On Bivariate Weibull-Geometric Distribution

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It is a soccer data from the UK Champion’s League for 2004-2005 & 2005-2006.

Consider matches where (i) at least one goal scored by the home team (ii) at least one goal scored directly from a kick (penalty or any other direct free-kick)

\[ X_1 = \text{time of the 1-st kick goal scored by any team} \]
\[ X_2 = \text{time of the 1-st goal scored by the home team}. \]
In this case all possibilities are there

\[ X_1 < X_2, \quad X_1 > X_2, \quad X_1 = X_2 \]

- Marshall-Olkin bivariate exponential model has been used
- Empirical hazard functions are not constant
- Empirical hazard function of \( X_2 \) is an increasing function.
- Empirical hazard function of \( X_1 \) is a monotone function.
Existing Bivariate Singular Model

- Marshall-Olkin bivariate exponential model.
- Marshall-Olkin bivariate Weibull model.
- Generalized bivariate exponential model.
- Generalized bivariate proportional reversed hazard model.
Basic Idea of Modeling

1. It should be simple.
2. It should be flexible.
3. Should have some physical interpretations.
4. Analysis should not be too difficult to do.
Motivation

Marshall-Olkin Univariate Model

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Basic Formulation

Marshall and Olkin (1997, Biometrika) introduced the following univariate model:
Suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables with common distribution function $F(\cdot)$, and $N$ is a geometric random variable with parameter $0 < \theta \leq 1$, and the following probability mass function

$$P(N = n) = \theta (1 - \theta)^{n-1}; \quad n = 1, 2, \ldots.$$ 

Consider a new random variable $Y$, as

$$Y = \min\{X_1, \ldots, X_N\}$$
Survival Function

The survival function of $Y$ can be obtained as

$$
\tilde{G}(y) = P(Y \geq y) = \sum_{n=1}^{\infty} P(Y \geq y | N = n) P(N = n)
$$

$$
= \sum_{n=1}^{\infty} \bar{F}(y)^n \theta (1 - \theta)^{n-1}
$$

$$
= \frac{\theta \bar{F}(y)}{1 - (1 - \theta) \bar{F}(y)}
$$
The distribution function becomes:

$$G(y) = 1 - \tilde{G}(y) = 1 - \frac{\theta \bar{F}(y)}{1 - (1 - \theta) \bar{F}(y)}$$

$$= \frac{F(y)}{1 - (1 - \theta) \bar{F}(y)}$$

The associated probability density function becomes:

$$g(y; \theta) = \frac{\theta f(y)}{(1 - (1 - \theta) \bar{F}(y))^2}$$
Now consider the case when $\theta \geq 1$. Suppose $X_1, X_2, \ldots$ are independent and identically distributed random variables with common distribution function $F(\cdot)$, and $N$ is a geometric random variable with parameter $0 < 1/\theta \leq 1$, i.e. it has the following probability mass function

$$P(N = n) = \frac{1}{\theta} \left(1 - \frac{1}{\theta}\right)^{n-1}; \quad n = 1, 2, \ldots.$$ 

Consider a new random variable $Y$, as

$$Y = \max\{X_1, \ldots, X_N\}$$
The distribution function of $Y$ can be obtained as

$$G(y) = P(Y \leq y) = \sum_{n=1}^{\infty} P(Y \leq y | N = n)P(N = n)$$

$$= \sum_{n=1}^{\infty} F(y)^n \frac{1}{\theta} \left(1 - \frac{1}{\theta}\right)^{n-1}$$

$$= \frac{F(y)}{\theta - F(y)(\theta - 1)}$$

$$= \frac{F(y)}{1 - (1 - \theta)\bar{F}(y)}$$
The distribution function becomes same for $0 < \theta < 1$ and also for $1 < \theta < \infty$.

The associated probability density function becomes:

$$g(y; \theta) = \frac{\theta f(y)}{(1 - (1 - \theta)\bar{F}(y))^2}$$
Special Case: Exponential

If $X_i$’s are i.i.d. exponential random variables with PDF $f(x; \lambda) = \lambda e^{-\lambda x}$, then

\[ g(y; \theta, \lambda) = \frac{\lambda \theta e^{-\lambda y}}{(1 - (1 - \theta)e^{-\lambda y})^2} \]

\[ \tilde{G}(y; \theta, \lambda) = \frac{\theta e^{-\lambda y}}{1 - (1 - \theta)e^{-\lambda y}} \]

\[ h(y; \theta, \lambda) = \frac{\lambda}{1 - (1 - \theta)e^{-\lambda y}} \]
Some Basic Properties

- Exponential distribution is a special case. \((\theta = 1)\)
- PDF can take variety of shapes
  - PDF can be a decreasing function.
  - PDF can be a unimodal function.
- Hazard function can take different shapes
  - It can be decreasing
  - It can be increasing
- It behaves very similar to Gamma, Weibull or Generalized exponential distribution.
- Generation is also very simple.
Special Case: Weibull

If $X_i$'s are i.i.d. Weibull random variables with PDF $f(x; \alpha, \lambda) = \alpha \lambda x^{\alpha-1} e^{-\lambda x^\alpha}$, then

\[ g(y; \alpha, \theta, \lambda) = \frac{\alpha \lambda \theta y^{\alpha-1} e^{-\lambda y^\alpha}}{(1 - (1 - \theta)e^{-\lambda y^\alpha})^2} \]

\[ \bar{G}(y; \alpha, \theta, \lambda) = \frac{\theta e^{-\lambda y^\alpha}}{1 - (1 - \theta)e^{-\lambda y^\alpha}} \]

\[ h(y; \alpha, \theta, \lambda) = \frac{\alpha \lambda y^{\alpha-1}}{1 - (1 - \theta)e^{-\lambda y^\alpha}} \]
Some Basic Properties

- Weibull distribution is a special case. ($\theta = 1$)
- PDF can take variety of shapes
  - PDF can be a decreasing function.
  - PDF can be a unimodal function.
- Hazard function can take different shapes
- It behaves very similar to the three-parameter exponentiated Weibull distribution.
- Generation is also very simple.
Special Case: Generalized Exponential Distribution

If $X_i$’s are i.i.d. Generalized Exponential random variables with PDF $f_{GE}(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}$, then

$$g(y; \alpha, \theta, \lambda) = \frac{\alpha \lambda \theta e^{-\lambda y} (1 - e^{-\lambda y})^{\alpha-1}}{(1 - (1 - \theta)(1 - (1 - e^{-\lambda y})^{\alpha-1}))^2}$$

$$G(y; \theta, \lambda) = \frac{(1 - e^{-\lambda y})^\alpha}{1 - (1 - \theta)(1 - (1 - e^{-\lambda y})^\alpha)}$$
Some Basic Properties

- Generalized exponential distribution is a special case. ($\theta = 1$)
- PDF can take variety of shapes
  - PDF can be a decreasing function.
  - PDF can be a unimodal function.
- Hazard function can take different shapes
- It behaves very similar to the three-parameter exponentiated Weibull distribution.
- Generation is also very simple.
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6. Bivariate Weibull-Geometric Model
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Suppose $U_0$, $U_1$ and $U_2$ are three independent exponential random variables with parameters $\lambda_0$, $\lambda_1$ and $\lambda_2$, respectively. Consider the following new bivariate random variable

$$Y_1 = \min\{U_0, U_1\} \quad \text{and} \quad Y_2 = \min\{U_0, U_2\}.$$ 

The joint survival function of $Y_1$ and $Y_2$ can be written as

$$P(Y_1 > y_1, Y_2 > y_2) = P(U_0 > \max\{y_1, y_2\}, U_1 > y_1, U_2 > y_2) = \begin{cases} e^{-(\lambda_0 + \lambda_1)y_1 - \lambda_2y_2} & \text{if } y_1 \geq y_2 \\ e^{-(\lambda_0 + \lambda_2)y_2 - \lambda_1y_1} & \text{if } y_2 > y_1 \end{cases}$$
The associated joint PDF becomes

\[ f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 
  f_1(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
  f_2(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
  f_0(y) & \text{if } 0 < y_1 = y_2 = y < \infty,
\end{cases} \]

where

\begin{align*}
  f_1(y_1, y_2) &= (\lambda_0 + \lambda_1)e^{-(\lambda_0+\lambda_1)y_1} \lambda_2 e^{-\lambda_2y_2}, \\
  f_2(y_1, y_2) &= \lambda_1 e^{-\lambda_1 y_1} (\lambda_0 + \lambda_2)e^{-(\lambda_0+\lambda_2)y_2}, \\
  f_0(y) &= \lambda_0 e^{-(\lambda_0+\lambda_1+\lambda_2)y},
\end{align*}
Basic Properties

- The marginals are exponential distributions.

- It has a singular component on $y_1 = y_2$ axis.

- It has an interesting physical interpretations (shock model).
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Definition

Suppose $U_0$, $U_1$ and $U_2$ are three independent Weibull random variables with parameters $(\alpha, \lambda_0)$, $(\alpha, \lambda_1)$ and $(\alpha, \lambda_2)$, respectively. Consider the following new bivariate random variable

\[ Y_1 = \min\{U_0, U_1\} \quad \text{and} \quad Y_2 = \min\{U_0, U_2\}. \]

The joint survival function of $Y_1$ and $Y_2$ can be written as

\[
P(Y_1 > y_1, Y_2 > y_2) = P(U_0 > \max\{y_1, y_2\}, U_1 > y_1, U_2 > y_2) = \begin{cases} e^{-\left(\lambda_0 + \lambda_1\right)y_1^\alpha - \lambda_2y_2^\alpha} & \text{if } y_1 \geq y_2 \\ e^{-\left(\lambda_0 + \lambda_2\right)y_2^\alpha - \lambda_1y_1^\alpha} & \text{if } y_2 > y_1 \end{cases}
\]
Joint PDF

The associated joint PDF becomes

\[ f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 
  f_1(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
  f_2(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
  f_0(y) & \text{if } 0 < y_1 = y_2 = y < \infty, 
\end{cases} \]

where

\[ f_1(y_1, y_2) = \alpha^2 y_1^{\alpha-1} y_2^{\alpha-2} (\lambda_0 + \lambda_1) e^{-(\lambda_0 + \lambda_1)y_1^\alpha} \lambda_2 e^{-\lambda_2 y_2^\alpha}, \]
\[ f_2(y_1, y_2) = \alpha^2 y_1^{\alpha-1} y_2^{\alpha-2} \lambda_1 e^{-\lambda_1 y_1^\alpha} (\lambda_0 + \lambda_2) e^{-(\lambda_0 + \lambda_2)y_2^\alpha}, \]
\[ f_0(y) = \alpha \lambda_0 y^{\alpha-1} e^{-(\lambda_0 + \lambda_1 + \lambda_2)y^\alpha}, \]
Basic Properties

- The marginals are Weibull distributions.
- It has a singular component on $y_1 = y_2$ axis.
- It has an interesting physical interpretations (shock model).
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Definition

Suppose $U_0$, $U_1$ and $U_2$ are three independent generalized exponential random variables with parameters $(\alpha_0, \lambda)$, $(\alpha_1, \lambda)$ and $(\alpha_2, \lambda)$, respectively. Consider the following new bivariate random variable

$$Y_1 = \max\{U_0, U_1\} \quad \text{and} \quad Y_2 = \max\{U_0, U_2\}.$$ 

The joint distribution function of $Y_1$ and $Y_2$ can be written as

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = P(U_0 \leq \min\{y_1, y_2\}, U_1 \leq y_1, U_2 \leq y_2)$$

$$= \begin{cases} (1 - e^{-\lambda y_1})^{\alpha_0 + \alpha_1} (1 - e^{-\lambda y_2})^{\alpha_2} & \text{if } y_1 \leq y_2 \\ (1 - e^{-\lambda y_1})^{\alpha_1} (1 - e^{-\lambda y_2})^{\alpha_0 + \alpha_2} & \text{if } y_2 > y_1 \end{cases}$$
Joint PDF

The associated joint PDF becomes

\[
f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 
  f_1(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
  f_2(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
  f_0(y) & \text{if } 0 < y_1 = y_2 = y < \infty,
\end{cases}
\]

where

\[
\begin{align*}
  f_1(y_1, y_2) &= f_{GE}(y_1; (\alpha_0 + \alpha_1), \lambda)f_{GE}(y_2; \alpha_2, \lambda), \\
  f_2(y_1, y_2) &= f_{GE}(y_1; \alpha_1, \lambda)f_{GE}(y_2; (\alpha_0 + \alpha_2), \lambda), \\
  f_0(y) &= \alpha_0 f_{GE}(y; (\alpha_0 + \alpha_1 + \alpha_2), \lambda),
\end{align*}
\]
Basic Properties

- The marginals are generalized exponential distributions.
- It has a singular component on $y_1 = y_2$ axis.
- It has an interesting physical interpretations (maintenance model).
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Suppose \( \{(X_{1n}, X_{2n}); n = 1, 2, \ldots \} \) is a sequence of i.i.d. bivariate random variables with common joint distribution function \( F(., .) \), and \( N \) is an independent geometric random variable. Consider a bivariate random variable \( (Y_1, Y_2) \)

\[
Y_1 = \min\{X_{11}, \ldots, X_{1N}\} \quad \text{and} \quad Y_2 = \min\{X_{21}, \ldots, X_{2N}\}.
\]
The joint survival function of $Y_1$ and $Y_2$ becomes

$$P(Y_1 > y_1, Y_2 > y_2)$$

$$= \sum_{n=1}^{N} P(\min\{X_{11}, \ldots, X_{1n}\} > y_1, \min\{X_{21}, \ldots, X_{2n}\} > y_2 \mid N = n) \times P(N = n)$$

$$= \sum_{n=1}^{\infty} \bar{F}^n(y_1, y_2) \theta (1 - \theta)^{n-1}$$

$$= \frac{\theta \bar{F}(y_1, y_2)}{1 - (1 - \theta)\bar{F}(y_1, y_2)}$$
The marginal survival function of $Y_1$ and $Y_2$ become

\[
P(Y_1 > y_1) = \frac{\theta \bar{F}_1(y_1)}{1 - (1 - \theta) \bar{F}_1(y_1)}
\]

\[
P(Y_2 > y_2) = \frac{\theta \bar{F}_2(y_2)}{1 - (1 - \theta) \bar{F}_2(y_2)}
\]
The joint survival function of $Y_1$ and $Y_2$ becomes

$$P(Y_1 > y_1, Y_2 > y_2) = \begin{cases} 
\theta e^{-(\lambda_0 + \lambda_1)y_1^{\alpha} - \lambda_2 y_2^{\alpha}} \\
1 - (1 - \theta) e^{-(\lambda_0 + \lambda_1)y_1^{\alpha} - \lambda_2 y_2^{\alpha}} & \text{if } y_1 \geq y_2 \\
\theta e^{-(\lambda_0 + \lambda_1)y_2^{\alpha} - \lambda_1 y_1^{\alpha}} \\
1 - (1 - \theta) e^{-(\lambda_0 + \lambda_2)y_2^{\alpha} - \lambda_1 y_1^{\alpha}} & \text{if } y_1 < y_2
\end{cases}$$
The associated joint PDF becomes

\[ g_{Y_1, Y_2}(y_1, y_2) = \begin{cases} 
  g_1(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
  g_2(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
  g_0(y) & \text{if } 0 < y_1 = y_2 = y < \infty,
\end{cases} \]
Shapes of the AC Part of PDF

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Shapes of the AC Part of PDF
Generation

1. Generate $n$ from $\text{GM}(\theta)$.
2. Generate $\{v_{01}, \ldots, v_{0n}\}$ from $\text{WE}(\alpha, \lambda_0)$, similarly, $\{v_{11}, \ldots, v_{1n}\}$ from $\text{WE}(\alpha, \lambda_1)$ and $\{v_{21}, \ldots, v_{2n}\}$ from $\text{WE}(\alpha, \lambda_2)$.
3. Obtain $u_{1k} = \min\{v_{0k}, v_{1k}\}$ and $u_{2k} = \min\{v_{0k}, v_{2k}\}$, for $k = 1, \ldots, n$.
4. Compute the desired $(y_1, y_2)$ as $y_1 = \min\{u_{11}, \ldots, u_{1n}\}$ and $y_2 = \min\{u_{21}, \ldots, u_{2n}\}$. 
Some Basic Properties

Let \((Y_1, Y_2) \sim BWG(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2)\), then

1. \(Y_1 \sim UWG(\theta, \alpha, \lambda_0 + \lambda_1)\)
2. \(Y_2 \sim UWG(\theta, \alpha, \lambda_0 + \lambda_2)\)
3. \(\min\{Y_1, Y_2\} \sim UWG(\theta, \alpha, \lambda_0 + \lambda_1 + \lambda_2)\)
4. \(P(Y_1 < Y_2) = \frac{\lambda_1}{\lambda_0 + \lambda_1 + \lambda_2}\).
{(y_{11}, y_{21}), \ldots, (y_{1m}, y_{2m})} is a bivariate random sample of size m from BWG(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2). We use the following notation

\begin{align*}
l_0 &= \{i : y_{1i} = y_{2i} = y_i\}, & l_1 &= \{i : y_{1i} > y_{2i}\}, & l_2 &= \{i ; y_{1i} < y_{2i}\},
\end{align*}

and suppose m_0, m_1 and m_2 denote the number of observations in the set l_0, l_1 and l_2, respectively. The log-likelihood function can be written as

\begin{align*}
l(\theta, \alpha, \lambda_0, \lambda_1, \lambda_2) &= \sum_{i \in l_0} \ln g_0(y_i) + \sum_{i \in l_1} \ln g_1(y_{1i}, y_{2i}) + \sum_{i \in l_2} \ln g_2(y_{1i}, y_{2i}),
\end{align*}
Suppose along with \((Y_1, Y_2)\), we observe \(N\) also. The joint PDF of \((Y_1, Y_2, N)\) can be obtained from the following fact:

\[(Y_1, Y_2 | N = n) \sim \text{MOBW}(\alpha, n\lambda_0, n\lambda_1, n\lambda_2).\]

The complete data is as follows: \[\{(y_{1i}, y_{2i}, n_i); i = 1, \ldots, m\}\].
Since \(\theta\) is assumed to be known, the MLEs of \(\alpha, \lambda_0, \lambda_1\) and \(\lambda_2\) can be obtained by maximizing the conditional log-likelihood function, and it is as follows:

\[
l_1(\alpha, \lambda_0, \lambda_1, \lambda_2) = \sum_{i \in I_0} \ln f_{0n_i}(y_i) + \sum_{i \in I_1} \ln f_{1n_i}(y_{1i}, y_{2i}) + \sum_{i \in I_2} \ln f_{2n_i}(y_{1i}, y_{2i}).
\]
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Prior Assumptions on $\lambda_0$, $\lambda_1$ and $\lambda_2$

\[
\pi_0(\lambda|a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}
\]

\[
\pi\left(\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda} | \lambda, a_0, a_1, a_2\right) = c \left(\frac{\lambda_0}{\lambda}\right)^{a_0-1} \left(\frac{\lambda_1}{\lambda}\right)^{a_1-1} \left(\frac{\lambda_2}{\lambda}\right)^{a_2-1}
\]

\[
\pi(\lambda_0, \lambda_1, \lambda_2|a, b, a_0, a_1, a_2) = c(b\lambda)^{a-a_0-a_1-a_2} \times \lambda_0^{a_0-1} e^{-b\lambda_0} \times \lambda_1^{a_1-1} e^{-b\lambda_1} \times \lambda_2^{a_2-1} e^{-b\lambda_2}
\]
Gamma-Dirichlet Prior

1. \[ \pi(\lambda_0, \lambda_1, \lambda_2|a, b, a_0, a_1, a_2) \sim GD(a, b, a_0, a_1, a_2) \]

2. If \( a = a_0 + a_1 + a_2 \), \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) are independent

3. Correlation between \( \lambda_i \) and \( \lambda_j \) can be positively or negatively correlated.
Prior Assumption on $\alpha$

No specific prior has been assumed on $\alpha$. It is assumed that the prior on $\alpha$ has a support on $(0, \infty)$, the PDF is log-concave. It is independent of the prior on $(\lambda_1, \lambda_2)$.

Note that several standard distribution functions have log-concave PDF, for example (i) log-normal, (ii) gamma (shape parameter greater than 1), (iii) Weibull (shape parameter greater than 1), (iii) generalized exponential distribution (shape parameter greater than 1) etc.
On $\theta$ either uniform or Beta prior can be assumed. Based on these prior assumptions Gibbs sampling method may be applied for estimating the unknown parameters.
Suppose \( \{(X_{1n}, X_{2n}); n = 1, 2, \ldots\} \) is a sequence of i.i.d. bivariate random variables with common joint distribution function \( F(., .) \), and \( N \) is an independent geometric random variable. Consider a bivariate random variable \((Y_1, Y_2)\)

\[
Y_1 = \max\{X_{11}, \ldots, X_{1N}\} \quad \text{and} \quad Y_2 = \max\{X_{21}, \ldots, X_{2N}\}.
\]
The joint distribution function of $Y_1$ and $Y_2$ becomes

$$
P(Y_1 \leq y_1, Y_2 \leq y_2) = \sum_{n=1}^{N} P(\max\{X_{11}, \ldots, X_{1n}\} \leq y_1, \max\{X_{21}, \ldots, X_{2n}\} \leq y_2 | N = n) \times P(N = n) \times \sum_{n=1}^{\infty} F^n(y_1, y_2) \theta(1 - \theta)^{n-1} = \frac{\theta F(y_1, y_2)}{1 - (1 - \theta) \bar{F}(y_1, y_2)}$$
Thank You