An Optimal Bayesian Sampling Plan for Two Parameter Exponential Distribution Under Type-I Hybrid Censoring

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Abstract

The Bayesian sampling plan for two parameter exponential distribution has been considered by Lam (1990) under the conventional Type-II censoring. Lin et al. (2008b) have obtained an exact Bayesian sampling plan for one parameter exponential distribution under Type-I and Type-II hybrid censoring schemes. In this paper, we obtain an optimal Bayesian sampling plan for the two parameter exponential distribution under Type-I hybrid censoring scheme based on a four parameter conjugate prior, introduced by Varde (1969). Bayes risk expressions of Lam (1990) for the conventional Type-II censoring scheme can be obtained as special cases of the Type-I hybrid censoring scheme. The optimal Bayesian sampling plan cannot be obtained analytically, we provide a numerical algorithm to compute the optimal Bayesian sampling plan. Different optimal Bayesian sampling plans have been reported.

Key words: Exponential distribution; Type-I hybrid censoring; Bayesian sampling plan; Bayes risk; conjugate priors; optimal sampling plan.

AMS Subject Classifications: 62F10, 62H12, 65D30.

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1 Introduction

The quality of a manufactured unit is very important for manufacturers as it directly affects their profits and market sale. A sampling plan defines a criterion for determining whether and how a lot of manufactured units has to be accepted or rejected. Various sampling schemes for acceptance sampling plan have been discussed in the literature. The Bayesian sampling plan is quite popular and realistic as it is based on the minimization of the Bayes risk of a decision function among all the sampling plans.

Several authors, see for example, Wetherill and Campling (1966); Hald (1968); Fertig and Mann (1974), have considered Bayesian sampling plans based on a linear loss function. Lam (1988) and Lam and Cheung (1993) developed a sampling plan with a polynomial loss function based on the assumption that the lifetimes of the experimental units follow normal distribution. Lam (1990) further discussed the Bayesian variable sampling plan for both one (scale) parameter and two (scale and location) parameters exponential distributions under the conventional Type-II censoring. He also showed that optimal sampling plan can be obtained in finite number of searching steps. Some more work on Bayesian sampling plan for exponentially distributed lifetimes has been developed by Lam (1994), Lam and Choy (1995), Huang and Lin (2002), Lin et al. (2008a,b, 2010, 2011) and see the references cited therein. The decision function used by these authors is also based on the maximum likelihood estimator of the mean lifetime of the experimental units. Lin et al. (2002), Huang and Lin (2004), Chen et al. (2004a), Chen et al. (2004b), Liang and Yang (2013), Tsai et al. (2014) and Yang et al. (2017) obtained acceptance sampling plans using different decision functions. Several authors considered others distributions also, see for example, Jianwei and Lam (1999), Aminzadeh (2003), Chen et al. (2004a), Jun et al. (2006), Chen et al. (2007), Balakrishnan et al. (2007), Amin and Salem (2012) and Yang et al. (2017).

Many consumer durables, like refrigerator, computer and many other electrical gadgets, etc, are designed in a way, so that they perform without any failure for at least some pre specified time period, that is, their lifetimes can be assumed to be shifted random variables and two parameter exponential distribution can be used in their modeling. This kind of manufactured units are expensive and have larger lifetimes and therefore, it is quite meaningless to observe the complete life data. Because of the restrictions on time and cost, Type-I and Type-II hybrid censoring schemes introduced by Epstein (1954) and Childs et al. (2003), respectively, are natural options. Hybrid censoring schemes are mixture of Type-I and Type-II censoring schemes. For a preassigned integer $r \geq 1$ and censoring time $T > 0$, the experiment is terminated by the time $\tau = min\{X_{(r)}, T\}$ under a Type-I hybrid censoring scheme. Where, $X_{(r)}$ denotes the r^{th} order statistic of sample $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ of size n. Till now, the main focus has been given mainly to one parameter exponential distribution for various censoring schemes. Lam (1990) considered the Bayesian sampling plan for two parameter exponential distribution under the conventional Type-II censoring.

The main focus of the paper is to obtain optimal Bayesian sampling plans for the two parameter exponential distribution based on Type-I hybrid censoring. We have considered a very flexible prior on (λ, μ), originally introduced by Varde (1969). Based on the conjugate prior and for a quadratic loss function, the Bayes risk is obtained. The same prior and quadratic loss function have been considered by Lam (1990) also. Here, the Bayes risk cannot be minimized analytically and we provide a numerical algorithm to minimize and to obtain optimal sampling plans. It has been proved that the algorithm converges in a finite number of steps.

The rest of the paper is organized as follows. Model building and description about the prior and loss function are included in Section 2. Bayes risk expressions are derived in Section 3. The graphical behavior of the Bayes risk and different optimal sampling plans are provided in Sections 4 and 5, respectively. Finally, the paper is concluded in Section 6. All the proofs and detailed derivations are provided in the Appendices.

2 Model Description and Priors

Suppose a lot of manufactured units for inspection has been given to make a decision whether to accept or reject the lot. For this, a sample of n units has been put on test for an experiment to inspect the quality of the lot. Suppose the quality of a unit is measured by its lifetime and the sampling is based on Type-I hybrid censoring. Let $(X_1, X_2, ..., X_n)$ be a random sample denoted by $\underline{X}$ with common probability density function (pdf)

$$f(x; \lambda, \mu) = \lambda e^{-\lambda(x-\mu)}, x \geq \mu, \lambda > 0, \mu > 0, \tag{1}$$

where $X_1, X_2, ..., X_n$ denote the lifetimes of n units and the parameters λ and μ are unknown.

To obtain an optimal Bayesian sampling plan, we need to design a decision function for accepting or rejecting the lot. We consider a decision function based on the maximum likelihood estimator of the mean lifetime. Suppose that the mean lifetime is denoted by W and its maximum likelihood estimator is denoted by $\widehat{W}$. Let the decision function

$$\delta(\underline{X}) = \begin{cases} 1, & \text{if } \widehat{W} \geq \xi \\ 0, & otherwise, \end{cases}$$

with $\xi \geq 0$, where 1 and 0 denote acceptance and rejection of the lot, respectively.

A reasonable quadratic loss function of the life testing experiment under discussion is given as:

$$\ell(\delta(\underline{X}), \lambda, \mu) = nC_s + \tau C_T - \ (n - D)r_s + \delta(\underline{X})\Big(\sum_{0 \leq i+j \leq 2} C_{ij}\lambda^i \mu^j\Big) + (1 - \delta(\underline{X}))C_r,$$

where

C_s: per unit cost due to inspection, C_T: per unit cost on time, D: number of failed units in the experiment, r_s: per unit salvage value, C_r: cost due to rejection of the lot, and the quadratic term denotes the cost due to acceptance of the lot with constant coefficients C_{ij}'s such that $\sum_{0\leq i+j\leq 2} C_{ij}\lambda^i\mu^j \geq 0$.

Optimal Bayesian sampling plan minimizes Bayes risk of decision function $\delta(\underline{X})$ under the loss function $\ell(\delta(\underline{X}), \lambda, \mu)$ based on a prior on the parameters λ and μ among all the sampling plans.

It is assumed that (λ, μ) follow a prior distribution introduced by Varde (1969). The prior density function is given by

$$p(\lambda, \mu) = \frac{1}{A}\lambda^{\alpha-1} exp[-\lambda(\beta - \gamma\mu)], \lambda > 0, 0 < \mu \leq \eta, \tag{2}$$

where,

$$\alpha > 0, \beta > 0, \alpha \leq \gamma < \frac{\beta}{\eta}$$

and

$$A = \begin{cases} \frac{\Gamma(\alpha-1)}{\gamma}[\frac{1}{(\beta-\gamma\eta)^{\alpha-1}} - \frac{1}{\beta^{\alpha-1}}], & \alpha \neq 1 \\ \frac{1}{\gamma}[ln\beta - ln(\beta - \gamma\eta)], & \alpha = 1, \end{cases}$$

with known α, β, γ and η. In practice, an uniform and an exponential or a gamma independent priors are taken for location μ and scale λ. (See Bayoud (2012) and Kundu et al. (2013)). The followings are true for the four parameter prior distribution given in equation (2):

(a) Prior is a conjugate prior.

(b) λ and μ are dependent which is a general case.

The four parameter conjugate prior was originally suggested by Varde (1969) and used by Lam (1990) in case of Type-II censoring.

Under this set up, a sampling plan is a 4-tuple quantity which is (n, r, T, ξ) and an optimal Bayesian sampling plan is obtained by choosing n, r, T and ξ such that the corresponding Bayes risk is minimum among all the sampling plans. Further, in order to obtain the optimal Bayesian sampling plan, it is necessary to determine the decision function $\delta(\underline{X})$ and its Bayes risk. The explicit expression of the Bayes risk will be provided in the next section.

3 Bayes Risk

Under Type-I hybrid censoring we stop the experiment by time $\tau = min\{X_{(r)}, T\}$ for a preassigned integer $r \geq 1$ and censoring time $T > 0$. Since D is the number of X_i's such that $X_i \leq \tau$, therefore $D = 0, 1, ..., r-1$ for $T < X_{(r)}$ and $D = r$ for $X_{(r)} \leq T$. Under the Type-I hybrid censoring scheme, the likelihood function is given by

$$L(\lambda, \mu|\underline{x}) = \begin{cases} \frac{n!}{(n-D)!}\lambda^D e^{-\lambda\left[\sum_{i=1}^{D}(x_{(i)}-\mu)+(n-D)(T-\mu)\right]}, & \text{if } T < X_{(r)} \\ \frac{n!}{(n-r)!}\lambda^r e^{-\lambda\left[\sum_{i=1}^{r}(x_{(i)}-\mu)+(n-r)(x_{(r)}-\mu)\right]}, & \text{if } X_{(r)} \leq T. \end{cases} \tag{3}$$

Childs et al. (2012) have given MLEs of the parameters λ and μ under Type-I HCS (hybrid censoring scheme). Assuming $\theta = \frac{1}{\lambda}$, it can be seen that MLE of (θ, μ) does not exist for any r satisfying $1 \leq r \leq n$ when $D = 0$. When $r = 1$, MLE of (θ, μ) does not exist for $D = 1$ but it is quite obvious to consider $(0, X_{(1)})$ as an estimator in this particular case. When $r \geq 2$, MLE of (θ, μ) exists and is given by

$$(\widehat{\theta}, \widehat{\mu}) = \begin{cases} \left(\frac{\sum_{i=1}^{D} X_{(i)}+(n-D)T-nX_{(1)}}{D}, X_{(1)}\right), & \text{if } D = 1, 2, ..., r-1 \\ \left(\frac{\sum_{i=1}^{r} X_{(i)}+(n-r)X_{(r)}-nX_{(1)}}{r}, X_{(1)}\right), & \text{if } D = r. \end{cases}$$

The estimator of the mean lifetime W will be $\widehat{\mu}+\widehat{\theta}$, *i.e.*, for $r=1$ and $D>0$, it is $X_{(1)}$ and for $r \geq 2$ and $D>0$, it is given by

$$\widehat{W}=\begin{cases} X_{(1)}+\frac{1}{D}\Big[\sum_{i=1}^{D} X_{(i)}+(n-D)T-nX_{(1)}\Big], & \text{if } D=1,2,...,r-1 \\ X_{(1)}+\frac{1}{r}\Big[\sum_{i=1}^{r} X_{(i)}+(n-r)X_{(r)}-nX_{(1)}\Big], & \text{if } D=r. \end{cases} \tag{4}$$

Note that the decision function is based on the estimator $\widehat{W}$, and we need to find the probability density function of $\widehat{W}$ in order to compute the Bayes risk. To find the conditional probability distribution of $\widehat{W}$, we first need the conditional moment generating function (mgf) of $\widehat{W}$ given $D \geq 1$ (*i.e.*, $D>0$). Define

$$M_{\widehat{W}}(s)=E(e^{s\widehat{W}}|D \geq 1).$$

Lemma 3.1. *The conditional moment generating functions of $\widehat{W}$ given $D>0$ for various cases are as follows*

Case (i): $r=1$ *and* $n \geq 1$

$$M_{\widehat{W}}(s)=(1-q^n)^{-1}\left\{\frac{e^{s\mu}}{(1-\frac{s}{\lambda n})}-q^n\frac{e^{sT}}{(1-\frac{s}{\lambda n})}\right\},\ |s|<n\lambda.$$

Case (ii): $r=2$ *and* $n=2$

$$M_{\widehat{W}}(s)=\frac{1}{1-q^2}\left\{2q(1-q)e^{sT}+\frac{e^{s\mu}}{(1-\frac{s}{2\lambda})^2}+q^2\frac{e^{sT}}{(1-\frac{s}{2\lambda})^2}-2q\frac{e^{(\mu+T)/2}}{(1-\frac{s}{2\lambda})^2}\right\},\ |s|<2\lambda.$$

Case (iii): $r \geq 2$ *and* $n \geq 3$

$$M_{\widehat{W}}(s)=\frac{1}{1-q^n}\left[\sum_{d=1}^{r-1}\sum_{k=0}^{d-1}\left\{c_{dk}\frac{e^{s(\mu+\mu_{dk})}}{(1-\frac{s}{\lambda d})^{d-1}(1-\frac{s}{\lambda_{dk}})}-e_{dk}\frac{e^{sT}}{(1-\frac{s}{\lambda d})^{d-1}(1-\frac{s}{\lambda_{dk}})}\right\}\right.$$

$$+\frac{e^{s\mu}}{(1-\frac{s}{\lambda r})^{r-1}(1-\frac{s}{\lambda n})}-q^{n}\frac{e^{sT}}{(1-\frac{s}{\lambda r})^{r-1}(1-\frac{s}{\lambda n})}$$
$$+\sum_{k=0}^{r-2}\left\{c_k\frac{e^{s(\mu+\mu_k)}}{(1-\frac{s}{\lambda r})^{r-1}(1-\frac{s}{\lambda_k})}-e_k\frac{e^{sT}}{(1-\frac{s}{\lambda r})^{r-1}(1-\frac{s}{\lambda_k})}\right\}\Bigg],\ |s|\ \leq\lambda.$$

Here

$$q=e^{-\lambda(T-\mu)}$$
$$c_{dk}=(-1)^k\binom{n}{d}\binom{d}{k}q^{n-d+k}\ \textit{for}\ d=1,2,...,(r-1)\ \textit{and}\ k=0,1,...,(d-1),$$
$$e_{dk}=(-1)^k\binom{n}{d}\binom{d}{k}q^{n}\ \textit{for}\ d=1,2,...,(r-1)\ \textit{and}\ k=0,1,...,(d-1),$$
$$\lambda_{dk}=\frac{\lambda d(d-k)}{(2d-k-n)}\ \textit{for}\ d=1,2,...,(r-1)\ \textit{and}\ k=0,1,...,(d-1),$$
$$\mu_{dk}=\frac{1}{d}(n-d+k)(T-\mu)\ \textit{for}\ d=1,2,...,(r-1)\ \textit{and}\ k=0,1,...,(d-1),$$
$$c_k=(-1)^{k+1}r\binom{n}{r}\binom{r-1}{k}\frac{1}{n-r+k+1}q^{n-r+k+1}\ \textit{for}\ k=0,1,...,(r-2),$$
$$e_k=(-1)^{k+1}r\binom{n}{r}\binom{r-1}{k}\frac{1}{n-r+k+1}q^{n}\ \textit{for}\ k=0,1,...,(r-2),$$
$$\lambda_k=\frac{\lambda r(r-k-1)}{(2r-k-n-1)}\ \textit{for}\ k=0,1,...,(r-2),$$
$$\mu_k=\frac{1}{r}(n-r+k+1)(T-\mu)\ \textit{for}\ k=0,1,...,(r-2).$$

Proof. The proof of case (*iii*) can be obtained from Childs et al. (2012). They did not provide explicit expression for case (*i*) and case (*ii*) but they also can be obtained along same lines. □

Theorem 3.1. *The conditional probability density functions or survival functions of* $\widehat{W}$ *given* $D>0$ *for various cases are as follows*

Case (i): $r=1$ *and* $n\geq 1$

$$f_{\widehat{W}}(x)=\sum_{l=0}^{\infty}q^{nl}\big\{g(x-\mu;1,n\lambda)-q^{n}g(x-T;1,n\lambda)\big\},\ x>\mu,$$

where $g(z, a, b)$ *is the pdf of* $Gamma(a, b)$ *given by*

$$g(z; a, b) = \begin{cases} \frac{b^a}{\Gamma(a)} z^{a-1} e^{-bz}, & \textit{if} \quad z > 0, \\ 0, & \textit{otherwise}, \end{cases} \qquad a > 0, \; b > 0.$$

Case (ii): $r = 2$ *and* $n = 2$

$$P(\widehat{W} > x) = \begin{cases} 1, & \textit{if } x < \mu \\ \sum_{l=0}^{\infty} q^{2l} \Big[2q(1-q) S_1(x) + S_2(x - \mu; 2, 2\lambda) & \\ + q^2 S_2(x - T; 2, 2\lambda) - 2q S_2(x - \frac{\mu+T}{2}; 2, 2\lambda) \Big], & \textit{if } \mu \le x < T \\ 0, & \textit{if } T \le x, \end{cases}$$

where

$$S_1(t) = \begin{cases} 1, & \textit{if } t < T \\ 0, & \textit{if } t \ge T \end{cases} \qquad \textit{and} \qquad S_2(t; a, b) = \begin{cases} 1, & \textit{if } t < 0 \\ \frac{b^a}{\Gamma(a)} \int_t^{\infty} u^{a-1} e^{-bu} \, du, & \textit{if } t \ge 0. \end{cases}$$

Case (iii): $r \ge 2$ *and* $n \ge 3$

$$\begin{aligned} f_{\widehat{W}}(x) = \sum_{l=0}^{\infty} q^{nl} \Big[& c_{10} g(-x + \mu + \mu_{10}; 1, \frac{\lambda}{n-2}) - e_{10} g(-x + T; 1, \frac{\lambda}{n-2}) \\ & + I_{\{r \ge 3\}} \sum_{d=2}^{r-1} \sum_{k=0}^{d-1} \Big\{ c_{dk} g_3(x - \mu - \mu_{dk}; d-1; \lambda d, \lambda_{dk}) - e_{dk} g_3(x - T; d-1, \lambda d, \lambda_{dk}) \Big\} \\ & + g_1(x - \mu; r-1, \lambda r, \lambda n) - q^n g_1(x - T; r-1, \lambda r, \lambda n) \\ & + \sum_{k=0}^{r-2} \Big\{ c_k g_3(x - \mu - \mu_k; r-1, \lambda r, \lambda_k) - e_k g_3(x - T; r-1, \lambda r, \lambda_k) \Big\} \Big], x > \mu, \end{aligned}$$

where

$$g_3(z;a_1,b_1,b_2) = \begin{cases} g_1(z;a_1,b_1,b_2), & \text{if } b_2 > 0 \\ g_2(z;a_1,b_1,-b_2), & \text{if } b_2 < 0 \\ g(z,a_1,b_1), & \text{if } b_2 = \infty, \end{cases}$$

with $g_1(z;a_1,b_1,b_2)$ *and* $g_2(z;a_1,b_1,b_2')$ *as pdfs of* $X-Y_1$ *and* $X-Y_2$*, respectively, for* $X \sim Gamma(a_1,b_1)$*,* $Y_1 \sim Gamma(1,b_2)$ *and* $Y_2 \sim Gamma(1,b_2')$ *with integer valued* a_1*. Here* X *is independent with* Y_1 *and* Y_2*, i.e.,*

$$\begin{aligned} g_1(z;a_1,b_1,b_2) &= \frac{b_1^{a_1}b_2}{\Gamma(a_1)}e^{-b_2z}\int_0^z u^{a_1-1}e^{(b_2-b_1)u}du \\ &= \begin{cases} \sum_{s=0}^{a_1-1}(-1)^s\frac{b_1^{a_1}b_2}{(b_2-b_1)^{s+1}}\frac{z^{a_1-s-1}e^{-b_1z}}{(a_1-s-1)!} + \left(\frac{b_1}{b_1-b_2}\right)^{a_1}b_2e^{-b_2z}, & \text{if } b_2-b_1 > 0 \\ g(z;a_1+1,b_1), & \text{if } b_2-b_1 = 0 \end{cases} \end{aligned}$$

and

$$g_2(z;a_1,b_1,-b_2) = \begin{cases} \sum_{s=0}^{a_1-1}(-1)^s\frac{b_1^{a_1}b_2}{(b_2-b_1)^{s+1}}\frac{z^{a_1-s-1}e^{-b_1z}}{(a_1-s-1)!}, & \text{if } z > 0 \\ -\left(\frac{b_1}{b_1-b_2}\right)^{a_1}b_2e^{-b_2z}, & \text{if } z < 0. \end{cases}$$

Proof. The results can be easily obtained using the mgfs. The details are avoided. □

Remark 3.1. *It is immediate from the Theorem 3.1 that the pdfs of* $\widehat{W}$ *for* $r = 1$ *and* $r \geq 2$ *reduce to* $g(x-\mu;1,\lambda n)$ *and* $g_1(x-\mu;r-1,\lambda r,\lambda n)$*, respectively, as* $T \to \infty$*. It may be noted that these are the pdfs for the estimators of the mean lifetime of a two parameter exponential distribution in the case of Type-II censoring, see Lam (1990).* □

The MLE of W, $\widehat{W}$, does not exist when $D = 0$. But note that the event $\{D = 0\}$

indicates that the lot is of good quality, therefore, it is reasonable to accept the lot in this situation. So we redefine the decision function as follows:

$$\delta(\underline{X}) = \begin{cases} 1, & \text{if } D = 0 \textit{ or } \widehat{W} \geq \xi \\ 0, & \textit{otherwise.} \end{cases} \tag{5}$$

The loss function to be considered under Type-I HCS is

$$\ell(\delta(\underline{X}), \lambda, \mu, n, r, T) = nC_s + \tau C_T - (n - D)r_s + \delta(\underline{X})\Big(\sum_{0 \leq i+j \leq 2} C_{ij}\lambda^i\mu^j\Big) + (1 - \delta(\underline{X}))C_r, \tag{6}$$

as the number of units in the sample which have not failed after the completion of the experiment is $(n - D)$.

Theorem 3.2. *For $\alpha > 1$, the Bayes risk of $\delta(\underline{X})$ denoted by $r(n, r, T, \xi)$ is given as follows:*

$$\begin{aligned} r(n, r, T, \xi) = n(C_s - r_s) &+ C_r + C_T E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}(\tau|\lambda, \mu) + r_s E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}(D|\lambda, \mu) \\ &+ (C_{00} - C_r) E_{\lambda,\mu}\big(e^{-n\lambda(T-\mu)}\big) + \sum_{1 \leq i+j \leq 2} C_{ij} E_{\lambda,\mu}\big(\lambda^i\mu^j e^{-n\lambda(T-\mu)}\big) + \sum_{0 \leq i+j \leq 2} C'_{ij} I_{ij}, \end{aligned} \tag{7}$$

where $I_{ij} = E_{\lambda,\mu}\Big(\lambda^i\mu^j P_{\underline{X}|\lambda,\mu}(\widehat{W} \geq \xi|\lambda, \mu)\Big)$ *and* $C'_{ij} = \begin{cases} C_{00} - C_r, & \textit{if } i + j = 0 \\ C_{ij}, & \textit{if } 1 \leq i + j \leq 2. \end{cases}$

Proof. See appendix A. □

Remark 3.2. *In case $T \to \infty$ (Type-II censoring), it can be verified using equation* (7) *and the pdfs given in Remark 3.1 that the Bayes risk matches with Bayes risk derived by Lam (1990) for the cases $r = 1$ and $r \geq 2$ when $C_T = 0$ and $r_s = 0$.* □

4 Optimal Sampling Plan

The optimal Bayesian sampling plan is the plan which minimizes the Bayes risk among all the possible sampling plans. Suppose a sampling plan and the optimal Bayesian sampling plan are denoted by (n, r, T, ξ) and (n_0, r_0, T_0, ξ_0), respectively, then

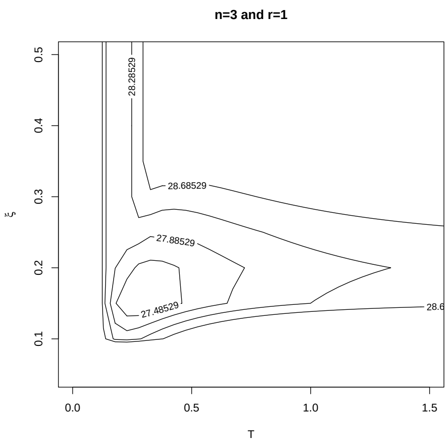
$$(n_0, r_0, T_0, \xi_0) = \underset{(n,r,T,\xi)}{\arg\min}\; r(n, r, T, \xi).$$

The Bayes risk function has complicated expression and therefore it is difficult to show analytically that the Bayes risk has an unique minimum. The Bayes risk is a function of n, r, T and ξ, and two of them are discrete. For fixed n and r, it is noted from graphs that the Bayes risk has an unique minimum with respect to T and ξ. Further, it is also shown that n_0 and r_0, optimal values of n and r, respectively, are bounded. Therefore, it suffices to provide contour plots of Bayes risk with respect to $T > 0$ and $\xi > 0$ for various values of n and r.

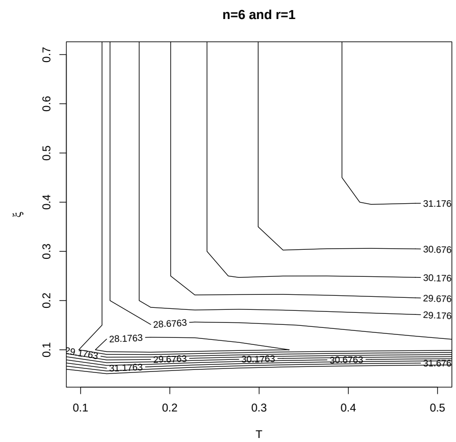
Figure 4.1 shows the contour plots of the Bayes risk for the hyper parameters $\alpha = 2.5, \beta = 0.8, \gamma = 2.5, \eta = 0.05$ and the coefficients $C_s = 0.5, C_r = 30, C_T = 5, C_{ij} = 2$ and $r_s = 0.3$. For convenience, only few contour plots have been given here (see Figure 4.1). Sub figure 4.1c contains the required contour plot having the minimum Bayes risk 25.4026 among all the contour plots with different values of n and r. Sub figure 4.1c also verifies the optimal Bayesian sampling plan obtained in Table 5.1, which is $(3, 2, 0.3268, 0.3000)$. The results of Table 5.1 have been obtained using the algorithm suggested in the next Section 5. The following result shows that n_0 and r_0 are finite.

Theorem 4.1. *n_0 and r_0 satisfy the following inequalities*

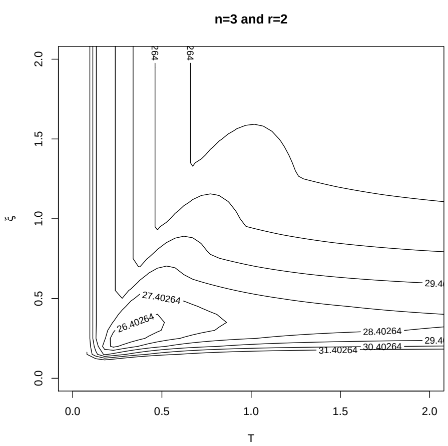
$$n_0 \leq \min\left\{ \frac{C_r}{C_s - r_s}, \frac{\sum\limits_{0\leq i+j\leq 2} C_{ij} E_{\lambda,\mu}\left(\lambda^i \mu^j\right)}{C_s - r_s} \right\} \text{ and } 1 \leq r_0 \leq n_0.$$



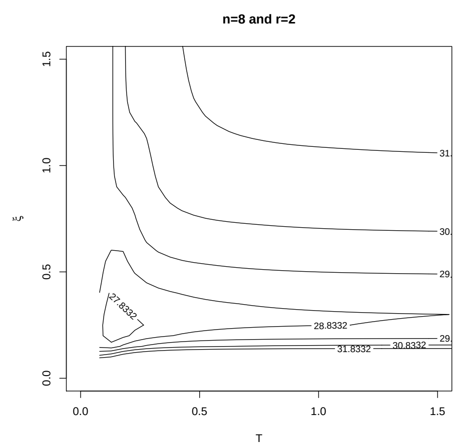
(a) 4.1a



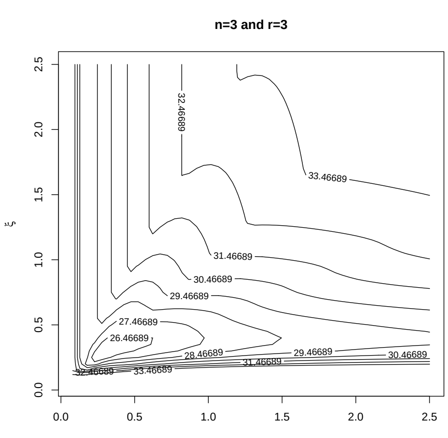
(b) 4.1b



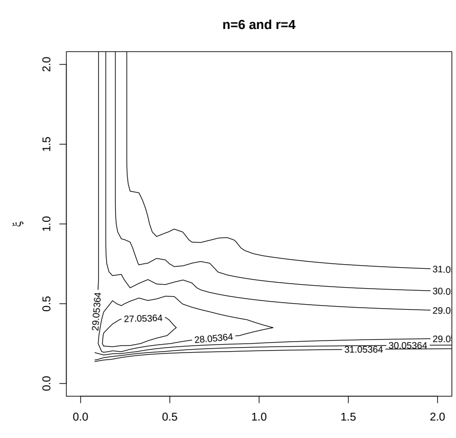
(c) 4.1c



(d) 4.1d



(e) 4.1e



(f) 4.1f

Figure 4.1: Contour plots of Bayes risk with $\alpha = 2.5, \beta = 0.8, \gamma = 2.5, \eta = 0.05$ and coefficients $C_s = 0.5, C_r = 30, C_T = 5, C_{ij} = 2$ and $r_s = 0.3$.

Proof. See appendix B. □

5 Numerical Results

The Bayes risk is obtained in explicit form. The Bayes risk has been minimized here to get an optimal Bayesian sampling plan. For finding an optimal Bayesian sampling plan, note that $r \geq 1$, $n \geq 1$, $T_L < T \leq T_U$ such that $P(X \leq T_L) = \nu/2 = P(X \geq T_U)$ for a given ν and $\xi > 0$. The following algorithm has been used to find the (n_0, r_0, T_0, ξ_0).

Algorithm:

Step-1: Fixing n and r $(1 \leq r \leq n)$, minimize $r(n, r, T, \xi)$ with respect to T and ξ.

Step-2: For the same n, repeat the above step for all $1 \leq r \leq n$ and minimize the Bayes risk over $1 \leq r \leq n$.

Step-3: Move to $n + 1$ and repeat Step-1 and Step-2.

Step-4: Then finally, by minimizing with respect to n, get the (n_0, r_0, T_0, ξ_0) such that the corresponding Bayes risk $r(n_0, r_0, T_0, \xi_0)$ is the smallest Bayes risk over all (n, r, T, ξ).

In the previous section, Theorem 4.1 shows that the optimal Bayesian sampling plan can be found in a finite number of searching steps. Some results have been obtained in Table 5.1 on optimal Bayesian sampling plans assuming $\nu = 0.05$ for some sets of hyper parameters with coefficients $C_s = 0.5$, $C_r = 30$, $C_T = 5$, $C_{ij} = 2$ and $r_s = 0.3$. In Table 5.1, only one hyperparameter is changed at a time keeping other hyperparameters fixed. The optimal Bayes risk decreases as the hyperparameter β increases when other hyperparameters are fixed and similarly, the optimal Bayes risk increases with respect to α, γ and η. Some more optimal Bayesian sampling plans have been shown varying C_s in Table 5.2. If one increases the inspection cost per unit C_s the optimal Bayes risk increases and at the same time optimal sample size n_0 decreases in order to make the Bayes risk minimum. In order to obtain numerical results, we have used grid search method.

Table 5.1: Optimal Bayesian sampling plan for various combinations of hyper parameters keeping the coefficients fixed.

α	β	γ	η	n_0	r_0	T_0	ξ_0	Bayes risk
2.5	0.6	2.5	0.001	3	2	0.2885	0.3000	27.8209
2.5	0.8	2.5	0.001	3	2	0.2236	0.2000	23.4082
2.5	1.0	2.5	0.001	3	2	0.0779	0.0500	17.0718
2.5	1.2	2.5	0.001	3	3	0.0934	0.0500	9.2588
2.5	0.6	2.5	0.050	3	2	0.3502	0.3500	28.9992
2.5	0.8	2.5	0.050	3	2	0.3268	0.3000	25.4026
2.5	1.0	2.5	0.050	3	2	0.1591	0.1500	19.4192
2.5	1.2	2.5	0.050	3	3	0.1124	0.1000	11.7658
2.5	1.2	2.5	0.050	3	3	0.1124	0.1000	11.7658
2.5	1.2	3.0	0.050	3	3	0.1117	0.1000	12.2938
2.5	1.2	3.5	0.050	3	2	0.1109	0.1000	12.8456
2.5	1.2	4.0	0.050	3	2	0.1102	0.1000	13.4158
2.5	0.8	2.5	0.001	3	2	0.2236	0.2000	23.4082
2.5	0.8	2.5	0.050	3	2	0.3268	0.3000	25.4026
2.5	0.8	2.5	0.080	3	2	0.3667	0.3500	26.5350
2.5	0.8	2.5	0.100	3	2	0.3594	0.3500	27.2385
1.8	0.8	4.0	0.050	3	2	0.2223	0.2000	18.3153
2.2	0.8	4.0	0.050	3	2	0.2726	0.2500	23.6387
2.7	0.8	4.0	0.050	3	3	0.3518	0.3500	27.6017
3.1	0.8	4.0	0.050	3	2	0.4059	0.4000	29.6131

Table 5.2: Optimal Bayesian sampling plans varying C_s for hyper parameters $\alpha = 2.5$, $\beta = 0.8, \gamma = 2.5$, and $\eta = 0.05$ keeping other coefficients $C_r = 30, C_T = 5$, $r_s = 0.3$ and $C_{ij} = 2$ fixed.

C_s	n_0	r_0	T_0	ξ_0	Bayes risk
0.300	5	3	0.2774	0.3000	24.6217
0.325	5	3	0.2774	0.3000	24.7467
0.350	5	3	0.2774	0.3000	24.8717
0.375	5	3	0.2774	0.3000	24.9967
0.400	3	2	0.3268	0.3000	25.1026
0.450	3	2	0.3268	0.3000	25.2526
0.500	3	2	0.3268	0.3000	25.4026
0.600	3	2	0.3268	0.3000	25.7026
1.000	3	2	0.3268	0.3000	26.9026
2.000	1	1	0.6727	0.4000	28.8057

Table 5.3: Bayes risk and optimal Bayesian sampling plan varying hyper parameters and keeping coefficients $C_s = 0.5, C_r = 30, C_T = 0, C_{ij} = 2$ and $r_s = 0$ fixed.

Plan	α	β	γ	η	n_0	r_0	T_0	ξ_0	Bayes risk
Plan A	2.5	0.4	2.5	0.001	1	1	1.31883	0.75000	29.62464
Plan B	2.5	0.4			2	2	1.12070	0.56030	29.81190
Plan A	2.5	0.6	2.5	0.001	3	3	0.81210	0.40000	27.43072
Plan B	2.5	0.6			3	3	0.85370	0.42680	27.81930
Plan A	3.5	0.8	3.5	0.001	3	3	0.98714	0.45000	29.24757
Plan B	3.5	0.8			2	2	1.00370	0.50190	29.36420

If $E(\mu)$ and $Var(\mu)$ are close to zero, the role of the location parameter can be ignored and the optimal Bayesian sampling plan becomes the optimal Bayesian sampling plan of one parameter exponential distribution. Therefore, the optimal Bayesian sampling plan obtained in this case should reduce to the optimal Bayesian sampling plan of Lin et al. (2010). Let us denote the proposed optimal Bayesian sampling plan and optimal Bayesian sampling plan of Lin et al. (2010) by plan A and plan B, respectively. Since the optimal Bayesian sampling plans obtained in this paper are based on a different prior it is clear that plan A and plan B are not exactly same. But it is observed that their optimal Bayes risks are quite close as it is observed in Table 5.3.

6 Conclusion

In this paper, an optimal Bayesian sampling plan is obtained for the two parameter exponential distribution for Type-I hybrid censoring scheme. A suitable four parameter conjugate prior distribution for the location and scale parameters has been considered. Considering a decision function based on the maximum likelihood estimator of the mean lifetime and based on a suitable loss function, the Bayes risk expression has been derived. An algorithm has been provided to find the optimal sampling plan. It has been shown that the algorithm stops in a finite number of steps. Although, in this paper we have mainly addressed the

Type-I hybrid censoring scheme, similar work for the two parameter exponential distribution under different censoring schemes can also be carried out. The work is in progress, it will be reported elsewhere.

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Declarations

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Appendices

A Proof of Theorem 3.2

$$
\begin{aligned}
r(n,r,T,\xi) =& E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}\Big[\ell(\delta(\underline{X}),\lambda,\mu,n,r,T)\Big] \\
=& E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}\Big[nC_s+\tau C_T-(n-D)r_s+\delta(\underline{X})\Big(\sum_{0\le i+j\le 2}C_{ij}\lambda^i\mu^j\Big)+(1-\delta(\underline{X}))C_r\Big] \\
=& n(C_s-r_s)+C_T E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)+r_s E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu)+C_r \\
&+E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}\Big[\delta(\underline{X})\Big(\sum_{0\le i+j\le 2}C_{ij}\lambda^i\mu^j-C_r\Big)\Big]
\end{aligned}
\tag{8}
$$

Equation (8) further can be simplified as follows

$$
\begin{aligned}
r(n,r,T,\xi) =& n(C_s-r_s)+C_r+C_T E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)+r_s E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu) \\
&+E_{\lambda,\mu}\Big[\Big(\sum_{0\le i+j\le 2}C_{ij}\lambda^i\mu^j-C_r\Big)\Big(P(D=0)+P(\widehat{W}\ge\xi|\lambda,\mu)\Big)\Big] \\
=& n(C_s-r_s)+C_r+C_T E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)+r_s E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu) \\
&+(C_{00}-C_r)E_{\lambda,\mu}\big(e^{-n\lambda(T-\mu)}\big)+\sum_{1\le i+j\le 2}C_{ij}E_{\lambda,\mu}\big(\lambda^i\mu^j e^{-n\lambda(T-\mu)}\big)+\sum_{0\le i+j\le 2}C'_{ij}I_{ij},
\end{aligned}
$$

where $I_{ij}=E_{\lambda,\mu}\Big(\lambda^i\mu^jP(\widehat{W}\ge\xi|\lambda,\mu)\Big)$. Further $E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)$, $E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu)$ and

$\sum_{0\le i+j\le 2} C_{ij} E_{\lambda,\mu}\big(\lambda^i \mu^j e^{-n\lambda(T-\mu)}\big)$ and I_{ij} have been obtained. In the remaining proof, it is to be noted that the Bayes risk in explicit forms has been provided for $\alpha > 1$.

Computation of I_{ij}:

Case (i): $r = 1$ *and* $n \ge 1$

Suppose notations η_1, ξ_μ^* and ξ_T^* are defined as $\eta_1 = min\{\eta, T\}$, $\xi_\mu^* = max\{\xi, \mu\}$ and $\xi_T^* = max\{\xi, T\}$, respectively, then

$$
\begin{aligned}
I_{ij} &= \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} f_{\widehat{W}}(x)\; dx d\lambda d\mu \\
&= \frac{1}{A}\sum_{l=0}^{\infty}\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{nl}\big\{g(x-\mu;1,n\lambda) - q^n g(x-T;1,n\lambda)\big\}\; dx d\lambda d\mu \\
&= \frac{1}{A}\sum_{l=0}^{\infty}\Bigg\{\int_0^{\eta_1}\int_{\xi_\mu^*}^{\infty}\int_0^{\infty} n\lambda^{\alpha+i-1}\mu^j e^{-\lambda\big(\beta-\gamma\mu+nl(T-\mu)\big)}\big(n\lambda e^{-n\lambda(x-\mu)}\big)\; d\lambda dx d\mu \\
&\quad - \int_0^{\eta_1}\int_{\xi_T^*}^{\infty}\int_0^{\infty} n\lambda^{\alpha+i-1}\mu^j e^{-\lambda\big(\beta-\gamma\mu+n(l+1)(T-\mu)\big)}\big(n\lambda e^{-n\lambda(x-T)}\big)\; d\lambda dx d\mu\Bigg\} \\
&= \frac{1}{A}\sum_{l=0}^{\infty}\Bigg\{\int_0^{\eta_1}\int_{\xi_\mu^*}^{\infty} \frac{n\Gamma(\alpha+i+1)\mu^j}{\big(\beta-\gamma\mu+nl(T-\mu)+n(x-\mu)\big)^{\alpha+i+1}} dx d\mu \\
&\quad - \int_0^{\eta_1}\int_{\xi_T^*}^{\infty} \frac{n\Gamma(\alpha+i+1)\mu^j}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)+n(x-T)\big)^{\alpha+i+1}} dx d\mu\Bigg\} \\
&= \frac{\Gamma(\alpha+i)}{A}\sum_{l=0}^{\infty}\Bigg\{\int_0^{\eta_1} \frac{\mu^j}{\big(\beta-\gamma\mu+nl(T-\mu)+n(\xi_\mu^*-\mu)\big)^{\alpha+i}}\; d\mu \\
&\quad - \int_0^{\eta_1} \frac{\mu^j}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)+n(\xi_T^*-T)\big)^{\alpha+i}}\; d\mu\Bigg\}.
\end{aligned}
$$

Case (*ii*): $r = 2$ *and* $n = 2$

In this case, I_{ij} can be expressed as

$$
\begin{aligned}
I_{ij} =& \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} P_{\underline{X}|\lambda,\mu}(\widehat{W} \geq \xi|\lambda,\mu)\ d\lambda d\mu \\
=& \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} \Bigg[I_{\{\xi<\mu\}} + I_{\{\mu\leq\xi\leq T\}} \Bigg\{ \sum_{l=0}^{\infty} q^{2l} \Big[2q(1-q)S_1(\xi) \\
& + S_2(\xi-\mu;2,2\lambda) + q^2 S_2(\xi-T;2,2\lambda) - 2qS_2(\xi - \frac{\mu+T}{2};2,2\lambda) \Big] \Bigg\} \Bigg]\ d\lambda d\mu \\
=& \frac{1}{A}\Bigg\{ M_0 + \sum_{l=0}^{\infty} 2(M_{l1} - M_{l4}) + M_{l2} + M_{l3} \Bigg\},
\end{aligned}
$$

where

$$
M_0 = \int_{\xi}^{\eta_1}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)}\ d\lambda d\mu = \Gamma(\alpha+i)\int_{\xi}^{\eta_1} \frac{\mu^j}{(\beta-\gamma\mu)^{\alpha+i}}\ d\mu,
$$

and also, with another notation $\eta_2 = min\{\xi, \eta_1\}$,

$$
\begin{aligned}
M_{l1} =& \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{2l+1}(1-q)S_1(\xi)\ d\lambda d\mu \\
=& \begin{cases} \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda\left(\beta-\gamma\mu+(2l+1)(T-\mu)\right)} \left(1-e^{-\lambda(T-\mu)}\right)\ d\lambda d\mu, & \text{if } \xi < T \\ 0, & \text{if } \xi \geq T \end{cases} \\
=& \begin{cases} \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda\left(\beta-\gamma\mu+(2l+1)(T-\mu)\right)}\ d\lambda d\mu & \\ -\int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda\left(\beta-\gamma\mu+2(l+1)(T-\mu)\right)}\ d\lambda d\mu, & \text{if } \xi < T \\ 0, & \text{if } \xi \geq T \end{cases} \\
=& \begin{cases} \Gamma(\alpha+i)\int_0^{\eta_2} \mu^j \left\{ \frac{1}{\left(\beta-\gamma\mu+(2l+1)(T-\mu)\right)^{\alpha+i}} - \frac{1}{\left(\beta-\gamma\mu+2(l+1)(T-\mu)\right)^{\alpha+i}} \right\}\ d\mu, & \text{if } \xi < T \\ 0, & \text{if } \xi \geq T \end{cases}
\end{aligned}
$$

as $\alpha+i>0$, $\beta-\gamma\mu+(2l+1)(T-\mu)>0$ and $\beta-\gamma\mu+2(l+1)(T-\mu)>0$,

$$
\begin{aligned}
M_{l2} =& \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{2l} S_2(\xi-\mu,2,2\lambda)\ d\lambda d\mu \\
=& \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu+2l(T-\mu))} \left\{ \frac{(2\lambda)^2}{\Gamma(2)} \int_{\xi-\mu}^{\infty} u e^{-2\lambda u}\ du \right\} d\lambda d\mu \\
=& 4\int_0^{\eta_2}\int_0^{\infty}\int_{\xi-\mu}^{\infty} \lambda^{\alpha+i+1}\mu^j u e^{-\lambda(\beta-\gamma\mu+2l(T-\mu)+2u)}\ du d\lambda d\mu.
\end{aligned}
$$

Note that $\beta-\gamma\mu+2l(T-\mu)+2u$ is strictly positive, therefore

$$
M_{l2} = 4\int_0^{\eta_2}\int_{\xi-\mu}^{\infty} \frac{\Gamma(\alpha+i+2)\mu^j u}{(\beta-\gamma\mu+2l(T-\mu)+2u)^{\alpha+i+2}}\ du d\mu.
$$

Let $x=2u/(\beta-\gamma\mu+2l(T-\mu))$ and also define $\xi_1^*(\mu)=2(\xi-\mu)/(\beta-\gamma\mu+2l(T-\mu))$, then

$$
\begin{aligned}
M_{l2} =& \Gamma(\alpha+i)\left\{ \frac{1}{B(2,\alpha+i)} \int_0^{\eta_2}\int_{\xi_1^*(\mu)}^{\infty} \frac{\mu^j}{(\beta-\gamma\mu+2l(T-\mu))^{\alpha+i}} \frac{x^{2-1}}{(1+x)^{\alpha+i+2}}\ dx d\mu \right\} \\
=& \Gamma(\alpha+i) D_1(2,\alpha+i,2l,\eta_2),
\end{aligned}
$$

where $D_s(a,b,y,x)=\int\limits_0^x \frac{\mu^j\left(1-I_{\xi_s^*(\mu)/(1+\xi_s^*(\mu))}(a,b)\right)}{(\beta-\gamma\mu+y(T-\mu))^{\alpha+i}}\ d\mu$ and $I_t(a,b)$ denotes the cdf of probability distribution Beta(a,b) at t. Now, thus and so

$$
\begin{aligned}
M_{l3} =& \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{2(l+1)} S_2(\xi-T,2,2\lambda)\ d\lambda d\mu \\
=& \begin{cases} \Gamma(\alpha+i)\int\limits_0^{\eta_2} \frac{\mu^j}{\left(\beta-\gamma\mu+2(l+1)(T-\mu)\right)^{\alpha+i}}\ d\mu, & \text{if } \xi<T \\ \Gamma(\alpha+i) D_2(2,\alpha+i,2(l+1),\eta_2), & \text{if } \xi\geq T \end{cases}
\end{aligned}
$$

and

$$M_{l4} = \int_0^{\eta_2}\int_0^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{2l+1} S_2(\xi - \frac{\mu+T}{2}, 2, 2\lambda)d\ d\lambda d\mu$$

$$=\Gamma(\alpha+i)\Bigg\{ \int\limits_{min\{\eta_2, max\{0,2\xi-T\}\}}^{\eta_2} \frac{\mu^j}{(\beta-\gamma\mu+(2l+1)(T-\mu))^{\alpha+i}}\ d\mu$$

$$+ D_3(2, \alpha+i, 2l+1, max\{0, min\{(2\xi-T), \eta_2\}\})\Bigg\}$$

with $\xi_2^*(\mu) = \frac{2(\xi-T)}{(\beta-\gamma\mu+2(l+1)(T-\mu))}$ and $\xi_3^*(\mu) = \frac{2(\xi-(\mu+T)/2)}{(\beta-\gamma\mu+(2l+1)(T-\mu))}$, respectively.

Case *(iii)*: $r \geq 2$ *and* $n \geq 3$

$$I_{ij} = \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} f_{\widehat{W}}(x)\ dx d\lambda d\mu$$

$$= \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} \Bigg\{ \sum_{l=0}^{\infty} q^{nl}\Big[c_{10} g(-x+\mu+\mu_{10}; 1, \frac{\lambda}{n-2})$$

$$- e_{10} g(-x+T; 1, \frac{\lambda}{n-2}) + I_{\{r\geq 3\}} \sum_{d=2}^{r-1}\sum_{k=0}^{d-1} \Big\{ c_{dk} g_3(x-\mu-\mu_{dk}; d-1; \lambda d, \lambda_{dk})$$

$$- e_{dk} g_3(x-T; d-1, \lambda d, \lambda_{dk})\Big\} + g_1(x-\mu; r-1, \lambda r, \lambda n) - q^n g_1(x-T; r-1, \lambda r, \lambda n)$$

$$+ \sum_{k=0}^{r-2} \Big\{ c_k g_3(x-\mu-\mu_k; r-1, \lambda r, \lambda_k) - e_k g_3(x-T; r-1, \lambda r, \lambda_k)\Big\}\Big]\Bigg\}\ dx d\lambda d\mu.$$

Define $a_{dk} = \frac{d-k}{2d-k-n}$ for $d = 2, 3, , \ldots, r-1,\ k = 0, 1, \ldots, d-1 \ni \lambda_{dk} = \lambda d a_{dk}$ and $a_k = \frac{r-k-1}{2r-k-n-1}$ for $k = 0, 1, \ldots, r-1 \ni \lambda_k = \lambda r a_k$. Note that $\lambda_{dk} > (<) 0 \iff 2d-k-n > (<) 0$, $\lambda_{dk} = \infty \iff 2d-k-n = 0$ and λ_k also behaves similarly. Therefore, define some sets as follows:

$A_0 = \{(d,k) \mid d = 2, 3, \ldots, r-1, k = 0, 1, \ldots, d-1\}$,

$A_1 = \{(d,k) \mid (d,k) \in A_0 \text{ and } 2d-k-n > 0\}$,

$A_2 = \{(d,k) \mid (d,k) \in A_0 \text{ and } 2d - k - n < 0\}$,

$A_3 = \{(d,k) \mid (d,k) \in A_0 \text{ and } 2d - k - n = 0\}$,

$B_0 = \{k \mid k = 0, 1, \ldots, r-2\}$,

$B_1 = \{k \mid k \in B_0 \text{ and } 2r - k - n - 1 > 0\}$,

$B_2 = \{k \mid k \in B_0 \text{ and } 2r - k - n - 1 < 0\}$,

$B_3 = \{k \mid k \in B_0 \text{ and } 2r - k - n - 1 = 0\}$.

Now replace q, c_{dk}, e_{dk}, c_k, e_k and g_3 by their respective expressions to have

$$
\begin{aligned}
I_{ij} =& \frac{1}{A}\sum_{l=0}^{\infty}\Bigg[n(K_{l1} - K_{l2}) + I_{\{r\geq 3\}}\Bigg\{\sum\sum_{A_1}(-1)^k\binom{n}{d}\binom{d}{k}(K_{l3} - K_{l4}) \\
&+ \sum\sum_{A_2}(-1)^k\binom{n}{d}\binom{d}{k}(K_{l5} - K_{l6}) + \sum\sum_{A_3}(-1)^k\binom{n}{d}\binom{d}{k}(K_{l7} - K_{l8})\Bigg\} \\
&+ (K_{l9} - K_{l10}) + \sum_{B_1}(-1)^{k+1}r\binom{n}{r}\binom{r-1}{k}\frac{1}{n-r+k+1}(K_{l11} - K_{l12}) \\
&+ \sum_{B_2}(-1)^{k+1}r\binom{n}{r}\binom{r-1}{k}\frac{1}{n-r+k+1}(K_{l13} - K_{l14}) \\
&+ \sum_{B_3}(-1)^{k+1}r\binom{n}{r}\binom{r-1}{k}\frac{1}{n-r+k+1}(K_{l15} - K_{l16})\Bigg],
\end{aligned}
$$

where $K_{l1}, K_{l2}, \ldots, K_{l16}$ denotes some expressions involving triple integrals. Moreover these integrals can be obtained along the same line as M_{l1} and M_{l2} have been obtained, so their proofs have been omitted here.

$$
\begin{aligned}
K_{l1} =& \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)-1} g(-x+\mu+\mu_{10}; 1, \lambda/(n-2)) \ dx d\lambda d\mu \\
=& \Gamma(\alpha+i) \int_0^{\eta_1} \mu^j \Bigg\{ \frac{1}{\big(\beta-\gamma\mu+(n(l+1)-1)(T-\mu)\big)^{\alpha+i}} \\
& - \frac{1}{\big(\beta-\gamma\mu+(n(l+1)-1)(T-\mu)-\frac{(min\{\xi_\mu^*, \mu+\mu_{10}\}-\mu-\mu_{10})}{n-2}\big)^{\alpha+i}} \Bigg\} \ d\mu, \\
K_{l2} =& \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)} g(-x+T; 1, \lambda/(n-2)) \ dx d\lambda d\mu \\
=& \Gamma(\alpha+i) \int_0^{\eta_1} \mu^j \Bigg\{ \frac{1}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)\big)^{\alpha+i}} \\
& - \frac{1}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)-\frac{(min\{\xi_\mu^*, T\}-T)}{n-2}\big)^{\alpha+i}} \Bigg\} \ d\mu, \\
K_{l3} =& \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)-d+k} g_1(x-\mu-\mu_{dk}; d-1, \lambda d, \lambda_{dk}) \ dx d\lambda d\mu \\
=& \Gamma(\alpha+i) \Bigg\{ \sum_{s=0}^{d-2} R_{dks} D_4(d-s-1, \alpha+i, n(l+1)-d+k, \eta_1) + \frac{1}{(1-a_{dk})^{d-1}} \\
& \times \int_0^{\eta_1} \frac{\mu^j}{\big(\beta-\gamma\mu+(n(l+1)-d+k)(T-\mu)+da_{dk}(max\{\xi, \mu+\mu_{dk}\}-\mu-\mu_{dk})\big)^{\alpha+i}} d\mu \Bigg\}
\end{aligned}
$$

where $R_{dks} = (-1)^s \frac{a_{dk}}{(a_{dk}-1)^{s+1}}$ and $\xi_4^*(\mu) = \frac{d(max\{\xi, \mu+\mu_{dk}\}-\mu-\mu_{dk})}{\beta-\gamma\mu+(n(l+1)-d+k)(T-\mu)}$,

$$
\begin{aligned}
K_{l4} =& \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)} g_1(x-T; d-1, \lambda d, \lambda_{dk}) \ dx d\lambda d\mu \\
=& \Gamma(\alpha+i) \Bigg\{ \sum_{s=0}^{d-2} R_{dks} D_5(d-s-1, \alpha+i, n(l+1), \eta_1) \\
& + \frac{1}{(1-a_{dk})^{d-1}} \int_0^{\eta_1} \frac{\mu^j}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)+da_{dk}(\xi_T^*-T)\big)^{\alpha+i}} d\mu \Bigg\}
\end{aligned}
$$

with $\xi_5^*(\mu) = \frac{d(\xi_T^* - T)}{(\beta - \gamma\mu + n(l+1)(T-\mu))}$,

$$K_{l5} = \int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)-d+k} g_2(x-\mu-\mu_{dk}; d-1, \lambda d, -\lambda_{dk})\ dxd\lambda d\mu$$

$$=\Gamma(\alpha+i)\Bigg[\sum_{s=0}^{d-2} R_{dks} D_4(d-s-1, \alpha+i, n(l+1)-d+k, \eta_1) + \frac{1}{(1-a_{dk})^{d-1}}$$

$$\times \int_0^{\eta_1} \mu^j \Bigg\{ \frac{1}{\big(\beta-\gamma\mu+(n(l+1)-d+k)(T-\mu)\big)^{\alpha+i}}$$

$$- \frac{1}{\big(\beta-\gamma\mu+(n(l+1)-d+k)(T-\mu)+da_{dk}(min\{\xi_\mu^*, \mu+\mu_{dk}\}-\mu-\mu_{dk})\big)^{\alpha+i}}\Bigg\} d\mu\Bigg],$$

$$K_{l6} = \int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)} g_2(x-T; d-1, \lambda d, -\lambda_{dk})\ dxd\lambda d\mu$$

$$=\Gamma(\alpha+i)\Bigg[\sum_{s=0}^{d-2} R_{dks} D_5(d-s-1, \alpha+i, n(l+1), \eta_1) + \frac{1}{(1-a_{dk})^{d-1}}$$

$$\times \int_0^{\eta_1} \mu^j \Bigg\{ \frac{1}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)\big)^{\alpha+i}}$$

$$- \frac{1}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)+da_{dk}(min\{\xi_\mu^*, T\}-T)\big)^{\alpha+i}}\Bigg\} d\mu\Bigg],$$

$$K_{l7} = \int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)-d+k} g(x-\mu-\mu_{dk}; d-1, \lambda d)\ dxd\lambda d\mu$$

$$=\Gamma(\alpha+i) D_4(d-1, \alpha+i, n(l+1)-d+k, \eta_1),$$

$$K_{l8} = \int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)} g(x-T; d-1, \lambda d)\ dxd\lambda d\mu$$

$$=\Gamma(\alpha+i) D_5(d-1, \alpha+i, n(l+1), \eta_1),$$

$$K_{l9} = \int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)} q^{nl} g_1(x-\mu; r-1, \lambda r, \lambda n)\ dxd\lambda d\mu$$

$$= \begin{cases} \Gamma(\alpha+i)\Big\{ \sum_{s=0}^{r-2} R_s D_6(r-s-1, \alpha+i, nl, \eta_1) \\ +\frac{1}{(1-\frac{n}{r})^{r-1}} \int_0^{\eta_1} \frac{\mu^j}{\big(\beta-\gamma\mu+nl(T-\mu)+n(\xi_\mu^*-\mu)\big)^{\alpha+i}} d\mu\Big\}, & \text{if } r<n \\ \Gamma(\alpha+i) D_6(r, \alpha+i, nl, \eta_1), & \text{if } r=n. \end{cases}$$

with $\xi_6^*(\mu) = \frac{r(\xi_\mu^* - \mu)}{(\beta - \gamma\mu + nl(T-\mu))}$ and $R_s = (-1)^s \frac{n/r}{((n/r)-1)^{s+1}}$,

$$K_{l10} = \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)} g_1(x-T; r-1, \lambda r, \lambda n)\ dx d\lambda d\mu$$

$$= \begin{cases} \Gamma(\alpha+i)\Big\{ \sum\limits_{s=0}^{r-2} R_s D_7(r-s-1, \alpha+i, n(l+1), \eta_1) \\ + \frac{1}{(1-\frac{n}{r})^{r-1}} \int\limits_0^{\eta_1} \frac{\mu^j}{\Big(\beta-\gamma\mu+n(l+1)(T-\mu)+n(\xi_T^*-T)\Big)^{\alpha+i}} d\mu \Big\}, & \text{if } r < n \\ \Gamma(\alpha+i) D_7(r, \alpha+i, n(l+1), \eta_1), & \text{if } r = n \end{cases}$$

with $\xi_7^*(\mu) = \frac{r(\xi_T^* - T)}{(\beta - \gamma\mu + n(l+1)(T-\mu))}$,

$$K_{l11} = \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)-r+k+1} g_1(x-\mu-\mu_k; r-1, \lambda r, \lambda_k)\ dx d\lambda d\mu$$

$$= \begin{cases} \Gamma(\alpha+i)\Big\{ \sum\limits_{s=0}^{r-2} R_{ks} D_8(r-s-1, \alpha+i, n(l+1)-r+k+1, \eta_1) \\ + \frac{1}{(1-a_k)^{r-1}} \int\limits_0^{\eta_1} \frac{\mu^j}{\Big(\beta-\gamma\mu+(n(l+1)-r+k+1)(T-\mu)+ra_k(max\{\xi,\mu+\mu_k\}-\mu-\mu_k)\Big)^{\alpha+i}} d\mu \Big\}, & \text{if } r < n \\ \Gamma(\alpha+i) D_8(r, \alpha+i, n(l+1)-r+k+1, \eta_1), & \text{if } r = n \end{cases}$$

with $\xi_8^*(\mu) = \frac{r(max\{\xi,\mu+\mu_k\}-\mu-\mu_k)}{(\beta - \gamma\mu + (n(l+1)-r+k+1)(T-\mu))}$,

$$K_{l12} = \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)} g_1(x-T; r-1, \lambda r, \lambda_k)\ dx d\lambda d\mu$$

$$= \begin{cases} \Gamma(\alpha+i)\Big\{ \sum\limits_{s=0}^{r-2} R_{ks} D_9(r-s-1, \alpha+i, n(l+1), \eta_1) \\ + \frac{1}{(1-a_k)^{r-1}} \int\limits_0^{\eta_1} \frac{\mu^j}{\Big(\beta-\gamma\mu+n(l+1)(T-\mu)+ra_k(\xi_T^*-T)\Big)^{\alpha+i}} d\mu \Big\}, & \text{if } r < n \\ \Gamma(\alpha+i) D_9(r, \alpha+i, n(l+1), \eta_1), & \text{if } r = n \end{cases}$$

with $\xi_9^*(\mu) = \frac{r(\xi_T^* - T)}{(\beta - \gamma\mu + n(l+1)(T-\mu))}$,

$$K_{l13} = \int_0^{\eta_1} \int_0^{\infty} \int_{\xi_\mu^*}^{\infty} \lambda^{\alpha+i-1} \mu^j e^{-\lambda(\beta-\gamma\mu)} q^{n(l+1)-r+k+1} g_2(x-\mu-\mu_k; r-1, \lambda r, -\lambda_k)\ dx d\lambda d\mu$$

$$
\begin{aligned}
=&\Gamma(\alpha+i)\Bigg[\sum_{s=0}^{r-2} R_{ks} D_8(r-s-1,\alpha+i,n(l+1)-r+k+1,\eta_1)+\frac{1}{(1-a_k)^{r-1}}\\
&\times\int_0^{\eta_1}\mu^j\Big\{\big(\beta-\gamma\mu+(n(l+1)-r+k+1)(T-\mu)\big)^{-\alpha-i}\\
&-\big(\beta-\gamma\mu+(n(l+1)-r+k+1)(T-\mu)+ra_k(min\{\xi_\mu^*,\mu+\mu_k\}-\mu-\mu_k)\big)^{-\alpha-i}\Big\}d\mu\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
K_{l14}=&\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty}\lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)}q^{n(l+1)}g_2(x-T;r-1,\lambda r,-\lambda_k)\ dxd\lambda d\mu\\
=&\Gamma(\alpha+i)\Bigg[\sum_{s=0}^{r-2} R_{ks} D_9(r-s-1,\alpha+i,n(l+1),\eta_1)+\frac{1}{(1-a_k)^{r-1}}\\
&\times\int_0^{\eta_1}\mu^j\Bigg\{\frac{1}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)\big)^{\alpha+i}}\\
&-\frac{1}{\big(\beta-\gamma\mu+n(l+1)(T-\mu)+ra_k(min\{\xi_\mu^*,T\}-T)\big)^{\alpha+i}}\Bigg\}d\mu\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
K_{l15}=&\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty}\lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)}q^{n(l+1)-r+k+1}g(x-\mu-\mu_k;r-1,\lambda r)\ dxd\lambda d\mu\\
=&\Gamma(\alpha+i)D_8(r-1,\alpha+i,n(l+1)-r+k+1,\eta_1)\text{ and}\\
K_{l16}=&\int_0^{\eta_1}\int_0^{\infty}\int_{\xi_\mu^*}^{\infty}\lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta-\gamma\mu)}q^{n(l+1)}g(x-T;r-1,\lambda r)\ dxd\lambda d\mu\\
=&\Gamma(\alpha+i)D_9(r-1,\alpha+i,n(l+1),\eta_1).
\end{aligned}
$$

Computation of $E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)$:

$$
\begin{aligned}
&E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)\\
&=\int_\mu\int_\lambda E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)p(\lambda,\mu)d\lambda d\mu\\
&=\frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu)\lambda^{\alpha-1}e^{-\lambda(\beta-\gamma\mu)}d\lambda d\mu\\
&=\frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}E_{\underline{X}|\lambda,\mu}\big[min\{X_{(r)},T\}|\lambda,\mu\big]\lambda^{\alpha-1}e^{-\lambda(\beta-\gamma\mu)}d\lambda d\mu\\
&=\frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}\Big\{\int_\mu^T yf_{X_{(r)}}(y)dy+TP_{\underline{X}|\lambda,\mu}\big[0\le D\le r-1|\lambda,\mu\big]\Big\}\lambda^{\alpha-1}e^{-\lambda(\beta-\gamma\mu)}d\lambda d\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{A}\sum_{l=0}^{r-1}(-1)^{r-1-l}(n-r+1)\binom{n}{r-1}\binom{r-1}{l} \\
&\times \Bigg[\frac{1}{n-l}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}\mu e^{-\lambda(\beta-\gamma\mu)}d\lambda d\mu - \frac{T}{n-l}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}e^{-\lambda(\beta+(n-l)T-(\gamma+n-l)\mu)}d\lambda d\mu \\
&- \frac{1}{(n-l)^2}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-2}e^{-\lambda(\beta+(n-l)T-(\gamma+n-l)\mu)}d\lambda d\mu + \frac{1}{(n-l)^2}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-2}e^{-\lambda(\beta-\gamma\mu)}d\lambda d\mu\Bigg] \\
&+ \frac{T}{A}\sum_{d=0}^{r-1}\sum_{l=0}^{d}(-1)^{d-l}\binom{n}{d}\binom{d}{l}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}e^{-\lambda(\beta+(n-l)T-(\gamma+n-l)\mu)}d\lambda d\mu
\end{aligned}
$$

Note that $\beta+(n-l)T-(\gamma+(n-l))\mu > 0$. Hence all the integrals except $\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-2}exp(-\lambda(\beta +(n-l)T-(\gamma+n-l)\mu))d\lambda d\mu$ and $\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-2}exp(-\lambda(\beta-\gamma\mu))d\lambda d\mu$ exist for the entire range of hyper parameter α. These two integrals do not exist when $0<\alpha\leq 1$, so this range is excluded and the Bayes risk has been provided only when $\alpha>1$. Hence

$$
\begin{aligned}
E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu) =& \frac{1}{A}\sum_{l=0}^{r-1}(-1)^{r-1-l}(n-r+1)\binom{n}{r-1}\binom{r-1}{l}\Bigg[\frac{1}{n-l}\int_0^{\eta_1}\Gamma(\alpha)\frac{\mu}{(\beta-\gamma\mu)^{\alpha}}d\mu \\
&- \frac{T}{n-l}\int_0^{\eta_1}\frac{\Gamma(\alpha)}{(\beta+(n-l)T-(\gamma+n-l)\mu)^{\alpha}}d\mu \\
&- \frac{1}{(n-l)^2}\int_0^{\eta_1}\frac{\Gamma(\alpha-1)}{(\beta+(n-l)T-(\gamma+n-l)\mu)^{\alpha-1}}d\mu \\
&+ \frac{1}{(n-l)^2}\int_0^{\eta_1}\frac{\Gamma(\alpha-1)}{(\beta-\gamma\mu)^{\alpha-1}}d\mu\Bigg] \\
&+ \frac{T}{A}\sum_{d=0}^{r-1}\sum_{l=0}^{d}(-1)^{d-l}\binom{n}{d}\binom{d}{l}\int_0^{\eta_1}\frac{\Gamma(\alpha)}{(\beta+(n-l)T-(\gamma+n-l)\mu)^{\alpha}}d\mu.
\end{aligned} \tag{9}
$$

Now there are two cases:

Case (1): When $\alpha\neq 2$, equation (9) is given by

$$
\begin{aligned}
E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu) = & \frac{1}{A}\sum_{l=0}^{r-1}(-1)^{r-l-1}(n-r+1)\binom{n}{r-1}\binom{r-1}{l}\Bigg\{\frac{\beta\ \Gamma(\alpha-1)}{(n-l)\gamma^2} \\
\times \Big(\frac{\beta^{\alpha-1}-(\beta-\gamma\eta_1)^{\alpha-1}}{\beta^{\alpha-1}(\beta-\gamma\eta_1)^{\alpha-1}}\Big) &- \frac{(\alpha-1)\ \Gamma(\alpha-2)}{(n-l)\gamma^2}\Big(\frac{\beta^{\alpha-2}-(\beta-\gamma\eta_1)^{\alpha-2}}{\beta^{\alpha-2}(\beta-\gamma\eta_1)^{\alpha-2}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{T\ \Gamma(\alpha-1)}{(n-l)(\gamma+n-l)}\Big(\frac{(\beta+(n-l)T)^{\alpha-1}-(\beta+(n-l)T-\eta_1(\gamma+n-l))^{\alpha-1}}{(\beta+(n-l)T)^{\alpha-1}(\beta+(n-l)T-\eta_1(\gamma+n-l))^{\alpha-1}}\Big)\\
&- \frac{\Gamma(\alpha-2)}{(n-l)^2(\gamma+n-l)}\Big(\frac{(\beta+(n-l)T)^{\alpha-2}-(\beta+(n-l)T-\eta_1(\gamma+n-l))^{\alpha-2}}{(\beta+(n-l)T)^{\alpha-2}(\beta+(n-l)T-\eta_1(\gamma+n-l))^{\alpha-2}}\Big)\\
&+ \frac{\Gamma(\alpha-2)}{(n-l)^2\gamma}\Big(\frac{\beta^{\alpha-2}-(\beta-\gamma\eta_1)^{\alpha-2}}{\beta^{\alpha-2}(\beta-\gamma\eta_1)^{\alpha-2}}\Big)\Bigg\} + \frac{T}{A}\sum_{d=0}^{r-1}\sum_{l=0}^{d}(-1)^{d-l}\binom{n}{d}\binom{d}{l}\frac{\Gamma(\alpha-1)}{(\gamma+n-l)}\\
&\times\Bigg\{\frac{(\beta+(n-l)T)^{\alpha-1}-(\beta+(n-l)T-\eta_1(\gamma+n-l))^{\alpha-1}}{(\beta+(n-l)T)^{\alpha-1}(\beta+(n-l)T-\eta_1(\gamma+n-l))^{\alpha-1}}\Bigg\}.
\end{aligned}
$$

Case (2): When $\alpha = 2$, simplifying equation (9) further the following expression is obtained:

$$
\begin{aligned}
E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu) = \frac{1}{A}\sum_{l=0}^{r-1}(-1)^{r-l-1}(n-r+1)\binom{n}{r-1}\binom{r-1}{l}\Bigg\{\frac{\eta_1}{(n-l)\gamma(\beta-\gamma\eta_1)}\\
- \frac{ln(\beta)-ln(\beta-\gamma\eta_1)}{(n-l)\gamma^2} - \frac{T}{(n-l)}\Big(\frac{\eta_1}{(\beta+(n-l)T)(\beta+(n-l)T-\eta_1(\gamma+n-l))}\Big)\\
- \frac{ln(\beta+(n-l)T)-ln(\beta+(n-l)T-\eta_1(\gamma+n-l))}{(n-l)^2(\gamma+n-l)} + \frac{ln(\beta)-ln(\beta-\gamma\eta_1)}{(n-l)^2\gamma}\Bigg\}\\
+ \frac{T}{A}\sum_{d=0}^{r-1}\sum_{l=0}^{d}(-1)^{d-l}\binom{n}{d}\binom{d}{l}\Bigg\{\frac{\eta_1}{(\beta+(n-l)T)(\beta+(n-l)T-\eta_1(\gamma+n-l))}\Bigg\}
\end{aligned}
$$

b. Computation of $E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu)$:

$$
\begin{aligned}
E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu) &= \int_0^{\eta_1}\int_0^{\infty} E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu)p(\lambda,\mu)d\lambda d\mu\\
&= \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}\Bigg\{\sum_{d=0}^{r-1} d\binom{n}{d}e^{-\lambda(n-d)(T-\mu)}\big(1-e^{-\lambda(T-\mu)}\big)^d\\
&\quad + r\sum_{k=r}^{n}\binom{n}{k}e^{-\lambda(n-k)(T-\mu)}\big(1-e^{-\lambda(T-\mu)}\big)^k\Bigg\}\lambda^{\alpha-1}e^{-\lambda(\beta-\gamma\mu)}d\lambda d\mu\\
&= \frac{1}{A}\sum_{d=0}^{r-1}\sum_{l=0}^{d}(-1)^{d-l}d\binom{n}{d}\binom{d}{l}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}e^{-\lambda\left[\beta+(n-l)T-(\gamma+n-l)\mu\right]}d\lambda d\mu\\
&\quad + \frac{r}{A}\sum_{k=r}^{n}\sum_{l=0}^{k}(-1)^{k-l}\binom{n}{k}\binom{k}{l}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}e^{-\lambda\left[\beta+(n-l)T-(\gamma+n-l)\mu\right]}d\lambda d\mu
\end{aligned}
$$

$$= \frac{1}{A}\sum_{d=0}^{n}\sum_{l=0}^{d}(-1)^{d-l}m\binom{n}{d}\binom{d}{l}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}e^{-\lambda\left[\beta+(n-l)T-(\gamma+n-l)\mu\right]}d\lambda d\mu$$

where $m = min\{d, r\}$. Hence

$$\begin{aligned}
E_{\lambda,\mu}E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu) &= \frac{1}{A}\sum_{d=0}^{n}\sum_{l=0}^{d}(-1)^{d-l}m\binom{n}{d}\binom{d}{l}\int_0^{\eta_1}\frac{\Gamma(\alpha)}{(\beta+(n-l)T-(\gamma+n-l)\mu)^{\alpha}}d\mu \\
&= \frac{1}{A}\sum_{d=0}^{n}\sum_{l=0}^{d}(-1)^{d-l}m\binom{n}{d}\binom{d}{l}\frac{\Gamma(\alpha-1)}{\gamma+n-l} \\
&\quad\times\left\{\frac{(\beta+(n-l)T)^{\alpha-1}-(\beta+(n-l)T-(\gamma+n-l)\eta_1)^{\alpha-1}}{(\beta+(n-l)T)^{\alpha-1}(\beta+(n-l)T-(\gamma+n-l)\eta_1)^{\alpha-1}}\right\}
\end{aligned}$$

c. Computation of $E_{\lambda,\mu}(e^{-n\lambda(T-\mu)})$:

$$\begin{aligned}
E_{\lambda,\mu}(e^{-n\lambda(T-\mu)}) &= \int_0^{\eta_1}\int_0^{\infty}e^{-n\lambda(T-\mu)}p(\lambda,\mu)d\lambda d\mu \\
&= \frac{1}{A}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha-1}e^{-\lambda(\beta+nT-(n+\gamma)\mu)}d\lambda d\mu \\
&= \frac{1}{A}\int_0^{\eta_1}\frac{\Gamma(\alpha)}{(\beta+nT-(n+\gamma)\mu)^{\alpha}}d\mu \\
&= \frac{1}{A}\frac{\Gamma(\alpha-1)}{n+\gamma}\left\{\frac{(\beta+nT)^{\alpha-1}-(\beta+nT-(n+\gamma)\eta_1)^{\alpha-1}}{(\beta+nT)^{\alpha-1}(\beta+nT-(n+\gamma)\eta_1)^{\alpha-1}}\right\}
\end{aligned}$$

d. Computation of $\sum_{1\le i+j\le 2}C_{ij}E_{\lambda,\mu}(\lambda^i\mu^j e^{-n\lambda(T-\mu)})$:

$$\begin{aligned}
&\sum_{1\le i+j\le 2}C_{ij}E_{\lambda,\mu}(\lambda^i\mu^j e^{-n\lambda(T-\mu)}) \\
&= \sum_{1\le i+j\le 2}C_{ij}\int_0^{\eta_1}\int_0^{\infty}\lambda^i\mu^j e^{-n\lambda(T-\mu)}p(\lambda,\mu)d\lambda d\mu \\
&= \frac{1}{A}\sum_{1\le i+j\le 2}C_{ij}\int_0^{\eta_1}\int_0^{\infty}\lambda^{\alpha+i-1}\mu^j e^{-\lambda(\beta+nT-(n+\gamma)\mu)}d\lambda d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{A}\sum_{1\le i+j\le 2} C_{ij}\int_0^{\eta_1} \frac{\Gamma(\alpha+i)}{(\beta+nT-(n+\gamma)\mu)^{\alpha+i}}\mu^j d\mu \\
&= \frac{1}{A}\sum_{1\le i+j\le 2} C_{ij}\frac{\Gamma(\alpha+i)}{(\beta+nT)^{\alpha+i}}\int_0^{\frac{(n+\gamma)\eta_1}{\beta+nT}} \left(\frac{\beta+nT}{(n+\gamma)}\right)^{j+1}\frac{u^j}{(1-u)^{\alpha+i}}du \\
&= \frac{1}{A}\sum_{1\le i+j\le 2} C_{ij}\frac{\Gamma(\alpha+i)}{(n+\gamma)^{j+1}(\beta+nT)^{\alpha+i-j-1}}\int_0^{\eta*} \frac{u^j}{(1-u)^{\alpha+i}}du \\
&= C_{01}\frac{\Gamma(\alpha)}{A(n+\gamma)^2(\beta+nT)^{\alpha-2}}\int_0^{\eta*}\frac{u}{(1-u)^{\alpha}}du + C_{10}\frac{\Gamma(\alpha+1)}{A(n+\gamma)(\beta+nT)^{\alpha}}\int_0^{\eta*}\frac{1}{(1-u)^{\alpha+1}}du \\
&\quad + C_{11}\frac{\Gamma(\alpha+1)}{A(n+\gamma)^2(\beta+nT)^{\alpha-1}}\int_0^{\eta*}\frac{u}{(1-u)^{\alpha+1}}du + C_{20}\frac{\Gamma(\alpha+2)}{A(n+\gamma)(\beta+nT)^{\alpha+1}} \\
&\quad \times \int_0^{\eta*}\frac{1}{(1-u)^{\alpha+2}}du + C_{02}\frac{\Gamma(\alpha)}{A(n+\gamma)^3(\beta+nT)^{\alpha-3}}\int_0^{\eta*}\frac{u^2}{(1-u)^{\alpha}}du,
\end{aligned}
$$

where $\eta* = \frac{(n+\gamma)\eta_1}{\beta+nT}$ and since $\int_0^{\eta*}\frac{u}{(1-u)^p}du = -\int_0^{\eta*}\frac{1}{(1-u)^{p-1}}du + \int_0^{\eta*}\frac{1}{(1-u)^p}du$ and $\int_0^{\eta*}\frac{u^2}{(1-u)^p}du = \int_0^{\eta*}\frac{1}{(1-u)^p}du + \int_0^{\eta*}\frac{1}{(1-u)^{p-2}}du - 2\int_0^{\eta*}\frac{1}{(1-u)^{p-1}}du$. Therefore,

$$
\begin{aligned}
&\sum_{1\le i+j\le 2} C_{ij}E_{\lambda,\mu}(\lambda^i\mu^j e^{-n\lambda(T-\mu)}) \\
&\quad = C_{01}\frac{\Gamma(\alpha)}{A(n+\gamma)^2(\beta+nT)^{\alpha-2}}\left\{-\int_0^{\eta*}\frac{1}{(1-u)^{\alpha-1}}du + \int_0^{\eta*}\frac{1}{(1-u)^{\alpha}}du\right\} \\
&\quad + C_{10}\frac{\Gamma(\alpha+1)}{A(n+\gamma)(\beta+nT)^{\alpha}}\int_0^{\eta*}\frac{1}{(1-u)^{\alpha+1}}du + C_{11}\frac{\Gamma(\alpha+1)}{A(n+\gamma)^2(\beta+nT)^{\alpha-1}} \\
&\quad \times \left\{-\int_0^{\eta*}\frac{1}{(1-u)^{\alpha}}du + \int_0^{\eta*}\frac{1}{(1-u)^{\alpha+1}}du\right\} + C_{20}\frac{\Gamma(\alpha+2)}{A(n+\gamma)(\beta+nT)^{\alpha+1}} \\
&\quad \times \int_0^{\eta*}\frac{1}{(1-u)^{\alpha+2}}du + C_{02}\frac{\Gamma(\alpha)}{A(n+\gamma)^3(\beta+nT)^{\alpha-3}} \times \left\{\int_0^{\eta*}\frac{1}{(1-u)^{\alpha}}du\right. \\
&\quad \left. + \int_0^{\eta*}\frac{1}{(1-u)^{\alpha-2}}du - 2\int_0^{\eta*}\frac{1}{(1-u)^{\alpha-1}}du\right\}.
\end{aligned}
$$

Case (1): Thus when $\alpha \notin \{2,3\}$, we get,

$$
\begin{aligned}
&\sum_{1\le i+j\le 2} C_{ij}E_{\lambda,\mu}(\lambda^i\mu^j e^{-n\lambda(T-\mu)}) \\
&\quad = C_{01}\frac{\Gamma(\alpha)}{A(n+\gamma)^2(\beta+nT)^{\alpha-2}}\left\{-\frac{1}{\alpha-2}\Big(\frac{1}{(1-\eta^*)^{\alpha-2}}-1\Big) + \frac{1}{\alpha-1}\Big(\frac{1}{(1-\eta^*)^{\alpha-1}}-1\Big)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_{10}\frac{\Gamma(\alpha+1)}{A(n+\gamma)(\beta+nT)^{\alpha}}\frac{1}{\alpha}\Big(\frac{1}{(1-\eta^*)^{\alpha}}-1\Big) + C_{11}\frac{\Gamma(\alpha+1)}{A(n+\gamma)^2(\beta+nT)^{\alpha-1}} \\
&\times \Bigg\{-\frac{1}{\alpha-1}\Big(\frac{1}{(1-\eta^*)^{\alpha-1}}-1\Big) + \frac{1}{\alpha}\Big(\frac{1}{(1-\eta^*)^{\alpha}}-1\Big)\Bigg\} + C_{20}\frac{\Gamma(\alpha+2)}{A(n+\gamma)(\beta+nT)^{\alpha+1}} \\
&\times \frac{1}{\alpha+1}\Big(\frac{1}{(1-\eta^*)^{\alpha+1}}-1\Big) + C_{02}\frac{\Gamma(\alpha)}{A(n+\gamma)^3(\beta+nT)^{\alpha-3}}\Bigg\{\frac{1}{\alpha-1}\Big(\frac{1}{(1-\eta^*)^{\alpha-1}}-1\Big) \\
&+ \frac{1}{\alpha-3}\Big(\frac{1}{(1-\eta^*)^{\alpha-3}}-1\Big) - 2\frac{1}{\alpha-2}\Big(\frac{1}{(1-\eta^*)^{\alpha-2}}-1\Big)\Bigg\} \\
&= \frac{\Gamma(\alpha-1)}{A(n+\gamma)^2(\beta+nT)^{\alpha-3}}\Big(\frac{C_{01}}{\beta+nT} - \frac{C_{11}\alpha}{(\beta+nT)^2} + \frac{C_{02}}{(n+\gamma)}\Big)\Big(\frac{1}{(1-\eta^*)^{\alpha-1}}-1\Big) \\
&- \frac{(\alpha-1)\Gamma(\alpha-2)}{A(n+\gamma)^2(\beta+nT)^{\alpha-3}}\Big(\frac{C_{01}}{\beta+nT} - \frac{2C_{02}}{(n+\gamma)}\Big)\Big(\frac{1}{(1-\eta^*)^{\alpha-2}}-1\Big) \\
&+ \frac{C_{02}\Gamma(\alpha)}{A(\alpha-3)(n+\gamma)^3(\beta+nT)^{\alpha-3}}\Big(\frac{1}{(1-\eta^*)^{\alpha-3}}-1\Big) + \frac{\Gamma(\alpha)}{A(n+\gamma)(\beta+nT)^{\alpha-1}} \\
&\times \Big(\frac{C_{10}}{\beta+nT} + \frac{C_{11}}{(n+\gamma)}\Big)\Big(\frac{1}{(1-\eta^*)^{\alpha}}-1\Big) + \frac{C_{20}\Gamma(\alpha+1)}{A(n+\gamma)(\beta+nT)^{\alpha+1}}\Big(\frac{1}{(1-\eta^*)^{\alpha+1}}-1\Big),
\end{aligned}
$$

Case (2): when $\alpha = 2$.

$$
\begin{aligned}
&\sum_{1\le i+j\le 2} C_{ij}E_{\lambda,\mu}(\lambda^i\mu^j e^{-n\lambda(T-\mu)}) \\
&= C_{01}\frac{\Gamma(2)}{A(n+\gamma)^2}\Bigg\{ln(1-\eta^*) + \Big(\frac{1}{(1-\eta^*)}-1\Big)\Bigg\} + C_{10}\frac{\Gamma(3)}{2A(n+\gamma)(\beta+nT)^2}\Big(\frac{1}{(1-\eta^*)^2}-1\Big) \\
&+ C_{11}\frac{\Gamma(3)}{A(n+\gamma)^2(\beta+nT)}\Bigg\{-\Big(\frac{1}{(1-\eta^*)}-1\Big) + \frac{1}{2}\Big(\frac{1}{(1-\eta^*)^2}-1\Big)\Bigg\} \\
&+ C_{20}\frac{\Gamma(4)}{A(n+\gamma)(\beta+nT)^3}\frac{1}{3}\Big(\frac{1}{(1-\eta^*)^3}-1\Big) + C_{02}\frac{\Gamma(2)}{A(n+\gamma)^3(\beta+nT)^{-1}} \\
&\times \Bigg\{\Big(\frac{1}{(1-\eta^*)}-1\Big) + \eta^* + 2ln(1-\eta^*)\Bigg\} \\
&= \frac{1}{A(n+\gamma)^2}\Big(C_{01} - \frac{2C_{11}}{\beta+nT} + \frac{C_{02}(\beta+nT)}{(n+\gamma)}\Big)\Big(\frac{\eta^*}{1-\eta^*}\Big) + \frac{1}{A(n+\gamma)^2}\Big(C_{01} + \frac{2C_{02}(\beta+nT)}{(n+\gamma)}\Big) \\
&\times ln(1-\eta^*) + \frac{1}{A(n+\gamma)(\beta+nT)}\Big(\frac{C_{10}}{\beta+nT} + \frac{C_{11}}{(n+\gamma)}\Big)\Big(\frac{1}{(1-\eta^*)^2}-1\Big) \\
&+ \frac{2C_{20}}{A(n+\gamma)(\beta+nT)^3}\Big(\frac{1}{(1-\eta^*)^3}-1\Big) + \frac{C_{02}(\beta+nT)}{A(n+\gamma)^3}\eta^*,
\end{aligned}
$$

Case (3): and when $\alpha = 3$.

$$
\begin{aligned}
&\sum_{1\le i+j\le 2} C_{ij}E_{\lambda,\mu}(\lambda^i\mu^j e^{-n\lambda(T-mu)}) \\
&= C_{01}\frac{\Gamma(3)}{A(n+\gamma)^2(\beta+nT)}\Big\{-\Big(\frac{1}{(1-\eta^*)}-1\Big)+\frac{1}{2}\Big(\frac{1}{(1-\eta^*)^2}-1\Big)\Big\} \\
&\quad + C_{10}\frac{\Gamma(4)}{A(n+\gamma)(\beta+nT)^3}\frac{1}{3}\Big(\frac{1}{(1-\eta^*)^3}-1\Big)+C_{11}\frac{\Gamma(4)}{A(n+\gamma)^2(\beta+nT)^2} \\
&\quad \times\Big\{-\frac{1}{2}\Big(\frac{1}{(1-\eta^*)^2}-1\Big)+\frac{1}{3}\Big(\frac{1}{(1-\eta^*)^3}-1\Big)\Big\}+C_{20}\frac{\Gamma(5)}{A(n+\gamma)(\beta+nT)^4} \\
&\quad \times\frac{1}{4}\Big(\frac{1}{(1-\eta^*)^4}-1\Big)+C_{02}\frac{\Gamma(3)}{A(n+\gamma)^3}\Big\{\frac{1}{2}\Big(\frac{1}{(1-\eta^*)^2}-1\Big)-ln(1-\eta^*)-2\Big(\frac{1}{(1-\eta^*)}-1\Big)\Big\} \\
&= \frac{1}{A(n+\gamma)^2}\Big(\frac{C_{01}}{\beta+nT}-\frac{3C_{11}}{(\beta+nT)^2}+\frac{C_{02}}{(n+\gamma)}\Big)\Big(\frac{1}{(1-\eta^*)^2}-1\Big)-\frac{1}{A(n+\gamma)^2}\Big(\frac{2C_{01}}{\beta+nT} \\
&\quad +\frac{C_{02}}{(n+\gamma)}\Big)\Big(\frac{\eta^*}{1-\eta^*}\Big)+\frac{2}{A(n+\gamma)(\beta+nT)^2}\Big(\frac{C_{10}}{\beta+nT}-\frac{C_{11}}{(n+\gamma)}\Big)\Big(\frac{1}{(1-\eta^*)^3}-1\Big) \\
&\quad +\frac{6C_{20}}{A(n+\gamma)(\beta+nT)^4}\Big(\frac{1}{(1-\eta^*)^4}-1\Big)-\frac{2C_{02}}{A(n+\gamma)^3}ln(1-\eta^*)
\end{aligned}
$$

B Proof of Theorem 4.1

Suppose $(0,0,0,0)$ denotes the sampling plan which accepts the lot without sampling ($n = 0$ or 1). It is easy to verify from the equation (8) that the corresponding Bayes risk

$$
r(0,0,0,0) = \sum_{0\le i+j\le 2} C_{ij}E_{\lambda,\mu}\Big(\lambda^i\mu^j\Big),
$$

for expressions of the required moments, see Lam (1990). Similarly if $(0,0,0,\infty)$ denotes the sampling plan which rejects the lot without sampling ($n = 0$), then the corresponding Bayes risk $r(0,0,0,\infty)$ will be C_r. Obviously the Bayes risk corresponding to the optimal

Bayesian sampling plan satisfies the inequality

$$
\begin{aligned}
& r(n_0, r_0, T_0, \xi_0) \leq min\{r(0,0,0,0), r(0,0,0,\infty)\} \\
\Longrightarrow\ & r(n_0, r_0, T_0, \xi_0) \leq min\Big\{\sum_{0\leq i+j\leq 2} C_{ij} E_{\lambda,\mu}\Big(\lambda^i \mu^j\Big), C_r\Big\}
\end{aligned} \tag{10}
$$

Also, by equation (8)

$$
\begin{aligned}
r(n_0, r_0, T_0, \xi_0) =& n_0(C_s - r_s) + C_T E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu) + r_s E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu) + C_r \\
& + E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}\Big[\delta(\underline{X})\Big(\sum_{0\leq i+j\leq 2} C_{ij}\lambda^i\mu^j - C_r\Big)\Big] \\
=& n_0(C_s - r_s) + C_T E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}(\tau|\lambda,\mu) + r_s E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}(D|\lambda,\mu) \\
& + C_r E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}\Big(1 - \delta(\underline{X})\Big) + E_{\lambda,\mu} E_{\underline{X}|\lambda,\mu}\Big[\delta(\underline{X})\Big(\sum_{0\leq i+j\leq 2} C_{ij}\lambda^i\mu^j\Big)\Big]
\end{aligned}
$$

In the right hand side of above equation, leaving out the first term, all others are positive terms. Therefore

$$
r(n_0, r_0, T_0, \xi_0) \geq n_0(C_s - r_s). \tag{11}
$$

Hence equations (10) and (11) together imply the first result and of course we have $1 \leq r_0 \leq n_0$.