Step-Stress Models and Associated Inference

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Outline

1. **Accelerated Life Test**
2. **Step Stress Test**
3. **Cumulative Exposure Model**
4. **Khamis and Higgins Model**
5. **Cumulative Risk Model**
6. **Stress Changes at Random Time**
7. **Different Other Issues and Open Problems**
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1 Accelerated Life Test
2 Step Stress Test
3 Cumulative Exposure Model
4 Khamis and Higgins Model
5 Cumulative Risk Model
6 Stress Changes at Random Time
7 Different Other Issues and Open Problems
Life testing experiments have gained popularity in recent times. The main aim of a life testing experiment is to measure one or more reliability characteristics of the product under consideration. In a very classical form of the life testing experiment certain numbers of identical items are placed on the test under normal operating conditions, and the time to failure of all the items are recorded.
However, due to substantial improvement of science and technology, the most of the items now-a-days are quite durable, and hence one of the major difficulties of the life testing experiments is the time duration of the experiment. To overcome this problem, the experimenters use accelerated life testing experiments, which ensure early failures of the experimental units.
What is an Accelerated Life Test?

Accelerated life test is a popular experimental strategy to obtain information on life distributions of highly reliable product. The main idea is to submit materials to higher than usual environmental conditions to ensure early failure. Data obtained from such an experiment need to be extrapolated to estimate lifetime distribution under normal conditions.
Accelerated Life Test

- **Stress Factors:**
  1. Single Stress
  2. Multiple Stress

- **Classical Stresses**
  1. Temperature
  2. Voltage
  3. Current
  4. Pressure
  5. Load
Accelerated Life Test

Advantages

1. If you increase the stress, reasonable number of failures are ensured
2. Reduce the experimental time

Disadvantages

1. Need to know the exact relation between the stress and lifetime
2. Model must take into account the effect of accumulation of stress
3. Model becomes more complex
4. Even in simple case analysis becomes difficult
One of major concerns with the accelerated life testing is to find the relationship between the lifetime under stress condition and lifetime under normal condition. A relationship between the lifetimes need to be established.
Some Definitions

- \( \theta(s) \): Stress Function at the stress level \( s \)
- \( X \): The time to failure under stress
- \( Y \): The time to failure under normal condition
- \( F_X \): The cumulative distribution function of \( X \)
- \( F_Y \): The cumulative distribution function of \( Y \)
- \( \phi_\theta \): The acceleration function

The acceleration function

\[
\phi_\theta : [0, \infty) \rightarrow [0, \infty).
\]

and

\[
Y = \phi_\theta(X)
\]
Relations Between \( X \) and \( Y \)

- **The Distribution Functions**

\[
F_Y(y) = F_X(\phi_\theta^{-1}(y))
\]

- **The Density Functions**

\[
f_Y(y) = f_X(\phi_\theta^{-1}(y)) \left| \frac{d}{dy} \phi_\theta^{-1}(y) \right|
\]

- **Hazard Functions**

\[
\lambda_Y(y) = \left| \frac{d}{dy} \phi_\theta^{-1}(y) \right| \lambda_X(\phi_\theta^{-1}(y))
\]
**Different Forms of $\phi(\theta)$**

Linear acceleration function

$$\phi(\theta) = \theta(s)X; \quad \theta(s) \geq 1.$$ 

In this case

$$\lambda_Y(y) = \frac{1}{\theta(s)}\lambda_X(x).$$

This is a Proportional Hazard Model

Power Transform

$$Y = AX^{\theta(s)}$$

With such an acceleration function, a Weibull($\alpha, \beta$) is transformed to another Weibull($A\alpha^{\theta}, \frac{\beta}{\theta}$).
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7. Different Other Issues and Open Problems
A Step-Stress life test is a particular accelerated life-test. You observe the failure times of the objects at a particular stress level, then change the stress level to a different level. Observe the failure times in the new stress level, and then the change the stress level again and so on.
Step-Stress Life Test

Several Variations are possible depending on the need

1. The Stress can be changed at the pre specified timing. In this case number of failures is random

2. The Stress can be changed at the pre specified number of failures. In this case failure time is random.
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Cumulative Exposure Model
It is the most popular model assumption for analyzing step-stress data. The main idea is to assume that the remaining lifetime of the specimens depends only on the current accumulated stress, regardless of how it has accumulated.
Let us consider simple step-stress model:
Suppose:
- $F_1$: The cumulative distribution function under stress $s_1$
- $F_2$: The cumulative distribution function under stress $s_2$
- $G$: The cumulative distribution function under a step-stress pattern

$G$ can be obtained from $F_1$ and $F_2$ under the assumptions that the lifetime $F_1$ (under stress $s_1$) at the time $t_1$ has an equivalent time $u_2$ of the distribution function $F_2$ (under stress $s_2$).
Graphical Representation

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Step-Stress Models and Associated Inference
Simple Step-Stress: Fixed Time

- \([0, \tau) \rightarrow \text{Stress is } s_1\)
- \([\tau, \infty) \rightarrow \text{Stress is } s_2\)
Data: Step-stress Life Tests

- $n$: Number of items put on the test.
- $s_1, s_2$: Stress levels (Simple SSLT).
- $\tau$: Stress changing time (Pre-fixed).
Data: Step-stress Life Tests

- \( n \): Number of items put on the test.
- \( s_1, s_2 \): Stress levels (Simple SSLT).
- \( \tau \): Stress changing time (Pre-fixed).

\[
\begin{align*}
& t_1:n \\
& t_2:n \\
& \ldots \\
& t_{N_1}:n \\
& t_{N_1+1}:n \\
& \ldots \\
& t_{n}:n \\
\end{align*}
\]
Data: Step-stress Life Tests

- **Generalization**
  - \( n \): No of items placed on the test.
  - \( s_1, s_2, s_3, \ldots, s_{m+1} \): Stress levels.
  - \( \tau_1 < \tau_2 < \ldots < \tau_m \): Stress changing times (Pre-fixed).

![Diagram showing step-stress life tests]

\( n \)

\( \tau_1 \)

\( \tau_m \)

\( s_1 \)

\( s_m \)

\( s_{m+1} \)
The effect of change of stress from $s_1$ to $s_2$ is to change the lifetime distribution at stress level $s_2$ from $F_2(\tau)$ to $F_2(\tau - h)$, where the shifting time $h$ is such that

$$F_2(\tau + h) = F_1(\tau).$$
Model Formulation: Simple Step-Stress

Based on the following assumptions:

1. $\theta_1$ and $\theta_2$ are the scale parameters associated with $F_1$ and $F_2$, respectively.
2. $F_1$ and $F_2$ belong to the same scale family.
3. $F_1$ and $F_2$ are absolute continuous

The shifting time $h$ becomes

$$h = \frac{\theta_2}{\theta_1} \tau - \tau$$
The cumulative distribution function of the lifetime under the cumulative exposure model becomes

\[
G(t) = \begin{cases} 
G_1(t) &= F_1(t) \quad \text{for } 0 < t < \tau \\
G_2(t) &= F_2\left(t + \frac{\theta_2}{\theta_1} \tau - \tau\right) \quad \text{for } \tau < t < \infty
\end{cases}
\]

The corresponding probability density function becomes;

\[
g(t) = \begin{cases} 
g_1(t) &= f_1(t) \quad \text{for } 0 < t < \tau \\
g_2(t) &= f_2\left(t + \frac{\theta_2}{\theta_1} \tau - \tau\right) \quad \text{for } \tau < t < \infty
\end{cases}
\]
The cumulative distribution function of the lifetime under the cumulative exposure model becomes

\[
G(t) = \begin{cases} 
F_1(t) & \text{for } 0 < t < \tau_1 \\
F_k(t + a_{k-1} - \tau_{k-1}) & \text{for } \tau_{k-1} < t < \tau_k, \quad k = 2, \ldots, m \\
F_{m+1}(t + a_m - \tau_m) & \text{for } \tau_m \leq t < \infty,
\end{cases}
\]

where

\[
a_{k-1} = \theta_k \sum_{i=1}^{k-1} \left( \frac{\tau_i - \tau_{i-1}}{\theta_i} \right) \quad \text{for } k = 2, \ldots, m + 1,
\]

with \(\tau_0 = 0\) and \(\theta_i\) is the scale parameter of \(F_i\) for \(i = 1, \ldots, m + 1\).
If we assume that the lifetime distributions are exponential at all stress levels, in case of simple step-stress model the cumulative distribution function takes the form:

\[
G(t) = \begin{cases} 
G_1(t) & = 1 - e^{-t/\theta_1} \quad \text{for } 0 < t < \tau \\
G_2(t) & = 1 - e^{-\left(t+\frac{\theta_2}{\theta_1}\tau-\tau\right)/\theta_2} \quad \text{for } \tau < t < \infty 
\end{cases}
\]

The corresponding probability density function becomes:

\[
g(t) = \begin{cases} 
g_1(t) & = \frac{1}{\theta_1} e^{-t/\theta_1} \quad \text{for } 0 < t < \tau \\
g_2(t) & = \frac{1}{\theta_2} e^{-\left(t+\frac{\theta_2}{\theta_1}\tau-\tau\right)/\theta_2} \quad \text{for } \tau < t < \infty 
\end{cases}
\]
The maximum likelihood estimator of $\theta_1$ does not exist if $N_1 = 0$, the MLE of $\theta_2$ does not exist if $N_1 = n$, and the MLEs of $\theta_1$ and $\theta_2$ exist only when $1 \leq N_1 \leq n - 1$, and they are

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{N_1} t_{i:n} + (n - N_1)\tau}{N_1}$$

$$\hat{\theta}_2 = \frac{\sum_{i=N_1+1}^{n} (t_{i:n} - \tau)}{n - N_1}$$

Clearly, these MLEs are conditional MLEs, conditional on $1 \leq N_1 \leq n - 1$. 

**Maximum Likelihood Estimators**
The conditional densities of $\hat{\theta}_1$ and $\hat{\theta}_2$ are

$$f_{\hat{\theta}_1}(t) = \sum_{j=1}^{n-1} \sum_{k=0}^{j} c_{j,k} f_G \left( t - \tau_{j,k}; j, \frac{j}{\hat{\theta}_1} \right)$$

and

$$f_{\hat{\theta}_2}(t) = \sum_{j=1}^{n-1} w_j f_G \left( t; j, \frac{j}{\hat{\theta}_2} \right)$$
The conditional densities of $\hat{\theta}_1$ and $\hat{\theta}_2$ can be obtained under different censoring schemes:

1. Type-I censoring scheme
2. Type-II censoring scheme
3. Type-I Hybrid censoring scheme
4. Type-II Hybrid censoring scheme
5. Progressive Type-II censoring scheme
Confidence Intervals

Under the assumptions that the conditional probabilities $P_{\theta_1}(\hat{\theta}_1 \geq a)$ and $P_{\theta_2}(\hat{\theta}_2 \geq a)$ are increasing functions of $\theta_1$ and $\theta_2$, respectively, the exact confidence intervals of $\theta_1$ and $\theta_2$ can be obtained by solving two non-linear equations:

\[
P_{\hat{\theta}_1}(\hat{\theta}_1 \geq \theta_L) = 1 - \frac{\alpha}{2} \quad P_{\hat{\theta}_1}(\hat{\theta}_1 \geq \theta_U) = \frac{\alpha}{2}.
\]

\[
P_{\hat{\theta}_2}(\hat{\theta}_2 \geq \theta_L) = 1 - \frac{\alpha}{2} \quad P_{\hat{\theta}_2}(\hat{\theta}_2 \geq \theta_U) = \frac{\alpha}{2}.
\]
The basic idea of the step-stress experiment is to increase the stress so that early failure is observed. It means it is assumed that the expected lifetime at the stress level $s_2$ should be smaller than the expected lifetime at the stress level $s_1$. Therefore, the natural restriction on the parameter space becomes

$$\theta_1 \geq \theta_2.$$ 

The inference on $\theta_1$ and $\theta_2$ should be performed under that restriction. It becomes quite difficult. The exact results are not available.
Bayesian approach seems to be a natural choice under this order restriction case namely $\theta_1 \geq \theta_2$. Let us consider

$$\alpha = \frac{\theta_2}{\theta_1} \quad \text{or} \quad \theta_1 = \frac{\theta_2}{\alpha} \quad 0 < \alpha < 1.$$ 

The following priors can be taken: $\pi_1(\theta_2)$ as inverted gamma and $\pi_2(\alpha)$ as beta. The Bayes estimates and the associated credible intervals also can be obtained.

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Step-Stress Models and Associated Inference
Different other lifetime models have been used for analyzing step-stress data.

1. Weibull distribution
2. log-logistic distribution
3. generalized exponential distribution
4. two-parameter exponential distribution
5. gamma distribution
Since for the CEM

\[ G(t) = \begin{cases} F_1(t) & \text{if } 0 < t < \tau \\ F_2(a(t)) & \text{if } \tau < t < \infty \end{cases} \]

where \( a(t) = \left( \frac{\theta_2}{\theta_1} \right) \tau + t - \tau \). Therefore, the PDF of Weibull distribution becomes

\[ g(t) = \begin{cases} \frac{\beta}{\theta_1} t^{\beta_1} e^{-\left(\frac{t^{\beta}}{\theta_1}\right)} & \text{if } 0 < t < \tau \\ \frac{\beta}{\theta_2} (a(t))^{\beta-1} e^{-\left(a(t)\right)^{\beta}/\theta_2} & \text{if } \tau < t < \infty. \end{cases} \]
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The CEM in case of Weibull distribution becomes quite complicated. Due to this reason Khamis and Higgins proposed the following model. The hazard function of CEM in case of exponential distribution takes the following form:

\[
    h(t) = \begin{cases} 
    \frac{1}{\theta_1} & \text{if } 0 < t < \tau \\
    \frac{1}{\theta_2} & \text{if } \tau < t < \infty 
    \end{cases}
\]
Khamis and Higgins Model: Weibull Distribution

It is assumed that at the two different stress levels the lifetime distributions follow Weibull distribution with the same shape parameter, but different scale parameter. At the stress level $s_i$, the lifetime becomes;

$$f(t) = \frac{\beta}{\theta_i} t^{\beta-1} e^{-t^\beta/\theta_i}; \quad t > 0, \beta > 0, \theta_i > 0$$
Khamis and Higgins proposed the following step-stress model for Weibull distribution:

\[
h(t) = \begin{cases} 
\frac{\beta}{\theta_1} t^{\beta-1} & \text{if } 0 < t < \tau \\
\frac{\beta}{\theta_2} t^{\beta-1} & \text{if } \tau < t < \infty 
\end{cases}
\]

The corresponding survival function becomes:

\[
S(t) = \begin{cases} 
\exp \left( -\frac{t^\beta}{\theta_1} \right) & \text{if } 0 < t < \tau \\
\exp \left( -\frac{t^\beta - \tau^\beta}{\theta_2} - \frac{\tau^\beta}{\theta_1} \right) & \text{if } \tau < t < \infty 
\end{cases}
\]
Khamis and Higgins Model: MLEs

- The MLEs cannot be obtained in explicit forms. Non-linear equations need to be solved to compute the MLEs.
- Bootstrap or asymptotic distributions can be used for constructing confidence intervals.
- Order restricted inference is also possible, but computationally it is quite demanding.
In case of order restricted inference i.e. when $\theta_1 > \theta_2$, Bayesian inference seems to be a natural choice in this case also. Similar prior assumptions can be made: The following priors can be taken: $\pi_1(\theta_2)$ as inverted gamma, $\pi_2(\alpha)$ as beta, take $\pi_3(\alpha)$ as a log-concave prior. The Bayes estimates and the associated credible intervals also can be obtained.
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Cumulative Risk Model

One of the problems for CEM model is that the hazard function is not continuous. The cumulative risk model has been suggested to overcome this problem. In case of exponential distribution the CRM takes the following form:

$$h(t) = \begin{cases} 
\theta_1 & \text{if } 0 < t < \tau_1 \\
 a + bt & \text{if } \tau_1 \leq t < \tau_2 \\
\theta_2 & \text{if } t > \tau_2 
\end{cases}$$

The parameters $a$ and $b$ are such that $h(t)$ becomes continuous. Therefore

$$a + b\tau_1 = \theta_1 \quad \text{and} \quad a + b\tau_2 = \theta_2.$$
Cumulative Risk Model: PDF

In case of exponential distribution, the PDF becomes

\[
f(t) = \begin{cases} 
(a + b\tau_1)e^{-(a+b\tau_1)t} & \text{if } 0 < t < \tau_1 \\
(a + bt)e^{-(at+b(t^2+\tau_1^2)/2)} & \text{if } \tau_1 < t < \tau_2 \\
(a + b\tau_2)e^{-(a+b\tau_2)t-b(\tau_1^2-\tau_2^2)/2} & \text{if } t > \tau_2
\end{cases}
\]
The associated cumulative hazard function becomes

\[ H(t) = \begin{cases} 
  h_1(t) & \text{if } 0 < t < \tau_1 \\
  h_2(t) & \text{if } \tau_1 < t < \tau_2 \\
  h_3(t) & \text{if } t > \tau_2 
\end{cases} \]

where

\[ h_1(t) = (a + b\tau_1)t \]
\[ h_2(t) = (a + b\tau_1)\tau_1 + (t - \tau_1)^2 b/2 \]
\[ h_3(t) = (a + b\tau_1)\tau_1 + (\tau_2 - \tau_1)^2 b/2 + (t - \tau_2)(a + b\tau_2) \]
The MLEs of the unknown parameters cannot be obtained in closed form. But the least squares estimators which can be obtained by minimizing

\[ \sum_{i=1}^{n} \left( H(t_i) - \hat{H}(t_i) \right)^2 \]

with respect to the unknown parameters \( a \) and \( b \) are in explicit forms. Here

\[ \hat{H}(t_i) = -\ln(\hat{S}(t_i)) = \ln n - \ln(n - i + 1) \]
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So far we have discussed the step-stress model where the stress changes at a pre-specified time. In this set up the stress changes at a time where the pre-specified number of failures take place. Suppose it is assumed that the stress changes at the $n_1$-th failure. Therefore we observe the failures at

$$t_{1:n} < \ldots < t_{n_1:n} < t_{n_1+1:n} < t_{n:n}.$$
Under the assumptions of cumulative exposure model if the lifetime distributions are exponential then

\[ \hat{\theta}_1 = \frac{T_1}{n_1}, \quad \hat{\theta}_2 = \frac{T_2}{n - n_1}, \]

where

\[ T_1 = \sum_{i=1}^{n_1} t_{i:n} + (n - n_1)t_{n_1:n} \]

\[ T_2 = \sum_{i=n_1+1}^{n} (t_{i:n} - t_{n_1:n}) \]
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How to choose $\tau$ so that some optimality criterion is satisfied? What can be a good optimality criterion?

1. Minimize the trace of the inverse of the Fisher information matrix.

2. Maximize the determinant of the Fisher information matrix.

3. Minimize the variance of the $p$-th percentile estimator of the distribution function under normal stress.
Step-Stress Model with Cure Fraction

In a cure rate model it is assumed that the population consists of two types of items: susceptible and immune. Susceptibles are those who are subject to failures and immunes are those who are not subject to failures. In this case the survival function of the population

\[ S(t) = p + (1 - p)S_0(t) \]

In case of step-stress set up it is assumed that \( S_0(t) \) has one of these step-stress models.
**Step-Stress Model with Competing Causes of Failures**

In many life testing experiments one observes more than one cause of failures. When the time of failure and the associated cause of failure is also observed it is known as competing risk data. In this case it is assumed that the observed failure time $T$ is as follows:

$$T = \min\{T_1, \ldots, T_M\},$$

where $T_1, \ldots, T_M$ are the failure times for different causes which are observable. A competing risk data is as follows;

$$(T, \Delta)$$

$\Delta$ denotes the cause of failure.
So far we have mainly discussed the simple step-stress model. In case of multiple step-stress model the stress changes at

\[ \tau_1 < \tau_2 < \ldots < \tau_k. \]

It is usually assumed that there is a link function of the different parameters at the different stress levels. For example the assumption of the log-link function is very common, \( i.e. \)

\[ \ln \theta_i = a + bx_i; \quad i = 1, \ldots, k. \]

Here \( x_i \) is some function of \( x_i \) and it is known.
Open Problems: General

1. Goodness of fit tests
2. Comparison of the different models
3. Bayesian Prior choice
5. Develop step-stress models with covariates
Open Problems: Specific

1. Develop order restricted inference for Weibull
2. Develop inference for three parameter Weibull
3. Bayesian inference for Weibull distribution
4. Develop step-stress models with covariates
Thank You