Bivariate Generalized Exponential Distribution

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Part of this work is going to appear in J. Mult. Anal.

Outline of the Talk

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- Univariate Generalized Exponential Distribution

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- Basic Properties

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- Generalizations

Univariate GE Distribution

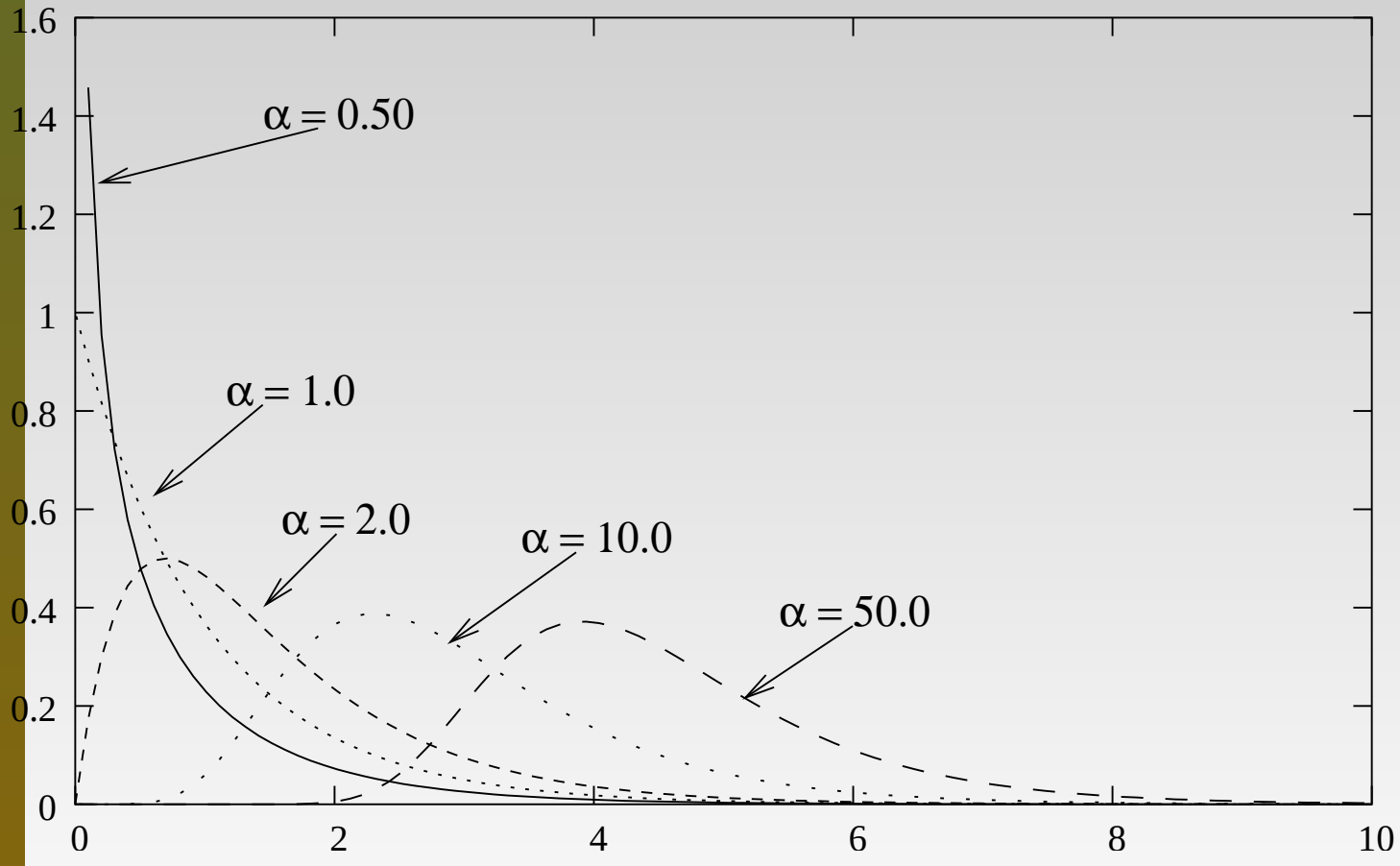
The random variable $X \sim$ GE(α, λ) if it has the following CDF;

$$F_{GE}(x;\alpha,\lambda) = \begin{cases} \left(1-e^{-\lambda x}\right)^{\alpha} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

The corresponding PDF becomes;

$$f_{GE}(x;\alpha,\lambda) = \begin{cases} \alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

PDF's of GE distribution for different α



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- PDFs are very similar to Weibull and gamma PDFs
- Hazard function can be increasing, decreasing or constant
- It enjoys several ordering properties
- It is a member of the proportional reversed hazard model
- It is closer to gamma distribution than Weibull distribution

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- It can be used for generating gamma random deviate
- It can be used for approximating normal CDF

Bivariate Singular Data

It is a soccer data from the UK Champion's League for 2004-2005 & 2005-2006.

Consider matches where (i) at least one goal scored by the home team (ii) at least one goal scored directly from a kick (penalty or any other direct free-kick)

X_1 = time of the 1-st kick goal scored by any team
X_2 = time of the 1-st goal scored by the home team.

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- Marshall-Olkin bivariate exponential model has been used
- Empirical hazard functions are not constant
- Empirical hazard function of X_2 is an increasing function.
- GE can be used quite effectively for fitting the marginals

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- Idea: Came from the formulation of the Marshall-Olkin bivariate exponential model
- Formulation:

$$U_1 \sim \text{GE}(\alpha_1, \lambda),\ U_2 \sim \text{GE}(\alpha_2, \lambda),\ U_3 \sim \text{GE}(\alpha_2, \lambda)$$

$$X_1 = \max\{U_1, U_3\}, \quad X_2 = \max\{U_2, U_3\}$$

$$(X_1, X_2) \sim \text{BVGE}(\alpha_1, \alpha_2, \alpha_3, \lambda)$$

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- (a) Stress Model

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- (a) Stress Model
- (b) Maintenance Model

Joint CDF

For $z = \min\{x_1, x_2\}$,

$$F_{X_1,X_2}(x_1, x_2) =$$
$$P(X_1 \leq x_1, X_2 \leq x_2) =$$
$$F_{GE}(x_1; \alpha_1) F_{GE}(x_2; \alpha_2) F_{GE}(z; \alpha_3) =$$
$$\begin{cases} F_{GE}(x_1; \alpha_1 + \alpha_3) F_{GE}(x_2; \alpha_2) & \text{if} \quad x_1 < x_2 \\ F_{GE}(x_1; \alpha_1 + \alpha_3) F_{GE}(x_2; \alpha_2) & \text{if} \quad x_1 > x_2 \\ F_{GE}(x; \alpha_1 + \alpha_2 + \alpha_3) & \text{if} \quad x_1 = x_2 = x \end{cases}$$

JOINT PDF

$$f_{X_1,X_2}(x_1,x_2) =$$

$$\begin{cases} f_{GE}(x_1;\alpha_1+\alpha_3)f_{GE}(x_2;\alpha_2) & \text{if} \quad x_1 < x_2 \\ f_{GE}(x_1;\alpha_1)F_{GE}(x_2;\alpha_2+\alpha_3) & \text{if} \quad x_1 > x_2 \\ \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{GE}(x;\alpha_1+\alpha_2+\alpha_3) & \text{if} \quad x_1 = x_2 = x \end{cases}$$

Note that the 1-st two terms are densities with respect to two dimensional Lebesgue measure and the 3-rd term is the density function with respect to one dimensional Lebesgue measure.

Decomposition of CDF

The joint CDF can be written as

$$F_{X_1,X_2}(x_1,x_2) = p\, F_a(x_1,x_2) + (1-p)F_s(x_1,x_2)$$

where

$$p = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3}$$

and for $z = \min\{x_1, x_2\}$

$$F_s(x_1,x_2) = (1-e^{-z})^{\alpha_1+\alpha_2+\alpha_3}.$$

$F_a(x_1,x_2)$ can be obtained by subtraction.

Different Shapes

Figure 1(a)

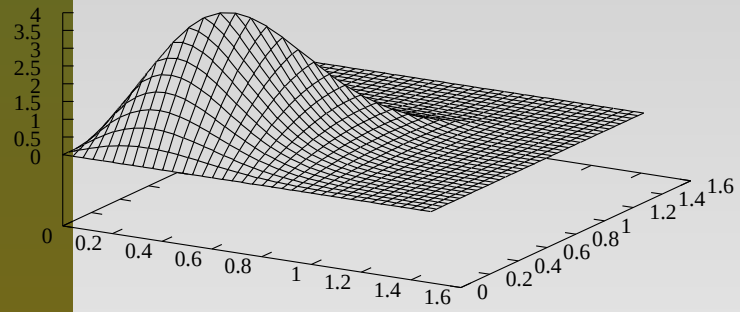


Figure 1(b)

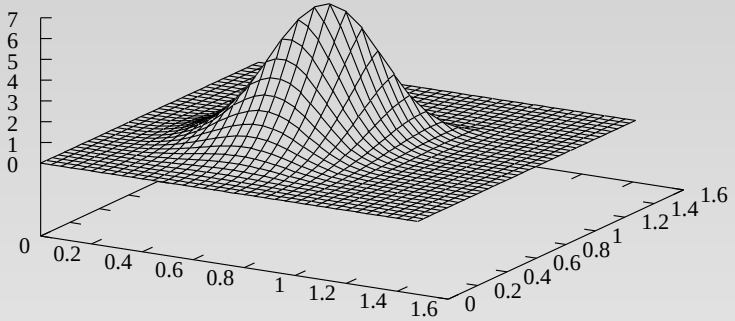


Figure 1(c)

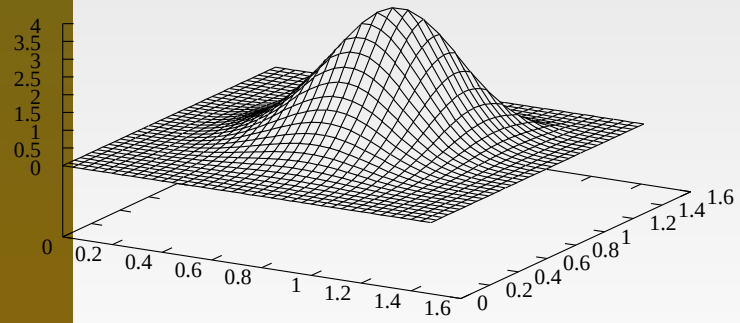
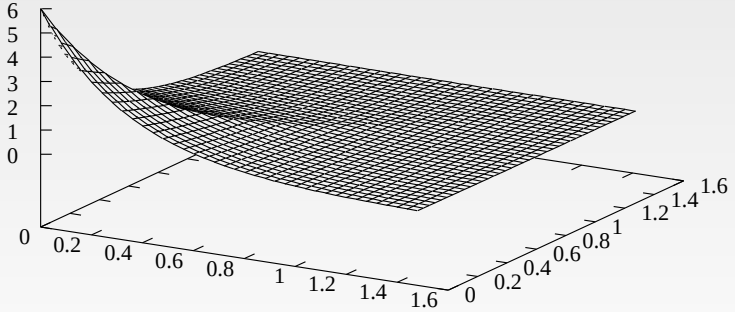


Figure 1(d)



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- Problems associated with the MLEs?
- MLEs may not always exist.
- Even if they exist they have to be obtained by solving a four dimensional optimization problem.

EM Algorithm

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- E-Step: Replace the missing observations by their conditional expectation to form the pseudo likelihood function
- M-Step: Maximize the pseudo likelihood function

EM Algorithm (cont.)

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- Observations: $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$.

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- They may be grouped as follows;

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$$I_1 = \{i; x_{1i} < x_{2i}\}, I_2 = \{i; x_{1i} > x_{2i}\}.$$

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- What are missing?

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- Let's go back to our original formulation of bivariate GE
- We have the following;

$$U_1 \sim \mathrm{GE}(\alpha_1, \lambda), \quad U_2 \sim \mathrm{GE}(\alpha_2, \lambda), \quad U_3 \sim \mathrm{GE}(\alpha_3, \lambda).$$

$$X_1 = \max\{U_1, U_3\}, \qquad X_2 = \max\{U_2, U_3\}$$

EM Algorithm (cont.)

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- All possible ordering of U_1, U_2, U_3

Ordering	(X_1, X_2)	Group
$U_1 < U_2 < U_3$	(U_3, U_3)	I_0
$U_1 < U_3 < U_2$	(U_3, U_2)	I_1
$U_2 < U_1 < U_3$	(U_3, U_3)	I_0
$U_2 < U_3 < U_1$	(U_1, U_3)	I_2
$U_3 < U_2 < U_1$	(U_1, U_2)	I_2
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$U_2 < U_3 < U_1$	(U_1, U_3)	I_2
$U_3 < U_2 < U_1$	(U_1, U_2)	I_2
$U_3 < U_1 < U_2$	(U_1, U_2)	I_1

- Suppose we know U_1, U_2, U_3 along with X_1, X_2

EM Algorithm (cont.)

The likelihood contribution of any observation from I_0

$$f_{GE}(x;\alpha_3,\lambda)F_{GE}(x;\alpha_1+\alpha_2,\lambda)$$

The likelihood contribution of any observation from I_1

$$f_{GE}(x_1;\alpha_3,\lambda)f_{GE}(x_2;\alpha_2,\lambda)F_{GE}(x_1;\alpha_1,\lambda) \quad \text{or}$$

$$f_{GE}(x_1;\alpha_1,\lambda)f_{GE}(x_2;\alpha_2,\lambda)F_{GE}(x_1;\alpha_3,\lambda).$$

The likelihood contribution of any observation from I_2

$$f_{GE}(x_2;\alpha_3,\lambda)f_{GE}(x_1;\alpha_1,\lambda)F_{GE}(x_2;\alpha_2,\lambda) \quad \text{or}$$

$$f_{GE}(x_1;\alpha_1,\lambda)f_{GE}(x_2;\alpha_2,\lambda)F_{GE}(x_2;\alpha_3,\lambda).$$

EM Algorithm (cont.); E-Step

Therefore in this case the complete observation is of the following type

$$(X_1, X_2, \Lambda_1, \Lambda_2)$$

Here Λ_1 and Λ_2 are as follows;

$$\Lambda_1 = \begin{cases} 1 & \text{if} \quad X_1 = U_1 \\ 3 & \text{if} \quad X_1 = U_3 \end{cases}$$

$$\Lambda_2 = \begin{cases} 2 & \text{if} \quad X_2 = U_2 \\ 3 & \text{if} \quad X_2 = U_3 \end{cases}$$

EM Algorithm (cont.); E-Step

Therefore for the Group I_0 both Λ_1 and Λ_2 are known.

$$\Lambda_1 = \Lambda_2 = 3$$

For Group I_1 Λ_2 is known and Λ_1 is unknown.

$$\Lambda_1 = 1 \text{ or } 3, \quad \Lambda_2 = 2$$

For Group I_2 Λ_1 is known and Λ_2 is unknown.

$$\Lambda_1 = 1, \quad \Lambda_2 = 2 \text{ or } 3.$$

Moreover

$$P(\Lambda_1 = 1|I_1) = \frac{\alpha_1}{\alpha_1 + \alpha_3}, \quad P(\Lambda_2 = 2|I_2) = \frac{\alpha_2}{\alpha_2 + \alpha_3},$$

EM Algorithm (cont.); M-Step

With the above information the pseudo log-likelihood function can be written. For M-step we need to maximize the pseudo log-likelihood function with respect to the unknown parameters. The pseudo log-likelihood function can be written as a profile log-likelihood function of λ only and the maximization can be carried out as a one dimensional optimization method.

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- Bayesian inference

Thank You