Statistical inference on the Shannon and Rényi entropy measures of generalized exponential distribution under the progressive censoring

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Abstract

The aim of this paper is two-fold. First, the estimation of the Shannon and Rényi entropy measures of a generalized exponential distribution is discussed when data are progressively censored. The maximum likelihood estimates are obtained. The Bayes estimates with respect to three loss functions are proposed. It is assumed that the unknown parameters have independent gamma priors. The closed-form expressions of the Bayes estimates cannot be obtained. So, Lindley's approximation and importance sampling methods are employed. The asymptotic confidence intervals are computed. The normal approximation of the maximum likelihood estimate and the log-transformed maximum likelihood estimate are used. In addition, bootstrap algorithms are used to compute the confidence intervals. Highest posterior density credible intervals of the entropy functions are developed. A detailed numerical study is performed to compare the proposed estimates with respect to their average values and the mean squared errors. The confidence intervals are compared with respect to the average lengths. A real dataset is considered to illustrate the proposed methods. Further, different criteria are proposed for the comparison of various sampling schemes. Then, the optimal sampling scheme for a given criterion is obtained.

Keywords: Bayes estimates, Lindley's approximation, importance sampling, bootstrap algorithms, highest posterior density, optimal censoring scheme.

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1 Introduction

It is well known that each probability distribution has some kind of uncertainty. The entropy is a useful tool to measure this uncertainty. The concept of entropy was introduced by Shannon (1948). Let X be a nonnegative and absolutely continuous random variable with probability density function f_X and cumulative distribution function F_X. Then, the Shannon entropy of X is defined as

$$S(f) = -\int_0^\infty f_X(x) \ln f_X(x) dx.$$

It can be observed that a distribution has low entropy if it has sharp peak. Further, the entropy is higher if the probability is scattered. In this sense, $S(f)$ measures uncertainty of a probability distribution. The Shannon entropy has a good number of applications in various areas such as ecology, economy, hydrology and water resources. In ecology, the concept of entropy can be used to measure the diversity indices of different species. The entropy is used for earthquake forecasting (see Harte and Vere-Jones (2005)). To increase the stability of farmers' income, Eun, Jung, Lee and Bae (2012) studied entropy in prices of agriculture product for various situations. Rass and König (2018) considered entropy concepts for the measurement of the quality of password choice process by applying multi-objective game theory to password security problem. Besides the concept of Shannon's entropy, there are several generalizations which have been proposed by various authors. Among these, the most famous is the Rényi entropy (see Rényi (1961)). The Rényi entropy of the nonnegative random variable X is defined as

$$R_\beta(f) = \frac{1}{1-\beta} \ln \int_0^\infty f_X(x)^\beta dx, \ \beta > 0 \ (\neq 1). \tag{1.1}$$

Note that Rényi's definition of entropy is parameterized by a single parameter β. When β is allowed to approach unity, (1.1) reverts to the familiar Shannon entropy. The Rényi entropy given in (1.1) is known as the quadratic entropy when $\beta = 2$. We get a large reduction in the computation effort required to produce entropy estimates when $\beta = 2$. Like Shannon's entropy, the Rényi entropy has also been used by several authors since its introduction in the literature. In this direction, we refer to Zhou, Cai and Tong (2013) and Prabakaran (2017).

In molecular sciences, reliability and life testing and information theory, estimating entropy of various statistical distributions are of importance. Estimation of entropy of molecules plays an important role to understand various chemical and biological processes in molecular sciences (see Nalewajski (2002)). The concept of entropy is also used in software reliability to measure uncertainty (see Kamavaram and Goseva-Popstojanova (2002)). In estimating uncertainty of a system with several independent components, we need to estimate uncertainty in individual components. In economics, entropy estimation (see Golan, Judge and Miller (1996)) often allows the researchers to use data for the improvement of the assumptions on the parameters in econometric models. Further, entropy is an important tool in experimental design to obtain optimal model discrimination (see Nowak and Guthke (2016)).

Like mean, standard deviation, variance and quantile, entropy is also an important characteristic of a parametric family of distributions. Hence, it is of great interest to estimate it for further insights into the nature of the distribution. When the dataset is not censored, various authors have studied the problem of estimating Shannon's and Rényi's entropies of some probability models. One may refer to Kayal, Kumar and Vellaisamy (2015) and Patra, Kayal and Kumar (2018).

Usually, the life testing experiments often face termination before the failure of all items. It happens due to the time constraints or low fund. The observations that result from this kind of situations are said to be the censored sample. There are various censoring schemes which have been introduced in the literature for the evaluation of different life testing plans. Among these, progressive censoring scheme is very useful since it allows to remove prefixed number of surviving items at different epochs. There are mainly two types of progressive censoring schemes: (i) progressive type-I censoring and (ii) progressive type-II censoring. In this paper, we deal with the progressive type-II censored sample for the purpose of estimating two uncertainty measures. We present a brief description on the progressive type-II censored sample. Let n independent and identically distributed units are placed on a certain life testing experiment at time zero. We assume that R_1 surviving units are removed from the test just after the occurrence of the first failure. Let the first failure occurs at random time denoted by $X_{1:m:n}$. Further, let at random time $X_{2:m:n}$, second failure occurs and immediately, R_2 surviving units are removed from the ongoing experiment. This process continues until the mth failure occurs at time $X_{m:m:n}$. After this failure, $R_m = n - \sum_{j=1}^{m-1} R_j - m$ units are removed from the test. The set of random times $(X_{1:m:n}, \ldots, X_{m:m:n})$ is known as the progressive type-II censored sample. This censoring scheme is a generalized version of the conventional type-II censoring and complete sampling schemes. For a detailed account of various progressive censoring schemes, we refer the readers to Balakrishnan (2007) and Balakrishnan and Cramer (2014). Further, various inferential procedures based on progressive type-II censoring scheme have been introduced for some useful lifetime models. For instance, see Lee and Cho (2017) and Maiti and Kayal (2019, 2021) and the references contained therein.

The exponentiated exponential (EE) distribution also known as the generalized exponential (GE) distribution was introduced by Gupta and Kundu (1999). This is an extended version of the exponential distribution. Let X be a random variable following generalized exponential distribution with probability density function and cumulative distribution function given by

$$f_X(x; \alpha, \lambda) = \frac{\alpha}{\lambda} \exp\left\{-\frac{x}{\lambda}\right\} \left(1 - \exp\left\{-\frac{x}{\lambda}\right\}\right)^{\alpha-1}, \quad x > 0, \ \alpha, \lambda > 0, \tag{1.2}$$

and

$$F_X(x; \alpha, \lambda) = \left(1 - \exp\left\{-\frac{x}{\lambda}\right\}\right)^{\alpha}, \quad x > 0, \ \alpha, \lambda > 0, \tag{1.3}$$

respectively. Here, α and λ are respectively shape and scale parameters. The generalized exponential distribution with density function (1.2) reduces to the exponential distribution when $\alpha = 1$. The generalized exponential distribution is unimodal for $\alpha > 1$. The probability density function (1.2) is log-convex if $\alpha \leq 1$ and log-concave if $\alpha \geq 1$. The hazard rate function of this distribution can be increasing ($\alpha > 1$), decreasing ($\alpha < 1$) or constant ($\alpha = 1$) depending on the shape parameter α. The generalized exponential distribution is an alternative to the gamma or Weibull distribution when analyzing lifetime data. Further, the distribution function of the generalized exponential distribution is in closed form like the Weibull distribution. Thus, it can be used very easily when the data are censored, unlike a gamma distribution. Hereafter, we denote $X \sim GE(\alpha, \lambda)$ if X has the distribution function given by (1.3).

Due to wide applications of the generalized exponential distribution, many authors have studied estimation of parameters and important characteristics of it. Kundu and Pradhan (2009a) considered Bayesian estimation of the unknown parameters of the progressively censored generalized exponential distribution. To compute Bayes estimates, they used Lindley's approximation and importance sampling technique. In another paper, Kundu and Pradhan (2009b) proposed Bayes estimates of the unknown parameters of the generalized exponential distribution based on the importance sampling procedure. To illustrate the results, a dataset has been considered and analyzed. Pradhan and Kundu (2009) obtained maximum likelihood estimates (MLEs) of the unknown parameters of a generalized exponential distribution based on the progressively censored sample. Chen and Lio (2010) introduced MLEs, method of moments estimates of the parameters of a generalized exponential distribution based on the progressive type-I interval censored sample. Ismail (2012) obtained the MLEs and the confidence intervals of the model parameters of generalized exponential distribution under partially accelerated tests with progressive type-II censoring. The author studied performance of the estimates numerically for various parameter values and sample sizes. Mohie El-Din, Amein, Shafay and Mohamed (2017) derived maximum likelihood (ML) and Bayes estimates of the unknown parameters, reliability and hazard functions of the generalized exponential distribution when an adaptive type-II progressive censored sample is available. Guo and Gui (2018) developed various estimates of the reliability function for generalized exponential distribution under progressive type-II censoring schemes.

In the above paragraph, we have presented contributions related to the estimation of the unknown parameters, reliability and hazard functions of a generalized exponential distribution based on various sampling schemes. Now, we provide work on the estimation of entropy functions of various lifetimes distributions when the random sample are censored as well as uncensored. Kang, Cho, Han and Kim (2012) obtained estimates of the Shannon entropy of a double exponential distribution based on the multiply type-II censored samples. They proposed estimates based on the ML and approximate ML estimation procedures. Under the doubly generalized type-II censored sample, Cho, Sun and Lee (2014) derived ML, approximate ML and Bayes estimates of the Shannon entropy of the Rayleigh distribution. Simulation study has been carried out to look at the performance of the proposed estimates. Further, for the illustration purposes, they considered a real dataset. Cho, Sun and Lee (2015) proposed Bayes estimates of the Shannon entropy of a two-parameter Weibull distribution based on generalized progressive censored sample. They considered squared error, LINEX and generalized entropy loss functions. Lindley's approximation was used in this direction. From the above development, it is clear that nobody has attempted the problem of estimation of the entropy functions due to Shannon (1948) and Rényi (1961) of an generalized exponential distribution based on the progressive type-II censored sample.

In this paper, we consider estimation of the Shannon and Rényi entropies of the generalized exponential distribution when progressive type-II censored sample is available. The estimands under study are the Shannon and Rényi entropy measures of the $EE(\alpha,\lambda)$ distribution which are respectively given by

$$S(\alpha,\lambda) = \psi(\alpha+1) - \psi(1) - \ln\left(\frac{\alpha}{\lambda}\right) - \frac{\alpha-1}{\alpha} \tag{1.4}$$

and

$$R_\beta(\alpha,\lambda) = \frac{1}{1-\beta}\Big[\beta\ln\alpha - (\beta-1)\ln\lambda + \ln\Gamma(\beta(\alpha-1)+1) + \ln\Gamma(\beta) - \ln\Gamma(\alpha\beta+1)\Big], \tag{1.5}$$

where $\psi(.)$ is the digamma function and $\Gamma(.)$ is the complete gamma function. We obtain likelihood equations. These equations do not provide the MLEs in explicit forms. So, we use optimx package for the purpose of computation. The Bayes estimates can not be obtained in closed form. Thus, we use two approximation methods to compute the desired Bayes estimates. Various interval estimates are also proposed. We use normal approximation to the MLEs/log-transformed MLEs, Bootstrap-t/p methods. Highest posterior credible intervals are computed. It is an important practical problem to find any optimal sampling scheme. This problem has received considerable interest in the last few years. Here, we need to choose $(R_1 \ldots, R_m)$ for specified values of sample size n and the effective sample size m such that it provides the maximum information. In this paper, we treat the entropy measures as optimality criterion. Using this criterion, we propose a method to choose the optimal sampling scheme for progressively censored generalized exponential distribution. To study the proposed methods, we Monte Carlo simulation has been performed.

The rest of the paper is organized as follows. In Section 2, we derive MLEs for the Shannon and Rényi entropies. Section 3 provides Bayes estimates with respect to various loss functions. Two independent gamma priors are considered. Since the explicit expressions of the Bayes estimates do not exist, we use Lindley's approximation method and importance sampling technique. We construct confidence intervals using various techniques in Section 4. Simulation study and real data analysis are carried out in Section 5. Some observations on the performance of the proposed estimates are presented. In Section 6, we present methodology of constructing the optimal censoring schemes. Section 7 concludes the paper.

2 Maximum likelihood estimation

Denote $X_i = X_{i:m:n}$, where $i = 1,\ldots,m$. Let $\boldsymbol{X} = (X_1,\ldots,X_m)$ be a progressive type-II censored sample with associated scheme $(R_1,\ldots,R_m)$. We remark that the MLEs of the unknown parameters of the generalized exponential distribution when progressive type-II censored sample is available have been obtained before. However, for the sake of completeness, we briefly present the derivation of the MLEs below. Denote $\boldsymbol{x} = (x_1, \ldots, x_m)$, where $x_i = x_{i:m:n}$. The likelihood function is written as

$$L(\alpha, \lambda \mid \boldsymbol{x}) = \Upsilon \prod_{i=1}^{m} f_X(x_i; \alpha, \lambda)\big(1 - F_X(x_i; \alpha, \lambda)\big)^{R_i},$$

where $\alpha, \lambda > 0$ and $\Upsilon = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\cdots(n - \sum_{j=1}^{m-1}(R_j + 1))$. The density and distribution functions $f_X(x_i; \alpha, \lambda)$ and $F_X(x_i; \alpha, \lambda)$ are given by Equations (1.2) and (1.3), respectively. The log-likelihood function can be written as

$$l(\alpha, \lambda \mid \boldsymbol{x}) \propto m \ln\left(\frac{\alpha}{\lambda}\right) - \sum_{i=1}^{m} \frac{x_i}{\lambda} + (\alpha - 1)\sum_{i=1}^{m} \ln \xi(\lambda; x_i) + \sum_{i=1}^{m} R_i \ln\left(1 - \xi(\lambda; x_i)^{\alpha}\right), \quad (2.1)$$

where $\xi(\lambda; x) = 1 - \exp\{-x/\lambda\}$. The MLEs of the unknown parameters α and λ can be obtained after maximizing the log-likelihood function given by (2.1) with respect to α and λ, respectively. On differentiating (2.1) with respect to α, and then equating to zero, the normal equation of α is given by

$$\frac{m}{\alpha} - \sum_{i=1}^{m} \frac{R_i \xi(\lambda; x_i)^{\alpha} \ln \xi(\lambda; x_i)}{1 - \xi(\lambda; x_i)^{\alpha}} + \sum_{i=1}^{m} \ln \xi(\lambda; x_i) = 0. \quad (2.2)$$

Similarly, the normal equation of λ is

$$m\lambda - \alpha \sum_{i=1}^{m} \frac{R_i x_i (1 - \xi(\lambda; x_i))\xi(\lambda; x_i)^{\alpha - 1}}{(1 - \xi(\lambda; x_i)^{\alpha})} + (\alpha - 1)\sum_{i=1}^{m} \frac{x_i(1 - \xi(\lambda; x_i))}{\xi(\lambda; x_i)} - \sum_{i=1}^{m} x_i = 0. \quad (2.3)$$

Solving (2.2) and (2.3) simultaneously in α and λ, the MLEs of α and λ can be obtained. We point out that the explicit solutions of (2.2) and (2.3) do not exist. Thus, the MLEs of α and λ can not be obtained in closed form. We use numerical technique to get solutions to Equations (2.2) and (2.3). Denote the MLEs of α and λ by $\hat{\alpha}$ and $\hat{\lambda}$, respectively. Now, using invariance property, the MLEs of the Shannon and Rényi entropy measures can be obtained, which are given by

$$\hat{S} = \psi(\hat{\alpha} + 1) - \psi(1) - \ln\left(\frac{\hat{\alpha}}{\hat{\lambda}}\right) - \frac{(\hat{\alpha} - 1)}{\hat{\alpha}}$$

and

$$\hat{R}_{\beta} = \frac{1}{1 - \beta}\left[\beta \ln \hat{\alpha} - (\beta - 1)\ln \hat{\lambda} + \ln \Gamma(\beta(\hat{\alpha} - 1) + 1) + \ln \Gamma(\beta) - \ln \Gamma(\beta\hat{\alpha} + 1)\right].$$

3 Bayesian estimation

In the previous section, we obtain MLEs of the uncertainty measures given by (1.4) and (1.5). In this section, we focus on the Bayesian estimation of the Shannon and Rényi entropy measures. Note that to study estimation problems from the Bayesian point of view, loss functions as well as prior distributions play an important role. Three loss functions are considered in this paper. One of these is the squared error loss function. It is a balanced type (symmetric) loss function, which gives same loss when positive and negative errors of the same magnitude occur. There are also some situations related to various life-threatening consequences, where it can be worse to underestimate the potentiality of an event than to overestimate it. Further, in estimating the survival time of a reliability system (rocket, satellite), overestimation is severe than underestimation. These practical situations can be handled with the asymmetric loss functions. We also consider LINEX and entropy loss functions. Let δ be an estimator to approximate an unknown parameter θ. Then, the squared error, LINEX and (generalized) entropy loss functions are given by

$$\begin{aligned} L_s(\theta,\delta) &= (\delta-\theta)^2, \\ L_l(\theta,\delta) &= \exp\{p(\delta-\theta)\} - p(\delta-\theta) - 1,\ p \neq 0, \\ L_e(\theta,\delta) &= (\delta/\theta)^q - q\ln(\delta/\theta) - 1,\ q \neq 0, \end{aligned}$$

respectively. When $q > 0$, a positive error is more dangerous than a negative error, and vice-versa when $q < 0$. From the posterior distributions, the Bayes estimates of the unknown parameter θ under the loss functions $L_l(\theta,\delta)$ and $L_e(\theta,\delta)$ can be obtained respectively as

$$\hat{\theta}^l_{bs} = -p^{-1}\ln\left(E_\theta\left(\exp\{-p\theta\}|\ \boldsymbol{x}\right)\right),\ p \neq 0 \ \text{ and} \tag{3.1}$$

$$\hat{\theta}^e_{bs} = \left[E_\theta\left(\theta^{-q}|\ \boldsymbol{x}\right)\right]^{-\frac{1}{q}},\ q \neq 0. \tag{3.2}$$

The Bayes estimate with respect to the squared error loss function can be obtained substituting $q = -1$ in (3.2). Further, for $q = 1$ and -2, the Bayes estimate given by (3.2) reduces to that with respect to the usual entropy loss function and the precautionary loss function. Choosing a prior for the unknown parameters is a difficult task. Indeed, there is no clear methodology to choose an appropriate prior (see Arnold and Press (1983)) for the purpose of Bayesian estimation. However, due to flexibility, here we assume independent gamma priors for α and λ as

$$\begin{aligned} g_1(\alpha;a,b) &= \frac{b^a\alpha^{a-1}\exp\{-\alpha b\}}{\Gamma(a)},\ \alpha > 0,\ a,b > 0, \\ g_2(\lambda;c,d) &= \frac{d^c\lambda^{c-1}\exp\{-\lambda d\}}{\Gamma(c)},\ \lambda > 0,\ c,d > 0, \end{aligned}$$

respectively. Further, the gamma distribution fits the failure data quiet well. After some calculations, the joint posterior distribution of α and λ can be obtained as

$$\Pi(\alpha,\lambda \mid \boldsymbol{x}) = \frac{1}{k}\ \alpha^{m+a-1}\lambda^{c-m-1}\exp\{-(\alpha b+\lambda d)\}\ \prod_{i=1}^{m}\ (1-\xi(\lambda;x_i))\,(1-\xi(\lambda;x_i)^\alpha)^{R_i} \times \xi(\lambda;x_i)^{(\alpha-1)}, \tag{3.3}$$

where

$$k=\int_0^\infty\int_0^\infty \alpha^{m+a-1}\lambda^{c-m-1}\exp\{-(\alpha b+\lambda d)\}\prod_{i=1}^{m}(1-\xi(\lambda;x_i))\,(1-\xi(\lambda;x_i)^\alpha)^{R_i}\times\xi(\lambda;x_i)^{(\alpha-1)}d\alpha d\lambda.$$

Making use of (3.1), (3.2) and (3.3), the Bayes estimates of the Shannon entropy $S(\alpha,\lambda)$ with respect to the LINEX and entropy loss functions are respectively obtained as

$$\hat{S}^l_{bs}=-\frac{1}{p}\ln\left[\frac{1}{k}\int_0^\infty\int_0^\infty \alpha^{m+a-1}\lambda^{c-m-1}\exp\{-(\alpha b+\lambda d+pS(\alpha,\lambda))\}\prod_{i=1}^{m}\{\xi(\lambda;x_i)^{\alpha-1}\times(1-\xi(\lambda;x_i))\,(1-\xi(\lambda;x_i)^\alpha)^{R_i}\}d\alpha d\lambda\right] \tag{3.4}$$

and

$$\hat{S}^e_{bs}=\left[\frac{1}{k}\int_0^\infty\int_0^\infty S(\alpha,\lambda)^{-q}\alpha^{m+a-1}\lambda^{c-m-1}\exp\{-(b\alpha+\lambda d)\}\prod_{i=1}^{m}\{\xi(\lambda;x_i)^{\alpha-1}\times(1-\xi(\lambda;x_i))\,(1-\xi(\lambda;x_i)^\alpha)^{R_i}\}d\alpha d\lambda\right]^{-\frac{1}{q}}. \tag{3.5}$$

Substitution $q=-1$ in (3.5), the Bayes estimate of $S(\alpha,\lambda)$ with respect to the squared error loss function can be obtained. Using similar procedure, we can obtain the Bayes estimates of the Rényi entropy $R_\beta(\alpha,\lambda)$ with respect to the above loss functions. Note that the closed form solutions of the ratio of two integrals given by (3.4) and (3.5) are difficult to compute. Thus, we use two approximation techniques which are discussed in the following consecutive subsections. First, we consider Lindley's approximation method.

3.1 Lindley's approximation

Let us first illustrate the Lindley's approximation method (see Lindley (1980)). The Bayes estimate is computed as the expectation of the function $\nu(\theta_1,\theta_2)$, where θ_1 and θ_2 are unknown parameters. The expectation is taken with respect to the posterior distribution. If $\hat{\nu}_{bs}$ denotes the Bayes estimate of $\nu(\theta_1,\theta_2)$, then

$$\hat{\nu}_{bs}=\frac{\int_0^\infty\int_0^\infty \nu(\theta_1,\theta_2)\exp\{l(\theta_1,\theta_2|\boldsymbol{x})+p^*(\theta_1,\theta_2)\}d\theta_1 d\theta_2}{\int_0^\infty\int_0^\infty \exp\{l(\theta_1,\theta_2|\boldsymbol{x})+p^*(\theta_1,\theta_2)\}d\theta_1 d\theta_2}, \tag{3.6}$$

where $l(\theta_1,\theta_2|\boldsymbol{x})$ is the log-likelihood function and $p^*(\theta_1,\theta_2)$ is the logarithm of the joint prior distribution of θ_1 and θ_2, say $\pi(\theta_1,\theta_2)$. Using Lindley's approximation method, the estimate given by (3.6) can be approximated as

$$\hat{\nu}_{bs}=\nu(\theta_1,\theta_2)+\frac{1}{2}\Big(A+l_{30}B_{12}+l_{03}B_{21}+l_{21}C_{12}+l_{12}C_{21}+2p_1^*A_{12}+2p_2^*A_{21}\Big),$$

where $A = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\eta_{ij}$, $p_i^* = \frac{\partial p^*}{\partial \theta_i}$, $g_i = \frac{\partial \nu}{\partial \theta_i}$, $g_{ij} = \frac{\partial^2 \nu}{\partial \theta_i \partial \theta_j}$, $A_{ij} = g_i\eta_{ii} + g_j\eta_{ji}$, $B_{ij} = (g_i\eta_{ii} + g_j\eta_{ij})\eta_{ii}$, $C_{ij} = 3g_i\eta_{ii}\eta_{ij} + g_j(\eta_{ii}\eta_{jj} + 2\eta_{ij}^2)$, $l_{ij} = \frac{\partial^{i+j} l}{\partial \theta_1^i \partial \theta_2^j}$, $p^* = \ln \pi(\theta_1, \theta_2)$ and η_{ij} is the (i,j)th element of the matrix $\left[-\frac{\partial^2 l}{\partial \theta_1^i \partial \theta_2^j}\right]^{-1}$, $i, j = 1, 2$. These terms are evaluated at the MLEs. Below, we present approximate Bayes estimates of the Shannon entropy with respect to the LINEX and entropy loss functions. For the case of the LINEX loss function when estimating $S(\alpha, \lambda)$, we have $\nu(\alpha, \lambda) = \exp\{-pS(\alpha, \lambda)\}$. Further,

$$
\begin{aligned}
g_1 &= \frac{p\alpha^{p-2}}{\lambda^p}\left(1 + \alpha - \alpha^2\psi'(\alpha+1)\right)\eta(\alpha), \quad g_2 = -\frac{p\alpha^p}{\lambda^{p+1}}\eta(\alpha), \\
g_{11} &= \frac{p\alpha^{p-4}}{\lambda^p}\eta(\alpha)\big[-\alpha^4\psi''(\alpha+1) - \alpha(\alpha+2) + p\alpha^2\psi'(\alpha+1)(\alpha^2\psi'(\alpha+1) \\
&\quad - 2(\alpha+1)) + p(\alpha+1)^2\big], \\
g_{22} &= \frac{p(p+1)\alpha^p}{\lambda^{p+2}}\eta(\alpha), \quad g_{12} = g_{21} = \frac{p^2\alpha^{p-2}}{\lambda^{p+1}}\left(\alpha^2\psi'(\alpha+1) - \alpha - 1\right)\eta(\alpha),
\end{aligned}
$$

where $\eta(\alpha) = \exp\left\{p\left(1 + \psi(1) - \psi(\alpha+1) - \frac{1}{\alpha}\right)\right\}$. Thus, the Bayes estimate of $S(\alpha, \lambda)$ with respect to the LINEX loss function is obtained as

$$
\begin{aligned}
\hat{S}_{bs}^{l} &= -\frac{1}{p}\ln\big[\exp\{-p\hat{S}(\alpha,\lambda)\} + \frac{1}{2}[(g_{11}\eta_{11} + g_{22}\eta_{22} + 2g_{12}\eta_{12}) + l_{30}\eta_{11}(g_1\eta_{11} + g_2\eta_{12}) \\
&\quad + l_{03}\eta_{22}(g_2\eta_{22} + g_1\eta_{21}) + l_{21}(3g_1\eta_{11}\eta_{12} + g_2(\eta_{11}\eta_{22} + 2\eta_{12}^2)) + l_{12}(3g_2\eta_{22}\eta_{21} \\
&\quad + g_1(\eta_{22}\eta_{11} + 2\eta_{21}^2)) + 2p_1^*(g_1\eta_{11} + g_2\eta_{21}) + 2p_2^*(g_2\eta_{22} + g_1\eta_{12})]\big]. \qquad (3.7)
\end{aligned}
$$

For the case of entropy loss function, we have $\nu(\alpha, \lambda) = S(\alpha, \lambda)^{-q}$. Also,

$$
\begin{aligned}
g_1 &= -q\left(\psi'(\alpha+1) - \frac{\alpha+1}{\alpha^2}\right)S(\alpha,\lambda)^{-q-1}, \quad g_2 = -\frac{q}{\lambda}S(\alpha,\lambda)^{-q-1}, \\
g_{11} &= \frac{-q}{\alpha^4}\left[\alpha S(\alpha,\lambda)\left(\alpha^3\psi''(\alpha+1) + \alpha + 2\right) - (q+1)\left(\alpha + 1 - \alpha^2\psi'(\alpha+1)\right)^2\right]S(\alpha,\lambda)^{-q-2}, \\
g_{22} &= \frac{q}{\alpha\lambda^2}\left(-\alpha\left(\ln\left(\frac{\alpha}{\lambda}\right) + \psi(1)\right) + \alpha\psi(\alpha+1) + \alpha q + 1\right)S(\alpha,\lambda)^{-q-2}, \\
g_{12} &= g_{21} = \frac{q(q+1)}{\lambda}\left(\psi'(\alpha+1) - \frac{\alpha+1}{\alpha^2}\right)S(\alpha,\lambda)^{-q-2}.
\end{aligned}
$$

Therefore, with respect to the entropy loss function, the Bayes estimate of $S(\alpha, \lambda)$ is obtained as

$$
\begin{aligned}
\hat{S}_{bs}^{e} &= \big[\hat{S}(\alpha,\lambda)^{-q} + \frac{1}{2}[(g_{11}\eta_{11} + g_{22}\eta_{22} + 2g_{12}\eta_{12}) + l_{30}\eta_{11}(g_1\eta_{11} + g_2\eta_{12}) + l_{03}\eta_{22} \\
&\quad \times (g_2\eta_{22} + g_1\eta_{21}) + l_{21}(3g_1\eta_{11}\eta_{12} + g_2(\eta_{11}\eta_{22} + 2\eta_{12}^2)) + l_{12} \\
&\quad \times (3g_2\eta_{22}\eta_{21} + g_1(\eta_{22}\eta_{11} + 2\eta_{21}^2)) + 2p_1^*(g_1\eta_{11} + g_2\eta_{21}) \\
&\quad + 2p_2^*(g_2\eta_{22} + g_1\eta_{12})]\big]^{-\frac{1}{q}}. \qquad (3.8)
\end{aligned}
$$

We recall that the right hand side expressions of (3.7) and (3.8) are to be evaluated at the MLEs $\hat{\alpha}$ and $\hat{\lambda}$. The Bayes estimate of the Shannon entropy with respect to the squared error loss function can be deduced from (3.8) on substituting $q = -1$. The Bayes estimates of the Rényi entropy with respect to the three loss functions can be obtained similarly. We omit the detailed expressions for the sake of conciseness.

3.2 Importance sampling method

In this subsection, we study importance sampling method for the computation of Bayes estimates of the unknown parametric functions given by (1.4) and (1.5). In this method, we need to rewrite the joint posterior distribution of α and λ in (3.3). It is given by

$$\Pi(\alpha, \lambda \mid \boldsymbol{x}) \propto IG_{\lambda}\left(m - c, \sum_{i}^{m} x_i\right) G_{\alpha|\lambda}\left(m + a, b - \sum_{i=1}^{m} \ln \xi(\lambda; x_i)\right) \kappa(\alpha, \lambda),$$

where

$$\kappa(\alpha, \lambda) = \frac{\left(b - \sum_{i=1}^{m} \ln \xi(\lambda; x_i)\right)^{-(m+a)} \prod_{i=1}^{m} \left(1 - \xi(\lambda; x_i)^{\alpha}\right)^{R_i}}{\left(\sum_{i}^{m} x_i\right)^{m-c} \exp\left\{\lambda d + \sum_{i=1}^{m} \ln \xi(\lambda; x_i)\right\}}.$$

The following steps can be used to get Bayes estimates of a function say $g(\alpha, \lambda)$ with respect to the LINEX and entropy loss functions.

Step-1 Generate λ from an inverse gamma distribution with shape parameter $(m - c)$ and scale parameter $(\sum_{i=1}^{m} x_i)^{-1}$ denoted by $IG_{\lambda}(m - c, \sum_{i=1}^{m} x_i)$.

Step-2 For a given λ obtained in Step-1, generate α from a gamma distribution with shape parameter $(m + a)$ and scale parameter $(b - \sum_{i=1}^{m} \ln \xi(\lambda; x_i))^{-1}$ denoted by $G_{\alpha|\lambda}(m + a, b - \sum_{i=1}^{m} \ln \xi(\lambda; x_i))$.

Step-3 Repeat Step-1 and Step-2 for N times to obtain $(\alpha_1, \lambda_1), \ldots, (\alpha_N, \lambda_N)$.

Step-4 The Bayes estimates of a parametric function $g(\alpha, \lambda)$ under the LINEX and entropy loss functions are given by

$$\hat{g}_{bs}^{l} = -\frac{1}{p} \ln \left[\frac{\sum_{i=1}^{N} \exp\{-pg(\alpha_i, \lambda_i)\} k(\alpha_i, \lambda_i)}{\sum_{i=1}^{N} k(\alpha_i, \lambda_i)}\right] \tag{3.9}$$

and

$$\hat{g}_{bs}^{e} = \left[\frac{\sum_{i=1}^{N} g(\alpha_i, \lambda_i)^{-q} k(\alpha_i, \lambda_i)}{\sum_{i=1}^{N} k(\alpha_i, \lambda_i)}\right]^{-\frac{1}{q}}, \tag{3.10}$$

respectively. The Bayes estimate of $g(\alpha, \lambda)$ with respect to the squared error loss function can be obtained from (3.10) when $q = -1$. Further, under LINEX and entropy loss functions, the Bayes estimates of the Shannon and Rényi entropy functions can be computed from (3.9) and (3.10) when $g(\alpha, \lambda) = S(\alpha, \lambda)$ and $g(\alpha, \lambda) = R_{\beta}(\alpha, \lambda)$, respectively.

4 Interval estimation

In this section, we study interval estimation for the Shannon and Rényi entropies using different techniques. In particular, we obtain three types of interval estimates: asymptotic, bootstrap confidence intervals and highest posterior density credible intervals. First, we consider asymptotic confidence intervals.

4.1 Asymptotic confidence intervals

In this subsection, we derive asymptotic confidence intervals of the entropy functions using two approaches: normal approximation (NA) of the MLE and the normal approximation of the log-transformed (NL) MLE.

4.1.1 NA method

The NA method is a useful tool to obtain asymptotic confidence intervals. To obtain $100(1-\gamma)\%$ confidence intervals for $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$, we need the inverse of the observed Fisher information matrix of α and λ. This is given by

$$\hat{I}^{-1}(\hat{\alpha},\hat{\lambda}) = \begin{pmatrix} -l_{20} & -l_{11} \\ -l_{11} & -l_{02} \end{pmatrix}^{-1}_{(\alpha,\lambda)=(\hat{\alpha},\hat{\lambda})} = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\lambda}) \\ Cov(\hat{\alpha},\hat{\lambda}) & Var(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix},$$

where

$$\begin{aligned} l_{20} &= -\frac{m}{\alpha^2} - 2\sum_{i=1}^{m} R_i \frac{\xi(\lambda;x_i)^\alpha \ln\xi(\lambda;x_i)}{1-\xi(\lambda;x_i)^\alpha}\left(1+\frac{\xi(\lambda;x_i)^\alpha}{(1-\xi(\lambda;x_i)^\alpha)}\right), \\ l_{02} &= \frac{m}{\lambda^2} - \frac{\alpha}{\lambda^4}\sum_{i=1}^{m} R_i \frac{x_i^2(1-\xi(\lambda;x_i))\xi(\lambda;x_i)^{\alpha-1}}{(1-\xi(\lambda;x_i)^\alpha)}\left(\frac{\alpha(1-\xi(\lambda;x_i))\xi(\lambda;x_i)^{\alpha-1}}{(1-\xi(\lambda;x_i)^\alpha)} + \frac{2\lambda}{x_i} - 1\right. \\ &\quad \left. + \frac{(\alpha-1)(1-\xi(\lambda;x_i))}{\xi(\lambda;x_i)}\right) - (\alpha-1)\sum_{i=1}^{m}\frac{x_i^2(1-\xi(\lambda;x_i))}{\lambda^4\xi(\lambda;x_i)}\left(1+\frac{(1-\xi(\lambda;x_i))}{\xi(\lambda;x_i)} - \frac{2\lambda}{x_i}\right) \\ &\quad - 2\sum_{i=1}^{m}\frac{x_i}{\lambda^3}, \\ l_{11} &= \frac{1}{\lambda^2}\sum_{i=1}^{m}\frac{R_i x_i(1-\xi(\lambda;x_i))\xi(\lambda;x_i)^{\alpha-1}}{(1-\xi(\lambda;x_i)^\alpha)}\left(1+\frac{\alpha\xi(\lambda;x_i)^\alpha\ln\xi(\lambda;x_i)}{(1-\xi(\lambda;x_i)^\alpha)} + \alpha\ln\xi(\lambda;x_i)\right) \\ &\quad - \frac{1}{\lambda^2}\sum_{i=1}^{m}\frac{x_i(1-\xi(\lambda;x_i))}{\xi(\lambda;x_i)}. \end{aligned}$$

Further, to obtain the approximate confidence intervals of the Shannon and Rényi entropy measures, we apply delta method. We refer to Greene (2003) for a detailed account of this method. Assume

$$\tau_S^T = \left(\frac{\partial S(\alpha,\lambda)}{\partial\alpha}, \frac{\partial S(\alpha,\lambda)}{\partial\lambda}\right) \quad \text{and} \quad \tau_{R_\beta}^T = \left(\frac{\partial R_\beta(\alpha,\lambda)}{\partial\alpha}, \frac{\partial R_\beta(\alpha,\lambda)}{\partial\lambda}\right),$$

where

$$\frac{\partial S(\alpha,\lambda)}{\partial \alpha} = \psi'(\alpha+1) - \frac{\alpha+1}{\alpha^2}, \quad \frac{\partial S(\alpha,\lambda)}{\partial \lambda} = \frac{\partial R_\beta(\alpha,\lambda)}{\partial \lambda} = \frac{1}{\lambda},$$
$$\frac{\partial R_\beta(\alpha,\lambda)}{\partial \alpha} = \frac{\beta}{1-\beta}\left(\psi(\beta(\alpha-1)+1) - \psi(\beta\alpha+1) + \frac{1}{\alpha}\right).$$

Moreover, the approximate estimated variances of $\hat{S}$ and $\hat{R}_\beta$ are given by

$$\widehat{Var}(\hat{S}) = \left[\tau_S^T \hat{I}(\hat{\alpha},\hat{\lambda})^{-1}\tau_S\right] \text{ and } \widehat{Var}(\hat{R}_\beta) = \left[\tau_{R_\beta}^T \hat{I}(\hat{\alpha},\hat{\lambda})^{-1}\tau_{R_\beta}\right].$$

Therefore, $(\hat{S}-S)/\sqrt{\widehat{Var}(\hat{S})}$ and $(\hat{R}_\beta - R_\beta)/\sqrt{\widehat{Var}(\hat{R}_\beta)}$ asymptotically follow standard normal distribution. Thus, the $100(1-\gamma)\%$ asymptotic confidence intervals for the Shannon and Rényi entropies are respectively given by

$$\hat{S} \pm Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{S})} \text{ and } \hat{R}_\beta \pm Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{R}_\beta)},$$

where $Z_{\gamma/2}$ denotes the upper $(\gamma/2)$th percentile of the standard normal distribution.

4.1.2 NL method

The method presented in the above subsection has some demerits. For instance, NA based method does not perform well when the sample sizes are small. Sometimes, it gives negative lower bound for the unknown positive valued parameter. A different transformation of the MLE can be used to correct the inadequate performance of the NA method. The $100(1-\gamma)\%$ normal approximate confidence intervals for log-transformed MLE are obtained as

$$\ln \hat{S} \pm Z_{\gamma/2}\sqrt{\eta_{11}(\ln \hat{S})} \text{ and } \ln \hat{R}_\beta \pm Z_{\gamma/2}\sqrt{\eta_{22}(\ln \hat{R}_\beta)},$$

where $\eta_{11}(\ln \hat{S}) = Var(\ln \hat{S})$ and $\eta_{22}(\ln \hat{R}_\beta) = Var(\ln \hat{R}_\beta)$. Thus, based on the normal approximation of the log-transformed MLE, the $100(1-\gamma)\%$ confidence intervals for $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$ are computed as

$$\hat{S} \times \exp\left\{\pm \frac{Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{S})}}{\hat{S}}\right\} \text{ and } \hat{R}_\beta \times \exp\left\{\pm \frac{Z_{\gamma/2}\sqrt{\widehat{Var}(\hat{R}_\beta)}}{\hat{R}_\beta}\right\},$$

respectively.

4.2 Bootstrap algorithm and confidence intervals

In this subsection, we construct the percentile bootstarp (Boot-*p*) (see Efron and Tibshirani (1986)) and bootstarp-*t* (Boot-*t*) (see Hall (1988)) confidence intervals for the Shannon and Rényi entropy functions given by (1.4) and (1.5). The bootstrap based methods are commonly used to estimate confidence intervals, though it can be used to estimate bias and variance of an estimator or calibrate hypothesis tests. Below, we describe the steps required for the estimation of the confidence intervals using these methods. First we illustrate Boot-*p* method.

4.2.1 Boot-p method

Step-1 From Equations (2.2) and (2.3) under the original datasets $x_i, i = 1, \ldots, m$, we obtain $\hat{\alpha}$ and $\hat{\lambda}$, and then compute the estimates of the Shannon and Rényi entropy measures. The estimates for the estimands given by (1.4) and (1.5) are denoted by $\hat{S}$ and $\hat{R}_\beta$, respectively. The algorithm described in Balakrishnan and Sandhu (1995) is used for the generation purpose.

Step-2 Generate a bootstrap sample $\boldsymbol{x}^* = (x_1^*, \ldots, x_m^*)$ from Step-1 based on the pre-specified censoring scheme. Obtain $\hat{\alpha}^*$ and $\hat{\lambda}^*$ and compute the bootstrap estimates $\hat{S}^*$ and $\hat{R}_\beta^*$.

Step-3 Repeat Step-2 for $N = 1000$ times. Then, we obtain $\hat{S}_1^*, \ldots, \hat{S}_{1000}^*$ and $\hat{R}_{\beta 1}^*, \ldots, \hat{R}_{\beta 1000}^*$, where $\hat{S}_i^* = \hat{S}^*(\hat{\alpha}^{*i}, \hat{\lambda}^{*i})$ and $\hat{R}_{\beta i}^* = \hat{R}_\beta(\hat{\alpha}^{*i}, \hat{\lambda}^{*i})$ for $i = 1, \ldots, 1000$.

Step-4 Arrange $\hat{S}_i^*$ and $\hat{R}_{\beta i}^*$, $i = 1, \ldots, 1000$ in ascending order and denote $\hat{S}_{(1)}^*, \ldots, \hat{S}_{(1000)}^*$ and $\hat{R}_{\beta(1)}^*, \ldots, \hat{R}_{\beta(1000)}^*$, respectively.

Then, the $100(1-\gamma)\%$ boot-p confidence intervals for Shannon's and Rényi's entropies are respectively given by

$$\left(\hat{S}_{\left(i\frac{\gamma}{2}\right)},\ \hat{S}_{\left(i\left(1-\frac{\gamma}{2}\right)\right)}\right) \quad \text{and} \quad \left(\hat{R}_{\beta\left(i\frac{\gamma}{2}\right)},\ \hat{R}_{\beta\left(i\left(1-\frac{\gamma}{2}\right)\right)}\right).$$

Note that when $N = 1000$, the bootstrap percentile confidence intervals of $S(\alpha, \lambda)$ and $R_\beta(\alpha, \lambda)$ at 95% level of confidence are $(\hat{S}_{(25)}, \hat{S}_{(975)})$ and $(\hat{R}_{\beta(25)}, \hat{R}_{\beta(975)})$, respectively.

4.2.2 Boot-t method

An advantage of the Boot-p is its simplicity. However, when the sample size is small, the percentile approach is generally not as accurate as the Boot-t method. Next, we illustrate the algorithm for the Boot-t method.

Step-1 Similar to Step-1 and Step-2 as in Boot-p method, obtain the bootstrap estimates $\hat{S}^*$ and $\hat{R}_\beta^*$ for the Shannon and Rényi entropies.

Step-2 Compute the variance-covariance matrix $I^*(\hat{\alpha}^*, \hat{\lambda}^*)^{-1}$ based on the variance-covariance matrix as in Subsection 4.1.1. Define

$$T_{S_i}^* = \frac{\hat{S}_i^* - \hat{S}_i}{\sqrt{\widehat{Var}(\hat{S}_i^*)}} \quad \text{and} \quad T_{R_{\beta i}}^* = \frac{\hat{R}_i^* - \hat{R}_i}{\sqrt{\widehat{Var}(\hat{R}_i^*)}}$$

for $i = 1, \ldots, N$. Let $N = 1000$.

Step-3 Repeat Step-1 and Step-2 1000 times to obtain $T_{S_1}^*, \ldots, T_{S_{1000}}^*$ and $T_{R_{\beta 1}}^*, \ldots, T_{R_{\beta 1000}}^*$.

Step-4 Arrange $T^*_{S_i}$ and $T^*_{R_{\beta i}}$, $i = 1, \ldots, 1000$ in ascending order. Denote $T^*_{S_{(1)}}, \ldots, T^*_{S_{(1000)}}$ and $T^*_{R_{\beta(1)}}, \ldots, T^*_{R_{\beta(1000)}}$.

Thus, the $100(1-\gamma)\%$ boot-t confidence intervals for the Shannon and Rényi entropies are obtained as

$$\left(\hat{T}_{S_{(i\frac{\gamma}{2})}},\ \hat{T}_{S_{(i(1-\frac{\gamma}{2}))}}\right) \quad \text{and} \quad \left(\hat{T}_{R_\beta\left(i\frac{\gamma}{2}\right)},\ \hat{T}_{R_\beta\left(i\left(1-\frac{\gamma}{2}\right)\right)}\right),$$

respectively. For the above case, the boot-t confidence intervals of $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$ at 95% level of confidence can be obtained as $(\hat{T}_{S(25)}, \hat{T}_{S(975)})$ and $(\hat{T}_{R_\beta(25)}, \hat{T}_{R_\beta(975)})$, respectively.

4.3 HPD credible intervals

In this subsection we propose highest posterior density (HPD) credible intervals of the entropy functions given by (1.4) and (1.5). The method due to Chen and Shao (1999) is used in this purpose. Denote $S_i = S(\alpha^{(i)}, \lambda^{(i)})$, $R_{\beta i} = R_\beta(\alpha^{(i)}, \lambda^{(i)})$ and

$$t_i = \frac{\kappa(\alpha^{(i)}, \lambda^{(i)})}{\sum_{i=1}^{N} \kappa(\alpha^{(i)}, \lambda^{(i)})},\ i = 1, \ldots, N,$$

where $\lambda^{(i)}$ and $\alpha^{(i)}$ for $i = 1, \ldots, N$ are posterior samples generated from Step-1 and Step-2 in Subsection 3.2 for λ and α, respectively. Let $S_{(i)}$ and $R_{\beta(i)}$ be the ordered values of S_i and $R_{\beta i}$, respectively. Then, the pth quantile of $S(\lambda,\alpha)$ and $R_\beta(\alpha,\lambda)$ can be estimated respectively by

$$\hat{S}^{(p)} = \begin{cases} S_{(1)}, & p = 0 \\ S_{(i)}, & \sum_{j=1}^{i-1} t_j < p \le \sum_{j=1}^{i} t_j \end{cases}$$

and

$$\hat{R}_\beta^{(p)} = \begin{cases} R_{\beta(1)}, & p = 0 \\ R_{\beta(i)}, & \sum_{j=1}^{i-1} t_j < p \le \sum_{j=1}^{i} t_j \end{cases}.$$

Let [.] denote the greatest integer function. Then, among the intervals of the following forms

$$\left(\hat{S}^{(\frac{j}{N})}, \hat{S}^{\left(\frac{j+[N(1-\gamma)]}{N}\right)}\right) \text{ and } \left(\hat{R}_\beta^{(\frac{j}{N})}, \hat{R}_\beta^{\left(\frac{j+[N(1-\gamma)]}{N}\right)}\right),$$

where $j = 1, \ldots, N - [N(1-\gamma)]$, the $100(1-\gamma)\%$ HPD intervals for $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$ are the corresponding one having the smallest interval widths.

5 Simulation study and real-life data analysis

This section deals with the comparison of the proposed estimates in terms of their average values, mean squared errors and average lengths. A real dataset is further considered to illustrate the estimates derived in the previous sections.

5.1 Numerical comparisons

In this subsection, we carry out a numerical study for the comparison of the proposed estimates of the entropy functions given by (1.4) and (1.5). For the MLEs and Bayes estimates, the comparisons are made based on their average values and mean squared errors. However, for the interval estimates, we compare with respect to their average lengths. In this purpose, we utilize 1000 progressive type-II censored samples for each simulations. All the computations are performed using Monte Carlo simulations on the statistical software R-3.5.1. Further, for the generation of the progressive type-II censored samples we use the algorithm provided by Balakrishnan and Sandhu (1995). Here, two different sample sizes $n = 30, 40$ and four failure sample sizes $m = 20, 25, 30, 40$ are taken. For the simulation purpose, the parameter values are taken to be $\alpha = 0.5$ and $\lambda = 1$. The MLEs are computed using optimx package in R software. The mean squared errors of the estimates are calculated using the formula

$$MSE = \frac{1}{M}\sum_{i=1}^{M}(\hat{\theta}_k^{(i)} - \theta_k)^2, \quad k = 1, 2.$$

For the problem under study, $\theta_1 = S(\alpha, \lambda)$, $\theta_2 = R_\beta(\alpha, \lambda)$ and $M = 1000$. Further, the notation $(0^3, 2)$ used in tables denotes the censoring scheme $(0, 0, 0, 2)$. The Bayes estimates with respect to the LINEX loss function are computed for the values of $p = -0.5, 0.05$. For the case of entropy loss function, we take $q = -0.5, 0.5$. The values of the hyperparameters are taken as $a = 2$, $b = 3$, $c = 2$ and $d = 3$. Before pointing out the observations from the numerical values presented in tables, we mention that each table (Tables 1-3) contains 8 columns and 56 rows. For particular (n, m), each scheme has four rows. The average values and the MSEs (in parenthesis) of the MLEs are presented in the third column and that of the Bayes estimates are provided in the fourth-eight columns. In particular, the fourth column presents the same for the Bayes estimates with respect to the squared error loss function. The fifth-sixth and seventh-eighth columns respectively present the Bayes estimates with respect to the LINEX and entropy loss functions. For each censoring schemes in Tables $1-3$, the fourth-eighth columns have four rows. The first and second (third and fourth) rows represent the average values and their corresponding MSEs of the Bayes estimates obtained using Lindley's approximation (importance sampling) methods. Now, we provide observations made from the presented tables.

(1) Table 1 provides the average values and MSEs of the ML and Bayes estimates of the Shannon entropy $S(\alpha, \lambda)$ for different choices of (n, m) and censoring schemes. Note that when $(\alpha, \lambda) = (0.5, 1.0)$, $S(\alpha, \lambda) = 2.306853$. In general, we notice that the Bayes estimates have superior performance than the MLEs in terms of the MSEs. For the case of the LINEX loss function, the Bayes estimate seems to be a reasonable choice when $p = 0.05$. Further, the performance of the Bayes estimate under squared error loss function and that under LINEX loss function is similar for small values of p. When we consider entropy loss function, the Bayes estimate with $q = 0.5$ is superior than that for $q = -0.5$. It is also observed that the Bayes estimates obtained using Lindley's

approximation method have better performance than that obtained using importance sampling method in terms of the MSEs.

(2) In Tables 2 and 3, we present the average values and their MSEs of the estimates of the Rényi entropy $R_\beta(\alpha, \lambda)$ under various choices of (n, m) and censoring schemes. The values of β are considered to be 0.25 and 0.75 in Table 2 and Table 3, respectively. Let $(\alpha, \lambda) = (0.5, 1.0)$. Then, $R_\beta(\alpha, \lambda) = 1.680823$ when $\beta = 0.25$, and $R_\beta(\alpha, \lambda) = 0.648021$ when $\beta = 0.75$. Similar observations to the case of the Shannon entropy are noticed. It is worth pointing that similar pattern for various other censoring schemes is also observed. As the effective sampling sizes increase the average values tend to the actual value of the estimand whereas the MSEs tend to decrease.

(3) We present the average length of 95% confidence intervals of the Shannon entropy $S(\alpha, \lambda)$ and the Rényi entropy $R_\beta(\alpha, \lambda)$ in Table 4 for various choices of (n, m) and censoring schemes. In this table, each censoring scheme corresponds to three rows. The first row represents the average length of the confidence intervals for the Shannon entropy. The second and third rows present that for the Rényi entropy function when $\beta = 0.25$ and $\beta = 0.75$, respectively. Four methods have been used to compute the average interval lengths. Among two asymptotic confidence intervals, the performance of NA method is the best. On the other hand, when we use bootstrap algorithms, Boot-t method provides better confidence intervals than Boot-p method. However, when comparing all the four methods, it is easy to notice that HPD credible intervals are the best in terms of the average length. Similar behaviour is observed for other censoring schemes. Further, the average interval length decreases when the effective sample size increases.

5.2 Real-life data analysis

In this subsection, we consider a real-life dataset for the illustration of the proposed estimates. It represents the number of million revolutions before failure for each of 23 ball bearings in a life test. The dataset is due to Lawless (2011), which has been used recently by Maiti and Kayal (2021). The dataset is given below.

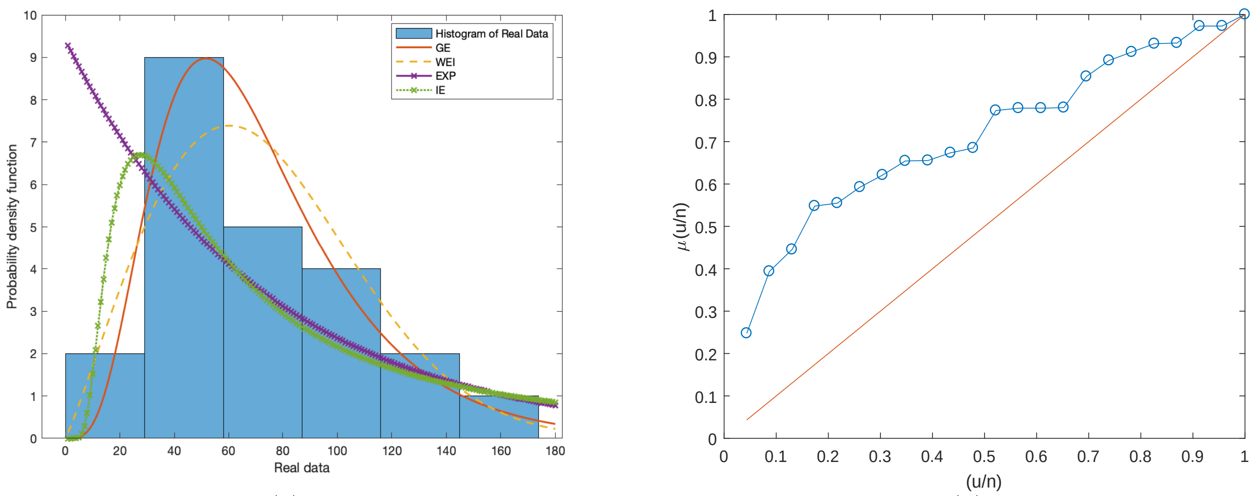
17.88	28.92	33	41.52	42.12	45.6	48.4	51.84	51.96	54.12
55.56	67.8	68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84
127.92	128.04	173.4							

To apply goodness of fit test, we consider various methods. These are Kolmogorov-Smirnov (KS) statistic, negative log-likelihood criterion, Akaikes-information criterion ($AIC = 2k - 2\ln L$), the associated second-order information criterion ($AICc = 2k - 2\ln L + \frac{2k(k+1)}{n-k-1}$) and Bayesian information criterion ($BIC = k\ln n - 2\ln L$), where k is the number of parameters of the model, n is the number of observations in the dataset, L is the maximum value of the likelihood function for the model. The values of the MLEs and five goodness of fit test statistics are presented in Table 5. The best distribution corresponds to the lowest

$BIC, AICc, AIC, -\ln L$, and KS statistic. The numerical values in Table 5 suggest that the generalized exponential distribution fits the data well compared to the other distributions (e.g. Weibull (WEI), exponential (EXP) and inverted exponential (IE) distributions). The histogram of the real dataset related to the number of million revolutions before failure and the density plots of four lifetime distributions are presented in Figure 1(a). The scaled total time on test (TTT) plot is useful to detect the shape of the hazard rate function, see, for instance, Mudholkar, Srivastava and Kollia (1996) and Mahmoudi (2011). Let $X_{(i)},\ i = 1, \ldots, n$ be the ith order statistic of the sample $X_{(1)}, \ldots, X_{(n)}$. Then, the scaled TTT transform is

$$\mu\left(\frac{u}{n}\right) = \frac{\sum_{i=1}^{u} X_{(i)} + (n-u)X_{(u)}}{\sum_{i=1}^{n} X_{(i)}},$$

where $u = 1, \ldots, n$. It is known that the hazard rate function is increasing, decreasing, bathtub and unimodal when the graph of $(u/n, \mu(u/n))$ is concave, convex, convex followed by concave and concave followed by convex, respectively. The scaled TTT plot of the given dataset has been depicted in Figure 1(b). The graph suggests that the hazard rate function of the number of million revolutions before failure for each of 23 ball bearings is increasing.



(a) (b)

Figure 1: (a) The histogram of the real data considered in Subsection 5.2 and the plots of the probability density functions of the fitted GE, WEI, EXP and IE models and (b) scaled TTT plot based on real dataset.

Now, we compute proposed estimates for the Shannon and Rényi entropy functions based on the real dataset. In Table 6, we consider progressive type-II censored sample with sample size $n = 23$, and the failure sample size $m = 17$. Further, we adopt various schemes for the purpose of computation such as $(6, 0^{16})$, $(0^{16}, 6)$ and $(0^8, 6, 0^8)$. In Tables 7, 8, 9 and 10, the first and second rows respectively represent the values of the Bayes estimates obtained using Lindley's approximation and importance sampling methods.

Table 1: Average values and MSEs of the ML and Bayes estimates of $S(\alpha, \lambda)$.

				$\hat{S}^l_{bs}$		$\hat{S}^e_{bs}$	
(n,m)	Scheme	$\hat{S}$	$\hat{S}^s_{bs}$	$p=-0.5$	$p=0.05$	$q=-0.5$	$q=0.5$
(30,20)	$10,0^{19}$	2.1380	2.4448	2.4737	2.4413	2.3757	2.3582
		(0.7591)	(0.3987)	(0.4116)	(0.3871)	(0.3203)	(0.3023)
			2.5823	2.6638	2.5747	2.5713	2.5543
			(0.5286)	(0.5448)	(0.5174)	(0.4561)	(0.4316)
	$0^{19},10$	2.0462	2.3711	2.4376	2.3648	2.3220	2.3208
		(1.0661)	(0.4263)	(0.4710)	(0.4185)	(0.4062)	(0.3709)
			2.5355	2.6445	2.5265	2.5164	2.5110
			(0.5604)	(0.6246)	(0.5547)	(0.5246)	(0.5169)
	$5,0^{18},5$	2.0863	2.4127	2.43846	2.4071	2.3973	2.3603
		(0.9170)	(0.3671)	(0.4154)	(0.3649)	(0.3487)	(0.3064)
			2.5465	2.6187	2.5319	2.5302	2.5037
			(0.5469)	(0.5597)	(0.5426)	(0.5105)	(0.5101)
(30,25)	$5,0^{24}$	2.1704	2.3736	2.4133	2.3685	2.3415	2.3296
		(0.7093)	(0.3577)	(0.4356)	(0.3482)	(0.3268)	(0.3178)
			2.4351	2.4689	2.4335	2.4320	2.4065
			(0.5164)	(0.5256)	(0.5069)	(0.5022)	(0.486)
	$0^{24},5$	2.134	2.3430	2.3893	2.3432	2.3301	2.3272
		(0.8225)	(0.3708)	(0.4719)	(0.3699)	(0.3032)	(0.2766)
			2.4236	2.4357	2.4067	2.3806	2.37623
			(0.5180)	(0.5346)	(0.5123)	(0.5052)	(0.4985)
	$3,0^{12},1,0^{10},1$	2.1570	2.3683	2.40643	2.358	2.3174	2.2919
		(0.7380)	(0.3597)	(0.4259)	(0.3644)	(0.2853)	(0.3510)
			2.4288	2.4437	2.4179	2.3781	2.3469
			(0.4960)	(0.5122)	(0.4878)	(0.4820)	(0.4778)
(30,30)	0^{30}	2.1805	2.3644	2.3761	2.3553	2.3090	2.2941
		(0.5776)	(0.2546)	(0.3286)	(0.2589)	(0.1775)	(0.1341)
			2.4177	2.4256	2.4137	2.4101	2.3946
			(0.3650)	(0.4252)	(0.3587)	(0.3466)	(0.3426)
(40,20)	$20,0^{19}$	2.1634	2.3729	2.4188	2.3715	2.3316	2.3037
		(0.6975)	(0.2170)	(0.2394)	(0.2207)	(0.1705)	(0.1159)
			2.4265	2.4448	2.4189	2.3765	2.3717
			(0.3596)	(0.3649)	(0.3546)	(0.3316)	(0.2946)
	$0^{19},20$	2.0206	2.3427	2.3957	2.3376	2.3129	2.2898
		(1.2376)	(0.2425)	(0.3025)	(0.2464)	(0.2161)	(0.1861)
			2.3647	2.3995	2.3577	2.3316	2.3264
			(0.3589)	(0.3604)	(0.3512)	(0.3366)	(0.3023)
	$10,0^{18},10$	2.0806	2.3850	2.4317	2.3713	2.3282	2.3181
		(0.9764)	(0.1806)	(0.1975)	(0.1889)	(0.1652)	(0.1648)
			2.3876	2.4385	2.3851	2.3651	2.3565
			(0.3385)	(0.3860)	(0.3256)	(0.3205)	(0.2865)
(40,30)	$5^2,0^{28}$	2.1999	2.3445	2.4212	2.3427	2.3201	2.3151
		(0.5210)	(0.1757)	(0.1927)	(0.1684)	(0.1641)	(0.1462)
			2.3764	2.4251	2.3754	2.3722	2.3576
			(0.2764)	(0.3164)	(0.2697)	(0.2360)	(0.2305)
	$0^{28},5^2$	2.1557	2.3222	2.4152	2.3184	2.2977	2.2890
		(0.6638)	(0.1825)	(0.2145)	(0.1874)	(0.2191)	(0.1736)
			2.3730	2.4178	2.3662	2.3517	2.3206
			(0.2545)	(0.2847)	(0.2461)	(0.2236)	(0.2077)
	$5,0^{14},5,0^{14}$	2.1855	2.3232	2.4100	2.3213	2.2982	2.2902
		(0.5411)	(0.1811)	(0.2444)	(0.1857)	(0.1571)	(0.1037)
			2.3846	2.4135	2.3827	2.3789	2.3624
			(0.2576)	(0.2871)	(0.2481)	(0.2416	(0.2159)
(40,40)	0^{40}	2.2293	2.3101	2.3648	2.3074	2.2935	2.2922
		(0.4102)	(0.1189)	(0.1647)	(0.1190)	(0.1511)	(0.1281)
			2.3344	2.3685	2.3279	2.3137	2.3088
			(0.1868)	(0.2136)	(0.1838)	(0.1569)	(0.1451)

Table 2: Average values and MSEs of the ML and Bayes estimates of $R_\beta(\alpha,\lambda)$ for $\beta = 0.25$.

				$\hat{R}^l_{\beta bs}$		$\hat{R}^e_{\beta bs}$	
(n,m)	Scheme	$\hat{R}_\beta$	$\hat{R}^s_{\beta bs}$	$p=-0.5$	$p=0.05$	$q=-0.5$	$q=0.5$
(30,20)	$10,0^{19}$	1.6036	1.8474	1.8872	1.8429	1.7605	1.7546
		(0.1196)	(0.0708)	(0.0760)	(0.0710)	(0.0593)	(0.0546)
			1.8900	1.9018	1.8888	1.8167	1.7971
			(0.0811)	(0.0836)	(0.0809)	(0.0790)	(0.0783)
	$0^{19},10$	1.5691	1.8556	1.8670	1.8541	1.7443	1.7246
		(0.19778)	(0.0809)	(0.0859)	(0.0808)	(0.0749)	(0.0726)
			1.8967	1.9135	1.8931	1.8860	1.8745
			(0.0817)	(0.0824)	(0.0816)	(0.0784)	(0.0774)
	$5,0^{18},5$	1.5854	1.8881	1.8951	1.8762	1.7168	1.7145
		(0.1586)	(0.0796)	(0.0834)	(0.0787)	(0.0723)	(0.0694)
			1.9034	1.9234	1.8993	1.8807	1.8798
			(0.0812)	(0.0821)	(0.0812)	(0.0794)	(0.0765)
(30,25)	$5,0^{24}$	1.6220	1.8094	1.8489	1.8138	1.7815	1.7754
		(0.0954)	(0.0624)	(0.0666)	(0.0615)	(0.0581)	(0.0580)
			1.8164	1.8645	1.8097	1.7862	1.7797
			(0.0786)	(0.0789)	(0.0777)	(0.0765)	(0.0761)
	$0^{24},5$	1.6101	1.7624	1.7742	1.7612	1.7534	1.7355
		(0.1223)	(0.0652)	(0.0675)	(0.0632)	(0.0579)	(0.0549)
			1.7966	1.8131	1.7854	1.7788	1.7465
			(0.0796)	(0.0798)	(0.0786)	(0.0755)	(0.0752)
	$3,0^{12},1,0^{10},1$	1.6176	1.777	1.7879	1.7760	1.7691	1.7531
		(0.1035)	(0.0628)	(0.0666)	(0.0624)	(0.0576)	(0.0535)
			1.7860	1.8036	1.7835	1.7800	1.7563
			(0.0774)	(0.0785)	(0.0768)	(0.0762)	(0.0746)
(30,30)	0^{30}	1.6307	1.7473	1.7561	1.7464	1.7406	1.7275
		(0.0731)	(0.0511)	(0.0566)	(0.0512)	(0.0506)	(0.0486)
			1.7568	1.7861	1.7526	1.7506	1.7288
			(0.0742)	(0.0754)	(0.0741)	(0.0740)	(0.0735)
(40,20)	$20,0^{19}$	1.6062	1.7808	1.8189	1.7764	1.7516	1.7392
		(0.1193)	(0.0464)	(0.0512)	(0.0462)	(0.0417)	(0.0412)
			1.7895	1.8421	1.7788	1.7700	1.7464
			(0.0717)	(0.7341)	(0.0708)	(0.0694)	(0.0679)
	$0^{19},20$	1.5478	1.7437	1.7482	1.7430	1.7401	1.7245
		(0.2740)	(0.0494)	(0.0535)	(0.0486)	(0.0446)	(0.0421)
			1.7599	1.7656	1.7535	1.7503	1.7469
			(0.0731)	(0.0744)	(0.0728)	(0.0715)	(0.0708)
	$10,0^{18},10$	1.5755	1.7908	1.8076	1.7864	1.7506	1.7454
		(0.1951)	(0.0409)	(0.0475)	(0.0404)	(0.0382)	(0.0360)
			1.8136	1.8319	1.8056	1.7765	1.7469
			(0.0725)	(0.0738)	(0.0718)	(0.0689)	(0.0679)
(40,30)	$5^2,0^{28}$	1.6320	1.7642	1.7973	1.7629	1.7616	1.7456
		(0.0733)	(0.0342)	(0.0357)	(0.0341)	(0.0319)	(0.0282)
			1.7706	1.8265	1.7685	1.7490	1.7347
			(0.0675)	(0.0695)	(0.0669)	(0.06581)	(0.0635)
	$0^{28},5^2$	1.6169	1.7303	1.7415	1.7291	1.7216	1.7045
		(0.1058)	(0.0364)	(0.0421)	(0.0364)	(0.0322)	(0.0304)
			1.7386	1.7486	1.7207	1.7164	1.7123
			(0.0665)	(0.0691)	(0.065733)	(0.0654)	(0.0647)
	$5,0^{14},5,0^{14}$	1.6274	1.7327	1.7419	1.7318	1.7257	1.7117
		(0.0799)	(0.0333)	(0.0351)	(0.0340)	(0.0328)	(0.0282)
			1.7509	1.7882	1.7535	1.7506	1.7346
			(0.0635)	(0.0645)	(0.0635)	(0.0615)	(0.0613)
(40,40)	0^{40}	1.6490	1.6873	1.7025	1.6806	1.6767	1.6613
		(0.0553)	(0.0230)	(0.0255)	(0.0241)	(0.0206)	(0.0162)
			1.6906	1.7317	1.6867	1.6803	1.6706
			(0.0618)	(0.0637)	(0.0617)	(0.0597)	(0.0563)

Table 3: Average values and MSEs of the ML and Bayes estimates of $R_\beta(\alpha, \lambda)$ for $\beta = 0.75$.

				$\hat{R}^l_{\beta bs}$		$\hat{R}^e_{\beta bs}$	
(n,m)	Scheme	$\hat{R}_\beta$	$\hat{R}^s_{\beta bs}$	$p=-0.5$	$p=0.05$	$q=-0.5$	$q=0.5$
(30,20)	$10,0^{19}$	0.5900	0.8207	0.8754	0.8270	0.8236	0.8035
		(0.1020)	(0.0412)	(0.0524)	(0.0415)	(0.0411)	(0.0409)
			0.8347	0.8980	0.8279	0.830672	0.823604
			(0.0496)	(0.0540)	(0.0473)	(0.0436)	(0.0402)
	$0^{19},10$	0.5691	0.7864	0.8338	0.7734	0.7710	0.7525
		(0.1347)	(0.0461)	(0.0502)	(0.0477)	(0.0471)	(0.0441)
			0.8164	0.8506	0.8076	0.7886	0.7760
			(0.0519)	(0.0546)	(0.0514)	(0.0488)	(0.0482)
	$5,0^{18},5$	0.5798	0.7734	0.8140	0.7693	0.7561	0.7435
		(0.1168)	(0.0455)	(0.0483)	(0.0454)	(0.04485)	(0.0446)
			0.7866	0.8340	0.7799	7746	0.7468
			(0.0514)	(0.0522)	(0.0507)	(0.0500)	(0.0486)
(30,25)	$5,0^{24}$	0.6051	0.7555	0.7862	0.7512	0.7255	0.7236
		(0.0806)	(0.0411)	(0.0432)	(0.0411)	(0.0410)	(0.0382)
			0.7588	0.8016	0.7564	0.7486	0.7432
			(0.0486)	(0.0509)	(0.0484)	(0.0477)	(0.045452)
	$0^{24},5$	0.5989	0.7314	0.7328	0.7313	0.7173	0.7054
		(0.0892)	(0.0439)	(0.0443)	(0.0435)	(0.0414)	(0.0402)
			0.7518	0.7787	0.7476	0.7461	0.7167
			(0.0479)	(0.0499)	(0.0475)	(0.0466)	(0.0446)
	$3,0^{12},1,0^{10},1$	0.6029	0.7326	0.7340	0.7325	0.716917	0.7134
		(0.0828)	(0.0438)	(0.0453)	(0.0437)	(0.0413)	(0.04017)
			0.7447	0.7699	0.7432	0.7300	0.7167
			(0.0467)	(0.0478)	(0.0466)	(0.0455)	(0.0436)
(30,30)	0^{30}	0.6139	0.6877	0.7199	0.6865	0.6817	0.6792
		(0.0643)	(0.0399)	(0.0411)	(0.0394)	(0.0359)	(0.0320)
			0.7260	0.7465	0.7242	0.7134	0.7115
			(0.0406)	(0.0419)	(0.0403)	(0.0385)	(0.0378)
(40,20)	$20,0^{19}$	0.5874	0.6949	0.7374	0.6934	0.6866	0.6813
		(0.1023)	(0.037)	(0.0406)	(0.0372)	(0.0346)	(0.0345)
			0.7289	0.7648	0.7176	0.6941	0.6870
			(0.0395)	(0.0408)	(0.0388)	(0.0384)	(0.0380)
	$0^{19},20$	0.5490	0.6898	0.7085	0.6936	0.6525	0.6521
		(0.1800)	(0.0386)	(0.0456)	(0.0378)	(0.0319)	(0.0303)
			0.7265	0.7470	0.7150	0.7061	0.6795
			(0.0416)	(0.0458)	(0.0414)	(0.0390)	(0.0365)
	$10,0^{18},10$	0.5692	0.6931	0.7052	0.6875	0.6611	0.6595
		(0.1363)	(0.0359)	(0.0385)	(0.0347)	(0.0314)	(0.0301)
			0.7469	0.7706	0.7425	0.7156	0.6806
			(0.0385)	(0.0395)	(0.0384)	(0.0381)	(0.0356)
(40,30)	$5^2,0^{28}$	0.6112	0.6609	0.7006	0.6584	0.6533	0.6515
		(0.0641)	(0.0236)	(0.0275)	(0.0237)	(0.0216)	(0.0202)
			0.6785	0.6845	0.6751	0.6549	0.6521
			(0.0317)	(0.0336)	(0.0307)	(0.0277)	(0.0264)
	$0^{28},5^2$	0.6090	0.6582	0.6907	0.6563	0.6524	0.6510
		(0.0750)	(0.0264)	(0.0286)	(0.0263)	(0.0225)	(0.0198)
			0.6791	0.6986	0.6768	0.6649	0.6459
			(0.0325)	(0.0360)	(0.0318)	(0.0278)	(0.0274)
	$5,0^{14},5,0^{14}$	0.6030	0.6597	0.6913	0.6578	0.6533	0.6509
		(0.0656)	(0.0234)	(0.0256)	(0.0234)	(0.0216)	(0.0205)
			0.6808	0.6974	0.6758	0.6655	0.6621
			(0.0285)	(0.0316)	(0.0283)	(0.0271)	(0.0246)
(40,40)	0^{40}	0.6257	0.6528	0.6769	0.6522	0.6502	0.6497
		(0.0473)	(0.0213)	(0.0230)	(0.0213)	(0.0204)	(0.0197)
			0.6649	0.6795	0.6632	0.6556	0.6530
			(0.0256)	(0.0291)	(0.0253)	(0.0248)	(0.0225)

Table 4: Average length of the interval estimates of $S(\alpha, \lambda)$ and $R_\beta(\alpha, \lambda)$.

(n,m)	Scheme	Asymptotic NA	NL	Boot-t	Boot-p	HPD
(30,20)	$10,0^{19}$	3.0374	3.5433	4.0246	4.3182	2.3480
		1.1853	1.2174	2.6069	2.7161	0.8642
		1.0922	1.2624	1.6085	1.8415	0.8215
	$0^{19},10$	3.4081	4.3389	4.5492	4.7315	2.5142
		1.4576	1.5285	3.2871	3.5126	1.1240
		1.2059	1.5651	1.8613	2.2153	0.7984
	$5,0^{18},5$	3.2376	3.9429	4.2935	4.4842	2.4441
		1.3290	1.3785	2.9792	3.3612	1.0745
		1.1406	1.3822	1.7131	2.4302	0.8146
(30,25)	$5,0^{24}$	3.0079	3.3051	3.8880	4.1643	2.2015
		1.1224	1.1547	2.4031	2.6419	0.7854
		1.0705	1.6388	1.5435	1.9245	0.6710
	$0^{24},5$	3.3536	3.7683	4.0224	4.2151	2.5314
		1.3058	1.3570	2.7270	3.0519	0.8462
		1.1418	1.8446	1.6186	1.7850	0.7716
	$3,0^{12},1,0^{10},1$	3.09980	3.4256	3.8752	4.2350	2.2941
		1.1762	1.2135	2.5210	2.6915	0.8046
		1.0892	1.6908	1.5684	1.7460	0.7256
(30,30)	0^{30}	2.9306	3.1460	3.3696	3.8405	2.1348
		1.0437	1.1096	2.0392	2.2642	0.6742
		1.0986	1.1826	1.4888	1.5106	0.6189
(40,20)	$20,0^{19}$	2.9902	3.4424	3.8587	4.2071	2.4155
		1.1914	1.2237	2.6148	2.6970	0.8216
		1.1016	1.2792	1.6054	2.0051	0.7615
	$0^{19},20$	3.6453	4.9418	4.9760	5.2340	3.1030
		1.6911	1.8179	3.7185	3.8201	1.0712
		1.3715	2.2267	2.1301	2.5120	0.9416
	$10,0^{18},10$	3.3575	4.1759	4.4104	4.6499	2.8192
		1.4629	1.5335	3.2750	3.5230	1.0045
		1.2193	1.5889	1.8628	2.1206	0.7649
(40,30)	$5^2,0^{28}$	3.0968	3.2274	3.2794	3.9421	2.2855
		1.1427	1.1585	2.0441	2.5109	0.7355
		1.0847	1.1634	1.4877	2.1240	0.6942
	$0^{28},5^2$	3.5177	3.6980	3.6275	4.0972	2.3846
		1.4070	1.4348	2.4216	3.1861	0.8046
		1.1998	1.2929	1.5708	2.2446	0.7422
	$5,0^{14},5,0^{14}$	3.1569	3.2895	3.3134	3.9995	2.5740
		1.1874	1.2047	2.1104	2.6609	0.7756
		1.1018	1.1812	1.4865	2.2281	0.7061
(40,40)	0^{40}	2.2096	2.3903	2.8715	3.3120	2.1611
		0.8331	0.8424	1.8138	2.0647	0.6438
		0.7577	0.7919	1.3792	1.7912	0.5765

Table 7 presents the ML and Bayes estimates of the Shannon entropy $S(\alpha, \lambda)$. Tables 8 and 9 provide the ML and Bayes estimates of the Rényi entropy $R_\beta(\alpha, \lambda)$ when $\beta = 0.25$ and $\beta = 0.75$, respectively. In Table 10, we provide the ML and Bayes estimates of the Shannon and Rényi entropies for $n = m$. The first two rows are assigned for the estimates of the Shannon entropy. The third and fourth rows are for the estimates of the Rényi entropy when $\beta = 0.25$. The last two rows are occupied for the estimates of Rényi's entropy when

$\beta = 0.75$. In Table 11, we present the average length of the interval estimates of $S(\alpha, \lambda)$ (first row) and $R_\beta(\alpha, \lambda)$ (second and third rows). The second row is for $\beta = 0.25$ and the third row is for $\beta = 0.75$. Table 12 represents the average length of confidence intervals when $n = m = 23$.

Table 5: The MLEs, BIC, AICc, AIC, $-\ln L$ and KS statistic for the real dataset.

Model	$\hat{\alpha}$	$\hat{\lambda}$	BIC	AICc	AIC	$-\ln L$	KS
GE	5.2961	30.9205	232.2267	230.5558	229.9558	112.9779	0.1055
WEI	2.1018	81.8745	233.6549	231.984	231.3839	113.6920	0.1510
EXP		0.0138	246.0030	245.0581	244.8675	121.4338	0.2622
IE		55.0551	246.5874	245.6423	245.4519	121.7259	0.3060

Table 6: Progressive type-II censored data for real dataset.

17.88	28.92	33	42.12	45.6	48.4	51.96	54.12	55.56
68.64	68.64	68.88	84.12	98.64	105.12	127.92	173.4	

Table 7: Average values of the estimates for the Shannon entropy $S(\alpha, \lambda)$.

				$\hat{S}^l_{bs}$		$\hat{S}^e_{bs}$	
(n,m)	Scheme	$\hat{S}$	$\hat{S}^s_{bs}$	$p = -0.5$	$p = 0.05$	$q = -0.5$	$q = 0.5$
(23,17)	$6,0^{16}$	3.2872	3.4361	3.5216	3.3954	3.3401	3.3173
			3.4534	3.5720	3.4462	3.4067	3.3645
	$0^{16},6$	4.6909	4.7406	4.7841	4.7383	4.6899	4.6668
			4.8131	4.8340	4.8073	4.7714	4.7645
	$0^8,6,0^8$	3.3698	3.4053	3.4268	3.4028	3.3752	3.3316
			3.4126	3.4451	3.4083	3.3846	3.355

Table 8: Average values of the estimates of the Rényi entropy $R_\beta(\alpha, \lambda)$ at $\beta = 0.25$.

				$\hat{R}^l_{\beta bs}$		$\hat{R}^e_{\beta bs}$	
(n,m)	Scheme	$\hat{R}_\beta$	$\hat{R}^s_{\beta bs}$	$p = -0.5$	$p = 0.05$	$q = -0.5$	$q = 0.5$
(23,17)	$6,0^{16}$	5.4985	5.5514	5.5642	5.5463	5.5162	5.4806
			5.5940	5.6131	5.5870	5.5511	5.5416
	$0^{16},6$	6.3404	6.3761	6.3884	6.3588	6.3164	6.2746
			6.3948	6.4172	6.3850	6.3561	6.3126
	$0^8,6,0^8$	5.5886	5.6349	5.7156	5.6198	5.6084	5.5946
			5.6810	5.7167	5.6756	5.6458	5.6097

Table 9: Average values of the estimates of the Rényi entropy $R_\beta(\alpha,\lambda)$ at $\beta = 0.75$.

(n,m)	Scheme	$\hat{R}_\beta$	$\hat{R}^s_{\beta bs}$	$\hat{R}^l_{\beta bs}$		$\hat{R}^e_{\beta bs}$	
				$p=-0.5$	$p=0.05$	$q=-0.5$	$q=0.5$
(23,17)	$6,0^{16}$	4.9853	5.1136	5.2409	5.1067	5.0846	5.0772
			5.1664	5.2654	5.1571	5.1431	5.1188
	$0^{16},6$	5.7754	5.8346	5.8485	5.8328	5.7982	5.7845
			5.8467	5.8560	5.8440	5.8164	5.7984
	$0^8,6,0^8$	5.0757	5.2648	5.3460	5.2553	5.2172	5.1976
			5.3246	5.3752	5.3168	5.3066	5.3015

Table 10: Average values of the estimates of $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$.

(n,m)	Scheme	MLE	L_s	L_l		L_e	
				$p=-0.5$	$p=0.05$	$q=-0.5$	$q=0.5$
(23,23)	0^{23}	3.2908	3.3461	3.3800	3.3426	3.3218	3.3205
			3.3626	3.4215	3.3571	3.3454	3.3404
		5.5245	5.5564	5.5672	5.5370	5.5238	5.5216
			5.6154	5.6344	5.6076	5.5644	5.5315
		5.0124	5.2237	5.2641	5.2095	5.1760	5.1680
			5.2315	5.2954	5.2270	5.1804	5.1798

Table 11: Interval estimates of $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$.

(n,m)	Scheme	Asymptotic NA	NL	Boot-t	Boot-p	HPD
(23,17)	$6,0^{16}$	(2.6163, 3.9581)	(2.6803, 4.0315)	(2.5671, 4.3392)	(2.7745, 4.8606)	(2.8152, 3.7217)
		(5.1027, 5.8944)	(5.1166, 5.9090)	(4.4682, 6.1295)	(5.0646, 6.7527)	(5.2645, 5.8249)
		(4.6024, 5.3682)	(4.6167, 5.3832)	(4.9301, 6.0973)	(4.5162, 6.3230)	(4.7752, 5.3333)
	$0^{16},6$	(3.6354, 5.7464)	(3.7457, 5.8746)	(3.5301, 5.8425)	(3.4456, 6.2592)	(3.7236, 5.1442)
		(5.8407, 6.8400)	(5.8599, 6.8602)	(5.7236, 6.9773)	(5.6006, 6.9140)	(5.9152, 6.6217)
		(5.3318, 6.2190)	(5.3484, 6.2365)	(5.1062, 6.2298)	(5.1023, 6.8888)	(5.4106, 6.0746)
	$0^8,6,0^8$	(2.6928, 4.0467)	(2.7564, 4.1195)	(2.4380, 4.2704)	(2.5785, 4.5022)	(2.7361, 3.6040)
		(5.1854, 5.9917)	(5.1997, 6.0066)	(5.0605, 6.2970)	(5.1204, 6.8004)	(5.2231, 5.8266)
		(4.6855, 5.4658)	(4.7002, 5.4812)	(4.4636, 6.1943)	(4.5002, 6.3028)	(4.7286, 5.3054)

Table 12: Interval estimates of $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$.

(n,m)	Scheme	Asymptotic NA	NL	Boot-t	Boot-p	HPD
(23,23)	0^{23}	(2.7096, 3.8682)	(2.7577, 3.9224)	(2.6482, 5.3892)	(2.4316, 5.2915)	(2.8464, 3.7259)
		(5.1832, 5.8637)	(5.1935, 5.8744)	(4.5221, 6.2223)	(4.3335, 6.0943)	(5.2680, 5.8201)
		(4.6817, 5.3411)	(4.6923, 5.3522)	(5.0864, 6.2911)	(4.9754, 6.4069)	(4.8565, 5.3973)

6 Optimal progressive type-II censoring scheme

In this section, we obtain optimal progressive censoring scheme from different censoring for which entropies (due to Shannon and Rényi) are minimum. For the specified total sample n, the effective sample m and censoring plan $R_i;\ i=1,\dots,m$, we know that $n = m+\sum_{i=1}^{m} R_i$. To determine the optimal censoring scheme $(R_1,\dots,R_m)$, our criterion is based on the precision of estimating logarithm of the entropy measure. The total possible number of censoring schemes is quite large when n and m are fixed. It is given by $\binom{n-1}{m-1}$. The total number of progressive censoring schemes is 92378 for $n=20$ and $m=10$. This is quite large. For computation purpose, we take five censoring schemes or plans, say P_1, P_2, P_3, P_4 and P_5, where $P_i = (R_1^i,\dots,R_m^i);\ i=1,\dots,m$ such that $n = m+\sum_{i=1}^{m} R_i^j$. Here, P_5 is a type-II censoring scheme. If P_1 reflects more information (less criterion value) than P_2, P_3, P_4 and P_5, then P_1 is considered to be the best among these schemes. For some references, one may look at the papers by Ng, Chan and Balakrishnan (2004) and Kundu (2009). It is noted that the posterior variances of $S(\alpha,\lambda)$ and $R_\beta(\alpha,\lambda)$ depend on the observed sample when model parameters are unknown. Thus, this should not be treated as a criterion for finding the optimal life testing plan. In this purpose, we consider criteria as

$$C_S(P) = \frac{E\left[V_{P(P)}(S(\alpha,\lambda))\right]}{E\left[V_{P(C)}(S(\alpha,\lambda))\right]} \quad \text{and} \quad C_{R_\beta}(P) = \frac{E\left[V_{P(P)}(R_\beta(\alpha,\lambda))\right]}{E\left[V_{P(C)}(R_\beta(\alpha,\lambda))\right]}, \tag{6.1}$$

where $V_{P(P)}(.)$ and $V_{P(c)}(.)$ are the posterior variances of the entropy for the censoring plan and complete sample, respectively. The expectations in (6.1) are evaluated with respect to the observed data. Based on the proposed criterion, P_1 is better than $P_i;\ i=2,3,4,5$ if $C_S(P_1) < C_S(P_i)$ when we consider the Shannon entropy. Similarly, for the other criterion for the Rényi entropy. It is easy to see that the explicit form of the solutions of (6.1) are hard to obtain. Therefore, we use Lindley's approximation method. In the proposed criterion, we compute approximation to $V_{P(P)}(S(\alpha,\lambda))$ using this method. Here,

$$V_{P(P)}(S(\alpha,\lambda)) = E_{P(P)}[S(\alpha,\lambda)]^2 - (E_{P(P)}[S(\alpha,\lambda)])^2. \tag{6.2}$$

In simulation purpose, we generate unknown parameters α and λ from the independent gamma priors with $a=c=2$ and $b=d=3$. We explain how to approximate $E_{P(P)}[.]$ under Monte Carlo simulation (see, Kundu (2008)). Both terms of the right hand side of (6.2) can be evaluated using Lindley's approximation method, which is discussed in Subsection 3.1. To approximate $E_{P(P)}[S(\alpha,\lambda)]^2$, we have $\nu(\alpha,\lambda) = (S(\alpha,\lambda))^2$. Here,

$$\begin{aligned}
g_1 &= \frac{2}{\alpha^2}\left(\alpha^2\psi'(\alpha+1)-\alpha-1\right)S(\alpha,\lambda), \quad g_{12}=g_{21}=\frac{2}{\alpha^2\lambda}\left(\alpha^2\psi'(\alpha+1)-\alpha-1\right)\\
g_{11} &= \frac{2}{\alpha^4}\bigg(\left(\alpha^2\left(-\psi'(\alpha+1)\right)+\alpha+1\right)^2+\alpha\left(\alpha^3\psi''(\alpha+1)+\alpha+2\right)S(\alpha,\lambda)\bigg),\\
g_2 &= \frac{2}{\lambda}S(\alpha,\lambda), \quad g_{22}=\frac{2}{\lambda^2}\left(2-S(\alpha,\lambda)\right).
\end{aligned}$$

Other terms in (3.13) remain same as in Subsection 3.1. Similarly, to evaluate $E_{P(P)}[S(\alpha,\lambda)]$, we have $\nu(\alpha,\lambda)=S(\alpha,\lambda)$. In this case,

$$g_1=\psi'(\alpha+1)-\frac{\alpha+1}{\alpha^2}, \quad g_{11}=\frac{\alpha+2}{\alpha^3}+\psi''(\alpha+1), \quad g_{12}=g_{21}=0, \quad g_2=\frac{1}{\lambda}, \quad g_{22}=-\frac{1}{\lambda^2}.$$

Similar procedure can be used to approximate $V_{P(P)}(R_\beta(\alpha,\lambda))$. To obtain $E_{P(P)}[R_\beta(\alpha,\lambda)]^2$, here we have $\nu(\alpha,\lambda) = (R_\beta(\alpha,\lambda))^2$. In this case,

$$
\begin{aligned}
g_1 &= \frac{2\beta}{(\beta-1)}\left(\psi((\alpha-1)\beta+1)-\psi(\alpha\beta+1)+\frac{1}{\alpha}\right)R_\beta(\alpha,\lambda), \quad g_2 = -\frac{2}{\lambda}R_\beta(\alpha,\lambda),\\
g_{11} &= \frac{2\beta}{(\beta-1)^2}\Bigg[(1-\beta)R_\beta(\alpha,\lambda)\Bigg(\beta\Bigg\{\frac{\Gamma((\alpha-1)\beta+1)\Gamma''((\alpha-1)\beta+1)-\Gamma'((\alpha-1)\beta+1)^2}{\Gamma((\alpha-1)\beta+1)^2}\\
&\quad +\frac{\Gamma'(\alpha\beta+1)^2-\Gamma(\alpha\beta+1)\Gamma''(\alpha\beta+1)}{\Gamma(\alpha\beta+1)^2}\Bigg\}-\frac{1}{\alpha^2}\Bigg)\\
&\quad +\beta\left(\psi((\alpha-1)\beta+1)-\psi(\alpha\beta+1)+\frac{1}{\alpha}\right)^2\Bigg],\\
g_{12} &= g_{21} = -\frac{2\beta}{(\beta-1)\lambda}\left(\psi((\alpha-1)\beta+1)-\psi(\alpha\beta+1)+\frac{1}{\alpha}\right), \quad g_{22} = \frac{2}{\lambda^2}(R_\beta-1).
\end{aligned}
$$

To approximate $E_{P(P)}[R_\beta(\alpha,\lambda)]$, here $\nu(\alpha,\lambda) = R_\beta(\alpha,\lambda)$. Further,

$$
\begin{aligned}
g_1 &= \frac{\beta}{1-\beta}\left(\psi((\alpha-1)\beta+1)-\psi(\alpha\beta+1)+\frac{1}{\alpha}\right), \quad g_2 = \frac{1}{\lambda}, \quad g_{22} = -\frac{1}{\lambda^2}, \quad g_{12} = g_{21} = 0,\\
g_{11} &= \frac{\beta}{1-\beta}\Bigg(\beta\Bigg(\frac{\Gamma((\alpha-1)\beta+1)\Gamma''((\alpha-1)\beta+1)-\Gamma'((\alpha-1)\beta+1)^2}{\Gamma((\alpha-1)\beta+1)^2}\\
&\quad +\frac{\Gamma'(\alpha\beta+1)^2-\Gamma(\alpha\beta+1)\Gamma''(\alpha\beta+1)}{\Gamma(\alpha\beta+1)^2}\Bigg)-\frac{1}{\alpha^2}\Bigg).
\end{aligned}
$$

Table 13 represents various schemes and the corresponding values of $C_S(P)$ and $C_{R_\beta}(P)$. From Table 13, we observe that plan P_3 is better compared to other plans based on the criterion $C_S(P)$ since it has more information. So, P_3 plan is optimal censoring scheme. Under the criterion $C_{R_\beta}(P)$, P_2 is optimal censoring scheme. Further, in plan P_5, we get the maximum value for each criterion.

Table 13: Optimal censoring scheme of $S(\alpha, \lambda)$ and $R_\beta(\alpha, \lambda)$ at $\beta = 0.25, 0.75$.

				$C_{R_\beta}(P)$	
(n,m)	Plan	Scheme	$C_S(P)$	$\beta = 0.25$	$\beta = 0.75$
(20, 10)	P_1	$10, 0^9$	2.1635	2.6081	2.7292
	P_2	$5^2, 0^8$	1.8459	2.4671	2.6558
	P_3	$5, 0^6, 2^2, 1$	1.7261	2.5506	2.6992
	P_4	$0^8, 5^2$	2.0654	2.6423	2.8241
	P_5	$0^9, 10$	2.2187	2.8451	2.8806
(20, 15)	P_1	$5, 0^{14}$	2.0394	2.2183	2.5314
	P_2	$3, 2, 0^{13}$	1.7758	1.8695	2.1068
	P_3	$3, 0^6, 1, 0^6, 1$	1.6899	2.1051	2.3486
	P_4	$0^{13}, 2, 3$	1.9654	2.3006	2.5781
	P_5	$0^{14}, 5$	2.1059	2.4529	2.6409
(30, 10)	P_1	$20, 0^9$	1.7609	1.9428	2.2473
	P_2	$10, 10, 0^8$	1.5481	1.7286	2.1736
	P_3	$5, 0^3, 10, 0^4, 5$	1.4228	1.8942	2.2153
	P_4	$0^8, 10, 10$	1.61060	2.1010	2.4725
	P_5	$0^9, 20$	1.8216	2.2534	2.5281
(30, 15)	P_1	$15, 0^{14}$	1.4086	1.7287	2.1010
	P_2	$10, 5, 0^{13}$	1.2806	1.6817	1.8753
	P_3	$5, 0^8, 10, 0^5$	1.1407	1.8400	1.9638
	P_4	$0^{13}, 5, 10$	1.3282	1.9226	2.2317
	P_5	$0^{14}, 15$	1.4681	2.0156	2.3453

7 Concluding remarks

In this paper, we focus on different methods for the estimation of the Shannon and Rényi entropy measures of generalized exponential distribution based on the progressive type-II censored sample. This type of censored sample has been considered by many experimenters as an effective approach of minimizing the cost and the time consumed. For the purpose of estimation, we derive maximum likelihood and Bayes estimates. To derive Bayes estimates, three loss functions are considered. The confidence intervals are also constructed. The MLEs are computed using optim function in R software. We have observed that the proposed Bayes estimates can not be obtained in explicit forms. Because of this, two methods such as Lindley's approximation and importance sampling are used. The confidence intervals are obtained using asymptotic distributions and parametric bootstrap methods. HPD credible intervals for the entropy functions are also obtained. To compare the performance of the proposed estimates, we carry out detailed simulation study using Monte Carlo Simulation technique. It is observed that the Bayes estimates outperform the MLEs. The HPD intervals perform the best among all the confidence intervals. For the illustrative purposes, a real dataset is considered to show an application of the proposed theoretical results. We also proposed criterion based on the entropy measures for choosing the optimal censoring scheme.

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