Bayesian Inference and Optimal Censoring Scheme Under Progressive Censoring

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Abstract In recent time an extensive work has been done related to the different aspects of the progressive censoring schemes. Here we deal with the statistical inference of the unknown parameters of a three-parameter Weibull distribution based on the assumption that the data are progressively Type-II censored. The maximum likelihood estimators of the unknown parameters do not exist due to the presence of the location parameter,. Therefore, Bayesian approach seems to be a reasonable alternative. We assume here that the location parameter follows an uniform prior and the shape parameter follows a log-concave prior density function. We further assume that the scale parameter has a conjugate gamma prior given the shape and the location parameters. Based on these priors the Bayes estimate of any function of the unknown parameters under the squared error loss function and the associated highest posterior density credible interval are obtained using Gibbs sampling technique. We have also used one precision criterion to compare two different censoring schemes, and it can be used to find the optimal censoring scheme. Since finding the optimal censoring scheme is a challenging problem from the computational view point, we propose sub-optimal censoring scheme, which can be obtained quite conveniently. We have carried out some Monte Carlo simulations to observe the performances of the proposed method, and for illustrative purposes, we presented the analysis of one data set.

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1 Introduction

In different areas of reliability and survival analysis researchers have used the Weibull distribution quite extensively. The main reason about the overwhelming popularity of an Weibull distribution is due to the fact that its probability density function (PDF) can take various shapes, and the cumulative distribution function (CDF) can be expressed in a closed and compact form. Since the CDF of a Weibull distribution can be written in a closed form, the Weibull distribution has been used quite conveniently when the data are censored. For a detailed discussions on Weibull distribution and for different inferential issues see for example e Johnson et al. (1995, chap. 21) or Murthy et al. (2004). In any life testing experiment often the data are censored. Type-I and Type-II are the two most common censoring schemes one encounters in practice. In the last two decades, since the appearance of the book by Balakrishnan and Aggarwala (2000), progressive censoring scheme becomes quite popular. The review article by Balakrishnan (2007) and the recent book by Balakrishnan and Cramer (2014) provided the details about the development of this topic during this period.

In this article we consider the inference of the unknown parameters of a three-parameter Weibull distribution based on Type-II progressively censored data. The three-parameter Weibull distribution with the shape, scale and location parameter as $\alpha > 0$, $\lambda > 0$ and $-\infty < \mu < \infty$, respectively, has the PDF

$$f(x;\alpha,\lambda,\mu) = \begin{cases} \alpha\lambda(x-\mu)^{\alpha-1}e^{-\lambda(x-\mu)^{\alpha}} & \text{if } x > \mu, \\ 0 & \text{if } x \leq \mu. \end{cases} \tag{1}$$

From now on a three-parameter Weibull distribution with the PDF (1) will be denoted by $\text{WE}(\alpha,\lambda,\mu)$. It may be mentioned that although the two-parameter Weibull distribution (when $\mu = 0$), satisfies the regularity condition in the sense of Rao (1945), the three-parameter Weibull distribution does not satisfy the regularity conditions. In fact it can be shown very easily that if all the three parameters are unknown, the maximum likelihood estimators (MLEs) do not exist even for complete sample. Due to this reason, several alternative estimators have been proposed in the literature, see for example Nagatsuka et al. (2013) and the references cited therein. They may not be as efficient as the MLEs. Due to this reason Bayesian inference seems to be a reasonable alternative.

In this paper we provide the Bayesian inference of a three-parameter Weibull when all the parameters are unknown, based on a Type-II progressively censored data . It may be mentioned that when $\mu = 0$, the three-parameter Weibull distribution reduces to a two-parameter Weibull distribution. For a two-parameter Weibull distribution, the classical and Bayesian inference of the unknown parameters, as well as the reliability sampling plans in presence of progressive censoring, have been considered by several authors. See for example, Viveros and Balakrishnan (1994), Balasooriya et al. (2000) and Kundu (2008). Therefore, the present paper can be seen as an extension of these work to the three-parameter case.

One needs to assume some priors on the unknown parameters to perform the Bayesian inference. In this article, we have taken a set of fairly flexible priors on the shape (α), scale (λ) and location (μ) parameters. It can be easily verified that if the shape parameter α and the location parameter μ are known, then the scale parameter λ has a conjugate gamma prior. Therefore it is quite natural to take a gamma prior on λ, for known α and μ, Even for known μ, there does not exist any conjugate joint priors on α and λ. Hence, based on the suggestion of Berger and Sun (1993), no specific form of prior on α is assumed. It is simply assumed that α has a prior whose support is on $(0,\infty)$, and it has a log-concave PDF. It may be mentioned that many well known common density functions such as, normal, log-normal, gamma (when the shape parameter is greater than one) and Weibull (when the shape parameter is greater than one) have log-concave PDFs. For the location parameter, it is assumed that μ has a uniform prior over a finite interval.

In this paper, based on the above priors on (α,λ,μ), we provide the posterior analysis of the unknown parameters. The Bayes estimator of any function of the unknown parameters cannot be obtained in explicit form under the squared error loss function. In this case, it is possible to generate samples directly from the joint posterior distribution of (α,λ,μ) given the $data$. Hence, the Gibbs sampling technique can be used quite conveniently to compute the simulation consistent Bayes estimate of any function of the parameters with respect to the squared error loss function, and also to construct the highest posterior density (HPD) credible interval. The second aim of this paper is to provide a methodology to compare two different progressive censoring schemes, hence, it can be used to compute the optimal sampling scheme, for a given set of prior distribution functions. During the last few years, Zhang and Meeker (2005) and Kundu (2008) discussed the Bayesian life testing plans for one-parameter (assuming the shape parameter to be known) Weibull and two-parameter Weibull distributions, respectively. In this article we extend the work to the three-parameter case. It is observed that finding the optimal censoring scheme is a discrete optimization problem, and it is computationally quite challenging. Due to this fact, we have suggested a sub-optimal plan, which can be obtained very easily. Therefore, the implementation of our proposed procedure is quite simple in practice.

The rest of the paper is organized as follows. In Section 2, we provide the details of the model assumptions and prior distributions. Simulation consistent Bayes estimate and the associated HPD credible interval are provided in Section 3. In Section 4, we provide the simulation results and the analysis of a progressively censored data set. The construction of optimal progressive censoring scheme is provided in Section 5, and finally in Section 6, we conclude the paper.

2 Model Assumptions and Prior Distributions

2.1 Model Assumptions

It is assumed that n units which are identical in nature are put on a life testing experiment at the time point 0. We denote their life times as $T_1, \ldots, T_n$, where T_i's are independent identically distributed (i.i.d.) random variables with PDF (1). Further, the integer $0 < m < n$, and also $R_1, \ldots, R_m$ are pre-fixed integers such that

$$R_1 + \ldots + R_m = n - m.$$

The experiment starts at the time point 0, and at the time of the first failure, say, t_1, R_1 out of the remaining $(n-1)$ units are chosen randomly and they are removed from the experiment. Similarly, at the time of the second failure, say, t_2, R_2 of the remaining units are chosen at random and they are removed, and so on. Finally, when the m-th failure takes place, the rest of the remaining R_m units are removed, and the experiment stops. Therefore, the usual Type-II censoring scheme can be obtained as a special case of the Type-II progressive censoring scheme. In a Type-II progressive censoring scheme the observed data will be of the form

$$data = \{t_1, \ldots, t_m\}$$

for a given $\{R_1, \ldots, R_m\}$.

2.2 Prior Distributions

In this section we provide the prior distributions on the set of parameters. It is known that the scale parameter λ has a conjugate gamma prior when the shape and location parameters are known. Hence, it is quite reasonable to assumed that the prior on λ, $\pi_1(\lambda|a,b) \sim \text{Gamma}(a,b)$, with PDF

$$\pi_1(\lambda|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda \leq 0 \end{cases} \tag{2}$$

Here the shape parameter $a > 0$ and the scale parameter $b > 0$ are the hyperparameters, and

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx,$$

is the gamma function. Note that the above gamma prior is a very flexible prior as it can take variety of shapes depending on the shape parameter. In most practical applications with proper informative information on the Weibull scale parameter, the prior variance is usually assumed to be finite, see Congdon (2006).

In practice, the shape parameter is usually unknown, and in this case the joint conjugate prior on α and λ do not exist even when μ is known, see for example Kaminskiy and Krivtsov (2005). In this case we do not take any specific prior on α. It is assumed that α has a prior $\pi_2(\alpha)$ which has a support on the $(0,\infty)$, and it has a PDF which is log-concave. This was originally suggested by Berger and Sun (1993), see also Kundu (2008). For specific calculation, in this paper we have assumed that $\pi_2(\alpha|c,d) \sim$ Gamma(c,d). Finally following the approach of Smith and Naylor (1987) it is assumed that μ has a prior $\pi_3(\mu|e,f) \sim$ U(e,f), i.e., the PDF of $\pi_3(\mu|e,f)$ is

$$\pi_3(\mu|e,f) = \begin{cases} \frac{1}{f-e} & \text{if } e \leq \mu \leq f \\ 0 & \text{if } \mu < e \text{ or } \mu > f \end{cases} \tag{3}$$

3 Bayes Estimates and Credible Intervals

In this section we obtain the Bayes estimate of any function of the unknown parameters and the associated HPD credible interval, with respect to the set of prior distributions mentioned in Section 2. In calculating the Bayes estimates although it is assumed the loss function is squared error, any other loss function also can be easily incorporated. Based on the observed data for a given $R_1,\ldots,R_m$, the likelihood function is

$$\mathscr{L}(data|\alpha,\lambda,\mu) \propto \alpha^m\lambda^m\prod_{i=1}^{m}(t_i-\mu)^{\alpha-1}\, exp\left(-\lambda\sum_{i=1}^{m}(1+R_i)(t_i-\mu)^{\alpha}\right); \tag{4}$$

for $\alpha > 0, \lambda > 0, \mu < t_1$, and 0, otherwise. From (4), it is easily observed that if we take λ = 1, α = 0.5, and $\mu \uparrow t_1$, then $\mathscr{L}(data|\alpha,\lambda,\mu) \uparrow \infty$. It shows that when all the three parameters are unknown, then the MLEs do not exist. Therefore, Bayesian inference seems to be a reasonable choice in this case.

Based on the prior distribution mentioned above on α, λ and μ, the joint distribution of the $data$, α, λ and μ can be written as

$$\mathscr{L}(data,\alpha,\lambda,\mu) = \mathscr{L}(data|\alpha,\lambda,\mu)\cdot\pi_1(\lambda)\cdot\pi_2(\alpha)\cdot\pi_3(\mu). \tag{5}$$

Based on (5), the joint posterior distribution of the unknown parameters is given by

$$\mathscr{L}(\alpha,\lambda,\mu|data) = \frac{\mathscr{L}(data|\alpha,\lambda,\mu)\cdot\pi_1(\lambda)\cdot\pi_2(\alpha)\cdot\pi_3(\mu)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathscr{L}(data|\alpha,\lambda,\mu)\cdot\pi_1(\lambda|\alpha,\mu)\cdot\pi_2(\alpha|\mu)\cdot\pi_3(\mu)}. \tag{6}$$

Therefore, the Bayes estimate of any function of α, λ and μ, say $g(\alpha,\lambda,\mu)$, under the squared error loss function is

$$\begin{aligned}\widehat{g}_B(\alpha,\lambda,\mu) &= E_{\alpha,\lambda,\mu|data}(g(\alpha,\lambda,\mu)) \\ &= \int_0^\infty \int_0^\infty \int_e^f g(\alpha,\lambda,\mu)\mathscr{L}(\lambda|\alpha,\mu,data)\cdot\mathscr{L}(\alpha|\mu,data)\cdot\mathscr{L}(\mu|data) \\ &\qquad d\mu d\lambda d\alpha. \end{aligned} \tag{7}$$

It is quite obvious that it will not be possible to compute (7) analytically for general $g(\alpha,\lambda,\mu)$. We propose to use Gibbs sampling technique to generate samples directly from the joint posterior distribution function (6), and based on the generated sample we provide a simulation consistent estimate of (7) and the associated highest posterior density credible interval of $\widehat{g}_B(\alpha,\lambda,\mu)$.

We need the following results for further development.

THEOREM 1: The conditional density of λ given α, μ and $data$ is

$$\mathscr{L}(\lambda|\alpha,\mu,data)\Gamma\left(a+m,\ b+\sum_{i=1}^m(1+R_i)(t_i-\mu)^\alpha\right).$$

PROOF: It is simple, hence the details are avoided.

THEOREM 2: The conditional density of α given μ and $data$, $\mathscr{L}(\alpha|\mu,data)$, is

$$\mathscr{L}(\alpha|\mu,data) = \kappa(\mu)\frac{\alpha^{c+m-1}\cdot e^{-d\alpha}\prod_{i=1}^m(t_i-\mu)^{\alpha-1}}{[b+\sum_{i=1}^m(1+R_i)(t_i-\mu)^\alpha]^{a+m}}, \tag{8}$$

and (8) is log-concave. Here $\kappa(\mu)$ is the normalizing constant, i.e.

$$\kappa(\mu) = \left[\int_0^\infty \frac{\alpha^{c+m-1}\cdot e^{-d\alpha}\prod_{i=1}^m(t_i-\mu)^{\alpha-1}}{[b+\sum_{i=1}^m(1+R_i)(t_i-\mu)^\alpha]^{a+m}}d\alpha\right]^{-1}.$$

PROOF: See in the Appendix.

THEOREM 3: The conditional density of μ given $data$ is

$$\mathscr{L}(\mu|data) \propto \kappa(\mu),$$

if $\mu \in [e,f]\cap(-\infty,t_1]$, and 0, otherwise.

PROOF: It is trivial.

Now using Theorems 1-3, and following the idea of Geman and Geman (1984), we propose the following algorithm to generate samples (λ,α,μ) directly from the posterior distribution (6). The generated samples can be used to compute simulation consistent Bayes estimate of any function of the unknown parameters, and the associated HPD credible interval.

ALGORITHM 1:

Step 1: Given the data $\{t_1,\ldots,t_m\}$, compute $\max\{\kappa(\mu)\}$ for $\mu \in [e,f]\cap(-\infty,t_1]$.

Step 2: Using acceptance rejection principle generate μ from $\mathscr{L}(\mu|data)$.

Step 3: Generate $\alpha \sim \mathscr{L}(\alpha|\mu, data)$ using the method proposed by Devroye (1984) or Kundu (2008).
Step 4: Generate $\lambda \sim \mathscr{L}(\lambda|\alpha, \mu, data)$
Step 5: Repeat Step 2 to Step 4, N times to generate $\{(\lambda_1, \alpha_1, \mu_1), \ldots, (\lambda_N, \alpha_N, \mu_N)\}$ and obtain $g_i = g(\lambda_i, \alpha_i, \mu_i)$, for $i = 1, \ldots, N$.
Step 6: A simulation consistent estimate of $\widehat{g}_B(\alpha, \lambda, \mu)$ is

$$\widehat{g} = \frac{1}{N}\sum_{i=1}^{N} g_i.$$

Step 7: A 100(1-β)% credible interval of $g(\alpha, \lambda, \mu)$ can be obtained as

$$(g_j, g_{j+[\beta N]}); \quad j = 1, \ldots, N - [\beta N].$$

Step 8: The HPD 100(1-β)% credible interval of $g(\alpha, \lambda, \mu)$ can be obtained as

$$(g_{j^*}, g_{j^*+[\beta N]}); \quad \text{such that} \quad g_{j^*+[\beta N]} - g_{j^*} \leq g_{j+[\beta N]} - g_j; \quad j = 1, \ldots, N - [\beta N].$$

4 Simulation & Data Analysis

4.1 Simulation

In this section we perform some simulation experiments to show the effectiveness of the proposed methods. All the simulations are performed using the statistical software **R**. We use different sample sizes n, different effective sample sizes m, different sets of parameter values and different sampling schemes. We use the following set of parameter values; Set 1: $(\alpha = 1, \lambda = 1, \mu = 0)$ and Set 2: $(\alpha = 1.5, \lambda = 1, \mu = 1)$, and the following censoring schemes.

1. Scheme 1: $n = 20$, $m = 10$, $R_1 \ldots R_{10} = 1$,
2. Scheme 2: $n = 20$, $m = 10$, $R_1 = 5, R_2 \ldots R_9 = 0, R_{10} = 5$,
3. Scheme 3: $n = 30$, $m = 20$, $R_1 \ldots R_{19} = 0, R_{20} = 10$.

Further we have used three different sets of hyperparameters to see the effects of the priors on the Bayes estimates and the associated credible intervals.

1. Prior 0: $a = b = c = d = 0.0001$ and e = -1.0, $f = t_1$
2. Prior 1: $a = b = c = d = 5$ and e = -1.0, $f = t_1$
3. Prior 2: $a = b = 5$, $c = 2.25$, $d = 1.5$ and e= -1.0, $f = t_1$

Note that Prior 0 is a non-informative prior; Prior 1 is an informative prior when we use Set 1, and and Prior 2 is an informative prior when we use Set 2. Prior 1 is an informative prior as the means of the prior distributions of the shape and scale parameters are same with the original distribution when Set 1 is used. In addition, it has a low variance. Similarly, for Set 2, Prior 2 ensures that the means of the prior distributions for the scale and the shape parameters are same with the the original distribution.

In each case we generated progressively censored samples based on the algorithm proposed by Balakrishnan and Aggarwala (2000) and we computed simulation consistent Bayes estimate and the corresponding mean squared errors (MSEs). The results are presented in Tables 1 - 6. Some of the points are quite clear from this simulation results. It is observed that in all the cases considered as expected the biases and the MSEs of the Bayes estimators based on informative priors perform better than the non-informative priors. Moreover, as the effective sample size increases the biases and MSEs decrease in all cases considered.

Par	Prior 0	Prior 1
α	1.1571	1.0886
	(0.6642)	(0.1180)
λ	0.8299	1.0809
	(0.0988)	(0.0459)
μ	-0.0514	-0.0455
	(0.0017)	(0.0015)

Table 1 Average Bayes estimates and the MSEs for α, λ, μ: Set 1, Scheme 1

Par	Prior 0	Prior 1
α	1.3840	1.2025
	(0.7501)	(0.0842)
λ	1.3714	1.0213
	(0.1499)	(0.0935)
μ	-0.0498	-0.0428
	(0.0016)	(0.0015)

Table 2 Average Bayes estimates and the MSEs for α, λ, μ: Set 1, Scheme 2

Par	Prior 0	Prior 1
α	1.2917	1.2015
	(0.6696)	(0.2172)
λ	0.9364	0.9947
	(0.1092)	(0.0611)
μ	-0.0505	-0.0449
	(0.0016)	(0.0015)

Table 3 Average Bayes estimates and the MSEs for α, λ, μ: Set 1, Scheme 3

We have also computed the 95% highest posterior density (HPD) credible interval based on Prior 0, and we present the average lengths of the HPD credible intervals and the associated coverage percentages based on 10000 MCMC samples.

Par	Prior 0	Prior 1
α	1.5306 (0.1708)	1.4678 (0.1102)
λ	0.8083 (0.0727)	1.0305 (0.0526)
μ	0.9124 (0.0087)	0.9822 (0.0041)

Table 4 Average Bayes estimates and the MSEs for α, λ, μ: Set-2, Scheme -1

Par	Prior 0	Prior 1
α	1.7221 (0.3221)	1.5912 (0.1529)
λ	0.8617 (0.0866)	0.9694 (0.0633)
μ	0.9872 (0.0041)	0.9919 (0.0039)

Table 5 Average Bayes estimates and the MSEs for α, λ, μ: Set 2, Scheme 2

Par	Prior 0	Prior 1
α	1.5716 (0.2328)	1.6179 (0.1315)
λ	1.1093 (0.0852)	1.0376 (0.0691)
μ	0.9917 (0.0041)	1.0249 (0.0003)

Table 6 Average Bayes estimates and the MSEs for α, λ, μ: Set 2 Scheme 3

Set	Scheme	μ	α	λ
μ=0,α=1,λ=1	(20,10,10*1)	0.1191 0.91	0.9387 0.96	0.8349 0.92
	(20,10,5,8*0,5)	0.1200 0.92	1.1022 0.95	1.0421 0.96
	(20,15,14*0,5)	0.1010 0.93	0.9074 0.96	0.8118 0.93
μ=1,α=1.5,λ=1	(20,10,10*1)	0.1970 0.96	1.5352 0.98	0.6852 0.84
	(20,10,5,8*0,5)	0.1986 0.93	1.4811 0.95	1.0539 0.96
	(20,15,14*0,5)	0.1955 0.94	1.0916 0.96	0.0512 0.96

Table 7 Average HPD Bayes Interval Length and Coverage Percentage

The results are presented in Table 7. From the simulation results it is clear that the proposed method is working quite satisfactorily in all the cases considered. It is observed that the coverage percentages of the HPD credible intervals are quite close to the nominal values in all the cases considered. Moreover as the effective sample size increases, the average lengths of the HPD credible intervals decrease.

4.2 Data Analysis

In this section we perform the analysis of a data set for illustrative purposes. The original data is from Lawless(1982), which represents the failure times for 36 appliances which are subject to an automatic life test. A progressively censored sample with the following censoring scheme: n = 36, m = 10, $R_1 = \ldots = R_9$ = 2, R_{10} = 8 was generated and and analyzed by Kundu and Joarder (2006) using an exponential model. The data are presented below.

$$\{11, 35, 49, 170, 329, 958, 1925, 2223, 2400, 2568\}.$$

Kundu (2008) used the same data set and analyzed it using a two-parameter Weibull model. Here, we use a three-parameter Weibull distribution to analyze this data set. For computational convenience, we divided all the values by 100 and it is not going to affect the analysis.

Since we do not have any prior knowledge of the parameters we have assumed non-informative priors, i.e. $a = b = c = d$ = 0.0001, as suggested by Congdon (2006). We have assumed $\mu \sim \mathrm{U}(-0.5, 0.011)$. Based on the above priors we compute the Bayes estimates and the associated 95% HPD credible intervals. The Bayes estimates of α, λ and μ are 0.6915, 0.2515, -0.0189, respectively. The associated 95% HPD credible intervals are (0.5213, 0.7913), (0.1317,3112), (-0214,-0.0079), respectively.

5 Optimal Censoring Scheme

In practice, it is very important to choose the ‘optimum’ censoring scheme among the class of all possible censoring schemes. Here possible censoring schemes mean, for fixed n and m, all possible choices of $\{R_1, \ldots, R_m\}$, such that

$$\sum_{i=1}^{m} R_i + m = n.$$

In this section we use a similar precision criterion as used by Zhang & Meeker(2008) and Kundu (2008) to choose the optimal censoring scheme from a class of possible schemes. We say that a censoring scheme $\Psi_1 = \{R_1^1 \ldots R_m^1\}$ is superior to $\Psi_2 = \{R_1^2 \ldots R_m^2\}$, if the information derived from Ψ_1 is more accurate than the information derived from Ψ_2. The method we employ here will be to examine the best censoring scheme for different combinations of n and m. The number of such possible schemes is $\binom{n-1}{m-1}$. Therefore, even when n = 25 and m = 12, the total number of possible schemes is 2,496,144, which is quite large. Hence, we search for a ‘sub-optimum’ censoring scheme by finding the best censoring scheme among those schemes where the entire weight $n-m$ is on a single R_j; $j = 1, \ldots, m$. By doing so, we obtain a convex hull of all possible censoring schemes, and we search only along the corner points.

Note that the p^{th} quantile point, T_p, of the three-parameter Weibull distribution is given by,

$$T_p = \mu + \left[-\frac{1}{\lambda} \cdot ln(1-p) \right]^{\frac{1}{\alpha}}.$$

The precision criterion we use here is the posterior variance of $\ln(T_p)$, see for example Kundu (2008) or Zhang and Meeker (2005). Since the posterior variance of $\ln(T_p)$ depends on the observed sample when the parameters are unknown, we eliminate any sample bias by taking the average posterior variance of $\ln(T_p)$ for multiple samples obtained from the same joint prior distribution. The precision criteria used is given as,

$$C(\Psi) = \mathbb{E}_{\pi(\alpha,\lambda,\mu)} \left(\mathbb{V}_{posterior}(ln(T_p)|\Psi) \right).$$

Where, $\Psi = \{R_1 \ldots R_m\}$ is the censoring scheme, and $\mathbb{V}_{posterior}(ln(T_p)|\Psi)$ is the posterior variance of the plan Ψ. We expect to minimize the posterior variance of the p^{th} quantile, and hence if $C(\Psi_1) < C(\Psi_1)$, we say that Ψ_1 is superior to Ψ_2. Therefore, we would like to choose that Ψ, a particular censoring scheme, for which $C(\psi)$ is minimum.

There are no explicit expressions known for the quantity $\mathbb{V}_{posterior}(ln(T_p)|\Psi)$, hence we proceed to compute the value of the precision criterion using Monte Carlo method. The following algorithm can be used for that purpose.

ALGORITHM:

1. Fix n, m and the priors a,b,c,d,e,f
2. Generate $(\alpha_i, \lambda_i, \mu_i)$ from $\pi(\alpha, \lambda, \mu | a,b,c,d,e,f)$
3. Generate a progressively censored sample $\{t_{(1)} \ldots t_{(m)}\}$ from $(\alpha_i, \lambda_i, \mu_i)$
4. For the data $\{t_{(1)} \ldots t_{(m)}\}$, estimate $\mathbb{V}_{posterior}(ln(T_p)|\Psi)$ using Gibbs sampling method as suggested in Section 3.
5. Repeat Steps 1-4 for N iterations and their average will provide an estimate of $C(\Psi)$.

We have considered two different sets namely ($n = 20$, $m = 10$) and ($n = 20$, $m = 15$). We compute the precision criterion based on 4 different quantiles namely for $p = 0.50$, $p = 0.75$, $p = 0.95$, $p = 0.99$ for Prior 1, and the results are presented in Tables 8 and 9.

6 Conclusion

In this paper we have provided the Bayesian inference of a three-parameter Weibull distribution based on the Type-II progressively censored data. It is well known that when all the three parameters are unknown the MLEs do not exist. Hence, Bayesian inference seems to be a reasonable choice. We have provided the Bayes estimates of the unknown parameters based on the squared error loss function when the shape

n=20,m=10	C(Ψ)	C(Ψ)	C(Ψ)	C(Ψ)
Scheme	p=0.50	p=0.75	p=0.95	p=0.99
10,0,0,0,0,0,0,0,0,0	2.287547	3.033706	3.456304	3.931685
0,10,0,0,0,0,0,0,0,0	1.409668	2.886745	3.584613	4.690873
0,0,10,0,0,0,0,0,0,0	1.569803	2.13762	3.883272	4.941062
0,0,0,10,0,0,0,0,0,0	1.816804	2.382721	4.044178	4.995784
0,0,0,0,10,0,0,0,0,0	2.69913	2.621744	4.173664	5.163259
0,0,0,0,0,10,0,0,0,0	2.268952	3.078297	4.330179	5.200208
0,0,0,0,0,0,10,0,0,0	2.51713	3.623896	4.667893	5.300237
0,0,0,0,0,0,0,10,0,0	2.15651	4.002551	4.954882	5.540816
0,0,0,0,0,0,0,0,10,0	1.573069	3.407709	5.326496	6.04542
0,0,0,0,0,0,0,0,0,10	1.396492	3.374444	4.979677	6.405011

Table 8 Optimal Censoring Scheme for $n = 20$, $m = 10$

n=20,m=15	C(Ψ)	C(Ψ)	C(Ψ)	C(Ψ)
Scheme	p=0.50	p=0.75	p=0.95	p=0.99
15,0,0,0,0,0,0,0,0,0,0,0,0,0,0	1.430799	2.0387188	2.945988	3.527577
0,15,0,0,0,0,0,0,0,0,0,0,0,0,0	1.588445	1.472637	3.909428	3.913904
0,0,15,0,0,0,0,0,0,0,0,0,0,0,0	1.096379	2.397453	6.1389017	4.038037
0,0,0,15,0,0,0,0,0,0,0,0,0,0,0	0.9984362	2.195926	3.237427	4.480111
0,0,0,0,15,0,0,0,0,0,0,0,0,0,0	0.630354	1.699538	4.917337	4.73495
0,0,0,0,0,15,0,0,0,0,0,0,0,0,0	0.8723458	2.405792	3.751554	4.7763508
0,0,0,0,0,0,15,0,0,0,0,0,0,0,0	0.8889859	2.637907	4.623516	5.238868
0,0,0,0,0,0,0,15,0,0,0,0,0,0,0	1.083057	3.242228	4.1193264	5.339358
0,0,0,0,0,0,0,0,15,0,0,0,0,0,0	1.410338	1.8871604	3.4573745	5.4188804
0,0,0,0,0,0,0,0,0,15,0,0,0,0,0	1.832797	2.494387	4.5991795	5.980342
0,0,0,0,0,0,0,0,0,0,15,0,0,0,0	1.083604	1.647242	3.170819	6.203538
0,0,0,0,0,0,0,0,0,0,0,15,0,0,0	1.289311	1.896631	3.816518	6.402096
0,0,0,0,0,0,0,0,0,0,0,0,15,0,0	1.011111	2.999492	3.665278	6.69655
0,0,0,0,0,0,0,0,0,0,0,0,0,15,0	0.7898133	2.0053203	4.8234677	6.92173
0,0,0,0,0,0,0,0,0,0,0,0,0,0,15	0.8662691	2.150556	5.021647	8.8057628

Table 9 Optimal Censoring Scheme for $n = 20$, $m = 15$

and scale parameters have gamma priors and the location parameter has an uniform prior over a fixed interval. It is not possible to obtain the Bayes in closed form, hence, we have suggested to use Gibbs sampling procedure to compute the Bayes estimates and the associated HPD credible intervals. Some simulation experiments have been performed and it is observed that the Bayes estimates and the associated HPD credible intervals perform quite satisfactorily. It may be mentioned that although we have considered the squared error loss function our method can be easily generalized for any other loss function also.

We have further considered the problem of finding the optimum censoring scheme among the class of all possible censoring schemes based on a similar criterion proposed by Zhang and Meeker (2005) and Kundu (2008). Since the total number of possible censoring schemes is quite high, we have provided an algorithm to find the obtained sub-optimum censoring scheme. Some sub-optimum sampling schemes have been presented. Finding an optimum censoring scheme from the all possible censoring schemes remains an open problem from the computational point of view. Efficient algorithm is needed to find the optimum censoring scheme particularly when n is large and $m \approx n/2$. More work is needed along that direction.

Appendix

PROOF OF THEOREM 2: To prove Theorem 2, it is enough to prove that

$$\frac{d^2}{d\alpha^2} \ln[b + \sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha}] > 0.$$

Now consider

$$g(\alpha) = b + \sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha};$$

then

$$g'(\alpha) = \frac{d}{d\alpha} g(\alpha) = \sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha} \ln(t_i - \mu)$$

and

$$g''(\alpha) = \frac{d^2}{d\alpha^2} g(\alpha) = \sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha} (\ln(t_i - \mu))^2.$$

Since

$$\begin{aligned} &\left(\sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha} (\ln(t_i - \mu))^2\right) \left(\sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha}\right) \\ &- \left(\sum_{i=1}^{m} (1+R_i)(t_i - \mu)^{\alpha} \ln(t_i - \mu)\right)^2 \\ &= \sum_{1 \le i < j \le m} (R_i + 1)(R_j + 1) \left(\ln(t_i - \mu) - \ln(t_j - \mu)\right)^2 \ge 0, \end{aligned}$$

the result follows.

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