Bayesian Analysis of Progressively Censored Competing Risks Data

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This is a joint work with Dr. Biswabrata Pradhan, and part of this work is going to appear in Sankhya, B.

December 30, 2011
Outline

1. Competing Risks
2. Progressive Censoring
3. Model Formulation and Prior Assumption
4. Posterior Analysis
5. Optimal Censoring Scheme
6. Open Problems
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What is a competing risk model?

In many life-testing studies often the failure of items may be associated to more than one cause. These risks factors in some sense compete with each other for the failure of the experimental unit. The models which can be used to analyze such data are known as the competing risk models.
Typically competing risk data looks like \((T, \Delta)\). Here \(T\) denotes the lifetime of the item and \(\Delta\) denotes the cause of failure.

The cause of failure may be assumed to be dependent or independent.

Although the assumption of dependence seems more reasonable, there is some identifiability issue.

It is not possible to test the assumption of independence of the failure time distributions without the presence of covariates.
Cox (1959, JRSS B) proposed the latent failure time modeling to analyze competing risk data. The latent failure time model has the following form:

\[ T_i = \min\{X_{i1}, \cdots, X_{iM}\} \]

here \( T_i \) denotes the lifetime of the \( i \)-th individual. \( X_{i1}, \cdots, X_{iM} \) are the latent failure times of the \( M \) different causes for the \( i \)-th individual, and they are assumed to be independent.
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Progressive censoring can be described as follows:

- Put \( n \) items on test.
- \( m \) and \( R_1, \ldots, R_m \) are pre-fixed integers such that
  \[
  R_1 + \cdots + R_m = n - m.
  \]
- At the time of the first failure, \( R_1 \) of the remaining units are randomly chosen and removed. Similarly, at the time of the second failure \( R_2 \) of the remaining units are chosen and removed. Finally at the time of the \( m \)-th failure rest of the \( R_m = n - m - R_1 - \cdots - R_{m-1} \) units are removed.
Competing Risks
Progressive Censoring
Model Formulation and Prior Assumption
Posterior Analysis
Optimal Censoring Scheme
Open Problems

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Progressively Censored Competing Risks Data

It is assumed we have a progressively censored competing risks data. Therefore, a progressively censored competing risks data will be as follows:

\[(t_1, \delta_1, R_1), \ldots, (t_m, \delta_m, R_m)\]

\[\{(t_i, 1); i \in I_1\}, \ldots, \{(t_i, M); i \in I_M\},\]

here

\[I_1 = \{i; \delta_i = 1\}, \ldots, I_M = \{i; \delta_i = M\}.\]
Here it is assumed that $M = 2$. The latent failure times have distributions with the same shape parameter but different scale parameters, i.e. $X_1 \sim \text{WE}(\alpha, \lambda_1)$ and $X_2 \sim \text{WE}(\alpha, \lambda_2)$ and they are independent. $\text{WE}(\alpha, \lambda)$ has the PDF

$$f(x; \alpha, \lambda) = \alpha \lambda x^{\alpha - 1} e^{-\lambda x^\alpha}; \quad x > 0.$$ 

It is well known that Weibull distribution is a very flexible lifetime distribution. Moreover, $\min\{X_1, X_2\} \sim \text{WE}(\alpha, \lambda)$, $\lambda = \lambda_1 + \lambda_2$. 
Prior Assumptions on $\lambda_1$ and $\lambda_2$

\[ \pi_0(\lambda|a_0, b_0) = \frac{b_0^{a_0}}{\Gamma(a_0)} \lambda^{a_0-1} e^{-b_0 \lambda} \]

\[ \pi(\lambda_1/\lambda|a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \left( \frac{\lambda_1}{\lambda} \right)^{a_1-1} \left( 1 - \frac{\lambda_1}{\lambda} \right)^{a_2-1} \]

\[ \pi(\lambda_1, \lambda_2|a_0, b_0, a_1, a_2) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_0)} (b_0 \lambda)^{a_0-a_1-a_2} \times \]

\[ \frac{b_0^{a_1}}{\Gamma(a_1)} \lambda_1^{a_1-1} e^{-b_0 \lambda_1} \]

\[ \times \frac{b_0^{a_2}}{\Gamma(a_2)} \lambda_2^{a_2-1} e^{-b_0 \lambda_2} \]
Beta-Dirichlet Prior

1. \[ \pi(\lambda_1, \lambda_2|a_0, b_0, a_1, a_2) \sim BD(b_0, a_0, a_1, a_2) \]

2. If \( a_0 = a_1 + a_2 \), \( \lambda_1 \) and \( \lambda_2 \) are independent.

3. \( \lambda_1 \) and \( \lambda_2 \) positively or negatively correlated if \( a_0 > a_1 + a_2 \) or \( a_0 < a_1 + a_2 \) respectively.
If \((\lambda_1, \lambda_2) \sim BD(b_0, a_0, a_1, a_2)\), then for \(i = 1, 2\)

\[
E(\lambda_i) = \frac{a_0 a_i}{b_0 (a_1 + a_2)}
\]

\[
V(\lambda_i) = \frac{a_0 a_i}{b_0^2 (a_1 + a_2)} \times \left\{ \frac{(a_i + 1)(a_0 + 1)}{a_1 + a_2 + 1} - \frac{a_0 a_i}{a_1 + a_2} \right\}
\]
Prior Assumption on $\alpha$

No specific prior has been assumed on $\alpha$. It is assumed that the prior on $\alpha$ has a support on $(0, \infty)$, the PDF is log-concave. It is independent of the prior on $(\lambda_1, \lambda_2)$.

Note that several standard distribution functions have log-concave PDF, for example (i) log-normal, (ii) gamma (shape parameter greater than 1), (iii) Weibull (shape parameter greater than 1), (iii) generalized exponential distribution (shape parameter greater than 1) etc.
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Shape Parameter Known

In this case

\[ \pi(\lambda_1, \lambda_2 | \text{Data}) \sim \text{BD} (B_0, A_0, A_1, A_2), \]

here

\[ B_0 = b_0 + \sum_{i=1}^{m} (R_i + 1) t_i^\alpha \]
\[ A_0 = a_0 + m_1 + m_2 \]
\[ A_1 = a_1 + m_1 \]
\[ A_2 = a_2 + m_2. \]
Bayes Estimates

The Bayes estimate under the squared error loss function is the posterior mean. Therefore, the Bayes estimates of $\lambda_1$ and $\lambda_2$ can be obtained as

$$\hat{\lambda}_{1B} = \frac{A_0 A_1}{B_0 (A_1 + A_2)}$$

$$\hat{\lambda}_{2B} = \frac{A_0 A_2}{B_0 (A_1 + A_2)}$$

The corresponding posterior variances also can be easily obtained in terms of $B_0, A_0, A_1, A_2$. 
A set $C_\beta$ is called a level 100(1-$\beta$)% credible set if

$$P \left( (\lambda_1, \lambda_2) \in C_\beta \mid (\lambda_1, \lambda_2) \sim \pi(B_0, A_0, A_1, A_2) \right) = 1 - \beta$$

$(\lambda_1, \lambda_2 \mid Data) \sim BD(B_0, A_0, A_1, A_2)$, therefore

$$\lambda_1 + \lambda_2 \mid Data \sim Gamma(A_0, B_0)$$

and

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} \mid Data \sim Beta(A_1, A_2)$$

and they are independently distributed.
Choose $\beta_1$ and $\beta_2$ such that $(1-\beta) = (1-\beta_1)(1-\beta_2)$. Further choose $(l_1, u_1)$ and $(l_2, u_2)$ such that

\[
P(l_1 \leq \lambda_1 + \lambda_2 \leq u_1) = 1 - \beta_1
\]

\[
P(l_2 \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \leq u_2) = 1 - \beta_2.
\]

\[
C_{\beta} = \left\{ (\lambda_1, \lambda_2); l_1 \leq \lambda_1 \leq u_1, l_2 \leq \frac{\lambda_1}{\lambda_1 + \lambda_2} \leq u_2 \right\}
\]
C_{\beta} = \text{The area between the four straight lines:}

\begin{align*}
\lambda_1 &= u_1, \\
\lambda_1 &= l_1, \\
\lambda_1(1 - u_2) &= u_2\lambda_2 \\
\lambda_1(1 - l_2) &= l_2\lambda_2.
\end{align*}
Common Shape Parameter is Unknown

In this case the joint posterior distribution of $\alpha$, $\lambda_1$ and $\lambda_2$ given the Data can be written as follows:

$$l(\alpha, \lambda_1, \lambda_2|Data) = l(\lambda_1, \lambda_2|\alpha, Data) \times l(\alpha|Data),$$

where

$$l(\lambda_1, \lambda_2|\alpha, Data) \sim BD(B_0, A_0, A_1, A_2)$$

$$l(\alpha|Data) \propto \pi_2(\alpha)\alpha^m \prod_{i=1}^{m} t_i^{\alpha} \times \frac{1}{B_{A_0}^{A_0}}$$
It is not possible to compute the Bayes estimates or the associated credible intervals in closed form for $\alpha$, $\lambda_1$, and $\lambda_2$. The following result becomes useful:

**Theorem** The conditional PDF of $\alpha$ given the data is log-concave.
Algorithm


2. Step 2: Given $\alpha$ generate $\lambda_1, \lambda_2$ from a BD($B_0, A_0, A_1, A_2$).

3. Step 3: Once the posterior samples are obtained, then simulation consistent Bayes estimates and the associated credible intervals can be constructed.
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What is an Optimal Censoring Scheme? For a given $n$ and $m$ what should be the choice of $R_1, \cdots, R_m$ such

$$R_1 + \cdots + R_m = n - m$$

and it provides the maximum *Information* of the unknown parameters.
Information measure

How to define Information measure of a censoring scheme? If it is only one parameter, the variance of the estimators can be defined as an information measure. But for more than one parameters, the choice is not clear. Trace or determinant of the Fisher information matrix has been used, but they are not scale invariant.
Information measure

Variance of the $p$-th percentile estimator can be taken as a possible measure, Zhang and Meeker (2005, Metrika). In this case

$$C(R) = w \text{Var}(\ln \hat{T}_{p,1}) + (1 - w) \text{Var}(\ln \hat{T}_{p,2}),$$

where $0 < w < 1$, and

$$T_{p,1} = \left[-\frac{1}{\lambda_1} \ln(1 - p)\right]^{1/\alpha} \quad T_{p,2} = \left[-\frac{1}{\lambda_2} \ln(1 - p)\right]^{1/\alpha}$$
The proposed criterion takes the following form:

\[ C^B(R) = wE_{data} \left[ V_{posterior}(\ln \hat{T}_{p,1}) \right] + (1-w)E_{data} \left[ V_{posterior}(\ln \hat{T}_{p,2}) \right] \]

It depends on the percentile point \( p \). The following more general criterion has been proposed.
Criterion

\[
C^B(R) = wE_{data} \left[ \int_0^1 V_{posterior}(\ln \hat{T}_{p,1}) dW_1(p) \right] \\
+ (1 - w)E_{data} \left[ \int_0^1 V_{posterior}(\ln \hat{T}_{p,2}) dW_2(p) \right]
\]
In all the cases the optimization has to be performed numerically. It is a discrete optimization problem. For a given $n$ and $m$, the optimum censoring scheme with respect to a given criterion can be found by exhaustive search for all possible values of $R_i$’s satisfying

$$R_1 + \cdots + R_m = n - m.$$
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Open problems

1. $M > 2$.

2. Other censoring schemes, like Type-I progressive censoring scheme, (ii) Type-I hybrid censoring scheme, (iii) Type-II hybrid censoring scheme etc.

3. Other proportional hazard models.

4. Data analysis in presence of covariates.
Thank You