}^{d}\sum_{k=j}^{m}(R_k+1), \quad d = 1, \cdots, m.$$

Childs et al. (2008), by using the conditional PDF in (117), discussed the exact confidence intervals for θ once again under the assumption that $P_\theta(\widehat{\theta} > b)$ is a monotone increasing function of θ.

In the case of Type-I PHCS, for the exponential distribution with mean θ, Park et al. (2011) showed that the Fisher information can be expressed as

$$\frac{1}{\theta^2}\sum_{i=1}^{m} F_{i:m:n}\left(\frac{T}{\theta}\right),$$

where $F_{i:m:n}(x)$ is the CDF of the i-th progressively Type-II censored order statistic.

Open Problem 2: Even though the exact confidence interval for θ has been developed in this case under the assumption that $P_\theta(\widehat{\theta} > b)$ is a monotone increasing function of θ, a formal proof of this monotonicity result remains open!

7.2 Type-II PHCS

To overcome the obvious drawback of the Type-I PHCS that the MLE may not always exist, Childs et al. (2008) introduced the *Type-II progressive hybrid censoring scheme* (*Type-II PHCS*). With the notation introduced in the preceding subsection, the Type-II PHCS involves the termination of the life-test at time $T^* = \max\{x_{m:n}, T\}$. Here again, let D denote the number of failures that occur before time T. Then, if $x_{m:n} > T$, the experiment would terminate at the m-th failure, with the withdrawal of units occurring after each failure according to the pre-fixed progressive censoring scheme $(R_1, \cdots, R_m)$. However, if $x_{m:n} < T$, then instead of terminating the experiment by removing all remaining R_m units after the m-th failure, the experiment would continue to observe failures without any further withdrawals up to time T. Thus, in this case, we have $R_m = R_{m+1} = \cdots = R_D = 0$.

Based on the above Type-II PHCS, the observed data will be one of the following two forms:

Case I: $\{x_{1:n} < \cdots < x_{m:n} < x_{m+1:n} < \cdots < x_{D:n}\}$ if $x_{m:n} < T$;

Case II: $\{x_{1:n} < \cdots < x_{m:n}\}$ if $x_{m:n} \geq T$,

where D denotes the number of failures before time T.

Under the assumption that the lifetimes of the units are exponential with mean θ, the MLE of θ always exists in this case and is given by

$$\widehat{\theta} = \begin{cases} \frac{1}{D}\left\{\sum_{i=1}^{D}(R_i+1)x_{i:n}\right\} & \text{if} \quad D = 0, 1, \cdots, m-1, \\ \frac{1}{D}\left\{\sum_{i=1}^{m}(R_i+1)x_{i:n} + \sum_{j=m+1}^{D} x_{j:n} + R'_D T\right\} & \text{if} \quad D = m, \cdots, n - \sum_{i=1}^{m-1} R_i, \end{cases} \tag{118}$$

where $R'_D = n - D - \sum_{k=1}^{m-1} R_k$ and $R_m = 0$, if $D \geq m$.

Childs et al. (2008) then derived the exact density of $\widehat{\theta}$, by using the moment generating function approach, as

$$\begin{aligned} f_{\widehat{\theta}}(x) &= \sum_{d=0}^{m-1}\sum_{i=0}^{d} c'(n,d) q^{R^*_{d-i+c}}\, c_{i,d}(R_1+1,\cdots,R_d+1) g\left(x - R^*_{d-i+1}T/m; \frac{1}{\theta}, m\right) \\ &+ \sum_{d=m}^{n-R_1-\cdots-R_m-1} c'(n,d) \sum_{i=0}^{d} c_{i,d}((R_1+1,\cdots,R_d+1) \\ &\quad \times q^{R'_d + b_{i,d}} g\left(x - [R'_d + b_{i,d}(R_1+1,\cdots,R_d+1)]T/d; \frac{1}{\theta}, d\right), \end{aligned} \tag{119}$$

where $g(x;\cdot)$ is the gamma density as defined before, and for the vector $(a_1,\cdots,a_r)$

$$b_{i,r}(a_1,\cdots,a_r) = \sum_{k=r-i+1}^{r} a_k,$$

with the usual convention that $\sum_{j=i}^{i-1} d_j = 0$.

Open Problem 3: Even though Childs et al. (2008) used the PDF in (119) to obtain exact confidence intervals for θ using the assumption that $P_\theta(\widehat{\theta} > b)$ is a monotone increasing function of θ, here again, a formal proof of the monotonicity of this tail probability remains open!

In the case of Type-II PHCS, for the exponential distribution with mean θ, Park et al. (2011) showed that the Fisher information can be expressed as

$$\frac{1}{\theta^2}\left\{\sum_{i=1}^{m} F_{i:m:n}\left(\frac{T}{\theta}\right) + \sum_{i=n-R_m+1}^{n} F_{i:n}\left(\frac{T}{\theta}\right)\right\},$$

where $F_{i:m:n}(x)$ and $F_{i:n}(x)$ are the CDF of the i-th progressively Type-II censored order statistic and the i-th order statistic, respectively.

Recently, Mokhtari et al. (2011) have developed inferential methods for the parameters of a Weibull distribution based on progressively hybrid censored data. They have examined the performance of the MLEs and AMLEs as well as Bayesian estimates for the model parameters. In addition, they have evaluated the performance of confidence intervals obtained by using the asymptotic property of the MLEs and the Bayesian credible intervals.

8 Adaptive Progressive Censoring Scheme

Ng et al. (2009) introduced a new censoring scheme, which is a mixture of Type-I and Type-II progressive censoring schemes, called the *adaptive progressive censoring scheme*. In this case, the effective sample size m is fixed in advance and the corresponding progressive scheme is given, but the number of items progressively removed from the experiment at each failure may change during the experiment. If the experimental time exceeds a pre-fixed time T, but the number of observed failures has not reached m yet, the experiment should be terminated as soon as possible by adjusting the number of items progressively removed from the experiment at each failure in such a way that the desired level of efficiency of the estimate(s) of the parameter(s) can be achieved.

Such an adaptive progressive censoring scheme can be defined as follows. Let n items be placed on a life-test, and the effective sample size $m < n$ be fixed in advance. Moreover, let the progressive censoring scheme $(R_1, \cdots, R_m)$ be set before the start of the experiment, but the values of some of the R_i's may change in an adaptive manner during the experiment. Suppose the experimenter fixes a time T, which is an ideal test duration, but the test itself may be allowed to run over time T. Let us denote the m completely observed failure times by $X_{i:m:n}$, $i = 1, \cdots, m$. If the m-th progressively censored failure time occurs before time T (i.e., $X_{m:m:n} < T$), the experiment will be terminated at time $X_{m:m:n}$. Otherwise, once the experimental time passes time T, but the number of observed failures has not yet reached m, terminate the experiment as soon as possible satisfying certain inferential efficiency criterion. Thus, this setting can be viewed as a design, in which one would ideally like to have m

observed failure times for efficiency of inference, and at the same time have the total time on test to be not too far away from the ideal test duration T.

Ng et al. (2009), after introducing the above adaptive progressive censoring scheme, developed inferential methods for the case when the lifetime distribution is exponential. These authors observed that the MLE always exist in this case, although the exact distribution of the MLE becomes intractable. For this reason, examination of the finite-sample properties of the MLE can be made only through Monte Carlo simulations. Asymptotic distribution of the MLE has been established using which approximate confidence intervals for the exponential parameter have also been developed. Bayesian inference has been developed based on the assumption of a conjugate gamma prior. The Bayesian estimate and the associate credible interval can be obtained explicitly. Extensive Monte Carlo simulations have been performed by the authors for comparing the performance of these methods, through which it has been observed that the Bayesian estimate with non-informative prior works quite well and is computationally easier as well. Lin et al. (2009) considered the adaptive progressive censoring scheme when the lifetime distribution is Weibull, and discussed the corresponding inferential issues. As one would expect, the MLEs of the Weibull parameters can not be obtained explicitly, and would require the use of some numerical method such as Newton-Raphson. They have also discussed confidence intervals for the model parameters through the use of the asymptotic distribution of the MLEs as well as by the bootstrap method.

9 Hybrid Censoring Schemes for Step-Stress Model

In many situations, it may be difficult to collect data on the lifetime of a product under normal operating conditions as the product may be highly reliable under normal operating conditions. For this reason, reliability engineers often resort to accelerated life-testing (ALT) experiments in order to force the test units to fail more quickly than under normal operating conditions. In this situation, they may increase stress on the units by using factors such as temperature, voltage and load. A special class of ALT is *step-stress testing* wherein the experimenter changes the stress in incremental steps during the experiment. Extensive work has been done on a simple step-stress model under different censoring schemes; see, for example, Balakrishnan et al. (2009), Balakrishnan et al. (2007), Bagdonavicius and Nikulin (2002), and the recent review article by Balakrishnan (2009).

Type-I and Type-II hybrid censoring schemes for step-stress models have been discussed by Balakrishnan and Xie (2007a, 2007b). Suppose n identical units are placed on a life-test at an initial stress level s_0, under an m-step-stress model, and that the successive failure times are recorded. The stress levels are changed to $s_1, \cdots, s_m$ at pre-fixed times, say, $T_1 < \cdots < T_m$. Consider a censoring scheme in which the experiment is terminated at time point T_{m+1} when the r-th failure does not occur before time point T_{m+1}, and otherwise the experiment gets terminated at the r-th failure (for $1 \leq r \leq n$), wherein T_{m+1} ($> T_m$) is a pre-fixed time. Thus, the termination time of this experiment is $T^*_{m+1} = \min\{X_{r:n}, T_{m+1}\}$, and this is indeed Type-I HCS for a step-stress model. Instead, if the termination time of the experiment is $T^{**}_{m+1} = \max\{X_{r:n}, T_{m+1}\}$, then it is the Type-II HCS for a step-stress model.

By assuming the cumulative exposure model of Nelson (1980), Balakrishnan and Xie (2007a, 2007b) considered simple step-stress models (in which stress level changes only once in the experiment) for Type-I and Type-II HCS. For the case when the lifetime distribution of units is exponential at both stress levels, they obtained the MLEs of the model parameters explicitly and then derived the exact distributions of the MLEs by using the conditional moment generating function (by conditioning on the event that at least one failure occurs at each stress level, which is the condition required for the existence of the MLEs of the two parameters of the model) approach. Based on the exact distributions of the MLEs, they also developed exact confidence intervals for the model parameters in both cases. They also studied confidence intervals obtained by using the asymptotic distributions of the MLEs, and by the use of bootstrap method. Through an extensive Monte Carlo simulation study, they observed that both the asymptotic and bootstrap approaches are not satisfactory in case of small sample sizes and in this case the exact method is to be preferred. For large sample sizes, however, the exact method becomes computationally involved and in this case the bootstrap method is seen to be quite satisfactory and is therefore to be preferred.

It is important to mention here that while developing exact confidence intervals for the model parameters, Balakrishnan et al. (2007), Balakrishnan and Xie (2007a, 2007b) and Balakrishnan et al. (2009) made use of the exact conditional density functions and then assumed the tail probabilities $P_{\theta_1}(\widehat{\theta_1} > a)$ and $P_{\theta_2}(\widehat{\theta_2} > a)$ to be monotone increasing functions of θ_1 and θ_2, respectively. In the case of Type-I and Type-II censoring, this monotonicity property was formally proved recently by Balakrishnan and Iliopoulos (2010).

Open Problem 4: Though the monotonicity property has been established for the Type-I and Type-II censoring situations, the required monotonicity property for the tail probabilities in the cases when the data are Type-I and Type-II HCS remain as open problems!

Kateri and Balakrishnan (2008) generalized the results for the exponential distribution by considering the case of Weibull distributed lifetimes. They then discussed the maximum likelihood estimation of the model parameters under a simple step-stress model for the case when the available data are Type-II censored.

Open Problem 5: For Weibull distributed lifetimes, inferential results for the model parameters of a simple step-stress model have not been developed yet for the case when the available data are hybrid censored! In fact, the same holds for many other important lifetime distributions such as gamma and log-normal!

10 Some Additional Topics

10.1 Optimal Censoring Scheme and Reliability Plans

In this subsection, we shall discuss how the optimal Type-I and Type-II HCS can be determined. Kundu (2007) and Banerjee and Kundu (2008) have provided such a discussion for the case of Weibull distribution. In practice, it is quite common to consider choosing the 'optimal censoring scheme' from a class of possible censoring schemes. Here, possible schemes mean, for a fixed sample size n, different choices of r and T, where $1 \leq r \leq n$ and $0 < T < \infty$. For convenience in notation, let us denote, for fixed n and for a given r and T, the corresponding HCS shortly by (r, T). Now, to compare two different schemes, say (r_1, T_1) and (r_2, T_2), we could consider the scheme (r_1, T_1) to be better than (r_2, T_2) if, for example, (r_1, T_1) provides more information than (r_2, T_2) about the model parameters. It is, therefore, important to define an information measure for a given censoring scheme. Note that if only one parameter is present, then it can be defined as a measure which is inversely proportional to the asymptotic variance of the unknown parameter; see Zhang and Meeker (2005). But, if the lifetime distribution has more than one parameter, then there is no unique and natural way to define the information. Some of the common choices that have been discussed in the literature are the trace and determinant of the Fisher information matrix. But, unfortunately, they are not scale-invariant; see, for example, Gupta and Kundu (2006).

One way to define the information measure for a particular sampling scheme is as the inverse of the asymptotic variance of the estimator of the $100p$-th quantile obtained from that particular censoring scheme; see Zhang and Meeker (2005). Since this information measure depends on p, Gupta and Kundu (2006) [see also Kundu (2008)] proposed the following information measure:

$$I\{(r,T)\} = \left[\int_0^1 \mathrm{Var}((r,T)_p)dp\right]^{-1}, \tag{120}$$

where $\mathrm{Var}((r,T)_p)$ denotes the asymptotic variance of the estimator of the $100p$-th quantile obtained from the hybrid censoring scheme (r,T). A more general information measure of a censoring scheme (r,T) can be defined as

$$I^*\{(r,T)\} = \left[\int_0^1 w(p)\mathrm{Var}((r,T)_p)dp\right]^{-1}, \tag{121}$$

where $w(p)$ is a non-negative weight function defined over $(0,1)$. It is easy to see that the information measure of Zhang and Meeker (2005) is a particular case of (121).

Among the different HCS, if one wants to choose that particular HCS which has maximum (120) or (121), obviously the choice would be $r = n$ and $T = \infty$. Therefore, for getting a non-trivial solution, some restriction needs to be posed involving the duration of the experiment. One way to bring the time into consideration is to introduce a cost factor on time, and then to maximize

$$I\{(r,T)\} - C \times \text{expected duration of the experiment} \tag{122}$$

or

$$I^*\{(r,T)\} - C \times \text{expected duration of the experiment}, \tag{123}$$

with respect to different HCS (r,T). Here, the constant $C > 0$ is the known cost per unit time of the experiment. This approach has been used under the Bayesian framework for the exponential distribution under Type-I censoring by Yeh (1994) and Lin et al. (2002). Alternatively, one may want to choose that scheme which provides the maximum $I\{(r,T)\}$ or $I^*\{(r,T)\}$, among the schemes with a given maximum expected duration of the experiment. Therefore, for a given expected duration of the experiment, one first chooses different schemes which have the same expected duration and this can be done by a discrete search on r. Then among those schemes, one needs to choose that scheme which has the maximum $I\{(r,T)\}$ or $I^*\{(r,T)\}$, which once again has to be performed by discrete optimization.

As mentioned above, Yeh (1994) and Lin et al. (2002) discussed the construction of reliability sampling plans based on Type-I censored data from an exponential distribution in the Bayesian framework by assuming a polynomial loss function. Following the work of Yeh (1994), Huang and Lin (2002) proposed a new decision function and considered the cost of unit time in the loss function. Lin et al. (2007) considered the construction of an optimal Bayesian sampling plan once again for the exponential distribution based on Type-I censored sample by basing it on the exact conditional PDF of the MLE $\widehat{\theta}$, and assuming a quadratic loss function. They then showed that the minimum Bayes risk of their sampling plan is less than that of Yeh's sampling plan. This work has been subsequently generalized by Lin et al. (2008a) to the case of Type-I HCS and Type-II HCS [see also the work of Liang and Yang (2012)], and to the more general case of Type-I PHCS and Type-II PHCS by Lin et al. (2011) recently.

10.2 Acceptance Sampling Plans

Acceptance sampling plans play a key role in reliability analysis and quality control. An acceptance sampling plan uses life-test procedures to suggest rules for either accepting or rejecting a lot of units based on the observed lifetime data from the sample. Jeong et al. (1996) developed hybrid sampling plans for the exponential distribution which meets the producer's and consumer's risks simultaneously. In any acceptance sampling plan, two kinds of risks exist, one is the producer's risk (denoted by α) and the other is the consumer's risk (denoted by β). In the case of exponential distribution, the problem can be formulated as follows.

Based on the data from the life-test, we want to test the following hypotheses on the mean lifetime θ:

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1 \ (< \theta_0). \tag{124}$$

Jeong et al. (1996) proposed the following acceptance sampling plan based on a HCS. Suppose n identical units are placed on a life-test at time 0. For a given r and T, if the r-th failure occurs before time T, then H_0 is rejected, while if the r-th failure does not occur before time T, then H_0 is not rejected. Clearly, r and T should be so chosen that the following two conditions are satisfied:

$$P(\text{Accept } H_0 | \theta = \theta_0) \geq 1 - \alpha \quad \text{and} \quad P(\text{Accept } H_0 | \theta = \theta_1) \leq \beta. \tag{125}$$

Jeong et al. (1996) provided an algorithm for determining r and T for a given n, α and β, and also provided extensive tables of values of r and T for different choices of n, α and β.

10.3 Test of Exponentiality Against Weibull

Das and Nag (2002) considered the test of exponentiality (with constant hazard rate) against Weibull alternatives, which may have increasing or decreasing hazard rate depending on the value of the shape parameter. They discussed this testing problem assuming that the observed data are Type-I HCS. Considering the Weibull distribution with PDF

$$f(x;\beta,\lambda) = \frac{\beta}{\lambda} x^{\beta-1} e^{-x^\beta/\lambda}, \quad x > 0,\ \beta > 0,\ \lambda > 0, \tag{126}$$

the problem becomes one of the following two testing problems:

$$H_0 : \beta = 1 \ \text{ vs. } \ H_1 : \beta < 1 \text{ (decreasing failure rate)} \tag{127}$$

or

$$H_0 : \beta = 1 \ \text{ vs. } \ H_1 : \beta > 1 \text{ (increasing failure rate).} \tag{128}$$

Das and Nag (2002) proposed the following test procedure under the assumption that the scale parameter λ is known (without loss of generality, it can be taken to be 1). Terminate the test as soon as the pre-fixed number of failures (r) occur, or a pre-fixed time $T_{r,\alpha}$ (which ever is sooner) is reached. Here, α denotes the size of the test, and the pre-fixed time $T_{r,\alpha}$ depends on both r and α. Das and Nag (2002) have provided extensive tables for optimal values of $(r, T_{r,\alpha})$ for different sample sizes varying from 5 to 200, and for the choices of α as 0.01, 0.05 and 0.10.

Open Problem 6: Though there are many different forms of formal goodness-of-fit tests based on the conventional Type-I, Type-II and progressively censored samples for exponential as well as other lifetime distributions [see D'Agostino and Stephens (1986) for an overview, and also the recent works of Balakrishnan and Lin (2003), Balakrishnan et al. (2004), Balakrishnan et al. (2007), Lin et al. (2008b), Habibi Rad et al. (2011), Pakyari and Balakrishnan (2012a,b)], the development of efficient goodness-of-fit procedures for the case of HCS has not been done yet, and it seems much work remains to be done in this direction!

10.4 Bayesian Prediction

In a series of papers, Balakrishnan and Shafay (2012) and Shafay and Balakrishnan (2012a, 2012b) recently developed Bayesian prediction intervals based on different forms of hybrid censored data. They considered both one- and two-sample Bayesian prediction intervals based on Type-I, Type-II and generalized Type-I HCS.

In many practical problems, one often would like to use the past data to predict a future observation from the same population. One way to do it is to present an interval that will contain the future observation with a specified probability, and such an interval is called the *prediction interval*. AL-Hussaini (1999) provided a number of references on the applications of Bayesian prediction in different areas of applied statistics. There are primarily two types of prediction problems that are of practical interest: (a) One-sample prediction, and (b) Two-sample prediction. They can be briefly described as follows. Let $T_{(1)} < \cdots < T_{(r)}$

and $T_{(r+1)} < \cdots < T_{(n)}$ represent the informative sample and a future sample, respectively. A one-sample prediction problem involves the prediction of the future order statistics $T_{(k)}$ (for $r < k \leq n$), based on the observed order statistics $T_{(1)} < \cdots < T_{(r)}$. On the other hand, if $T_{(1)} < \cdots < T_{(r)}$ and $Y_{(1)} < \cdots < Y_{(m)}$ represent the informative sample and an independent future ordered sample, respectively, then a two-sample prediction problem involves the prediction of the order statistics $Y_{(1)} < \cdots < Y_{(m)}$ of a future sample arising from the same distribution.

Balakrishnan and Shafay (2012) and Shafay and Balakrishnan (2012a, 2012b) assumed a fairly general form of the PDF of T_i's as

$$f(x|\theta) = \lambda'(x;\theta)e^{-\lambda(x;\theta)}, \tag{129}$$

where θ is either a scalar or a vector, and $\lambda(x;\theta)$ is continuous, monotonically increasing and differentiable function, with $\lambda(x;\theta) \to 0$ as $x \to -\infty$ and $\lambda(x;\theta) \to \infty$ as $x \to \infty$. Moreover, $\lambda'(x;\theta)$ is the derivative of $\lambda(x;\theta)$ with respect to x. It may be noted that several well-known distributions can be obtained as special cases of the PDF in (129). For example, exponential, Weibull, Pareto and Burr Type XII can be obtained as special cases of (129) by taking $\lambda(x;\theta) = x\theta$, $\lambda(x;\theta) = \alpha x^{\beta}$, $\lambda(x;\theta) = -\alpha \ln(\beta/x)$ and $\lambda(x;\theta) = \alpha \ln(1 + x^{\beta})$, respectively.

Following the suggestion of Al-Hussaini (1999), Balakrishnan and Shafay (2012) and Shafay and Balakrishnan (2012a, 2012b) considered the following conjugate prior of θ

$$\pi(\theta;\delta) \propto C(\theta,\delta)e^{-D(\theta;\delta)}, \tag{130}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution under consideration and δ is the vector of hyper-parameters.

Based on the prior in (130), Balakrishnan and Shafay (2012) and Shafay and Balakrishnan (2012a, 2012b) developed a general procedure for determining the Bayesian prediction intervals for both one-sample and two-sample problems based on different forms of HCS. They have illustrated their procedure for the one-parameter exponential and the two-parameter Pareto distributions for which the conjugate priors exist. In all cases considered, the Bayesian prediction intervals for the one-sample problem can not be obtained in explicit forms. For this reason, they used the Markov Chain Monte Carlo (MCMC) technique to compute predictive intervals in this case, and showed that the MCMC technique performs very well. In the two-sample case, however, the Bayesian prediction intervals can be derived in explicit forms.

Open Problem 7: The derivation of likelihood-based prediction intervals in the case of HCS remains open, however!

11 Illustrative Examples

In this section, we present a few numerical examples in order to illustrate some of the inferential results described in the preceding sections and also to highlight the usefulness of some of the forms of hybrid censoring scheme discussed in this article.

Example 11.1 Chen and Bhattacharyya (1988) considered the following failure times [taken from Barlow et al. (1968)] which involved $n = 10$ units on a time-censored life-testing experiment with $T = 50$:

$$4, 9, 11, 18, 27, 38.$$

With these data, by taking $T = 50$ and $r = 4, 6$ and 8, they produced Type-I HCS. Then, by assuming a one-parameter exponential distribution as the underlying model for the data, they found the exact 95% and 90% lower confidence bounds for the mean parameter θ. Their results contained some errors which were subsequently corrected by Childs et al. (2003), and these are presented in Table 11.1.

Table 11.1. MLEs of θ, their standard errors (SE) and the 95% and 90% lower confidence bounds for the Type-I HCS

r	$\hat{\theta}$	SE	95%	90%
4	37.50	19.78	19.35	22.45
6	43.17	23.64	24.64	27.93
8	51.17	31.11	28.46	32.12

Example 11.2 In the case of one-parameter exponential distribution, maximum likelihood estimation of the parameter θ in the case of Type-II censored data is simple and its exact distribution is also known precisely. For example, based on a Type-II censored sample of size r, the distribution of $\frac{2r\hat{\theta}}{\theta}$ is a chi-square with $2r$ degrees of freedom; see Balakrishnan and Cohen (1991). In this case, one may find it convenient to treat a Type-I HCS to be simply a Type-II censored sample of size r. In this case, a $100(1-\alpha)\%$ lower confidence bound for θ will be $\frac{2r\hat{\theta}}{\chi^2_{2r,\alpha}}$. For the data considered in Example 11.1, when $r = 6$, we would have $\hat{\theta} = 43.167$ and the 95% and 99% lower confidence bounds for θ to be 24.636 and 27.925, respectively. Note that these results are almost the same as those obtained exactly by treating the observed sample correctly as Type-I HCS.

Instead, had we treated the case $r = 8$ as a Type-II censored sample, then the corresponding lower confidence bounds would have been exactly the same as above (as in the case of $r = 6$). However, by correctly assuming that the data are indeed Type-I HCS with $T = 50$ and $r = 8$, we determine from the exact density of the MLE $\hat{\theta}$ corresponding to this case that $P_{24.636}(\hat{\theta} > 51.17) = 0.0178$, frow which we observe that the 95% lower confidence bound is in fact a 98.22% lower confidence bound. Similarly, we determine that the 90% lower confidence bound is in fact a 95.6% lower confidence bound; see Childs et al. (2003). This shows that treating a Type-I HCS as a Type-II censored sample could result in very liberal confidence bounds.

Example 11.3 Bartholomew (1963) presented the following data arising as lifetimes of

$n = 20$ units on a life-test for a pre-fixed time of 150 hours:

$$3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, 138.$$

From these data, Childs et al. (2003) considered two Type-II HCS with $T = 50$ and $r = 7$ and $r = 15$. By assuming a one-parameter exponential distribution for these data and using the exact results for the case of Type-II HCS presented earlier, Childs et al. (2003) obtained the results presented in Table 11.2.

Table 11.2. MLEs of θ, their standard errors (SE) and the 95% and 90% lower confidence bounds for the Type-II HCS

r	$\hat{\theta}$	SE	95%	90%
7	89.89	30.96	53.56	59.54
15	101.80	26.28	69.77	75.86

Example 11.4 As in Example 11.2, we look at the effect of treating the Type-II HCS in this case as simply a Type-II censored sample. In the case when $r = 7$, since 9 failure times would have been observed, the 95% lower confidence bound for θ in this case would be $\frac{2\times 9\times 89.89}{28.8693} = 56.046$. Similarly, the 90% lower confidence bound would be 62.256. Now, once again by using the exact results for the Type-II HCS, we determine

$$P_{56.046}(\hat{\theta} > 89.89) = 0.0684 \quad \text{and} \quad P_{62.256}(\hat{\theta} > 89.89) = 0.1288,$$

which imply that the 95% and 90% lower confidence bounds are in fact 93.16% and 87.12% lower confidence bounds; see Childs et al. (2003). This shows that treating a Type-II HCS as a Type-II censored sample could result in somewhat conservative confidence bounds.

Example 11.5 With the data presented in Example 11.3, Chandrasekar et al. (2004) used time $T = 50$ and the choices of $k = 12, r = 15$, $k = 5, r = 11$ and $k = 4, r = 7$ to produce three generalized Type-I HCS. Then, with the assumption of one-parameter exponential distribution for the data and the exact results for the case of generalized Type-I **Example**

11.5 With the data presented in Example 11.3, Chandrasekar et al. (2004) used time $T = 50$ and the choices of $k = 12, r = 15$, $k = 5, r = 11$ and $k = 4, r = 7$ to produce three generalized Type-I HCS. Then, with the assumption of one-parameter exponential distribution for the data and the exact results for the case of generalized Type-I HCS, these authors determined the MLEs of θ and the 95% and 90% lower confidence bounds for θ. These results are presented in Table 11.3.

Table 11.3. MLEs of θ, their standard errors (SE) and the 95% and 90% lower confidence bounds for the generalized Type-I HCS

k	r	$\hat{\theta}$	SE	95%	90%
12	15	100.92	28.94	66.51	72.96
5	11	89.89	37.00	53.56	59.54
4	7	95.29	45.30	56.32	63.33

Example 11.6 With the data presented in Example 11.3, Chandrasekar et al. (2004) used times $T_1 = 50$ and $T_2 = 100$ and the choices of $r = 7, 13, 15$ to produce three generalized Type-II HCS. Then, with the assumption of one-parameter exponential distribution for the data and the exact results for the case of generalized Type-II HCS, these authors determined the MLEs of θ and the 95% and 90% lower confidence bounds for θ. These results are presented in Table 11.4.

Table 11.4. MLEs of θ, their standard errors (SE) and the 95% and 90% lower confidence bounds for the generalized Type-II HCS

r	$\hat{\theta}$	SE	95%	90%
7	89.89	31.14	53.57	59.54
13	98.69	33.33	65.01	71.14
15	99.23	32.79	64.90	70.97

Example 11.7 Viveros and Balakrishnan (1994) generated the following progressively Type-II censored sample from the data reported by Nelson (1982, p. 228) on times to breakdown

of insulating fluids tested at various voltages:

i	1	2	3	4	5	6	7	8
$x_{i:8:19}$	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.335
R_i	0	0	3	0	3	0	0	5

By choosing the time $T = 6.0$ and $m = 6, 8$, Childs et al. (2008) considered two different Type-I PHCS and by assuming one-parameter exponential distribution for the data and using the exact inferential results for the case of Type-I PHCS, these authors determined the MLEs of θ and the 95% and 90% lower confidence bounds for θ. These results are presented in Table 11.5. Note that in the case when $m = 6$, we take $R_m = 7$.

Table 11.5. MLEs of θ, their standard errors (SE) and the 95% and 90% lower confidence bounds for the Type-I PHCS

r	$\hat{\theta}$	SE	95%	90%
6	9.3400	4.786	5.330	6.042
8	10.6817	5.833	6.004	6.766

Example 11.8 For the data in Example 11.7, and with the same choice of $T = 6.0$ and $m = 6, 8$, Childs et al. (2008) considered two Type-II PHCS. Then, by using the corresponding exact inferential results for the case of Type-II PHCS, they determined the MLEs of θ and the 95% and 90% lower confidence bounds for θ. These results are presented in Table 11.6. Note that in the case when $m = 6$ in this scenario, we take $R_m = 0$.

Table 11.6. MLEs of θ, their standard errors (SE) and the 95% and 90% lower confidence bounds for the Type-II PHCS

r	$\hat{\theta}$	SE	95%	90%
6	10.6817	4.045	6.019	6.781
8	9.0863	3.078	5.513	6.157

Of course, a number of other numerical examples are present in many of the articles cited in the text, illustrating different forms of hybrid censoring from many different lifetime models. All these examples will further demonstrate the importance and usefulness of hybrid censoring schemes and inferential procedures based on them.

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