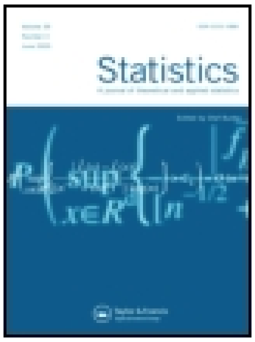

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Debasis Kundu

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ASYMPTOTIC THEORY OF THE LEAST SQUARES ESTIMATORS OF SINUSOIDAL SIGNAL

DEBASIS KUNDU

Indian Institute of Technology Kanpur, India

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The consistency and the asymptotic normality of the least squares estimators are derived of the sinusoidal model under the assumption of stationary random error. It is observed that the model does not satisfy the standard sufficient conditions of Jennrich (1969), Wu (1981) or Kundu (1991). Recently the consistency and the asymptotic normality are derived for the sinusoidal signal under the assumption of normal error (Kundu; 1993) and under the assumptions of independent and identically distributed random variables in Kundu and Mitra (1996). This paper will generalize them. Hannan (1971) also considered the similar kind of model and establish the result after making the Fourier transform of the data for one parameter model. We establish the result without making the Fourier transform of the data. We give an explicit expression of the asymptotic distribution of the multiparameter case, which is not available in the literature. Our approach is different from Hannan's approach. We do some simulations study to see the small sample properties of the two types of estimators.

Keywords and Phrases: Asymptotic distribution; strong consistency; least squares estimators and stationary distribution

AMS Subject Classifications (1985): 62J02, 62C05

1. INTRODUCTION

The least squares method plays an important role in drawing the inferences about the parameters in the nonlinear regression model. In this paper we consider the least squares estimators (LSE's) of the following sinusoidal time series regression model:

$$Y(t) = A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) + X(t); \quad t = 1, \dots, N \tag{1}$$

Here A_0 and B_0 are unknown fixed constants, ω_0 is an unknown frequency lying between 0 and π. $X(t)$'s are stationary time series satisfying the following assumption:

Assumption 1

$$X(t) = \sum_{j=-\infty}^{\infty} \alpha(j)\varepsilon(t-j), \quad \sum_{j=-\infty}^{\infty} |\alpha(j)| < \infty \tag{2}$$

where $\varepsilon(t)$'s are independent and identically distributed (i.i.d.) random variables with mean zero and finite variance $\sigma^2 > 0$. Here '=' means $X(t)$ has that almost sure representation.

This is an important and well studied model in Time Series and Signal Processing literature. See for example Stoica (1993) for an extensive list of references for different estimation procedures. Hannan (1971, 1973), Walker (1969, 1971), Kundu (1993, 1995), Kundu and Mitra (1995, 1996) also considered this or similar kind of model to study the asymptotic properties of the different estimators and some of the computational issues have been discussed in Rice and Rosenblatt (1988). Walker (1971) considered the approximate least squares estimators (ALSE's) and proved the strong consistency and the asymptotic normality of the ALSE's under the assumptions that the errors are i.i.d. random variables with mean zero and finite variance. The result has been extended by Hannan (1971, 1973) to the case when the errors are stationary random variables with continuous spectrum. Kundu (1993) also considered a similar model and proved directly the consistency and the asymptotic normality of the LSE's under the assumption that $X(t)$'s are i.i.d. with mean zero and finite variance and they are normally distributed. The result was extended to the case of general mean zero and finite variance i.i.d. errors in Kundu and Mitra (1996). In this paper we generalize the result of Kundu and Mitra (1996) to the case when the errors are coming from a mean zero and finite variance stationary process. We prove directly the consistency and the asymptotic normality of the LSE's when the $X(t)$'s satisfy Assumption 1. It is important to observe that we do not need the continuity assumption of the spectrum. Our approach is straight forward and different from that of Walker (1969, 1971) or Hannan (1971, 1973). Hannan (1971, 1973) obtained the result for the one parameter case after making the Fourier transform of the data. We observe that it is not necessary to make the Fourier transform of the data. We also consider the multiparameter case and obtained the explicit expression of the asymptotic covariance matrix, which is not available in the literature. We also perform some numerical experiments to compare the small sample behavior of the ALSE's and the exact LSE's. In this paper the almost sure convergence means with respect to the usual Lebesgue measure and it will be denoted by a.s.. Also the notation $a = O(N^b)$ means $|a/N^b|$ is bounded for all N.

The rest of the paper is organised as follows, in Section 2 we prove the consistency of the LSE's and establish the asymptotic normality results in Section 3. The results for the several Harmonic case are obtained in Section 4. Some numerical results are presented in Section 5 and finally we draw conclusion in Section 6.

2. CONSISTENCY OF THE LSE's

Let's denote $\hat{\theta}_N = (\hat{A}_N, \hat{B}_N, \hat{\omega}_N)$ to be the LSE of $\theta_0 = (A_0, B_0, \omega_0)$, obtained by minimizing

$$Q_N(\theta) = \sum_{t=1}^{N} (Y(t) - A\cos(\omega t) - B\sin(\omega t))^2 \tag{3}$$

with respect to $\theta = (A, B, \omega)$. It is important to observe that the existence and the uniqueness of a respective measurable function satisfying (3) follows along the same line of Jennrich (1969). To prove the consistency results we need the following lemma.

LEMMA 1 *Let $X(t)$ be a stationary sequence which satisfies Assumption 1, then*

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N} \sum_{t=1}^{N} X(t)\cos(t\theta) \right| = 0 \text{ a.s.} \tag{4}$$

Before giving the proof in details, we would like to give a sketch of the main idea. First we show that (4) holds for the subsequence N^3. Then we show that

$$\sup_{\theta} \sup_{N^3 < K \leqslant (N+1)^3} \left| \frac{1}{N^3} \sum_{t=1}^{N} X(t)\cos(t\theta) - \frac{1}{K} \sum_{t=1}^{N} X(t)\cos(t\theta) \right| \tag{5}$$

converges to zero a.s. as N tends to ∞.

Proof of Lemma 1

$$\frac{1}{N}\sum_{t=1}^{N} X(t)\cos(t\theta) = \frac{1}{N}\sum_{t=1}^{N}\sum_{j=-\infty}^{\infty} \alpha(j)\varepsilon(t-j)\cos(t\theta)$$

$$= \frac{1}{N}\sum_{t=1}^{N}\sum_{j=-\infty}^{\infty} \alpha(j)\varepsilon(t-j)\{\cos((t-j)\theta)\cos(j\theta) - \sin((t-j)\theta)\sin(j\theta)\}$$

$$= \frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)\cos(j\theta)\sum_{t=1}^{N}\varepsilon(t-j)\cos((t-j)\theta)$$

$$- \frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)\sin(j\theta)\sum_{t=1}^{N}\varepsilon(t-j)\sin((t-j)\theta) \tag{6}$$

Therefore

$$\sup_{\theta}\left|\frac{1}{N}\sum_{t=1}^{N} X(t)\cos(t\theta)\right|$$

$$\leqslant \sup_{\theta}\left|\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)\cos(j\theta)\sum_{t=1}^{N}\varepsilon(t-j)\cos((t-j)\theta)\right|$$

$$+ \sup_{\theta}\left|\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)\sin(j\theta)\sum_{t=1}^{N}\varepsilon(t-j)\sin((t-j)\theta)\right| \quad \text{a.s..} \tag{7}$$

We would like to prove that both the terms on the right hand side of (7) converges to zero as N tends to infinity. Now observe that

$$\left\{E\sup_{\theta}\left|\frac{1}{N}\sum_{j=-\infty}^{\infty}\left|\alpha(j)\cos(j\theta)\sum_{t=1}^{N}\varepsilon(t-j)\cos((t-j)\theta)\right|^{2}\right\}^{1/2}$$

$$\leqslant \frac{1}{N} \sum_{j=-\infty}^{\infty} |\alpha(j)| \left\{ E \sup_{\theta} \left| \sum_{t=1}^{N} \varepsilon(t-j) \cos((t-j)\theta) \right|^2 \right\}^{1/2}$$

$$\leqslant \frac{1}{N} \sum_{j=-\infty}^{\infty} |\alpha(j)| \left\{ N + \sum_{t=-N+1}^{N} E[|\Sigma_m \varepsilon(m)\varepsilon(m+t)|]^{1/2} \right\} \tag{8}$$

where the sum $\Sigma_{t=-N+1}^{N}$ omits the term $t=0$ and the term Σ_m is over $N-|t|$ term (dependent on j). Since

$$\sum_{t=-N+1}^{N} E[|\Sigma_m \varepsilon(m)\varepsilon(m+t)|]$$

$$\leqslant \sum_{t=-N+1}^{N} E[|\Sigma_m \varepsilon(m)\varepsilon(m+t)|^2]^{1/2} = O(N^{3/2}) \tag{9}$$

(uniformly in j) therefore (8) is $O(N^{-1/4})$. Let $M = N^3$. Therefore

$$E \sup_{\theta} \left| \frac{1}{M} \sum_{j=-\infty}^{\infty} \alpha(j) \cos(j\theta) \sum_{t=1}^{N} \cos((t-j)\theta) \right|^2 = O(M^{-3/2}) \tag{10}$$

Similarly the result is true if the cosine function is replaced by the sine also. Therefore

$$\sup_{\theta} \left| \frac{1}{M} \sum_{t=1}^{M} X(t)\cos(t\theta) \right| \to 0 \quad \text{a.s.} \tag{11}$$

when $M = N^3$. Now

$$\sup_{\theta} \sup_{N^3 < K \leqslant (N+1)^3} \left| \frac{1}{N^3} \sum_{t=1}^{N} X(t)\cos(t\theta) - \frac{1}{K} \sum_{t=1}^{N} X(t)\cos(t\theta) \right|$$

$$= \sup_{\theta} \sup_{N^3 < K \leqslant (N+1)^3} \left| \frac{1}{N^3} \sum_{t=1}^{N} X(t)\cos(t\theta) - \frac{1}{N^3} \sum_{t=1}^{K} X(t)\cos(t\theta) \right.$$

$$\left. + \frac{1}{N^3} \sum_{t=1}^{K} X(t)\cos(t\theta) - \frac{1}{K} \sum_{t=1}^{K} X(t)\cos(t\theta) \right|$$

$$\leqslant \frac{1}{N^3} \sum_{t=N^3+1}^{(N+1)^3} |X(t)| + \sum_{t=1}^{(N+1)^3} |X(t)| \left[\frac{1}{N^3} - \frac{1}{(N+1)^3} \right] \quad \text{a.s..} \tag{12}$$

The mean squared of the first quantity on the right hand side of (12) is dominated by $(K/N^6)\,[(N+1)^3 - N^3]^2 = O(N^{-2})$. Similarly the mean squared of the second quantity on the right hand side of (12) is dominated by $K(N^6/N^8) = O(N^{-2})$. Therefore both will converge to zero almost surely, which proves the lemma.

COROLLARY 1 *The result is true if the cosine function is replaced by the sine function.*

COROLLARY 2 *It can be proved similarly that if $X(t)$ is a sequence which satisfies Assumption* 1, *then*

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N^3} \sum_{t=1}^{N} t^2 X(t) \cos(t\theta) \right| = 0 \quad \text{a.s.} \tag{13}$$

Now consider

$$\frac{1}{N}[Q_N(\theta) - Q_N(\theta_0)]$$

$$= \frac{1}{N} \sum_{t=1}^{N} \{(Y(t) - A\cos(\omega t) - B\sin(\omega t))^2 - X(t)^2\}$$

$$= \frac{1}{N} \sum_{t=1}^{N} (A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) - A\cos(\omega t) - B\sin(\omega t))^2$$

$$+ \frac{2}{N} \sum_{t=1}^{N} X(t)(A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) - A\cos(\omega t) - B\sin(\omega t))$$

$$= f_N(A, B, \omega) + g_N(A, B, \omega). \tag{14}$$

Now with the help of lemma 1, we can easily conclude that

$$\lim_{N\to\infty} \sup_{\theta\in S_{\delta,M}} g_N(A, B, \omega) = 0 \quad \text{a.s.} \tag{15}$$

where the set $S_{\delta,M}$ for $\delta > 0$, is as follows;

$$S_{\delta,M} = \{(A,B,\omega),\ |A - A_0| \geqslant \delta, |A| \leqslant M, |B| \leqslant M$$

$$\text{or} \quad |B - B_0| \geqslant \delta, |A| \leqslant M, |B| \leqslant M$$

$$\text{or} \quad |\omega - \omega_0| \geqslant \delta,\ |A| \leqslant M, |B| \leqslant M\} \tag{16}$$

therefore for all $\delta > 0$,

$$\lim \inf_{S_{\delta,M}} \frac{1}{N} [Q_N(\theta) - Q_N(\theta_0)] = \lim_{N\to\infty} \sup_{\theta \in S_{\delta,M}} f_N(A,B,\omega) > 0. \tag{17}$$

(17) follows easily from Kundu and Mitra (1996). Here lim means limit infimum. Now suppose $(\hat{A}_N, \hat{B}_N, \hat{\omega}_N)$ be the LSE's of (A_0, B_0, ω_0) and they are not consistent. Therefore either

Case I For all subsequences $\{N_K\}$ of $\{N\}$, $|\hat{A}_{N_K}| + |\hat{B}_{N_K}|$ tends to infinity or

Case II There exists a $\delta > 0$ and a $M < \infty$ and a subsequence $\{N_K\}$ such that $(\hat{A}_{N_K}, \hat{B}_{N_K}, \hat{\omega}_{N_K}) \in S_{\delta,M}$, for all $K = 1,2,\ldots$.

Now

$$\hat{Q}_{N_K}(\hat{A}_{N_K}, \hat{B}_{N_K}, \hat{\omega}_{N_K}) - Q_{N_K}(A_0, B_0, \omega_0) \leqslant 0 \tag{18}$$

as $(\hat{A}_{N_K}, \hat{B}_{N_K}, \hat{\omega}_{N_K})$ is the LSE of (A_0, B_0, ω_0), when $N = N_K$. Observe that as $K \to \infty$, for both the cases, the left hand side of (18) converges to a number which is strictly positive, that is a contradiction. Therefore the LSE's of the model (1) have to be strongly consistent. Therefore we can state the following theorem:

THEOREM 1 *If $\hat{\theta}_N = (\hat{A}_N, \hat{B}_N, \hat{\omega}_N)$ is the LSE of the nonlinear regression model* (1), *then it is a strongly consistent estimator of* $\theta_0 = (A_0, B_0, \omega_0)$.

3. ASYMPTOTIC NORMALITY

In this section we prove the asymptotic normality of $\hat{\theta}_N$ by using the Taylor series expansion. Let's denote

$$Q'_N(\theta) = \left(\frac{\delta Q_N(\theta)}{\delta A}, \frac{\delta Q_N(\theta)}{\delta B}, \frac{\delta Q_N(\theta)}{\delta \omega}\right) \tag{19}$$

and $Q''_N(\theta)$ to be the corresponding 3×3 matrix which contains the double derivative of $Q_N(\theta)$. Therefore

$$Q'_N(\hat{\theta}_N) - Q'_N(\theta_0) = (\hat{\theta} - \theta_0) Q''_N(\bar{\theta}) \tag{20}$$

where $\bar{\theta} = (\bar{A}, \bar{B}, \bar{\omega})$ is a point in the line joining $\hat{\theta}_N$ and θ_0. Observe that although $\bar{\theta}$ depends on N, we omit it for brevity. Since $Q'_N(\hat{\theta}_N) = 0$, (20) implies

$$(\hat{\theta} - \theta_0) = -Q'_N(\theta_0)[Q''_N(\bar{\theta})]^{-1}. \tag{21}$$

Now

$$\frac{\delta Q_N(\theta_0)}{\delta A} = -2 \sum_{t=1}^{N} X(t) \cos(\omega_0 t) \tag{22}$$

$$\frac{\delta Q_N(\theta_0)}{\delta B} = -2 \sum_{t=1}^{N} X(t) \sin(\omega_0 t) \tag{23}$$

$$\frac{\delta Q_N(\theta_0)}{\delta \omega} = -2 \sum_{t=1}^{N} t X(t)(A_0 \sin(\omega_0 t) - B_0 \cos(\omega_0 t)). \tag{24}$$

Also

$$\frac{\delta^2 Q_N(\bar{\theta})}{\delta A^2} = 2 \sum_{t=1}^{N} \cos^2(\bar{\omega} t), \frac{\delta^2 Q_N(\bar{\theta})}{\delta B^2} = 2 \sum_{t=1}^{N} \sin^2(\bar{\omega} t),$$

$$\frac{\delta^2 Q_N(\bar{\theta})}{\delta \omega^2} = 2 \sum_{t=1}^{N} t^2 [(A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t) - \bar{A} \cos(\bar{\omega} t) - \bar{B} \sin(\bar{\omega} t)$$

$$+ X(t)) \times (\bar{A}\cos(\bar{\omega}t) + \bar{B}\sin(\bar{\omega}t)) + (\bar{A}\sin(\bar{\omega}t) - \bar{B}\cos(\bar{\omega}t))^2] \quad (25)$$

$$\frac{\delta^2 Q_N(\bar{\theta})}{\delta\omega\delta A} = 2\sum_{t=1}^{N} t[\sin(\bar{\omega}t)(A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) - \bar{A}\cos(\bar{\omega}t)$$

$$- \bar{B}\sin(\bar{\omega}t) + X(t)) - \cos(\bar{\omega}t)(\bar{A}\sin(\bar{\omega}t) - \bar{B}\cos(\bar{\omega}t))] \quad (26)$$

$$\frac{\delta^2 Q_N(\bar{\theta})}{\delta\omega\delta B} = -2\sum_{t=1}^{N} t[\cos(\bar{\omega}t)(A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) - \bar{A}\cos(\bar{\omega}t)$$

$$- \bar{B}\sin(\bar{\omega}t) + X(t)) - \sin(\bar{\omega}t)(\bar{A}\sin(\bar{\omega}t) - \bar{B}\cos(\bar{\omega}t))] \quad (27)$$

$$\frac{\delta^2 Q_N(\bar{\theta})}{\delta A\delta B} = 2\sum_{t=1}^{N} \sin(\bar{\omega}t)\cos(\bar{\omega}t). \quad (28)$$

Let's define

$$\sigma_{11} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \cos^2(\omega_0 t) = \frac{1}{2}$$

$$\sigma_{22} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \sin^2(\omega_0 t) = \frac{1}{2}$$

$$\sigma_{33} = \lim_{N\to\infty} \frac{1}{N^3}\sum_{t=1}^{N} t^2(A_0\sin(\omega_0 t) - B_0\cos(\omega_0 t))^2 = \frac{1}{6}(A_0^2 + B_0^2)$$

$$\sigma_{13} = \sigma_{31} = \lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} B_0 t\cos^2(\omega_0 t) = \frac{1}{4}B_0$$

$$\sigma_{23} = \sigma_{32} = -\lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} A_0 t\sin^2(\omega_0 t) = -\frac{1}{4}A_0$$

$$\sigma_{12} = \sigma_{21} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \sin(\omega_0 t)\cos(\omega_0 t) = 0.$$

Now observe that as $\bar{\omega} \to \omega_0$, $\bar{A} \to A_0$ and $\bar{B} \to B_0$ a.s., we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \cos^2(\bar{\omega}t) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \cos^2(\omega_0 t) = \frac{1}{2} \tag{29}$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \sin^2(\bar{\omega}t) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \sin^2(\omega_0 t) = \frac{1}{2} \tag{30}$$

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2N^3}\frac{\delta^2 Q_N(\bar{\theta})}{\delta\omega^2} &= \lim_{N\to\infty} \frac{1}{N^3}\sum_{t=1}^{N} t^2(\bar{A}\sin(\bar{\omega}t) + \bar{B}\cos(\bar{\omega}t))^2 \\ &= \lim_{N\to\infty} \frac{1}{N^3}\sum_{t=1}^{N} t^2(A_0\sin(\omega_0 t) - B_0\cos(\omega_0 t))^2 \\ &= \frac{1}{6}(A_0^2 + B_0^2)\end{aligned} \tag{31}$$

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2N^2}\frac{\delta^2 Q_N(\bar{\theta})}{\delta\omega\delta A} &= -\lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} t\cos(\bar{\omega}t)(\bar{A}\sin(\bar{\omega}t) - \bar{B}\cos(\bar{\omega}t)) \\ &= \lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} tB_0\cos^2(\omega_0 t) = \frac{1}{4}B_0\end{aligned} \tag{32}$$

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2N^2}\frac{\delta^2 Q_N(\bar{\theta})}{\delta\omega\delta B} &= -\lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} t\sin(\bar{\omega}t)(\bar{A}\sin(\bar{\omega}t) - \bar{B}\cos(\bar{\omega}t)) \\ &= -\lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} tA_0\sin^2(\omega_0 t) = -\frac{1}{4}A_0\end{aligned} \tag{33}$$

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N}\frac{\delta^2 Q_N(\bar{\theta})}{\delta A\delta B} &= \lim_{N\to\infty} \frac{2}{N}\sum_{t=1}^{N} \sin(\bar{\omega}t)\cos(\bar{\omega}t) \\ &= \lim_{N\to\infty} \frac{2}{N}\sum_{t=1}^{N} \sin(\omega_0 t)\cos(\omega_0 t) = 0.\end{aligned} \tag{34}$$

Let's define the 3×3 matrix $\mathbf{\Sigma} = ((\sigma_{ij}))$; $i, j = 1, 2, 3$ and also define the 3×3 diagonal matrix $\mathbf{D}$ as follows $\mathbf{D} = \operatorname{diag}\{N^{-1/2}, N^{-1/2}, N^{-3/2}\}$.

Rewrite (21) as

$$(\hat{\theta}-\theta_0)\mathbf{D}^{-1} = -Q'_N(\theta_0)\mathbf{D}[\mathbf{D}Q''_N(\bar{\theta})\mathbf{D}]^{-1}. \tag{35}$$

Now from (29)–(34) we obtain

$$\lim_{N\to\infty}\mathbf{D}Q''_N(\bar{\theta})\mathbf{D} = \lim_{N\to\infty}\mathbf{D}Q''_N(\theta_0)\mathbf{D} = 2\Sigma \tag{36}$$

where

$$\Sigma = \begin{bmatrix} 1/2 & 0 & 1/4B_0 \\ 0 & 1/2 & -(1/4)A_0 \\ 1/4 & -(1/4)A_0 & (1/6)(A_0^2+B_0^2) \end{bmatrix} \tag{37}$$

and Σ^{-1} exists if $(A_0^2+B_0^2)>0$ and it is as follows;

$$\Sigma^{-1} = \frac{4}{A_0^2+B_0^2}\begin{bmatrix} (1/2)A_0^2+2B_0^2 & -(3/2)A_0B_0 & -3B_0 \\ -(3/2)A_0B_0 & (1/2)B_0^2+2A_0^2 & 3A_0 \\ -3B_0 & 3A_0 & 6 \end{bmatrix}. \tag{38}$$

Now from the Central Limit theorem of Stochastic Process (see Fuller; 1976) it easily follows that $Q'_N(\theta_0)\mathbf{D}$ tends to a multivariate (3-variate) normal distribution as given below;

$$Q'_N(\theta_0)\mathbf{D} \to N_3(\mathbf{0}, 4\sigma^2 c\Sigma) \tag{39}$$

where

$$c = \left|\sum_{j=-\infty}^{\infty}\alpha(j)\cos(\omega_0 j)\right|^2 + \left|\sum_{j=-\infty}^{\infty}\alpha(j)\sin(\omega_0 j)\right|^2.$$

Therefore we have;

$$(\hat{\theta}_N-\hat{\theta}_0)\mathbf{D}^{-1} \to N_3(\mathbf{0}, \sigma^2 c\Sigma^{-1}). \tag{40}$$

Now we can state the result as the following theorem;

THEOREM 2 *Under the assumptions of Theorem* 1, $\{N^{1/2}(\hat{A}_N - A_0), N^{1/2}(\hat{B}_N - B_0), N^{3/2}(\hat{\omega}_N - \omega_0)\}$ *converges in distribution to a 3-variate normal distribution with mean vector zero and the dispersion matrix is given by* $\sigma^2 c \Sigma^{-1}$, *where* c *and* Σ^{-1} *are as defined before.*

4. MULTIPARAMETER CASE

In this section we will extend the results of Section 2 and Section 3 to the following model:

$$Y(t) = \sum_{K=1}^{M} A_0^K \cos(\omega_0^K t) + B_0^K \sin(\omega_0^K t) + X(t); \quad t = 1, \ldots, N, \tag{41}$$

where A_0^K, B_0^K are arbitrary real numbers and ω_0^K's are the distinct frequencies lying between 0 and π for $K = 1, \ldots, M$. $X(t)$'s satisfy Assumption 1.

Let us use the following notations $\mathbf{A} = (A^1, \ldots, A^M)$, $\mathbf{B} = (B^1, \ldots, B^M)$ and $\omega = (\omega^1, \ldots, \omega^M)$. Similarly $\mathbf{A}_0, \mathbf{B}_0$, ω_0 and $\hat{\mathbf{A}}_N$, $\hat{\mathbf{B}}_N$ and $\hat{\omega}_N$ are also defined. We would like to investigate the consistency and the asymptotic normality properties of the LSE's obtained by minimizing $R_N(\Phi) =$,

$$\sum_{t=1}^{N} \left(Y(t) - \sum_{K=1}^{M} [A^K \cos(\omega^K t) + B^K \sin(\omega^K t)] \right)^2$$

with respect to $\Phi = (\mathbf{A}, \mathbf{B}, \omega)$. Now we have the following result:

THEOREM 3 *If* $\hat{\Phi}_N = (\hat{\mathbf{A}}_N, \hat{\mathbf{B}}_N, \hat{\omega}_N)$ *is the LSE of* $\Phi_0 = (\mathbf{A}_0, \mathbf{B}_0, \omega_0)$, *then* $\hat{\Phi}_N$ *is a strongly consistent estimator of* Φ_0.

Proof of the Theorem 3 With the help of Lemma 1 and using the similar kind of techniques as that of (Kundu and Mitra; 1995), the results can be established.

Let's denote the $1 \times 3M$ vector $R'_N(\Phi)$ as follows:

$$R'_N(\Phi) = \left(\frac{\delta R_N(\Phi)}{\delta \mathbf{A}}, \frac{\delta R_N(\Phi)}{\delta \mathbf{B}}, \frac{\delta R_N(\Phi)}{\delta \omega} \right)$$

and $R''_N(\theta)$ denotes the $3M \times 3M$ matrix which contains the double derivative of $R_N(\Phi)$. Now we have

$$R'_N(\hat{\Phi}_N) - R'_N(\Phi_0) = (\hat{\Phi}_N - \Phi_0) R''_N(\bar{\Phi}) \tag{42}$$

where $\bar{\Phi} = (\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\omega})$ is a point in the line joining $\hat{\Phi}_N$ and Φ_0. Since $R'_N(\hat{\Phi}_N) = 0$, we have

$$(\hat{\Phi}_N - \Phi_0) = -R'_N(\Phi_0)[R''_N(\bar{\Phi})]^{-1}. \tag{43}$$

Let's define the $3M \times 3M$ diagonal matrix $\mathbf{V}$ whose first 2M diagonal elements are $N^{-1/2}$ and the last M diagonal elements are $N^{-3/2}$. Therefore we can write (43) as

$$(\hat{\Phi}_N - \Phi_0)\mathbf{V}^{-1} = -R'_N(\Phi_0)\mathbf{V}^{-1}[\mathbf{V}^{-1}R''_N(\bar{\Phi})\mathbf{V}^{-1}]^{-1}.$$

Now using the similar kind of arguments as of Section 3, we can say that

$$R'_N(\Phi_0)\mathbf{V} \to N_{3M}(\mathbf{0}, 4\sigma^2\mathbf{G})$$

where $\mathbf{G}$ is a $3M \times 3M$ matrix and it has the following structure

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{G}_{13} \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{G}_{23} \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{G}_{33} \end{bmatrix} \tag{44}$$

where each of the $\mathbf{G}_{ij}$ is a $M \times M$ matrix and

$$\mathbf{G}_{11} = \mathbf{G}_{22} = \text{diag}\left\{\frac{1}{2}c_1, \ldots, \frac{1}{2}c_M\right\}$$

$$\mathbf{G}_{13} = \mathbf{G}_{31} = \text{diag}\left\{\frac{1}{4}B_0^1 c_1, \ldots, \frac{1}{2}B_0^M c_M\right\}$$

$$\mathbf{G}_{23} = \mathbf{G}_{32} = -\text{diag}\left\{\frac{1}{4}A_0^1 c_1, \ldots, \frac{1}{2}A_0^M c_M\right\}$$

$$\mathbf{G}_{33} = \frac{1}{6}\text{diag}\{d_1, \ldots, d_M\}$$

$$\mathbf{G}_{12} = \mathbf{0} \tag{45}$$

here $c_K =$

$$\left| \sum_{j=-\infty}^{\infty} \alpha(j)\cos(\omega_0^K j) \right|^2 + \left| \sum_{j=-\infty}^{\infty} \alpha(j)\sin(\omega_0^K j) \right|^2$$

and $d_K = c_K[(A_0^K)^2 + (B_0^K)^2]$ for $K = 1, \ldots, M$. Observe that

$$\lim_{N\to\infty} \mathbf{V}R_N''(\Phi)\mathbf{V} = \lim_{N\to\infty} \mathbf{V}R_N''(\Phi_o)\mathbf{V} = 2\Gamma \tag{46}$$

here the $3M \times 3M$ matrix Γ is

$$\Gamma = \begin{bmatrix} (1/2)\mathbf{I}_M & \mathbf{0} & \mathbf{S}_1 \\ \mathbf{0} & (1/2)\mathbf{I}_M & \mathbf{S}_2 \\ \mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 \end{bmatrix} \tag{47}$$

where $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ are $M \times M$ diagonal matrices as follows;

$$\mathbf{S}_1 = \frac{1}{4}\text{diag}\{B_0^1, \ldots, B_0^M\}$$

$$\mathbf{S}_2 = -\frac{1}{4}\text{diag}\{A_0^1, \ldots, A_0^M\}$$

$$\mathbf{S}_3 = \frac{1}{6}\text{diag}\{d_1, \ldots, d_M\} \tag{48}$$

and $\mathbf{I}_M$ is the identity matrix of order M. Since

$$\Gamma^{-1} = 4\begin{bmatrix} (1/2)\mathbf{R}_4 + 2\mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \\ \mathbf{R}_2 & (1/2)\mathbf{R}_1 + 2\mathbf{R}_4 & \mathbf{R}_5 \\ \mathbf{R}_3 & \mathbf{R}_5 & \mathbf{R}_6 \end{bmatrix} \tag{49}$$

where

$$\mathbf{R}_1 = \text{diag}\left\{\frac{(B_0^1)^2}{d_1}, \ldots, \frac{(B_0^M)^2}{d_M}\right\}$$

$$\mathbf{R}_2 = -\frac{3}{2}\text{diag}\left\{\frac{A_0^1 B_0^1}{d_1}, \ldots, \frac{A_0^M B_0^M}{d_M}\right\}$$

$$\mathbf{R}_3 = -3\,\text{diag}\left\{\frac{B_0^1}{d_1}, \ldots, \frac{B_0^M}{d_M}\right\}$$

$$\mathbf{R}_4 = \text{diag}\left\{\frac{(A_0^1)^2}{d_1}, \ldots, \frac{(A_0^M)^2}{d_M}\right\}$$

$$\mathbf{R}_5 = 3\,\text{diag}\left\{\frac{A_0^1}{d_1}, \ldots, \frac{A_0^M}{d_M}\right\}$$

$$\mathbf{R}_6 = 6\,\text{diag}\left\{\frac{1}{d_1}, \ldots, \frac{1}{d_M}\right\} \tag{50}$$

we have

$$(\hat{\Phi}_N - \Phi_0)\mathbf{V}^{-1} \to N_{3M}(\mathbf{0}, \sigma^2\Gamma^{-1}\mathbf{G}\Gamma^{-1})$$

therefore we can state the result as the following theorem;

THEOREM 4 *Under the assumptions of Theorem 3,* $\{N^{1/2}(\hat{\mathbf{A}}_N - \mathbf{A}_0), N^{1/2}(\hat{\mathbf{B}}_N - \mathbf{B}_0), N^{3/2}(\hat{\omega}_N - \omega_0)\}$ *converges in distribution to a* $3M$*-variate normal distribution with mean vector zero and the dispersion matrix is given by* $\sigma^2\Gamma^{-1}\mathbf{G}\Gamma^{-1}$.

5. NUMERICAL EXPERIMENTS

In this section we perform some Monte Carlo simulations to see how the asymptotic results work for small sample. We considered the following model:

$$Y(t) = A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) + X(t); \quad t = 1, \ldots, N. \tag{51}$$

We took $A_0 = B_0 = 1.5$, $\omega_0 = .25\pi(\approx 0.735398)$, $.50\pi(\approx 1.570796)$ and $.75\pi(\approx 2.356194)$. $X(t) = \varepsilon(t) + .5\varepsilon(t-1)$, where $\varepsilon(t)$'s are i.i.d. normal random variables with mean zero and variance one. Numerical results are reported for $N = 10, 15, 25$. All these computations were performed at the Indian Institute of Technology Kanpur, using PC-486 and the random deviate generator proposed by Press *et al.* (1992). For a particular N and ω, 1000 different data sets were generated and for each data set we estimated the nonlinear parameters by two different methods, one (denoted by L.S.) by directly minimizing (3) with respect to the different parameters and the other one (denoted by A.L.S.) by first making the Fourier transform of the data as suggested by Hannan (1971, 1973), Walker (1971). We computed the average estimates and the average mean squared errors over 1000 replications. We reported the result in Table I for the frequency only because the others are quite similar in nature. The figures in the top denote the average estimates and the figures in the parenthesis below give the corresponding average mean squared errors. We also computed the 95% confidence interval for ω for each data sets. The results are reported in Table II. The first figure in the parenthesis is the average length of the confidence interval and the second figure is the coverage frequency over 1000 replications. From Table I and Table II, it is clear that although asymptotically both the methods are same but for small sample it is observed that the exact LSE's are better than the ALSE's. The average mean squared errors of ω are lower for the usual LSE's

TABLE I

ω	$N = 10$		$N = 15$		$N = 25$	
	L.S.	*A.L.S*	*L.S.*	*A.L.S*	*L.S.*	*A.L.S*
$.25\pi$	.7093	.6949	.7525	.7190	.7871	.7794
	(.1314)	(.1367)	(.0287)	(.0539)	(.0078)	(.0139)
$.50\pi$	1.3402	1.2555	1.4371	1.4543	1.4749	1.4497
	(.3387)	(.5072)	(.1287)	(.1467)	(.0975)	(.1436)
$.75\pi$	1.7772	1.6292	2.1143	2.0450	2.2501	2.1875
	(1.060)	(1.426)	(.4455)	(.6202)	(.1987)	(.3637)

TABLE II

ω	$N = 10$		$N = 15$		$N = 25$	
	L.S.	*A.L.S*	*L.S.*	*A.L.S*	*L.S.*	*A.L.S*
$.25\pi$	(.24, 46)	(.24, .41)	(.25, .73),	(.15, .53)	(.15, .88),	(.09, .81)
$.50\pi$	(.31, 53)	(.21, .42)	(.29, .78),	(.19, .71)	(.16, .87),	(.10, .73)
$.75\pi$	(.35, 62)	(.37, .49)	(.32, .86),	(.21, .80)	(.17, .94),	(.11, .86)

for almost all the sample sizes and for all ω's. About the confidence intervals, it is observed that for higher values of ω, the confidence interval of ω obtained by using the exact LSE's usually give higher coverage probability. It is also observed that for both the methods as N increases the average length decreases and the coverage probabability increases.

6. CONCLUSIONS

In this paper we considered the one parameter and multiparameter sinusoidal model under the assumption of additive stationary errors. We obtained the asymptotic properties of the LSE's directly without making the Fourier transform of the data. We also obtained the explicit expression of the covariance matrix for the multiparameter case, which is not available in the literature. From the numerical study it is observed that although asymptotically the two methods are equivalent but the exact LSE's are better than the ALSE's in terms of the mean squared errors. Since both the methods require the same amount of computations, therefore it is recommended not to Fourier transform the data at least for small samples to make any finite sample inference from the asymptotic results.

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Debasis Kundu,
Department of Mathematics,
Indian Institute of Technology,
Kanpur,
Pin 208016,
India.