Step-Stress Models and Associated Inference

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Outline

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Life Testing Experiments

Life testing experiments have gained popularity in recent times. The main aim of a life testing experiment is to measure one or more reliability characteristics of the product under consideration. In a very classical form of the life testing experiment certain numbers of identical items are placed on the test under normal operating conditions, and the time to failure of all the items are recorded.

Life Testing Experiments: Difficulties

However, due to substantial improvement of science and technology, the most of the items now-a-days are quite durable, and hence one of the major difficulties of the life testing experiments is the time duration of the experiment. To overcome this problem, the experimenters use accelerated life testing experiments, which ensure early failures of the experimental units.

What is an Accelerated Life Test?

Accelerated life test is a popular experimental strategy to obtain information on life distributions of highly reliable product. The main idea is to submit materials to higher than usual environmental conditions to ensure early failure. Data obtained from such an experiment need to be extrapolated to estimate lifetime distribution under normal conditions.

Accelerated Life Test

- Stress Factors:
  1. Single Stress
  2. Multiple Stress
- Classical Stresses
  1. Temperature
  2. Voltage
  3. Current
  4. Pressure
  5. Load

Accelerated Life Test

- Advantages
  1. If you increase the stress, reasonable number of failures are ensured
  2. Reduce the experimental time
- Disadvantages
  1. Need to know the exact relation between the stress and lifetime
  2. Model must take into account the effect of accumulation of stress
  3. Model becomes more complex
  4. Even in simple case analysis becomes difficult

Acceleration Model

One of major concerns with the accelerated life testing is to find the relationship between the lifetime under stress condition and lifetime under normal condition. A relationship between the lifetimes need to be established.

Some Definitions

- $\theta(s)$: Stress Function at the stress level s
- X : The time to failure under stress
- Y : The time to failure under normal condition
- F_X: The cumulative distribution function of X
- F_Y: The cumulative distribution function of Y
- ϕ_θ : The acceleration function

The acceleration function

$$\phi_\theta : [0, \infty) \to [0, \infty).$$

and

$$Y = \phi_\theta(X)$$

Relations Between X and Y

- The Distribution Functions

$$F_Y(y) = F_X(\phi_\theta^{-1}(y))$$

- The Density Functions

$$f_Y(y) = f_X(\phi_\theta^{-1}(y)) \left| \frac{d}{dy} \phi_\theta^{-1}(y)) \right|$$

- Hazard Functions

$$\lambda_Y(y) = \left| \frac{d}{dy} \phi_\theta^{-1}(y)) \right| \lambda_X(\phi_\theta^{-1}(y))$$

Different Forms of ϕ_θ

Linear acceleration function

$$\phi_\theta = \theta(s)X; \quad \theta(s) \geq 1.$$

In this case

$$\lambda_Y(y) = \frac{1}{\theta(s)}\lambda_X(x).$$

This is a Proportional Hazard Model
Power Transform

$$Y = AX^{\theta(s)}$$

With such an acceleration function, a Weibull(α, β) is transformed to another Weibull$(A\alpha^\theta, \frac{\beta}{\theta})$.

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What is a Step-Stress Life Test?

A Step-Stress life test is a particular accelerated life-test. You observe the failure times of the objects at a particular stress level, then change the stress level to a different level. Observe the failure times in the new stress level, and then the change the stress level again and so on.

Step-Stress Life Test

Several Variations are possible depending on the need

1. The Stress can be changed at the pre specified timing. In this case number of failures is random
2. The Stress can be changed at the pre specified number of failures. In this case failure time is random.

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Lifetime Distribution Under Step-Stress

Cumulative Exposure Model

It is the most popular model assumption for analyzing step-stress data. The main idea is to assume that the remaining lifetime of the specimens depends only on the current accumulated stress, regardless of how it has accumulated.

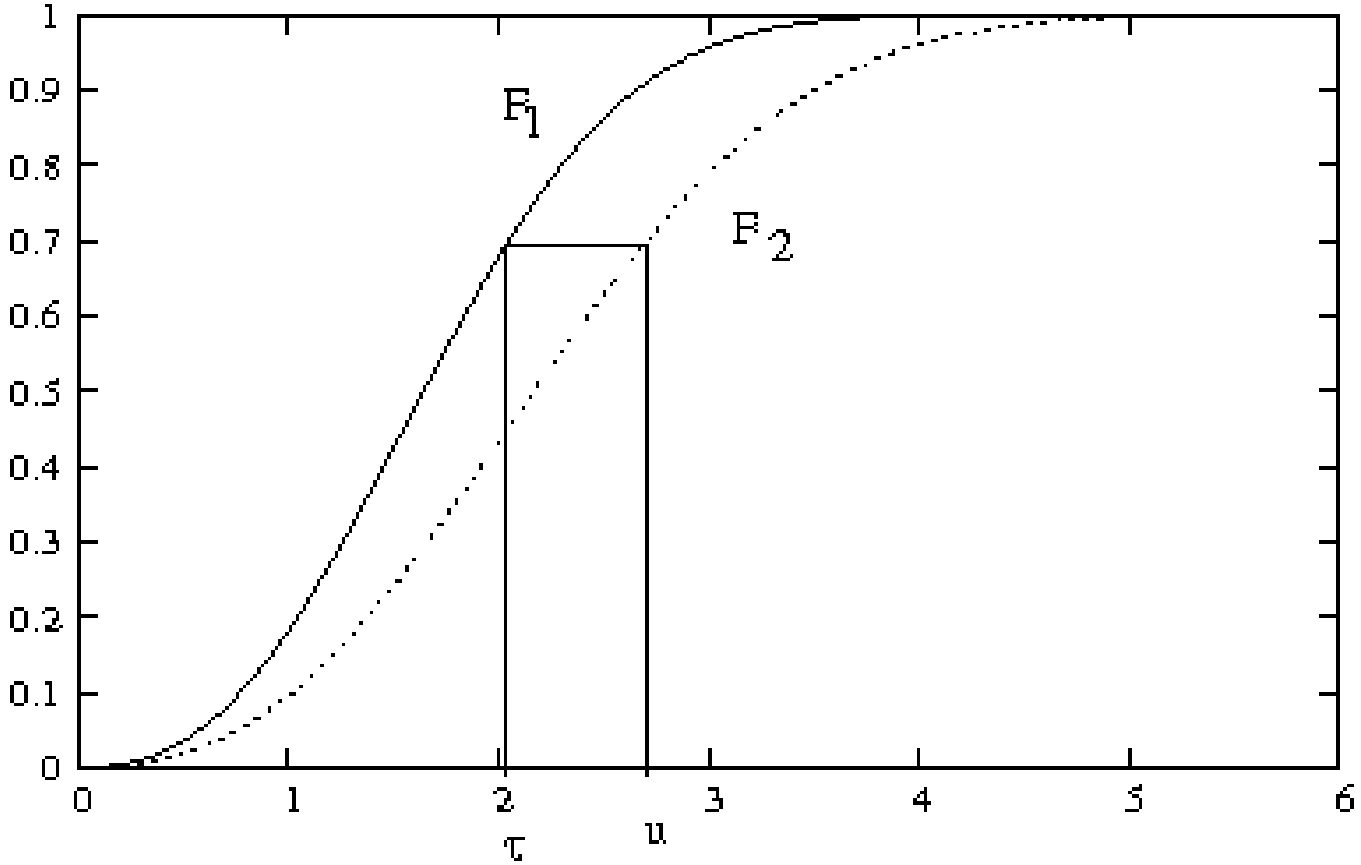
Cumulative Exposure Model

Let us consider simple step-stress model:
Suppose:

- F_1 : The cumulative distribution function under stress s_1
- F_2 : The cumulative distribution function under stress s_2
- G : The cumulative distribution function under a step-stress pattern

G can be obtained from F_1 and F_2 under the assumptions that the lifetime F_1 (under stress s_1) at the time t_1 has an equivalent time u_2 of the distribution function F_2 (under stress s_2).

Graphical Representation

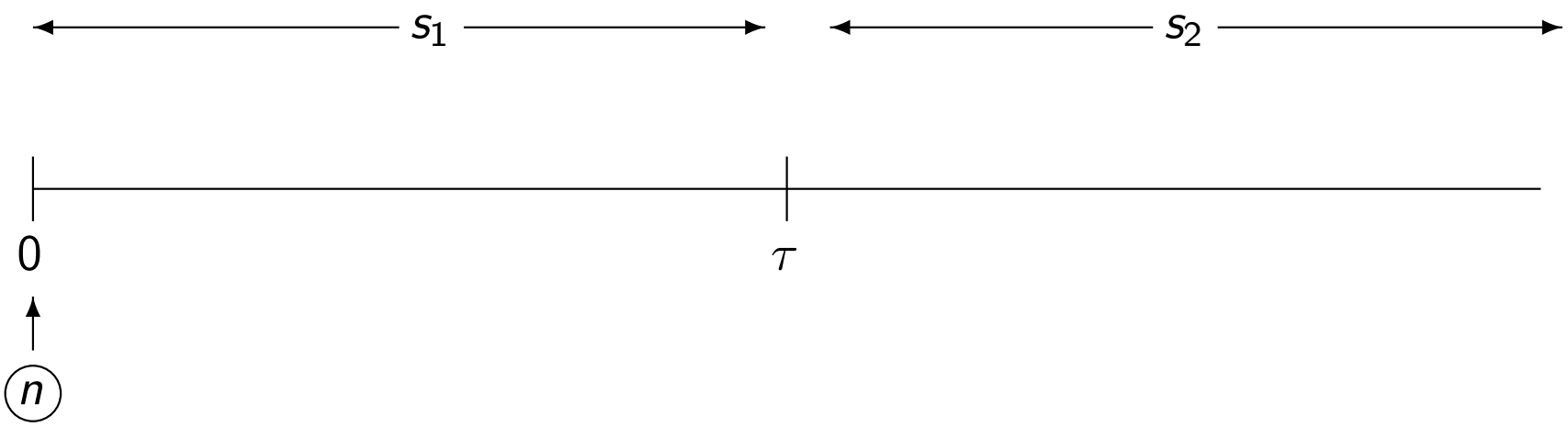


Simple Step-Stress: Fixed Time

- $[0, \tau) \rightarrow$ Stress is s_1
- $[\tau, \infty) \rightarrow$ Stress is s_2
- $\downarrow$

Data: Step-stress Life Tests

- n : Number of items put on the test.
- s_1, s_2 : Stress levels (Simple SSLT).
- τ : Stress changing time (Pre-fixed).



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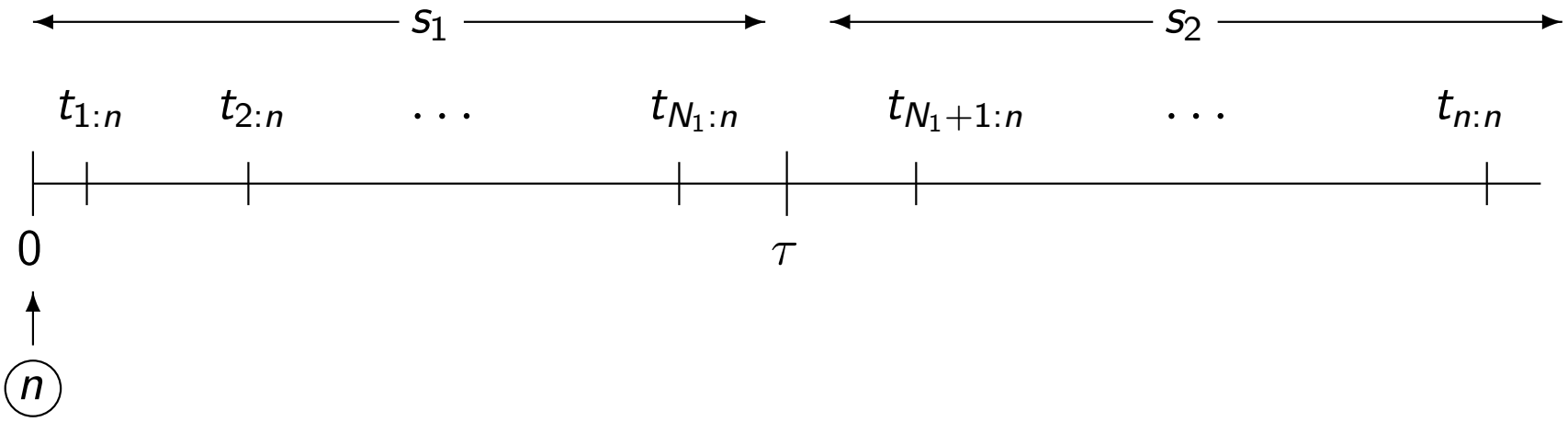
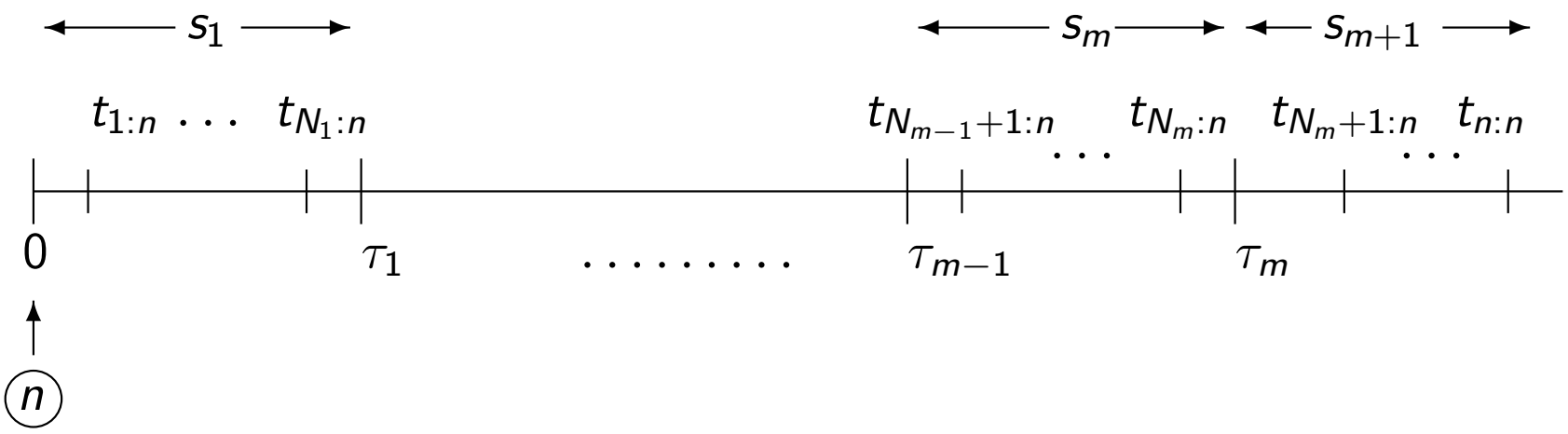


Data: Step-stress Life Tests

- Generalization
  - n : No of items placed on the test.
  - $s_1, s_2, s_3, \ldots, s_{m+1}$: Stress levels.
  - $\tau_1 < \tau_2 < \ldots < \tau_m$: Stress changing times (Pre-fixed).



Model Formulation: Simple Step-Stress

The effect of change of stress from s_1 tp s_2 is to change the lifetime distribution at stress level s_2 from $F_2(\tau)$ to $F_2(\tau - h)$, where the shifting time h is such that

$$F_2(\tau + h) = F_1(\tau).$$

Model Formulation: Simple Step-Stress

Based on the following assumptions:

1. θ_1 and θ_2 are the scale parameters associated with F_1 and F_2, respectively.
2. F_1 and F_2 belong to the same scale family.
3. F_1 and F_2 are absolute continuous

The shifting time h becomes

$$h = \frac{\theta_2}{\theta_1}\tau - \tau$$

CDF and PDF for CEM

The cumulative distribution function of the lifetime under the cumulative exposure model becomes

$$G(t) = \begin{cases} G_1(t) & = & F_1(t) & \text{for} & 0 < t < \tau \\ G_2(t) & = & F_2\left(t + \frac{\theta_2}{\theta_1}\tau - \tau\right) & \text{for} & \tau < t < \infty \end{cases}$$

The corresponding probability density function becomes;

$$g(t) = \begin{cases} g_1(t) & = & f_1(t) & \text{for} & 0 < t < \tau \\ g_2(t) & = & f_2\left(t + \frac{\theta_2}{\theta_1}\tau - \tau\right) & \text{for} & \tau < t < \infty \end{cases}$$

CDF and PDF for CEM

The cumulative distribution function of the lifetime under the cumulative exposure model becomes

$$G(t) = \begin{cases} F_1(t) & \text{for} \quad 0 < t < \tau_1 \\ F_k\left(t + a_{k-1} - \tau_{k-1}\right) & \text{for} \quad \tau_{k-1} < t < \tau_k, \quad k = 2, \ldots, m \\ F_{m+1}(t + a_m - \tau_m) & \text{for} \quad \tau_m \leq t < \infty, \end{cases}$$

where

$$a_{k-1} = \theta_k \sum_{i=1}^{k-1} \left(\frac{\tau_i - \tau_{i-1}}{\theta_i} \right) \quad \text{for} \quad k = 2, \ldots, m+1,$$

with $\tau_0 = 0$ and θ_i is the scale parameter of F_i for $i = 1, \ldots, m+1$.

Exponential Distribution

If we assume that the lifetime distributions are exponential at all stress levels, in case of simple step-stress model the cumulative distribution function takes the form:

$$G(t) = \begin{cases} G_1(t) &= 1 - e^{-t/\theta_1} & \text{for} \quad 0 < t < \tau \\ G_2(t) &= 1 - e^{-\left(t + \frac{\theta_2}{\theta_1}\tau - \tau\right)/\theta_2} & \text{for} \quad \tau < t < \infty \end{cases}$$

The corresponding probability density function becomes;

$$g(t) = \begin{cases} g_1(t) &= \frac{1}{\theta_1} e^{-t/\theta_1} & \text{for} \quad 0 < t < \tau \\ g_2(t) &= \frac{1}{\theta_2} e^{-\left(t + \frac{\theta_2}{\theta_1}\tau - \tau\right)/\theta_2} & \text{for} \quad \tau < t < \infty \end{cases}$$

Maximum Likelihood Estimators

The maximum likelihood estimator of θ_1 does not exist if $N_1 = 0$, the MLE of θ_2 does not exist if $N_1 = n$, and the MLEs of θ_1 and θ_2 exist only when $1 \leq N_1 \leq n-1$, and they are

$$\widehat{\theta}_1 = \frac{\sum_{i=1}^{N_1} t_{i:n} + (n - N_1)\tau}{N_1}$$

$$\widehat{\theta}_2 = \frac{\sum_{i=N_1+1}^{n} (t_{i:n} - \tau)}{n - N_1}$$

Clearly, these MLEs are conditional MLEs, conditional on $1 \leq N_1 \leq n-1$.

Conditional Densities

The conditional densities of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are

$$f_{\widehat{\theta}_1}(t) = \sum_{j=1}^{n-1} \sum_{k=0}^{j} c_{j,k} f_G\left(t - \tau_{j,k}; j, \frac{j}{\theta_1}\right)$$

and

$$f_{\widehat{\theta}_2}(t) = \sum_{j=1}^{n-1} w_j f_G\left(t; j, \frac{j}{\theta_2}\right)$$

Conditional Densities

The conditional densities of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be obtained under different censoring schemes:

1. Type-I censoring scheme
2. Type-II censoring scheme
3. Type-I Hybrid censoring scheme
4. Type-II Hybrid censoring scheme
5. Progressive Type-II censoring scheme

Confidence Intervals

Under the assumptions that the conditional probabilities $P_{\theta_1}(\widehat{\theta}_1 \geq a)$ and $P_{\theta_2}(\widehat{\theta}_2 \geq a)$ are increasing functions of θ_1 and θ_2, respectively, the exact confidence intervals of θ_1 and θ_2 can be obtained by solving two non-linear equations.

$$P_{\widehat{\theta}_1}(\widehat{\theta}_1 \geq \theta_L) = 1 - \frac{\alpha}{2} \qquad P_{\widehat{\theta}_1}(\widehat{\theta}_1 \geq \theta_U) = \frac{\alpha}{2}.$$

$$P_{\widehat{\theta}_2}(\widehat{\theta}_2 \geq \theta_L) = 1 - \frac{\alpha}{2} \qquad P_{\widehat{\theta}_2}(\widehat{\theta}_2 \geq \theta_U) = \frac{\alpha}{2}.$$

Order Restricted MLEs

The basic idea of the step-stress experiment is to increase the stress so that early failure is observed. It means it is assumed that the expected lifetime at the stress level s_2 should be smaller than the expected lifetime at the stress level s_1. Therefore, the natural restriction on the parameter space becomes

$$\theta_1 \geq \theta_2.$$

The inference on θ_1 and θ_2 should be performed under that restriction. It becomes quite difficult. The exact results are not available.

Bayesian Approach

Bayesian approach seems to be a natural choice under this order restriction case namely $\theta_1 \geq \theta_2$. Let us consider

$$\alpha = \frac{\theta_2}{\theta_1} \quad \text{or} \quad \theta_1 = \frac{\theta_2}{\alpha} \quad 0 < \alpha < 1.$$

The following priors can be taken: $\pi_1(\theta_2)$ as inverted gamma and $\pi_2(\alpha)$ as beta.
The Bayes estimates and the associated credible intervals also can be obtained.

Other Lifetime Distributions

Different other lifetime models have been used for analyzing step-stress data.

1. Weibull distribution
2. log-logistic distribution
3. generalized exponential distribution
4. two-parameter exponential distribution
5. gamma distribution

Weibull Distribution: CEM

Since for the CEM

$$G(t) = \begin{cases} F_1(t) & \text{if} \quad 0 < t < \tau \\ F_2(a(t)) & \text{if} \quad \tau < t < \infty, \end{cases}$$

where $a(t) = (\theta_2/\theta_1)\tau + t - \tau$. Therefore, the PDF of Weibull distribution becomes

$$g(t) = \begin{cases} \frac{\beta}{\theta_1} t^{\beta_1} e^{-(t^\beta/\theta_1} & \text{if} \quad 0 < t < \tau \\ \frac{\beta}{\theta_2}(a(t))^{\beta-1} e^{-(a(t))^\beta/\theta_2} & \text{if} \quad \tau < t < \infty. \end{cases}$$

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Khamis and Higgins Model

The CEM in case of Weibull distribution becomes quite complicated. Due to this reason Khamis and Higgins proposed the following model. The hazard function of CEM in case of exponential distribution takes the following form:

$$h(t) = \begin{cases} \frac{1}{\theta_1} & \text{if} \quad 0 < t < \tau \\ \\ \frac{1}{\theta_2} & \text{if} \quad \tau < t < \infty \end{cases}$$

Khamis and Higgins Model: Weibull Distribution

It is assumed that at the two different stress levels the lifetime distributions follow Weibull distribution with the same shape parameter, but different scale parameter. At the stress level s_i, the lifetime becomes;

$$f(t) = \frac{\beta}{\theta_i} t^{\beta-1} e^{-t^{\beta}/\theta_i}; \quad t > 0, \beta > 0, \theta_i > 0$$

Khamis and Higgins Model: Weibull Distribution

Khamis and Higgins proposed the following step-stress model for Weibull distribution:

$$h(t) = \begin{cases} \frac{\beta}{\theta_1} t^{\beta-1} & \text{if} \quad 0 < t < \tau \\ \\ \frac{\beta}{\theta_2} t^{\beta-1} & \text{if} \quad \tau < t < \infty \end{cases}$$

The corresponding survival function becomes:

$$S(t) = \begin{cases} \exp\left(-\frac{t^\beta}{\theta_1}\right) & \text{if} \quad 0 < t < \tau \\ \\ \exp\left(-\frac{t^\beta - \tau^\beta}{\theta_2} - \frac{\tau^\beta}{\theta_1}\right) & \text{if} \quad \tau < t < \infty \end{cases}$$

Khamis and Higgins Model: MLEs

- The MLEs cannot be obtained in explicit forms. Non-linear equations need to be solved to compute the MLEs.
- Bootstrap or asymptotic distributions can be used for constructing confidence intervals.
- Order restricted inference is also possible, but computationally it is quite demanding.

Khamis and Higgins Model: Bayesian Inference

In case of order restricted inference *i.e.* when $\theta_1 > \theta_2$, Bayesian inference seems to be a natural choice in this case also. Similar prior assumptions can be made:
The following priors can be taken: $\pi_1(\theta_2)$ as inverted gamma, $\pi_2(\alpha)$ as beta, take $\pi_3(\alpha)$ as a log-concave prior.
The Bayes estimates and the associated credible intervals also can be obtained.

Outline

Cumulative Risk Model

One of the problems for CEM model is that the hazard function is not continuous. The cumulative risk model has been suggested to overcome this problem. In case of exponential distribution the CRM takes the following form:

$$h(t) = \begin{cases} \theta_1 & \text{if} \quad 0 < t < \tau_1 \\ a + bt & \text{if} \quad \tau_1 \leq t < \tau_2 \\ \theta_2 & \text{if} \quad t > \tau_2 \end{cases}$$

The parameters a and b are such that $h(t)$ becomes continuous. Therefore

$$a + b\tau_1 = \theta_1 \qquad \text{and} \qquad a + b\tau_2 = \theta_2.$$

Cumulative Risk Model: PDF

In case of exponential distribution, the PDF becomes

$$f(t) = \begin{cases} (a + b\tau_1)e^{-(a+b\tau_1)t} & \text{if} \quad 0 < t < \tau_1 \\ (a + bt)e^{-(at+b(t^2+\tau_1 2)/2)} & \text{if} \quad \tau_1 < t < \tau_2 \\ (a + b\tau_2)e^{-(a+b\tau_2)t-b(\tau_1^1-\tau_2^2)/2} & \text{if} \quad t > \tau_2 \end{cases}$$

Cumulative Risk Model: CHF

The associated cumulative hazard function becomes

$$H(t) = \begin{cases} h_1(t) & \text{if} \quad 0 < t < \tau_1 \\ h_2(t) & \text{if} \quad \tau_1 < t < \tau_2 \\ h_3(t) & \text{if} \quad t > \tau_2 \end{cases}$$

where

$$h_1(t) = (a + b\tau_1)t$$

$$h_2(t) = (a + b\tau_1)\tau_1 + (t - \tau_1)^2 b/2$$

$$h_3(t) = (a + b\tau_1)\tau_1 + (\tau_2 - \tau_1)^2 b/2 + (t - \tau_2)(a + b\tau_2)$$

Cumulative Risk Model: Estimation

The MLEs of the unknown parameters cannot be obtained in closed form. But the least squares estimators which can be obtained by minimizing

$$\sum_{i=1}^{n} \left(H(t_i) - \widehat{H}(t_i) \right)^2$$

with respect to the unknown parameters a and b are in explicit forms. Here

$$\widehat{H}(t_i) = -\ln(\widehat{S}(t_i)) = \ln n - \ln(n - i + 1)$$

Outline

Step-Stress Model: Stress Changes at Random Time

So far we have discussed the step-stress model where the stress changes at a pre-specified time. In this set up the stress changes at a time where the pre-specified number of failures take place. Suppose it is assumed that the stress changes at the n_1-th failure. Therefore we observe the failures at

$$t_{1:n} < \ldots < t_{n_1:n} < t_{n_1+1:n} < t_{n:n}.$$

Stress Changes at Random Time: MLEs

Under the assumptions of cumulative exposure model if the lifetime distributions are exponential then

$$\widehat{\theta}_1 = \frac{T_1}{n_1} \qquad \widehat{\theta}_2 = \frac{T_2}{n - n_1},$$

where

$$T_1 = \sum_{i=1}^{n_1} t_{i:n} + (n - n_1) t_{n_1:n}$$

$$T_2 = \sum_{i=n_1+1}^{n} (t_{i:n} - t_{n_1:n})$$

Outline

Optimum τ

How to choose τ so that some optimality criterion is satisfied?
What can be a good optimality criterion?

1. Minimize the trace of the inverse of the Fisher information matrix.
2. Maximize the determinant of the Fisher information matrix.
3. Minimize the variance of the p-th percentile estimator of the distribution function under normal stress.

Step-Stress Model with Cure Fraction

In a cure rate model it is assumed that the population consists of two types of items: susceptible and immune. Susceptibles are those who are subject to failures and immunes are those who are not subject to failures. In this case the survival function of the population

$$S(t) = p + (1-p)S_0(t)$$

In case of step-stress set up it is assumed that $S_0(t)$ has one of these step-stress models.

Step-Stress Model with Competing Causes of Failures

In many life testing experiment one observes more than one causes of failures. When the time of failure and the associated cause of failure is also observed it is known that competing risk data. In this case it is assumed that the observed failure time T is as follows:

$$T = \min\{T_1, \ldots, T_M\},$$

where $T_1, \ldots, T_M$ are the failure times for different causes which are observable. A competing risk data is as follows;

$$(T, \Delta)$$

Δ denotes the cause of failure.

Multiple Step-Stress Model

So far we have mainly discussed the simple step-stress model. In case of multiple step-stress model the stress changes at

$$\tau_1 < \tau_2 < \ldots < \tau_k.$$

It is usually assumed that there is a link function of the different parameters at the different stress levels. For example the assumption of the log-link function is very common, *i.e.*

$$\ln \theta_i = a + bx_i; \quad i = 1, \ldots, k.$$

Here x_i is some function of x_i and it is known.

Open Problems: General

1. Goodness of fit tests
2. Comparison of the different models
3. Bayesian Prior choice
4. Optimal for simple step-stress and multiple step-stress.
5. Develop step-stress models with covariates

Open Problems: Specific

1. Develop order restricted inference for Weibull
2. Develop inference for three parameter Weibull
3. Bayesian inference for Weibull distribution
4. Develop step-stress models with covariates

Thank You