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Point and Interval Estimation for a Simple Step-Stress Model with Random Stress-Change Time

Debasis Kundu — *Indian Institute of Technology*
N. Balakrishnan — *McMaster University*

ABSTRACT In accelerated testing, the units are tested at varying stress levels. A special class of accelerated tests is step-stress test, that allows the experimenter to change the stress levels at pre-specified times during the experiment. It is observed that in the conventional step-stress testing, the parameters are not always estimable and even when the life time distributions are exponential, the exact confidence intervals are quite complicated. In this paper, we consider a simple step-stress model with a random stress-change time. In this set up the stress level changes at the time when a pre-specified number of failures take place. We derive the maximum likelihood estimators when the life time distributions are exponential and under the assumption of a cumulative exposure model. The joint distribution of the parameters is obtained. We provide the confidence intervals using the exact distribution and by two bootstrap methods. Bayes estimates and the corresponding credible intervals are also obtained. Monte Carlo simulations are performed to compare the performances of the different methods.

Keywords Accelerated testing; Step-stress test; Cumulative exposure model; Maximum likelihood estimator; Uniformly minimum variance unbiased estimator; Bootstrap confidence intervals; Credible intervals.

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□ Debasis Kundu (corresponding author) is affiliated with the Department of Mathematics and Statistics at Indian Institute of Technology, Kanpur Pin 208016, India; web page: http://www.iitk.ac.in/math/faculty/kundu. N. Balakrishnan is a Professor in the Department of Mathematics and Statistics at McMaster University, Hamilton, Ontario, Canada L8S 4K1; email: bala@mcmaster.ca, web page: http://icarus.math. mcmaster.ca/bala/bala.html. Professor Balakrishnan is a Fellow of both the American Statistical Association and the Institute of Mathematical Statistics. He is currently the Editor-in-Chief of the journal *Communications in Statistics/Series A and Series B* and the Executive Editor of *Journal of Statistical Planning and Inference*.
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1. Introduction

In reliability experiments, for highly reliable units, it is almost impossible to obtain adequate information about the lifetime distributions under the life-testing experiments using Type-I or Type-II censoring. Due to this, the experimenter uses the accelerated testing, where the units are subjected to higher stress levels than normal. In accelerated life-testing experiments, the units are tested at higher than usual levels of stress, such as temperature, voltage, load etc., to induce early failure. The data obtained from the accelerated experiments can be extrapolated to estimate the underlying lifetime distributions. For an excellent review on accelerated life-testing, the readers are referred to the book of Nelson [9].

A particular case of the accelerated life-testing is the step-stress testing. In step-stress testing, several units are placed on a life-test under an initial stress level x_0. At the pre-specified times, $\tau_1, \ldots, \tau_m$, the stress levels are changed to $x_1, \ldots, x_m$ respectively. In a simple step-stress model with Type-II censoring, the experiment stops when a pre-specified number of items fail. This model has been studied extensively in the literature, see for example Miller and Nelson [8], Bai *et al.* [1], DeGroot and Goel [4], Shaked and Singpurwalla [10], Khamis and Higgins [7], Xiong [11], Xiong and Milliken [13], Balakrishnan *et al.* [2] and see the references therein.

Unfortunately, in the conventional step-stress models as described above, the parameters associated with the failure time distributions at the different stress levels are not always estimable. Even when the lifetime distributions are exponential the exact confidence intervals are not very easy to obtain (see Balakrishnan *et al.* [2]). Because of this, we consider the following step-stress model where the the parameters associated with different parameters are always estimable. In this model the stress changes when a pre-specified number of failures takes place, which is random. This model was first considered by Xiong and Milliken [12] and recently by Xiong *et al.* [14]. The random stress-change time step-stress model with m levels can be described as follows: For the sample size n, prefixed $n_1, \ldots, n_m$, such that $n_1 + \ldots + n_m < n$. Let us denote $t_{1:n} < t_{2:n} < \ldots$ as the ordered failure times. Suppose n units are placed on a life-testing experiment and they are subjected to a stress level x_1. At the time of n_1-th failure $t_{n_1:n}$, the stress level is changed to x_2, then at the time of $(n_1 + n_2)$-th failure the stress level is changed to x_3 and so on. Finally at the time of $(n_1 + \ldots + n_{m-1})$-th failure, the stress level is changed to x_m and the test continues until a total of $(n_1 + \ldots + n_m)$ failures are observed.

It is further assumed that the data come from a cumulative exposure model as introduced by Nelson [9]. The cumulative exposure model relates the life distributions of the units at one stress level to the next stress level. The model assumes that the residual life of the experimental units depends only on cumulative exposures the units have experienced, with no memory of how this exposure was accumulated.

In this article, we mainly consider the above step-stress model with only two stress levels x_1 and x_2. The lifetime distributions at stress levels x_1 and x_2 are assumed to be exponential random variables, with means θ_1 and θ_2 respectively. The main purpose of this paper is to obtain the point and interval estimation of the different unknown parameters under this set up.

The rest of the paper is organized as follows. We define all the notations at the end of this section. In section 2, we provide the maximum likelihood estimators (MLEs) of the unknown parameters and find their joint distributions. Different confidence intervals are proposed in section 3. In section 4, we present the Bayes estimates and the corresponding credible intervals under the assumptions of inverted gamma priors on the unknown parameters. Monte Carlo simulation results are presented in section 5. Some related issues are discussed in section 6 and finally we conclude the paper in section 7.

We will be using the following notations in this paper.

ALT :	accelerated life-testing
HPD :	highest posterior density
PDF :	probability density function
MLE :	maximum likelihood estimator
MSE :	mean squared errors
$exp(\theta)$:	exponential random variable with PDF; $e^{-\frac{x}{\theta}}/\theta, \;\; x > 0$
$exp(\mu, \theta)$:	exponential random variable with PDF $e^{-\frac{x-\mu}{\theta}}/\theta, \;\; x > \mu$
$gamma(\alpha, \lambda)$:	gamma random variable with PDF; $\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \;\; x > 0$
$igamma(\alpha, \lambda)$:	inverted gamma random variable with PDF; $\frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{e^{-\frac{\lambda}{x}}}{x^{\alpha+1}}, \;\; x > 0$
$\gamma_{\alpha,\lambda}(\delta)$:	the lower δ-th percentile point of a $gamma(\alpha, \lambda)$ distribution
$\chi^2_k(\alpha)$:	the lower α-th percentile point of a χ^2 distribution with k degrees of freedom
[x] :	the integer part of x

2. MLEs and Their Joint Distribution

Suppose we observe the following sample;

$$\{t_{1:n} < \ldots < t_{n_1:n} < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n}\}. \tag{1}$$

Based on the assumptions of the cumulative exposure model, the joint density function of $t_{1:n} < \ldots < t_{n_1:n} < t_{n_1+1:n} < \ldots < t_{n_1+n_2:n}$ can be written as follows: The joint density of $\{t_{1:n} < \ldots < t_{n_1:n}\}$ is that of the first n_1 order statistics from a sample of size n from $exp(\theta_1)$ and the conditional density of $\{t_{n_1+1:n} < \ldots < t_{n_1+n_2:n}\}$ given $t_{1:n}, \ldots, t_{n_1:n}$ is that of the smallest n_2 order statistics from a sample of size $n - n_1$ from $exp(t_{n_1:n}, \theta_2)$. Therefore, the likelihood function of the observed data (1) is

$$l(\theta_1, \theta_2) = \frac{c}{\theta_1^{n_1}\theta_2^{n_2}} e^{-\frac{T_1}{\theta_1}} e^{-\frac{T_2}{\theta_2}}, \tag{2}$$

where $c = n(n-1)\dots(n-r+1)$, $r = n_1 + n_2$ and

$$T_1 = \sum_{i=1}^{n_1} t_{i:n} + (n-n_1)t_{n_1:n} \tag{3}$$

$$T_2 = \sum_{i=n_1+1}^{r} (t_{i:n} - t_{n_1:n}) + (n-r)(t_{r:n} - t_{n_1:n}). \tag{4}$$

From the likelihood function (2), it is clear that (T_1, T_2) is a complete sufficient statistic. The MLEs of θ_1 and θ_2 can be obtained as

$$\hat{\theta}_1 = \frac{T_1}{n_1} \quad \text{and} \quad \hat{\theta}_2 = \frac{T_2}{n_2}.$$

Now we will obtain the joint distribution of $\hat{\theta}_1$ and $\hat{\theta}_2$. The following results will be useful.

Lemma 1 *Suppose $Z_1, \dots, Z_{n_1+n_2}$ are (n_1+n_2) random variables with joint probability density function*

$$f_{Z_1,\dots,Z_{n_1+n_2}}(t_1,\dots,t_{n_1+n_2}) = \begin{cases} \frac{c}{\theta_1^{n_1}\theta_2^{n_2}} \times e^{-\frac{T_1}{\theta_1}} e^{-\frac{T_2}{\theta_2}}, & \text{for } 0 < t_1 < \dots < t_{n_1+n_2} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

where c, T_1 and T_2 are same as defined before. Consider the new set of random variables;

$$Y_1 = \frac{Z_1}{\theta_1}, \dots, Y_{n_1} = \frac{Z_{n_1}}{\theta_1}, Y_{n_1+1} = \frac{Z_{n_1+1} - Z_{n_1}}{\theta_2}, \dots, Y_{n_1+n_2} = \frac{Z_{n_1+n_2} - Z_{n_1}}{\theta_2},$$

then the joint probability density function of $Y_1, \dots, Y_{n_1+n_2}$ is

$$f_{Y_1,\dots,Y_{n_1+n_2}}(y_1,\dots,y_{n_1+n_2}) = e^{-(\sum_{i=1}^{n_1} y_i + (n-n_1)y_{n_1}) - \left(\sum_{i=n_1+1}^{n_1+n_2} y_i + (n-(n_1+n_2))y_{n_1+n_2}\right)},$$

for $0 < y_1 < \dots < y_{n_1} < \infty$, $0 < y_{n_1+1} < \dots < y_{n_1+n_2} < \infty$ and zero otherwise.

Proof. It is simple. □

Lemma 2 *The joint moment generating function of $\frac{\hat{\theta}_1}{\theta_1}$ and $\frac{\hat{\theta}_2}{\theta_2}$ is*

$$M(t,s) = Ee^{\left(t\frac{\hat{\theta}_1}{\theta_1} + s\frac{\hat{\theta}_2}{\theta_2}\right)} = \left(1 - \frac{t}{n_1}\right)^{-n_1} \left(1 - \frac{s}{n_2}\right)^{-n_2}.$$

Proof. Note that after simplifications

$$Ee^{\left(tn_1\frac{\hat{\theta}_1}{\theta_1} + sn_2\frac{\hat{\theta}_2}{\theta_2}\right)} = c\int_0^\infty \dots \int_{y_{n_1-1}}^\infty e^{-(1-t)\left(\sum_{i=1}^{n_1} y_i + (n-n_1)y_{n_1}\right)} dy_{n_1} \dots dy_1 \times$$

$$\int_0^{\infty} \cdots \int_{y_{n_1+n_2-1}}^{\infty} e^{-(1-s)\left(\sum_{i=n_1+1}^{n_1+n_2} y_i + (n-n_1-n_2) y_{n_1+n_2}\right)} dy_{n_1+n_2} \cdots d_{y_{n_1+1}}$$
$$= (1-t)^{-n_1}(1-s)^{-n_2}.$$

Therefore, the result follows. □

Now we can state the main result regarding the distribution of the MLEs.

Theorem 1 *$\frac{\hat{\theta}_1}{\theta_1}$ and $\frac{\hat{\theta}_2}{\theta_2}$ are distributed as gamma(n_1, n_1) and gamma(n_2, n_2) respectively and they are independent.*

Proof. It simply follows from the moment generating function of a gamma random variable. □

From theorem 1, it is clear that $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ_1 and θ_2 respectively. Since they are the functions of the complete sufficient statistics, they are uniformly minimum variance unbiased estimators also. Moreover,

$$V(\hat{\theta}_1) = \frac{\theta_1^2}{n_1} \quad \text{and} \quad V(\hat{\theta}_2) = \frac{\theta_2^2}{n_2}. \tag{6}$$

From (6), it is clear that the estimators of θ_1 and θ_2 become better as n_1 and n_2 increase. As $n_1 \to \infty$, $n_2 \to \infty$, $\hat{\theta}_1$, $\hat{\theta}_2$ are consistent estimators of θ_1, θ_2 respectively.

3. Confidence Intervals

The exact 100(1-α)% confidence intervals of θ_1 and θ_2 can be obtained using theorem 1. The 2-sided 100(1-α)% confidence intervals for θ_1 and θ_2 are

$$\left(\frac{\hat{\theta}_1}{\gamma_{n_1,n_1}(1-\alpha/2)}, \frac{\hat{\theta}_1}{\gamma_{n_1,n_1}(\alpha/2)}\right) \quad \text{and} \quad \left(\frac{\hat{\theta}_2}{\gamma_{n_2,n_2}(1-\alpha/2)}, \frac{\hat{\theta}_2}{\gamma_{n_2,n_2}(\alpha/2)}\right), \tag{7}$$

respectively, where $\gamma_{k,k}(\delta)$ is the lower δ-th percentile point of a *gamma*(k,k) distribution. Alternatively, (7) can be written as

$$\left(\frac{2T_1}{\chi^2_{2n_1}(1-\alpha/2)}, \frac{2T_1}{\chi^2_{2n_1}(\alpha/2)}\right) \quad \text{and} \quad \left(\frac{2T_2}{\chi^2_{2n_2}(1-\alpha/2)}, \frac{2T_2}{\chi^2_{2n_2}(\alpha/2)}\right), \tag{8}$$

respectively.

We are proposing two parametric bootstrap confidence intervals of θ_1 and θ_2 based on the percentile bootstrap (Boot-p) method of Efron [5] and the bootstrap-t (Boot-t) method of Hall [6]. We provide how to obtain 100(1-α)% bootstrap confidence intervals for θ_1, the corresponding intervals for θ_2 can be similarly obtained.

BOOT-P METHOD:

(1) Based on the sample obtain $\hat{\theta}_1$ and $\hat{\theta}_2$.

(2) Based on $\hat{\theta}_1$, $\hat{\theta}_2$, n, n_1 and n_2, obtain the bootstrap sample $\{t^*_{1:n} < \ldots < t^*_{r:n}\}$ from the joint density function (5) and compute $\hat{\theta}^*_1$, $\hat{\theta}^*_2$, the bootstrap estimates of θ_1 and θ_2

respectively. (The detail procedure to generate $t^*_{1:n} < \ldots < t^*_{r:n}$ from the joint density function (5) are provided in section 5).

(3) Repeat step (2) NBOOT times.

(4) Order the NBOOT $\hat{\theta}^*_1$ and let us denote them as $\hat{\theta}^*_{(1)} < \ldots < \hat{\theta}^*_{(NBOOT)}$. We suppress '1' for brevity.

(5) Now consider all possible 100(1-α)% confidence intervals of the type

$$(\hat{\theta}^*_{(i)}, \hat{\theta}^*_{([(1-\alpha)\times NBOOT]+i)}), i = 1, \ldots, NBOOT - [(1-\alpha) \times NBOOT].$$

(6) Choose that interval for which the length is minimum. That will be the 100(1-α)% shortest Boot-p confidence interval.

BOOT-T METHOD:

(1) Based on the sample obtain $\hat{\theta}_1$ and $\hat{\theta}_2$.

(2) Based on $\hat{\theta}_1$, $\hat{\theta}_2$, n, n_1 and n_2, obtain the bootstrap sample $\{t^*_{1:n} < \ldots < t^*_{r:n}\}$ from the joint density function (5) and compute $\hat{\theta}^*_1$, $\hat{\theta}^*_2$, the bootstrap estimates of θ_1 and θ_2 respectively. Consider the following statistic

$$T^* = \frac{\sqrt{n_1}(\hat{\theta}_1 - \hat{\theta}^*_1)}{\sqrt{V(\hat{\theta}^*_1)}}.$$

(3) Repeat step (2) NBOOT times.

(4) Order the NBOOT T^* and let us denote them as $T^*_{(1)} < \ldots < T^*_{(NBOOT)}$.

(5) Now consider all possible intervals of the type

$$(T^*_{(i)}, T^*_{([(1-\alpha)\times NBOOT]+i)}), i = 1, \ldots, NBOOT - [(1-\alpha) \times NBOOT].$$

(6) Choose that interval for which the length is minimum, suppose the interval is (T^*_L, T^*_U), then 100(1-α)% shortest Boot-t confidence interval of θ_1 will be

$$(\hat{\theta}_1 + n_1^{-\frac{1}{2}}\sqrt{V(\hat{\theta}_1)}T^*_L, \quad \hat{\theta}_1 + n_1^{-\frac{1}{2}}\sqrt{V(\hat{\theta}_1)}T^*_U).$$

4. Bayesian Analysis

In this section we compute the Bayes estimates and the corresponding credible intervals of θ_1 and θ_2 under the assumptions of independent inverted gamma priors on θ_1 and θ_2. It is assumed that the priors on θ_1 and θ_2 for $a_1, b_1, a_2, b_2 > 0$ are as follows;

$$p(\theta_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} \times \frac{e^{-\frac{b_1}{\theta_1}}}{\theta_1^{a_1+1}}; \qquad \theta_1 > 0, \tag{9}$$

$$p(\theta_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} \times \frac{e^{-\frac{b_2}{\theta_2}}}{\theta_2^{a_2+1}}; \qquad \theta_2 > 0. \tag{10}$$

Therefore, from (2), it follows that the joint density of the *data*, θ_1 and θ_2 is

$$p(data, \theta_1, \theta_2) = c \frac{1}{\theta_1^{a_1+n_1+1}} \times \frac{1}{\theta_2^{a_2+n_2+1}} \times e^{-\frac{1}{\theta_1}(T_1+b_1)} \times e^{-\frac{1}{\theta_2}(T_2+b_2)}, \tag{11}$$

where c is same as before. It is clear that the posterior density functions of θ_1 and θ_2 are independent and they are as follows;

$$p(\theta_1|data) \propto \frac{1}{\theta_1^{a_1+n_1+1}} \times e^{-\frac{1}{\theta_1}(T_1+b_1)}, \quad \theta_1 > 0 \tag{12}$$

$$p(\theta_2|data) \propto \frac{1}{\theta_2^{a_2+n_2+1}} \times e^{-\frac{1}{\theta_2}(T_2+b_2)}, \quad \theta_2 > 0. \tag{13}$$

Therefore, the Bayes estimates of θ_1 and θ_2 with respect to squared error loss are

$$\hat{\theta}_{1,Bayes} = \frac{T_1 + b_1}{a_1 + n_1 - 1} \quad \text{and} \quad \hat{\theta}_{2,Bayes} = \frac{T_2 + b_2}{a_2 + n_2 - 1}, \tag{14}$$

respectively if $a_1 + n_1 - 1 > 0$, $a_2 + n_2 - 1 > 0$ and the corresponding Bayes risks are

$$\frac{(T_1 + b_1)^2}{(a_1 + n_1 - 1)^2(a_1 + n_1 - 2)} \quad \text{and} \quad \frac{(T_2 + b_2)^2}{(a_2 + n_2 - 1)^2(a_2 + n_2 - 2)}, \tag{15}$$

respectively, if $a_1 + n_1 - 2 > 0$, $a_2 + n_2 - 2 > 0$. From (11), it follows that

$$p(data) \propto \frac{1}{(T_1 + b_1)^{a_1+n_1}(T_2 + b_2)^{a_2+n_2}}. \tag{16}$$

Therefore, the expected Bayes risks of $\hat{\theta}_{1,Bayes}$ and $\hat{\theta}_{2,Bayes}$ are

$$\frac{b_1^2}{(a_1 + n_1 - 1)(a_1 + n_1 - 2)(a_1 + n_1 - 3)} \quad \text{and} \quad \frac{b_2^2}{(a_2 + n_2 - 1)(a_2 + n_2 - 2)(a_2 + n_2 - 3)} \tag{17}$$

respectively, if $a_1 + n_1 > 3, a_2 + n_2 > 3, b_1 > 0, b_2 > 0$.

Interestingly, it can be easily observed that under the assumption of Jeffrey's prior on θ_1, θ_2 ($i.e.$ $p(\theta_1, \theta_2) \propto \frac{1}{\theta_1\theta_2}$) the Bayes estimates of θ_1 and θ_2 exist if $n_1 > 1$ and $n_2 > 1$ and then can be obtained from (14) by using $a_1 = a_2 = 0$ and $b_1 = b_2 = 0$. On the other hand if we take $0 - 1$ loss function as follows;

$$L(a,b) = \begin{cases} 0 & \text{if } |a - b| < \delta \\ 1 & \text{otherwise,} \end{cases} \tag{18}$$

for some $\delta > 0$, then the Bayes estimates of θ_1 and θ_2 corresponding to this $0-1$ loss function are same as the the corresponding MLEs.

Now using (12) and (13) the credible intervals for θ_1 and θ_2 can be constructed. Note that $\frac{1}{\theta_1}$ and $\frac{1}{\theta_2}$ follow gamma$(a_1 + n_1, T_1 + b_1)$ and *gamma*$(a_2 + n_2, T_2 + n_2)$ respectively. Therefore, the 100(1-α)% credible intervals for θ_1 and θ_2 are

$$\left(\frac{1}{\gamma_{a_1+n_1,T_1+b_1}(1-\alpha/2)}, \frac{1}{\gamma_{a_1+n_1,T_1+b_1}(\alpha/2)}\right), \quad \left(\frac{1}{\gamma_{a_2+n_2,T_2+b_2}(1-\alpha/2)}, \frac{1}{\gamma_{a_2+n_2,T_2+b_2}(\alpha/2)}\right) \tag{19}$$

respectively. If $2(a_1 + n_1)$ and $2(a_2 + n_2)$ are integers, then the (19) can be written as

$$\left(\frac{2(T_1 + b_1)}{\chi^2_{2(a_1+n_1)}(1-\alpha/2)}, \quad \frac{2(T_1 + b_1)}{\chi^2_{2(a_1+n_1)}(\alpha/2)} \right) \quad \text{and} \quad \left(\frac{2(T_2 + b_2)}{\chi^2_{2(a_2+n_2)}(1-\alpha/2)}, \quad \frac{2(T_2 + b_2)}{\chi^2_{2(a_2+n_2)}(\alpha/2)} \right), \tag{20}$$

respectively. Alternatively, it is possible to obtain the HPD credible intervals of θ_1 and θ_2 using the Gibbs sampling idea of Chen and Shao [3]. Note that although the Bayes estimates of θ_1 and θ_2 do not exist for $n_1 = 1$ and $n_2 = 1$ under the squared error loss function, but the corresponding credible intervals exist even when $n_1 = n_2 = 1$.

5. Monte Carlo Simulations

In this section we present some Monte Carlo simulation results to compare different methods. The exact confidence intervals based on the MLEs have 100(1-α)% coverage probabilities, but the expected length may not be minimum. On the other hand the credible intervals corresponding to the Jeffrey's priors have smaller lengths than the (7) but, the coverage probabilities are not known, at least for small samples. So these methods are compared with the the proposed bootstrap methods.

First, before proceeding any further, we will describe how to generate $\{t_{1:n}, \ldots, t_{n_1+n_2:n}\}$ using lemma 1. For a given, n, n_1, n_2, θ_1 and θ_2, generate n random variables from $exp(1)$ and choose the first n_1 order statistics. Suppose they are $y_1, \ldots, y_{n_1}$, then compute $t_{1:n} = \theta_1 y_1, \ldots, t_{n_1:n} = \theta_1 y_{n_1}$. Now generate $n - n_1$ random variables from $exp(1)$ and consider the first n_2 order statistics, suppose they are $y_{n_1+1}, \ldots, y_{n_1+n_2}$. Compute $t_{n_1+1:n} = \theta_2 y_{n_1+1} + y_{n_1}, \ldots, t_{n_1+n_2:n} = \theta_2 y_{n_1+n_2} + y_{n_1}$. Then $\{t_{1:n}, \ldots, t_{n_1+n_2:n}\}$ is the required sample.

Table 1a Confidence intervals of θ_1 for different n_1 when n = 10 and r = 6

n_1	Exact	Bayes	Boot-p	Boot-t
1	238.25(0.95)	143.61(0.95)	29.34(0.82)	1226.83(0.95)
2	47.94(0.95)	36.23(0.95)	19.12(0.88)	93.43(0.95)
3	26.42(0.96)	22.80(0.95)	15.14(0.90)	42.92(0.95)
4	19.19(0.95)	16.84(0.95)	12.60(0.90)	27.22(0.95)
5	15.38(0.95)	13.85(0.95)	11.19(0.92)	20.84(0.95)

Since all the results depend on the ratio $\frac{\theta_1}{\theta_2}$, we kept θ_1 fixed at 6 and consider different ranges of θ_2. We compute the coverage percentages and the expected confidence lengths with 95% nominal level, in all the four cases and the results are reported in the Tables 1a, 1b, 2a and 2b. In each case we report the expected length and the corresponding coverage percentages are reported within bracket. Some of the important findings are as follows. In all the cases it is observed that the exact, Bayes with non-informative prior and Boot-t confidence intervals maintain the nominal level, where as Boot-p confidence intervals can not maintain the required nominal level. The coverage percentages correspond to the Boot-p confidence intervals are significantly lower than the nominal level in all most all cases considered. It is observed that Boot-t confidence intervals have the largest lengths for both θ_1 and θ_2. Bayes credible intervals and the exact confidence lengths behave quite similarly for both θ_1 and θ_2. Although for very small sample sizes Bayes credible intervals are smaller than exact confidence intervals. Therefore, both the methods can be used for practical purposes.

Table 1b Confidence intervals of θ_2 for different n_1 when $n = 10$ and $r = 6$

n_1	θ_2	Exact	Bayes	Boot-p	Boot-t
1	1	2.61(0.95)	2.36(0.95)	1.92(0.93)	3.60(0.95)
	2	5.22(0.95)	4.72(0.95)	3.84(0.93)	7.19(0.95)
	3	7.83(0.95)	7.09(0.95)	5.76(0.93)	10.79(0.95)
2	1	3.22(0.96)	2.94(0.94)	2.16(0.91)	4.76(0.94)
	2	6.43(0.96)	5.87(0.94)	4.33(0.91)	9.52(0.94)
	3	9.64(0.96)	8.81(0.94)	6.49(0.91)	14.28(0.95)
3	1	4.04(0.95)	3.85(0.95)	2.48(0.91)	7.10(0.95)
	2	8.81(0.95)	7.69(0.95)	4.97(0.91)	14.19(0.95)
	3	13.21(0.95)	11.53(0.95)	7.45(0.91)	21.29(0.95)
4	1	7.89(0.96)	6.02(0.95)	3.01(0.87)	15.70(0.95)
	2	15.78(0.96)	12.03(0.95)	6.02(0.87)	31.40(0.95)
	3	23.68(0.96)	18.05(0.95)	9.03(0.87)	47.11(0.95)
5	1	38.00(0.95)	24.02(0.95)	4.73(0.81)	189.42(0.95)
	2	76.00(0.95)	48.04(0.95)	9.46(0.81)	378.74(0.95)
	3	114.00(0.95)	72.06(0.95)	14.19(0.81)	568.16(0.95)

Table 2a Confidence intervals of θ_1 for different n_1 when $n = 25$ and $r = 18$

n_1	Exact	Bayes	Boot-p	Boot-t
3	26.77(0.95)	22.40(0.95)	15.24(0.89)	42.60(0.93)
6	13.33(0.95)	12.35(0.95)	10.00(0.91)	16.86(0.95)
9	9.72(0.95)	9.47(0.94)	8.13(0.94)	11.46(0.95)
12	7.88(0.95)	7.95(0.95)	6.90(0.93)	8.93(0.95)
15	6.90(0.95)	6.79(0.94)	6.29(0.95)	7.71(0.95)

Table 2b Confidence intervals of θ_2 for different n_1 when $n = 10$ and $r = 6$

n_1	θ_2	Exact	Bayes	Boot-p	Boot-t
3	1	1.13(0.95)	1.14(0.95)	1.04(0.94)	1.29(0.95)
	2	2.26(0.95)	2.29(0.95)	2.08(0.94)	2.58(0.95)
	3	3.40(0.95)	3.43(0.95)	3.13(0.94)	3.87(0.95)

Table 2b (continued) Confidence intervals of θ_2 for different n_1 when n = 10 and r = 6

n_1	θ_2	Exact	Bayes	Boot-p	Boot-t
6	1	1.34(0.95)	1.31(0.95)	1.19(0.93)	1.54(0.94)
	2	2.67(0.95)	2.62(0.95)	2.37(0.93)	3.08(0.94)
	3	4.01(0.95)	3.92(0.95)	3.56(0.93)	4.62(0.94)
9	1	1.59(0.95)	1.58(0.94)	1.36(0.92)	1.91(0.95)
	2	3.19(0.95)	3.15(0.94)	2.72(0.92)	3.82(0.95)
	3	4.78(0.95)	4.73(0.94)	4.08(0.92)	5.73(0.95)
12	1	2.23(0.95)	2.04(0.95)	1.68(0.92)	2.86(0.95)
	2	4.47(0.95)	4.08(0.95)	3.37(0.92)	5.72(0.95)
	3	6.70(0.95)	6.12(0.95)	5.05(0.92)	8.59(0.95)
15	1	4.33(0.96)	3.75(0.95)	2.49(0.88)	7.13(0.94)
	2	8.67(0.96)	7.51(0.95)	4.99(0.88)	14.26(0.95)
	3	13.00(0.96)	11.26(0.95)	7.48(0.88)	21.39(0.95)

6. Some Related Issues

6.1 Optimum Choice of n_1

It is quite important to choose the *optimum* n_1 and n_2 for a given r and n. We answer this question using two competing criteria of variance- (Var) and determinant-(D) optimality.

Variance Optimality: A common purpose of an ALT experiment is to estimate parameters with maximum precision. In the step-stress set up, it can be written as the discrete minimization of $\tau_1(k)$, where

$$\tau_1(k) = \left\{ \frac{\theta_1^2}{k} + \frac{\theta_2^2}{r-k} \right\} . \tag{21}$$

The minimization is performed for $1 \le k \le (r-1)$, for a given θ_1 and θ_2. The optimum choice of n_1 is the value of k which minimizes $\tau_1(k)$. Unfortunately if there is no information available on θ_1 and θ_2, then this minimization is not possible. On the other hand if at least the ratio of θ_1 and θ_2 is known then it is possible to find optimum n_1 with respect to variance optimality criterion.

Suppose $\theta_1 = c\theta_2$ and c is known, then (24) can be written as

$$\tau_1(k) = \theta_1^2 \left\{ \frac{c^2}{k} + \frac{1}{r-k} \right\} . \tag{22}$$

Since $\tau_1(x)$ for $1 \le x \le (k-1)$ is a convex function, it has a unique minimum in that range. Therefore, the value k which minimizes $\tau_1(k)$ for any θ_1 can be obtained as follows. If $r/(1+\frac{1}{c})$ is an integer, then $k = r/(1+\frac{1}{c})$ minimizes (22). Otherwise consider the two consecutive integers $\left[r/(1+\frac{1}{c})\right]$ and $\left[r/(1+\frac{1}{c})\right]+1$, which ever minimizes (22) is the required k.

D- Optimality Yes another optimality criterion used in the context of planning constant stress ALT is based on the determinant of the information matrix. Note that the area of the joint confidence intervals of (θ_1, θ_2) is proportional to $\frac{\theta_1\theta_2}{\sqrt{n_1}\sqrt{n_2}}$, which is the inverse of the square root of the Fisher information matrix. Consequently, a smaller value of this determinant will provide higher precision of the estimates. Motivated by this, another criterion is to minimize $\tau_2(k)$, where

$$\tau_2(k) = \left\{ \frac{\theta_1^2\theta_2^2}{k(r-k)} \right\} \tag{23}$$

for $1 \le k \le (r-1)$. It is readily observed that if $\frac{r}{2}$ is an integer, the optimum value of n_1 is $\frac{r}{2}$, otherwise, it will be either $\left[\frac{r}{2}\right]$ or $\left[\frac{r}{2}\right] + 1$, whichever minimizes $\frac{1}{k(r-k)}$. Interestingly, in this case it is independent of c.

6.2 Comparing with the Traditional Step-Stress Model

In this subsection we present some numerical results to compare our proposed step-stress model with the traditional Type-II step-stress model. The two models are comparable when for fixed n, n_1, n_2, for the proposed model, and τ, the point where the stress changes for the traditional model satisfy the following relation;

$$\tau = E(t_{n_1:n}) = \theta_1 \sum_{i=1}^{n_1} \frac{1}{n-i+1}. \tag{24}$$

For comparison purposes, for fixed n, r, θ_1, θ_2 and τ as given in (24) we compute the biases and mean squared errors of the MLEs for both the traditional and the proposed step-stress models. Note that the exact biases and mean squared errors of the MLEs for the traditional step-stress model can be obtained from Balakrishnan *et al.* [2]. The results are reported in Tables 3 and 4. We report the biases of the MLEs of $\hat{\theta}_1$ for the traditional model. Since $E(\hat{\theta}_1/\theta_1)$ is independent of θ_1, we report $E(\hat{\theta}_1/\theta_1 - 1)$ for the traditional model. We also report the MSEs of $\hat{\theta}_1$ and $\hat{\theta}_2$ for the traditional model as well as the proposed model. Since in both the cases the MSEs of $\hat{\theta}_1/\theta_1$ and $\hat{\theta}_2/\theta_2$ are independent of θ_1 and θ_2 respectively, we report the normalized MSEs in both the cases. We consider different values of n_1 and n_2. We consider the cases when $r = n$ and $r = [2n/3]$. The results are reported in Tables 3 and 4. It is observed that the the MLEs of θ_1 are quite biased for the traditional model when n_1 is small. Moreover the MSEs of $\hat{\theta}_1$ for the traditional model are significantly larger than the corresponding MSEs of $\hat{\theta}_1$ for the proposed model particularly when n_1 is small. The MSEs of $\hat{\theta}_2$ for the traditional model are marginally larger than the corresponding MSEs of $\hat{\theta}_2$ for the proposed model for small n_2. When n_2 is large they are almost same.

7. Conclusions

In this paper we consider a simple step-stress model, where the stress changes at a random time point rather than a fixed time point. It is observed in this paper that under the cumulative exposure model assumption of Nelson [9] and when the life time distributions of the different units are exponential then the proposed step-stress model has several advantages over the traditional type-II step-stress model. Even when the lifetime distributions are not exponential it is expected that the proposed model will have the similar advantages over the traditional step-stress model. Another interesting point should be mentioned that although we have performed the analysis for the two-step model but the results can be easily extended to the k-step model also.

Table 3 Biases and mean squared errors of the different estimators for different sample sizes when $r = n$. *

$n_1 \rightarrow$ $n_2 \downarrow$	5	10	15	20	25
5	0.2173 0.6832(0.2000) 0.2158(0.2000)	0.0714 0.1569(0.1000) 0.2207(0.2000)	0.0377 0.0838(0.0667) 0.2233(0.2000)	0.0245 0.0566(0.0500) 0.2248(0.2000)	0.0174 0.0448(0.0400) 0.2257(0.2000)
10	0.2465 0.7184(0.2000) 0.1021(0.1000)	0.0893 0.1815(0.1000) 0.1031(0.2000)	0.0486 0.0879(0.0667) 0.1038(0.2000)	0.0315 0.0613(0.0500) 0.1043(0.1000)	0.0222 0.0459(0.0400) 0.1046(0.1000)
15	0.2644 0.7693(0.2000) 0.0674(0.0667)	0.0987 0.1782(0.1000) 0.0677(0.0667)	0.0544 0.0936(0.0667) 0.0679(0.0667)	0.0362 0.0613(0.0500) 0.0681(0.0667)	0.0252 0.0474(0.0400) 0.0683(0.0667)
20	0.2770 0.7661(0.2000) 0.0503(0.0500)	0.1066 0.1906(0.1000) 0.0505(0.0500)	0.0598 0.0945(0.0667) 0.0506(0.0500)	0.0378 0.0643(0.0500) 0.0507(0.0500)	0.0280 0.0477(0.0400) 0.0508(0.0500)
25	0.2798 0.7512(0.2000) 0.0402(0.0400)	0.1096 0.2054(0.1000) 0.0402(0.0400)	0.0609 0.0963(0.0667) 0.0403(0.0400)	0.0418 0.0648(0.0500) 0.0404(0.0400)	0.0295 0.0489(0.0400) 0.0404(0.0400)

* In each cell the first entry represents the $E(\hat{\theta}_1/\theta_1 - 1)$ for traditional step-stress model. Similarly the second and third entries represent the $E(\hat{\theta}_1/\theta_1 - 1)^2$ and $E(\hat{\theta}_2/\theta_2 - 1)^2$ respectively. The corresponding results for the new step-stress model are reported within brackets.

Table 4 Biases and mean squared errors of the different estimators for different sample sizes when $r = [\frac{2n}{3}]$.*

$n_1 \rightarrow$ $n_2 \downarrow$	5	10	15	20	25
5	0.2500 0.7127(0.2000) 0.2299(0.2000)	0.1041 0.1996(0.1000) 0.2429(0.2000)	0.0636 0.0915(0.0667) 0.2503(0.2000)	0.0470 0.0617(0.0500) 0.2532(0.2000)	0.0378 0.0464(0.0400) 0.2545(0.2000)

Table 4 (continued) Biases and mean squared errors of the different estimators for different sample sizes when $r = [\frac{2n}{3}]$.*

$n_1 \rightarrow$ / $n_2 \downarrow$	5	10	15	20	25
10	0.2685 0.7393(0.2000) 0.1036(0.1000)	0.1066 0.1907(0.1000) 0.1070(0.1000)	0.0614 0.0969(0.0667) 0.1105(0.1000)	0.0423 0.0646(0.0500) 0.1144(0.1000)	0.0320 0.0491(0.0400) 0.1176(0.1000)
15	0.2798 0.7513(0.2000) 0.0677(0.0667)	0.1138 0.2377(0.1000) 0.0685(0.0667)	0.0646 0.0989(0.0667) 0.0694(0.0667)	0.0448 0.0666(0.0500) 0.0702(0.0667)	0.0353 0.0501(0.0400) 0.0712(0.0667)
20	0.2803 0.7528(0.2000) 0.0505(0.0500)	0.1138 0.2181(0.1000) 0.0508(0.0500)	0.0673 0.1024(0.0667) 0.0511(0.0500)	0.0487 0.0685(0.0500) 0.0514(0.0500)	0.0360 0.0510(0.0400) 0.0515(0.0500)
25	0.2868 0.7904(0.2000) 0.0402(0.0400)	0.1169 0.2237(0.1000) 0.0404(0.0400)	0.0692 0.1041(0.0667) 0.0406(0.0400)	0.0485 0.0698(0.0500) 0.0407(0.0400)	0.0359 0.0494(0.0400) 0.0409(0.0400)

* In each cell the first entry represents the $E(\hat{\theta}_1/\theta_1 - 1)$ for traditional step-stress model. Similarly the second and third entries represent the $E(\hat{\theta}_1/\theta_1 - 1)^2$ and $E(\hat{\theta}_2/\theta_2 - 1)^2$ respectively. The corresponding results for the new step-stress model are reported within brackets.

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