INFERENCES ON STRESS-STRENGTH RELIABILITY FROM LINDLEY DISTRIBUTIONS

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Abstract

This paper deals with the estimation of the stress-strength parameter $R = P(Y < X)$ when X and Y are independent Lindley random variables with different shape parameters. The uniformly minimum variance unbiased estimator has explicit expression, however, its exact or asymptotic distribution is very difficult to obtain. The maximum likelihood estimator of the unknown parameter can also be obtained in explicit form. We obtain the asymptotic distribution of the maximum likelihood estimator and it can be used to construct confidence interval of R. Different parametric bootstrap confidence intervals are also proposed. Bayes estimator and the associated credible interval based on independent gamma priors on the unknown parameters are obtained using Monte Carlo methods. Different methods are compared using simulations and one data analysis has been performed for illustrative purposes.

Key Words and Phrases Lindley distribution; Maximum likelihood estimator; Asymptotic distribution; Uniformly minimum variance unbiased estimator; Prior distribution; Posterior analysis; Credible intervals.

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1. Introduction

The Lindley distribution was originally proposed by Lindley (1958) in the context of Bayesian statistics, as a counter example of fudicial statistics. The Lindley distribution has the following probability density function (PDF)

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, \qquad x > 0, \quad \theta > 0. \tag{1}$$

From now on if a random variable (RV) X has the PDF (1), then it will be denoted by Lindley(θ). The corresponding cumulative distribution function (CDF) and hazard rate function (HRF) are

$$F(x;\theta) = 1 - \left(1 + \frac{\theta}{1+\theta}x\right) e^{-\theta x}, \tag{2}$$

and

$$h(x;\theta) = \frac{f(x;\theta)}{1 - F(x;\theta)} = \frac{\theta^2(1+x)}{1+\theta(1+x)}, \tag{3}$$

respectively. Ghitany *et al.* (2008) showed that the PDF of the Lindley distribution is unimodal when $0 < \theta < 1$, and is decreasing when $\theta \geq 1$. They have also shown that the shape of the HRF of the Lindley distribution is an increasing function and the mean residual life function

$$\mu(x) = E(X - x|X > x) = \frac{1}{\theta} + \frac{1}{\theta(1+\theta+\theta x)}, \tag{4}$$

is a decreasing function of x. It may be mentioned that the Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distribution with shape parameter 2. Therefore, many properties of the mixture distributions can be translated for the Lindley distribution. Ghitany *et al.* (2008) also developed different properties and the necessary inferential procedure for the Lindley distribution.

The main aim of this paper is to develop the inferential procedure of the stress-strength parameter $R = P(Y < X)$, when X and Y are independent Lindley (θ_1) and Lindley (θ_2), respectively. Note that the stress-strength parameter plays an important role in the reliability analysis. For example if X is the strength of a system which is subjected to stress Y, then the parameter R measures the system performance and it is very common in the context of mechanical reliability of a system. Moreover, R provides the probability of a system failure, if the system fails whenever the applied stress is greater than its strength.

This problem has a long history starting with the pioneering work of Birnbaum (1956) and Birnbaum and McCarty (1958). The term stress-strength was first introduced by Church and Harris (1970). Since then significant amount of work has been done both from parametric and non-parametric point of view. A comprehensive treatment of the different stress-strength models till 2001 can be found in the excellent monograph by Kotz *et al.* (2003). Some of the recent work on the stress-strength model can be obtained in Kundu and Gupta (2005, 2006), Raqab and Kundu (2005), Kundu and Raqab (2009), Krishnamoorthy *et al.* (2007) and the references cited therein.

In this paper, first we obtain the distribution of sum of n independent identically distributed (IID) Lindley RVs, which is not available in the literature and it is useful to obtain the uniformly minimum variance unbiased estimator (UMVUE) of R. It is observed that the distribution of the sum of n IID RVs can be obtained as a mixture of n gamma RVs with different shape, but same scale parameters. It is possible to obtain the UMVUE of R in explicit form, however, the exact or asymptotic distribution of the UMVUE of R is very difficult to obtain, and it is not pursued further. On the other hand the MLE of R also can be obtained in explicit form and is very convenient to use in practice. The asymptotic distribution of the MLE of R can be easily obtained and based on that, we obtain the asymptotic confidence interval of R. We also propose three different parametric bootstrap confidence intervals of R, which are also very easy to use in practice.

We further consider the Bayes estimator of R and the associated credible interval under the assumption of independent gamma priors on the unknown parameters. We have restricted our attention mainly on the squared error loss function, although any other loss functions also can be easily incorporated. The Bayes estimator of R cannot be obtained in explicit form, we have proposed to use the Monte Carlo techniques to compute the Bayes estimate of R and the associated credible interval. Different methods are compared using Monte Carlo simulations and one data set has been analyzed for illustrative purposes.

The rest of the paper is organized as follows. In Section 2, we provide the UMVUE and MLE of R. Confidence interval based on the asymptotic distribution of the MLE and three other bootstrap confidence intervals are discussed in Section 3. In Section 4 we discuss Bayesian inference on R. Monte Carlo simulation results and data analysis are presented in Section 5 and Section 6, respectively. Finally, we conclude the paper in Section 7.

2. UMVUE and MLE of R

Suppose that X and Y are two independent Lindley RVs with respective parameters θ_1 and θ_2 having PDFs $f_X(\cdot)$ and $f_Y(\cdot)$. Then

$$\begin{aligned} R &= P(Y < X) \\ &= \int_0^\infty P(Y < X | Y = y)\ f_Y(y)\ dy \\ &= 1 - \frac{\theta_1^2[\theta_1(\theta_1+1) + \theta_2(\theta_1+1)(\theta_1+3) + \theta_2^2(2\theta_2+3) + \theta_2^3]}{(\theta_1+1)(\theta_2+1)(\theta_1+\theta_2)^3}. \end{aligned} \tag{5}$$

2.1 UMVUE of R

We need the following results for further development.

THEOREM 1: If $X_1, \cdots, X_n$ are IID RVs from Lindley (θ), then the PDF of $Z = X_1 + \cdots + X_n$ is

$$g(z; n, \theta) = \sum_{k=0}^{n} p_{k,n}(\theta)\ f_{GA}(z; 2n-k, \theta), \tag{6}$$

where $p_{k,n}(\theta) = \binom{n}{k} \dfrac{\theta^k}{(1+\theta)^n}$ and $f_{GA}(z; m, \theta) = \frac{\theta^m}{\Gamma(m)} z^{m-1} e^{-\theta z},\ z > 0$, is the PDF of gamma distribution with shape and scale parameters m and θ, respectively.

PROOF: Recall that X_1 has the PDF

$$f_{X_1}(x; \theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \quad = \quad \frac{\theta}{1+\theta}\ f_{GA}(x; 1, \theta) + \frac{1}{1+\theta}\ f_{GA}(x; 2, \theta).$$

The moment generating function (MGF) X_1 for $|t| < \theta$ is

$$M_{X_1}(t) = E\left(e^{tX_1}\right) = \frac{\theta}{1+\theta}\left(1 - \frac{t}{\theta}\right)^{-1} + \frac{1}{1+\theta}\left(1 - \frac{t}{\theta}\right)^{-2}.$$

Hence, the MGF of Z for $|t| < \theta$ is

$$\begin{aligned} M_Z(t) = E\left(e^{tZ}\right) &= \left\{\frac{\theta}{1+\theta}\left(1 - \frac{t}{\theta}\right)^{-1} + \frac{1}{1+\theta}\left(1 - \frac{t}{\theta}\right)^{-2}\right\}^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{\theta^k}{(1+\theta)^n} \left(1 - \frac{t}{\theta}\right)^{-(2n-k)}. \end{aligned}$$

Therefore, the result follows using the fact that $\left(1 - \dfrac{t}{\theta}\right)^{-(2n-k)}$ is the MGF of a gamma RV with shape and scale parameters $2n - k$ and θ, respectively. ∎

LEMMA 1: If $X_1, \cdots, X_n$ are n IID Lindley (θ) RVs, then the conditional PDF of X_1 given $Z = \sum_{i=1}^{n} X_i$, is

$$f_{X_1|Z=z}(x) = \frac{1+x}{A_n(z)} \sum_{k=0}^{n-1} C_{k,n}\ (z-x)^{2n-3-k}, \qquad 0 < x < z,$$

where $C_{k,n} = \frac{\binom{n-1}{k}}{\Gamma(2n-2-k)}$ and $A_n(u) = \sum_{j=0}^{n} \binom{n}{j} \frac{u^{2n-j-1}}{\Gamma(2n-j)}$.

PROOF: It mainly follows from the fact that

$$f_{X_1|Z=z}(x) = \frac{f(x;\theta)\; g(z-x;n-1,\theta)}{g(z;n,\theta)},$$

and using Theorem 1. ∎

THEOREM 2: Suppose $u = \sum_{i=1}^{n_1} x_i$ and $v = \sum_{j=1}^{n_2} y_j$.

(i) For $u \le v$, the UMVUE of R is

$$\widehat{R}_{UMVUE} = \frac{1}{A_{n_1}(u)\; A_{n_2}(v)} \sum_{m=0}^{n_1-1} \sum_{n=0}^{n_2-1} C_{m,n_1}\; C_{n,n_2}\; I_1(u,v,2n_1-3-m,2n_2-3-n),$$

where

$$I_1(u,v,k,\ell) = \sum_{t=0}^{\ell} \binom{\ell}{t} (v-u)^{\ell-t}\; u^{k+t+2} \frac{(k+2)(k+t+3)(k+t+4)+(k+t+4+u)(3k+t+6)u}{(k+1)(k+2)(k+t+2)(k+t+3)(k+t+4)}.$$

(ii) For $u > v$, the UMVUE of R is

$$\widehat{R}_{UMVUE} = \frac{1}{A_{n_1}(u)\; A_{n_2}(v)} \sum_{m=0}^{n_1-1} \sum_{n=0}^{n_2-1} C_{m,n_1}\; C_{n,n_2}\; I_2(u,v,2n_1-3-m,2n_2-3-n),$$

where

$$I_2(u,v,k,\ell) = \sum_{s=0}^{k+1} \binom{k+1}{s} (u-v)^{k-s+1}\; v^{\ell+s+1} \frac{(\ell+s+3)[(\ell+s+2)(k+u+2)+(2k+u+3)v]+2(k+1)v^2}{(k+1)(k+2)(\ell+s+1)(\ell+s+2)(\ell+s+3)}.$$

PROOF: Let us denote $U = \sum_{i=1}^{n_1} X_i$ and $V = \sum_{j=1}^{n_2} Y_j$. Since U and V are complete and sufficient statistics for θ_1 and θ_2, respectively, the UMVUE of R can be obtained as

$$\widehat{R}_{UMVUE} = E\left[\phi(X_1,Y_1) \mid U=u, V=v\right],$$

where

$$\phi(X_1,Y_1) = \begin{cases} 1, & \text{if} \quad Y_1 < X_1, \\ 0, & \text{if} \quad Y_1 > X_1. \end{cases}$$

Therefore, the UMVUE of R is given by

$$\widehat{R}_{UMVUE} = \int_0^{\min(u,v)} \int_y^u f_{X_1|U=u}(x)\ f_{Y_1|V=v}(y)\ dxdy.$$

From Lemma 1, we have

$$\begin{aligned}
\widehat{R}_{UMVUE} &= \int_0^{\min(u,v)} \int_y^u f_{X_1|U=u}(x)\ f_{Y_1|V=v}(y)\ dxdy \\
&= \frac{1}{A_{n_1}(u)\ A_{n_2}(v)} \sum_{m=0}^{n_1-1} \sum_{n=0}^{n_2-1} C_{m,n_1}\ C_{n,n_2}\ I(u,v,2n_1-3-m,2n_2-3-n),
\end{aligned}$$

where, using the substitution $z = u - x$,

$$\begin{aligned}
I(u,v,k,\ell) &= \int_0^{\min(u,v)} \int_y^u (1+x)(u-x)^k\ dx\ (1+y)(v-y)^\ell\ dy \\
&= \int_0^{\min(u,v)} \int_0^{u-y} (1+u-z)z^k\ dz\ (1+y)(v-y)^\ell\ dy \\
&= \int_0^{\min(u,v)} \Big(\frac{1+u}{k+1} - \frac{u-y}{k+2}\Big)\ (u-y)^{k+1}(1+y)(v-y)^\ell\ dy.
\end{aligned}$$

(i) For $u \le v$, we have, using the substitution $w = u - y$,

$$\begin{aligned}
I_1(u,v,k,\ell) &= \int_0^u \Big(\frac{1+u}{k+1} - \frac{w}{k+2}\Big)\ w^{k+1}(1+u-w)(v-u+w)^\ell\ dw \\
&= \sum_{t=0}^{\ell} \binom{\ell}{t}(v-u)^{\ell-t}\ v^{\ell+s+1} \int_0^u \Big(\frac{1+u}{k+1} - \frac{w}{k+2}\Big)\ w^{k+t+1}(1+u-w)\ dw.
\end{aligned}$$

Calculation of the last integral gives the required result.

(ii) For $u > v$, the proof is similar to part (i). ∎

Note that each of the expressions $I_1(u,v,k,\ell)$ and $I_2(u,v,k,\ell)$ is strictly positive.

2.2 MLE of R and its asymptotic distribution

Suppose that $x_1, \cdots, x_{n_1}$ and $y_1, \cdots, y_{n_2}$ are independent random samples from Lindley (θ_1) and Lindley (θ_2), respectively. Let $\bar{x}$ and $\bar{y}$ be the corresponding sample means.

Ghitany *et al.* (2008) showed that the MLEs of θ_1 and θ_2 are given by

$$\widehat{\theta}_1 = \frac{-(\bar{x}-1)+\sqrt{(\bar{x}-1)^2+8\bar{x}}}{2\bar{x}}, \tag{7}$$

$$\widehat{\theta}_2 = \frac{-(\bar{y}-1)+\sqrt{(\bar{y}-1)^2+8\bar{y}}}{2\bar{y}}. \tag{8}$$

Hence, using the invariance property of the MLE, the MLE $\widehat{R}$ of R can be obtained by substituting $\widehat{\theta}_k$ in place of θ_k, in (5) for $k = 1$ and 2. Ghitany *et al.* (2008) also showed that

$$\sqrt{n_k}\ (\widehat{\theta}_k - \theta_k) \xrightarrow{D} N(0, \frac{1}{\sigma_k^2}), \qquad \sigma_k^2 = \frac{\theta_k^2+4\theta_k+2}{\theta_k^2\ (\theta_k+1)^2}, \quad k = 1, 2.$$

Therefore, it easily follows as $n_1 \to \infty$ and $n_2 \to \infty$ that

$$\frac{\widehat{R}-R}{\sqrt{\frac{d_1^2}{n_1\ \sigma_1^2}+\frac{d_2^2}{n_2\ \sigma_2^2}}} \xrightarrow{D} N(0,1), \tag{9}$$

where

$$\begin{aligned} d_1 = \frac{\partial R}{\partial \theta_1} &= -\frac{\theta_1\theta_2^2\ [\theta_1^3+2\theta_1^2(\theta_2+3)+\theta_1(\theta_2+2)(\theta_2+6)+2(\theta_2^2+3\theta_2+3)]}{(\theta_1+1)^2(\theta_2+1)(\theta_1+\theta_2)^4}, \\ d_2 = \frac{\partial R}{\partial \theta_2} &= \frac{\theta_1^2\theta_2\ [6+\theta_1^2(\theta_2+2)+2\theta_1(\theta_2+1)(\theta_2+3)+\theta_2(\theta_2^2+6\theta_2+12)]}{(\theta_1+1)(\theta_2+1)^2(\theta_1+\theta_2)^4}. \end{aligned}$$

Although, $\widehat{R}$ can be obtained in explicit form, the exact distribution of $\widehat{R}$ is difficult to obtain. Due to this reason, to construct the confidence interval of R we mainly consider the confidence interval based on the asymptotic distribution of $\widehat{R}$ and different parametric bootstrap confidence intervals.

3. Confidence intervals of R

In this section we consider four different confidence intervals of R. First one is the confidence interval obtained using the asymptotic distribution of the MLE of R and three parametric bootstrap confidence intervals.

Asymptotic Confidence Interval:

Using the asymptotic distribution of $\widehat{R}$, $100(1-\alpha)\%$ confidence interval for R can be easily obtained as

$$\widehat{R} \mp z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{d}_1{}^2}{n_1\ \hat{\sigma}_1^2} + \frac{\hat{d}_2{}^2}{n_2\ \hat{\sigma}_2^2}}, \tag{10}$$

here $\hat{\sigma}_k^2$ and $\hat{d}_k$ are the MLEs of σ_k and d_k, respectively, for $k = 1,\ 2$.

Since, in this case $0 < R < 1$, a better confidence interval may be obtained (as suggested by the associate editor) by carrying out the large sample inference in the logit scale, and then switching back to the original scale. We have performed a detailed simulation experiments in the logit scale also in Section 5.

Bootstrap Confidence Interval of R

We propose to use the following method to generate parametric bootstrap samples, as suggested by Efron and Tibshirani (1998), of R, from the given independent random samples $x_1, \cdots, x_{n_1}$ and $y_1, \cdots, y_{n_2}$ obtained from Lindley (θ_1) and Lindley (θ_2), respectively.

Algorithm: (Parametric bootstrap sampling)

- Step 1: From the given samples $x_1, \cdots, x_{n_1}$ and $y_1, \cdots, y_{n_2}$ compute the MLE $(\widehat{\theta}_1, \widehat{\theta}_2)$ of (θ_1, θ_2).
- Step 2: Generate independent bootstrap samples $x_1^*, \cdots, x_{n_1}^*$ and $y_1^*, \cdots, y_{n_2}^*$ from Lindley$(\widehat{\theta}_1)$ and Lindley$(\widehat{\theta}_2)$, respectively. Compute the MLE $(\widehat{\theta}_1^*, \widehat{\theta}_2^*)$ of (θ_1, θ_2) as

well as $\widehat{R}^* = R(\widehat{\theta}_1^*, \widehat{\theta}_2^*)$ of R.

- Step 3: Repeat Step 2, B times to obtain a set of bootstrap samples of R, say $\{\widehat{R}_j^*, j = 1, \cdots, B\}$.

Using the above bootstrap samples of R we obtain three different bootstrap confidence interval of R. The ordered $\widehat{R}_j^*$ for $j = 1, \cdots, B$ will be denoted as:

$$\widehat{R}^{*(1)} < \cdots < \widehat{R}^{*(B)}.$$

(i) *Percentile bootstrap (p-boot) confidence interval*:

Let $\widehat{R}^{*(\tau)}$ be the τ percentile of $\{\widehat{R}_j^*, j = 1, 2, \ldots, B\}$, i.e. $\widehat{R}^{*(\tau)}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I(\widehat{R}_j^* \leq \widehat{R}^{*(\tau)}) = \tau, \qquad 0 < \tau < 1,$$

where $I(\cdot)$ is the indicator function.

A $100(1-\alpha)\%$ p-boot confidence interval of R is given by

$$(\widehat{R}^{*(\alpha/2)}, \widehat{R}^{*(1-\alpha/2)}).$$

(ii) *Student's t bootstrap (t-boot) confidence interval*:

Let $se^*(\widehat{R})$ be the sample standard deviation of $\{\widehat{R}_j^*, j = 1, 2, \ldots, B\}$, i.e.

$$se^*(\widehat{R}) = \sqrt{\frac{1}{B}\sum_{j=1}^{B}(\widehat{R}_j^* - \widehat{R}^*)^2}.$$

Also, let $\widehat{t}^{*(\tau)}$ be the τ percentile of $\{\frac{\widehat{R}_j^* - \widehat{R}}{se^*(\widehat{R})}, j = 1, 2, \ldots, B\}$, i.e. $\widehat{t}^{*(\tau)}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I(\frac{\widehat{R}_j^* - \widehat{R}}{se^*(\widehat{R})} \leq \widehat{t}^{*(\tau)}) = \tau \qquad 0 < \tau < 1.$$

A $100(1-\alpha)\%$ t-boot confidence interval of R is given by

$$\widehat{R} \pm \hat{t}^{*(\alpha/2)}\ se^{*}(\widehat{R}).$$

(iii) *Bias-corrected and accelerated bootstrap (BC_a-boot) confidence interval*:

Let $z^{(\tau)}$ and $\hat{z}_0$, respectively, be such that $z^{(\tau)} = \Phi^{-1}(\tau)$ and

$$\hat{z}_0 = \Phi^{-1}\Big(\frac{1}{B}\sum_{j=1}^{B} I(\widehat{R}_j^{*} \le \widehat{R})\Big),$$

where $\Phi^{-1}(.)$ is the inverse CDF of the standard normal distribution. The value $\hat{z}_0$ is called bias-correction. Also, let

$$\hat{a} = \frac{\sum_{i=1}^{n}\left(\widehat{R}_{(.)} - \widehat{R}_{(i)}\right)^3}{6\Big[\sum_{i=1}^{n}\left(\widehat{R}_{(.)} - \widehat{R}_{(i)}\right)^2\Big]^{3/2}}$$

where $\widehat{R}_{(i)}$ is the MLE of R based of $(n-1)$ observations after excluding the ith observation and $\widehat{R}_{(\cdot)} = \frac{1}{n}\sum_{i=1}^{n}\widehat{R}_{(i)}$. The value $\hat{a}$ is called acceleration factor.

A $100(1-\alpha)\%$ BC_a-boot confidence interval of R is given by

$$(\widehat{R}^{*(\nu_1)}, \widehat{R}^{*(\nu_2)}),$$

where

$$\nu_1 = \Phi\Big(\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha/2)}}{1-\hat{a}(\hat{z}_0 + z^{(\alpha/2)})}\Big), \qquad \nu_2 = \Phi\Big(\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha/2)}}{1-\hat{\alpha}(\hat{z}_0 + z^{(1-\alpha/2)})}\Big).$$

It may be mentioned that all the bootstrap confidence intervals can be obtained even in the logit scale also, and we have presented those results in Section 5.

4. Bayesian inference of R

In this section we provide the Bayesian inference of R. First we obtain the Bayes estimate and then we provide the associated credible interval of R. We have mainly considered the squared error loss function. It is assumed apriori that θ_1 and θ_2 are two independent gamma RVs, each with shape and scale parameters a_1 and b_1 (a_2 and b_2). Based on the observations, the likelihood function becomes

$$l(data|\theta_1,\theta_2) \;\propto\; \frac{\theta_1^{2n_1}\theta_2^{2n_2}}{(1+\theta_1)^{n_1}(1+\theta_2)^{n_2}}\;\exp\Big(-\theta_1\sum_{i=1}^{n_1}x_i \;-\; \theta_2\sum_{j=1}^{n_2}y_j\Big). \tag{11}$$

From (11) and using the prior density of θ_1 and θ_2, we obtained the posterior density function of θ_1 and θ_2 as

$$\begin{aligned} l(\theta_1,\theta_2|data) = \frac{c}{(1+\theta_1)^{n_1}(1+\theta_2)^{n_2}}\; & f_{GA}\Big(\theta_1; a_1+2n_1-1, b_1+\sum_{i=1}^{n_1}x_i\Big) \\ & f_{GA}\Big(\theta_2; a_2+2n_2-1, b_2+\sum_{i=j}^{n_2}y_j\Big), \end{aligned} \tag{12}$$

where c is the normalizing constant. It is clear from (12) that *a posteriori* θ_1 and θ_2 are independent. Let us denote

$$l(\theta_1,\theta_2|data) = \; l_1(\theta_1|data)\; l_2(\theta_2|data),$$

where

$$l_1(\theta_1|data) = \frac{c_1}{(1+\theta_1)^{n_1}}\; f_{GA}\Big(\theta_1; a_1+2n_1-1, b_1+\sum_{i=1}^{n_1}x_i\Big), \tag{13}$$

$$l_2(\theta_2|data) = \frac{c_2}{(1+\theta_2)^{n_2}}\; f_{GA}\Big(\theta_2; a_2+2n_2-1, b_2+\sum_{i=1}^{n_2}y_i\Big), \tag{14}$$

with c_1 and c_2 being normalizing constants. Therefore, the Bayes estimator $\widehat{R}_{Bayes}$ of $R = R(\theta_1,\theta_2)$, as defined in (5), under the squared error loss function, is given by the posterior mean:

$$\widehat{R}_{Bayes} = \frac{\int_0^\infty\int_0^\infty R(\theta_1,\theta_2)\; l_1(\theta_1|data)\; l_2(\theta_2|data)\; d\theta_1 d\theta_2}{\int_0^\infty l_1(\theta_1|data)\; d\theta_1 \;\; \int_0^\infty l_2(\theta_2|data)\; d\theta_2}. \tag{15}$$

Note that $\widehat{R}_{Bayes}$ cannot be obtained in explicit form and must be calculated numerically. Alternatively some approximation like Lindley's approximation may be used to approximate the ratio of two integrals. Although we can obtain the Bayes estimate using numerical integration or approximation, but we will not be able to obtain the credible interval of R.

Now we provide a simulation based consistent estimate of $\widehat{R}_{Bayes}$ using importance sampling procedure. We use the following algorithm assuming that a_1, b_1, a_2, b_2 are known apriori.

ALGORITHM: (Importance sampling)

- Step 1: Generate

$$\theta_{11} \sim f_{GA}\Big(\theta_1; a_1 + 2n_1 - 1, b_1 + \sum_{i=1}^{n_1} x_i\Big),$$

  and

$$\theta_{21} \sim f_{GA}\Big(\theta_2; a_2 + 2n_2 - 1, b_2 + \sum_{i=j}^{n_2} y_j\Big).$$

- Step 2: Repeat this procedure N times to obtain $(\theta_{11}, \theta_{21}), \cdots, (\theta_{1N}, \theta_{2N})$.

- Step 3: A simulation consistent estimate of $\widehat{R}_{Bayes}$ can be obtained as

$$\sum_{i=1}^{N} w_i \; R_i,$$

  where $w_i \equiv w(\theta_{1i}, \theta_{2i}) = \frac{h(\theta_{1i},\theta_{2i})}{\sum_{j=1}^{N} h(\theta_{1j},\theta_{2j})}$, $h(\theta_{1i}, \theta_{2i}) = \frac{1}{(1+\theta_{1i})^{n_1} \; (1+\theta_{2i})^{n_2}}$ and $R_i \equiv R(\theta_{1i}, \theta_{2i})$, as defined in (5), for $i = 1, \cdots, N$.

Next, we would like to construct a credible interval of R using the generated importance samples. Suppose that R_p is such that $P(R \leq R_p) = p$. Let $R_{(1)} \leq R_{(2)} \leq \ldots \leq R_{(N)}$ be the order statistics of $R_1, R_2, \ldots, R_N$ and $w_{(1)}, w_{(2)}, \ldots, w_{(N)}$ be the values of $w_1, w_2, \ldots, w_N$ associated with $R_{(1)}, R_{(2)}, \ldots, R_{(N)}$, i.e. $w_{(i)} = w_j$ when $R_{(i)} = R_j, i, j = 1, 2, \ldots, N.$. Note that $w_{(1)}, w_{(2)}, \ldots, w_{(N)}$ are not ordered, they are just associated with $R_{(i)}$. Then a simulation consistent estimate of R_p is $\widehat{R}_p = R_{(N_p)}$, where N_p is the integer satisfying

$$\sum_{i=1}^{N_p} w_{(i)} \leq p < \sum_{i=1}^{N_p+1} w_{(i)}.$$

Therefore, using the above procedure a symmetric 100(1-α)% credible interval of R can be obtained as $(\widehat{R}_{\alpha/2}, \;\; \widehat{R}_{1-\alpha/2})$. Here we are finding the percentiles of the weighted distribution of R, see Smith and Gelfand (1992).

An alternative approach that replaces the importance sampling algorithm is to sample directly from the joint posterior density of θ_1 and θ_2 using the acceptance-rejection method as given by Smith and Gelfand (1992). This is mainly possible since in equations (13) and (14), we have $\frac{1}{(1+\theta_1)^{n_1}} \leq 1$ and $\frac{1}{(1+\theta_2)^{n_2}} \leq 1$ which imply that

$$l(\theta_1|data) \leq c_1 \ f_{GA}\Big(\theta_1; a_1 + 2n_1 - 1, b_1 + \sum_{i=1}^{n_1} x_i\Big),$$

and

$$l(\theta_2|data) \leq c_2 \ f_{GA}\Big(\theta_2; a_2 + 2n_2 - 1, b_2 + \sum_{i=1}^{n_2} y_i\Big).$$

Once the posterior samples have been obtained, the Bayes estimate $\widehat{R}_{Bayes}$ of $R = (\theta_1, \theta_2)$, as defined by (5), and the associated credible interval can be easily constructed. It may be mentioned that for small values of n_1 and n_2, the acceptance rejection method can be very effective in this case. On the other hand, for very large values of n_1 and n_2 this method can be very inefficient, since in that case the proportion of rejection will be very large. Note that once we generate samples directly from the joint posterior distribution, it is quite simple to compute simulation consistent Bayes estimate with respect to other loss function also. For example, if we want to compute the Bayes estimate under the absolute error loss function, *i.e.* the posterior median, $\tilde{R}_{Bayes}$, it can be easily obtained from the generated samples obtained from the posterior distribution. Detailed comparisons of the different Bayes estimators have been presented in Section 5.

5. Simulation results

In this section we perform some simulation experiments to see the behavior of the proposed methods for various sample sizes and for different sets of parameter values. We have taken sample sizes namely (n_1, n_2) = (15, 20), (20, 15), (20, 20), (25, 30), (30, 25), (30, 30) and two different sets of parameter values namely; (θ_1, θ_2) = (1.0, 1.0) and (0.1, 1.0), so that the true reliability parameter values are 0.5 and 0.972, respectively.

For each pair of (n_1, n_2), we have generated samples from Lindley (θ_1) and Lindley (θ_2) accompanied with $B = 5000$ bootstrap samples over 10,000 replications. Then, we report for all sample sizes (a) the average bias and mean squared error of UMVUE and MLE (in the logit scale), parametric bootstrap (Boots), and (b) the coverage probability (average confidence length) of the simulated confidence intervals of $R = P(Y < X)$ for different values of θ_1 and θ_2. The calculations of those quantities in (a) and (b) above are given in Tables 1, 2, 6, and 7.

For Bayes estimates we have taken different hyper-parameters of the priors, namely:

Prior-1: $a_1 = b_1 = a_2 = b_2 = 0.0$,

Prior-2: $a_1 = 4.0, b_1 = 8.0, a_2 = b_2 = 4.0$,

Prior-3: $a_1 = b_1 = a_2 = b_2 = 4.0$.

While Prior-1 is a non-informative prior, Prior-2 and Prior-3 are informative priors. We have used both the algorithms mentioned in Section 4 to generate posterior samples, and we denote them as Bayes-1 and Bayes-2, respectively. In computing the Bayes estimate and the associated credible interval we have used $N = 5000$ importance sampling. In this case also, we have computed the average biases and the mean squared errors of the Bayes estimates and for the credible intervals we have computed the coverage percentages and average lengths of the simulated intervals of R over 10,000 replications. For comparison purposes, we have considered two different loss functions, namely squared error and absolute error loss functions. The Bayesian simulation results are reported in Tables 3, 4, 5, 8, 9, and 10.

Some of the points are quite clear from the simulation results. For all the methods, as the sample size increases the biases and the mean squared errors decrease. It verifies the consistency properties of all the methods. Comparing the results from Tables 1 and 6, it is observed that even for very small sample sizes the biases for all the estimators are quite small and the absolute value of the bias of $\widehat{R}_{UMVUE}$ is much smaller than $\widehat{R}_{MLE}$ and $\widehat{R}_{Boot}$. The bias of $\widehat{R}_{UMVUE}$ seems not large enough even for small sample sizes. Indeed, Table 1 (Table 6) shows that discrepancies among the biases of the $\widehat{R}_{UMVUE}$ are in the fourth (fifth) decimal places for all sample sizes.

$\widehat{R}_{UMVUE}$ is of course, unbiased by construction, and the very small biases that do show up are the consequence of round off and finite number of iterations. This observation was reported in Awad and Gharraf (1986) and Reiser and Guttman (1987) in the context of the UMVUE of R in the Burr XII and normal cases, respectively.

Now, we compare the MSE of $\widehat{R}$ using UMVUE, MLE and parametric bootstrapping.

From Table 1, where $R = 0.5$, we observe that

$$MSE(\widehat{R}_{UMVUE}) > MSE(\widehat{R}_{MLE}) > MSE(\widehat{R}_{Boot}).$$

From Table 6, where $R = 0.972$, we observe that

$$MSE(\widehat{R}_{UMVUE}) < MSE(\widehat{R}_{MLE}) < MSE(\widehat{R}_{Boot}).$$

Reiser and Guttman (1987) reported similar simulation results for estimating R using UMVUE and MLE in the normal case.

From the Tables 3, 4, 8, and 9, it is clear that both the Bayes estimators (Bayes-1 and Bayes-2) behave quite similarly for each of the square and absolute error loss functions. As expected, in most of the cases, it is observed that the Bayes estimators for Prior 2 or Prior 3 have a lower mean squared errors than the Bayes estimators obtained under Prior 1. Since the credible intervals of R are based on its posterior distribution, the coverage probability and average credible length do not depend on the error loss function used.

Table 1: Average bias (mean squared error) of different estimators of $R = P(Y < X)$ when $\theta_1 = 1.0$ and $\theta_2 = 1.0$. All entries of the table are multiplied by 10^{-5}.

n_1, n_2	UMVUE	MLE	Boots
15, 20	54.4 (806.2)	-114.8 (764.2)	-274.8 (727.9)
20, 15	-51.5 (786.5)	117.7 (745.6)	278.9 (710.2)
20, 20	-13.4 (697.4)	-13.2 (666.1)	-13.9 (637.9)
25, 30	19.6 (496.9)	-48.4 (480.4)	-114.5 (465.4)
30, 25	-15.6 (506.6)	52.4 (489.8)	118.1 (474.2)
30, 30	20.5 (453.8)	20.3 (440.1)	21.0 (427.2)

The performances of the MLEs are very similar with the corresponding Bayes estimators when the non-informative prior is used, and for both the loss functions used here. Interestingly when the prior means are exactly equal to the true means, then the performances of the Bayes estimators are slightly better than the MLEs, otherwise they are usually worse.

Examining Tables 2, 5, 7, and 10, it is clear that the performances of the different confidence and credible intervals are quite satisfactory. Most of the confidence intervals reported here are able to maintain the nominal levels even for small sample sizes. Although, for some cases, mainly for small sample sizes, the coverage percentages of the credible intervals based on non-informative priors are slightly lower than the nominal level. Among the confidence intervals, the performances of the p-boot confidence intervals are the best.

Table 2: Coverage probability (average confidence length) of the simulated 95% confidence intervals of $R = P(Y < X)$ when $\theta_1 = 1.0$ and $\theta_2 = 1.0$.

n_1, n_2	MLE	p-boot	t-boot	BC_a-boot
15, 20	0.950	0.957	0.950	0.952
	(0.326)	(0.337)	(0.331)	(0.331)
20, 15	0.950	0.948	0.950	0.951
	(0.326)	(0.325)	(0.331)	(0.332)
20, 20	0.948	0.951	0.949	0.949
	(0.304)	(0.308)	(0.308)	(0.309)
25, 30	0.948	0.952	0.949	0.948
	(0.265)	(0.270)	(0.267)	(0.267)
30, 25	0.950	0.950	0.950	0.950
	(0.264)	(0.265)	(0.267)	(0.267)
30, 30	0.947	0.949	0.948	0.947
	(0.253)	(0.255)	(0.255)	(0.256)

Table 3: Average bias (mean squared error) of different Bayes estimates of $R = P(X < Y)$ when $\theta_1 = 1.0$ and $\theta_2 = 1.0$. The loss function is squared error loss function. All entries of the table are multiplied by 10^{-5}.

	Bayes-1			Bayes-2		
n_1, n_2	Prior-1	Prior-2	Prior-3	Prior-1	Prior-2	Prior-3
15, 20	112.4	398.2	501.5	145.6	365.1	576.2
	(573.1)	(511.1)	(387.4)	(512.5)	513.9	(394.5)
20, 15	331.8	-2318.3	392.5	298.6	-1873.5	367.8
	(498.5)	(456.6)	(398.7)	(476.5)	(298.5)	(401.1)
20, 20	127.7	-2483.6	318.3	106	- 2674.3	402.5
	(499.3)	(398.6)	(278.6)	(499.4)	(401.3)	(411.3)
25, 30	364.3	-2198.1	151.2	293.4	-2289.3	111.6
	(432.6)	(299.3)	(298.3)	(451.2)	(345.2)	(401.4)
30, 25	192.5	-167.3	197.5	198.5	-156.7	149.2
	(398.6)	(298.4)	(305.3)	(406.4)	(265.2)	(293.4)
30, 30	311.3	-1798.3	143.8	301.4	- 1789.3	131.2
	(321.8)	(292.3)	(287.5)	(267.8)	(298.1)	(287.9)

Table 4: Average bias (mean squared error) of Bayes estimates of $R = P(X < Y)$ when $\theta_1 = 1.0$ and $\theta_2 = 1.0$, and when the absolute error loss function is used. All entries of the table are multiplied by 10^{-5}.

	Bayes-1			Bayes-2		
n_1, n_2	Prior-1	Prior-2	Prior-3	Prior-1	Prior-2	Prior-3
15, 20	111.4 (641.9)	410.2 (525.4)	511.2 (398.3)	104.4 (664.3)	375.4 (499.4)	487.4 (412.5)
20, 15	325.7 (476.4)	1323.2 (463.2)	377.8 (397.4)	312.5 (512.4)	1898.5 (454.3)	412.4 (401.9)
20, 20	189.2 (476.8)	-1672.4 (422.2)	376.7 (376.3)	214.6 (525.4)	-1512.8 (434.6)	435.6 (398.7)
25, 30	323.4 (456.3)	-989.3 (359.9)	187.9 (376.5)	312.5 (487.6)	-1167.5 (345.6)	112.9 (351.4)
30, 25	98.9 (387.8)	-749.9 (312.2)	-303.8 (345.8)	111.3 (401.3)	-777.4 (287.9)	-275.4 (367.5)
30, 30	222.4 (321.2)	1098.2 (253.3)	289.3 (276.3)	198.7 (298.7)	1231.7 (214.5)	268.5 (245.6)

Table 5: Coverage probability (average credible length) of the simulated 95% credible intervals of different Bayes estimators of $R = P(Y < X)$ when $\theta_1 = 1.0$ and $\theta_2 = 1.0$.

	Bayes-1			Bayes-2		
n_1, n_2	Prior-1	Prior-2	Prior-3	Prior-1	Prior-2	Prior-3
15, 20	0.919 (0.253)	0.923 (0.251)	0.945 (0.246)	0.921 (0.256)	0.926 (0.253)	0.958 (0.248)
20, 15	0.928 (0.251)	0.942 (0.249)	0.942 (0.243)	0.931 (0.255)	0.946 (0.251)	0.943 (0.245)
20, 20	0.915 (0.230)	0.941 (0.223)	0.941 (0.225)	0.918 (0.228)	0.944 (0.220)	0.942 (0.227)
25, 30	0.916 (0.186)	0.936 (0.197)	0.931 (0.183)	0.919 (0.191)	0.913 (0.190)	0.937 (0.187)
30, 25	0.921 (0.185)	0.941 (0.183)	0.904 (0.191)	0.928 (0.181)	0.946 (0.185)	0.911 (0.189)
30, 30	0.928 (0.174)	0.932 (0.174)	0.940 (0.171)	0.931 (0.179)	0.939 (0.176)	0.942 (0.173)

Table 6: Average bias (mean squared error) of different estimators of $R = P(Y < X)$ when $\theta_1 = 0.1$ and $\theta_2 = 1.0$. All entries of the table are multiplied by 10^{-5}.

n_1, n_2	UMVUE	MLE	Boots
15, 20	-3.2 (13.4)	-237.4 (15.8)	-481.5 (19.6)
20, 15	-5.8 (13.3)	-194.3 (15.2)	-387.4 (17.8)
20, 20	2.2 (10.9)	-177.7 (12.5)	-362.2 (14.7)
25, 30	9.4 (8.0)	-130.0 (8.8)	-273.1 (10.1)
30, 25	-8.4 (7.7)	-131.0 (8.5)	-255.9 (9.5)
30, 30	-5.8 (7.2)	-124.9 (7.9)	-246.3 (8.8)

Table 7: Coverage probability (average confidence length) of the simulated 95% confidence intervals of $R = P(Y < X)$ when $\theta_1 = 0.1$ and $\theta_2 = 1.0$.

n_1, n_2	MLE	p-boot	t-boot	BC_a-boot
15, 20	0.950 (0.050)	0.951 (0.050)	0.952 (0.050)	0.951 (0.050)
20, 15	0.945 (0.049)	0.938 (0.047)	0.946 (0.048)	0.948 (0.050)
20, 20	0.949 (0.045)	0.947 (0.044)	0.952 (0.045)	0.950 (0.045)
25, 30	0.946 (0.037)	0.945 (0.037)	0.947 (0.037)	0.948 (0.037)
30, 25	0.950 (0.037)	0.945 (0.036)	0.951 (0.037)	0.949 (0.037)
30, 30	0.950 (0.035)	0.947 (0.035)	0.951 (0.035)	0.952 (0.035)

Table 8: Average bias (mean squared error) of different Bayes estimates of $R = P(X < Y)$ when $\theta_1 = 0.1$ and $\theta_2 = 1.0$, under squared error loss function. All entries of the table are multiplied by 10^{-5}.

	Bayes-1			Bayes-2		
n_1, n_2	Prior-1	Prior-2	Prior-3	Prior-1	Prior-2	Prior-3
15, 20	393.1 (21.1)	892.1 (15.1)	1091.2 (13.1)	493.9 (22.8)	801.3 (15.5)	1287.3 (12.9)
20, 15	498.3 (22.4)	932.1 (14.1)	1089.1 (13.7)	501.6 (23.2)	789.2 (13.6)	897.6 (13.1)
20, 20	412.6 (19.6)	712.3 (13.9)	812.5 (14.3)	387.5 (20.2)	723.4 (14.2)	816.3 (15.1)
25, 30	312.7 (16.7)	701.3 (11.6)	666.8 (10.9)	256.7 (16.1)	689.4 (12.1)	712.6 (11.4)
30, 25	356.4 (16.2)	545.7 (11.9)	643.3 (10.9)	333.4 (17.4)	416.8 (12.3)	612.1 (10.5)
30, 30	212.7 (12.6)	523.2 (9.8)	565.7 (8.7)	111.3 (11.9)	595.1 (9.4)	656.5 (9.1)

Table 9: Average bias (mean squared error) of Bayes estimates of $R = P(X < Y)$ when $\theta_1 = 0.1$ and $\theta_2 = 1.0$, under absolute error loss function. All entries of the table are multiplied by 10^{-5}.

	Bayes-1			Bayes-2		
n_1, n_2	Prior-1	Prior-2	Prior-3	Prior-1	Prior-2	Prior-3
15, 20	398.4 (27.2)	767.5 (21.3)	854.6 (20.5)	423.5 (25.5)	855.6 (19.5)	999.1 (16.7)
20, 15	525.3 (25.9)	778.6 (16.2)	811.1 (16.8)	512.9 (26.4)	884.5 (17.6)	878.2 (15.2)
20, 20	334.4 (22.8)	626.8 (15.2)	701.4 (15.4)	387.2 (21.3)	634.6 (14.7)	732.4 (15.6)
25, 30	316.7 (18.9)	577.3 (13.4)	499.6 (12.9)	276.3 (17.8)	622.3 (12.1)	577.4 (11.2)
30, 25	331.1 (13.9)	578.5 (9.4)	612.3 (9.8)	298.5 (13.3)	612.2 (10.4)	588.3 (9.2)
30, 30	197.4 (10.8)	387.5 (8.3)	478.9 (8.5)	201.3 (11.1)	476.4 (8.8)	523.4 (8.2)

Table 10: Coverage probability (average credible length) of the simulated 95% credible intervals of different Bayes estimators of $R = P(Y < X)$ when $\theta_1 = 0.1$ and $\theta_2 = 1.0$ under both squared and absolute error loss functions.

	Bayes-1			Bayes-2		
n_1, n_2	Prior-1	Prior-2	Prior-3	Prior-1	Prior-2	Prior-3
15, 20	0.924	0.948	0.947	0.926	0.951	0.943
	(0.049)	(0.054)	(0.055)	(0.045)	(0.058)	(0.041)
20, 15	0.947	0.950	0.944	0.948	0.949	0.942
	(0.051)	(0.054)	(0.054)	(0.053)	(0.051)	(0.057)
20, 20	0.919	0.946	0.930	0.913	0.943	0.933
	(0.044)	(0.047)	(0.047)	(0.042)	(0.045)	(0.049)
25, 30	0.904	0.943	0.941	0.908	0.945	0.945
	(0.035)	(0.037)	(0.037)	(0.038)	(0.039)	(0.037)
30, 25	0.922	0.936	0.945	0.919	0.940	0.943
	(0.035)	(0.037)	(0.037)	(0.033)	(0.039)	(0.035)
30, 30	0.921	0.928	0.935	0.923	0.931	0.937
	(0.033)	(0.036)	(0.034)	(0.035)	(0.038)	(0.036)

6. Data analysis

In this section we present the analysis of real data, partially considered in Ghitany *et al.* (2008), for illustrative purposes. The data represent the waiting times (in minutes) before customer service in two different banks. The data sets are presented in Tables 11 and 12. Note that $n_1 = 100$ and $n_2 = 60$. We are interested in estimating the stress-strength parameter $R = P(Y < X)$ where X (Y) denotes the customer service time in Bank A (B).

First we want to see whether Lindley distribution can be used to fit these data sets or not. Figure 1 shows the relative histograms of the two data sets. Clearly, both histograms indicate that the underlying PDFs are decreasing functions. Another useful graphical approach is to use the $Q - Q$ plot for each data set. This plot depicts the points $(F^{-1}(\frac{i}{n+1}; \hat{\theta}),\ x_{(i)}),\ i = 1, 2, \ldots, n$, where $x_{(i)}$ is the ith order statistic of the given data and $\hat{\theta}$ is the MLE of θ, see, for example, Arnold *et al.* (2008), p. 187. Since the Lindley distribution function $F(x; \theta)$ does not have an explicit inverse $F^{-1}(x; \theta)$, a numerical method is used to search the support

Table 11: Waiting time (in minutes) before customer service in Bank A.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5

Table 12: Waiting time (in minutes) before customer service in Bank B.

0.1	0.2	0.3	0.7	0.9	1.1	1.2	1.8	1.9	2.0
2.2	2.3	2.3	2.3	2.5	2.6	2.7	2.7	2.9	3.1
3.1	3.2	3.4	3.4	3.5	3.9	4.0	4.2	4.5	4.7
5.3	5.6	5.6	6.2	6.3	6.6	6.8	7.3	7.5	7.7
7.7	8.0	8.0	8.5	8.5	8.7	9.5	10.7	10.9	11.0
12.1	12.3	12.8	12.9	13.2	13.7	14.5	16.0	16.5	28.0

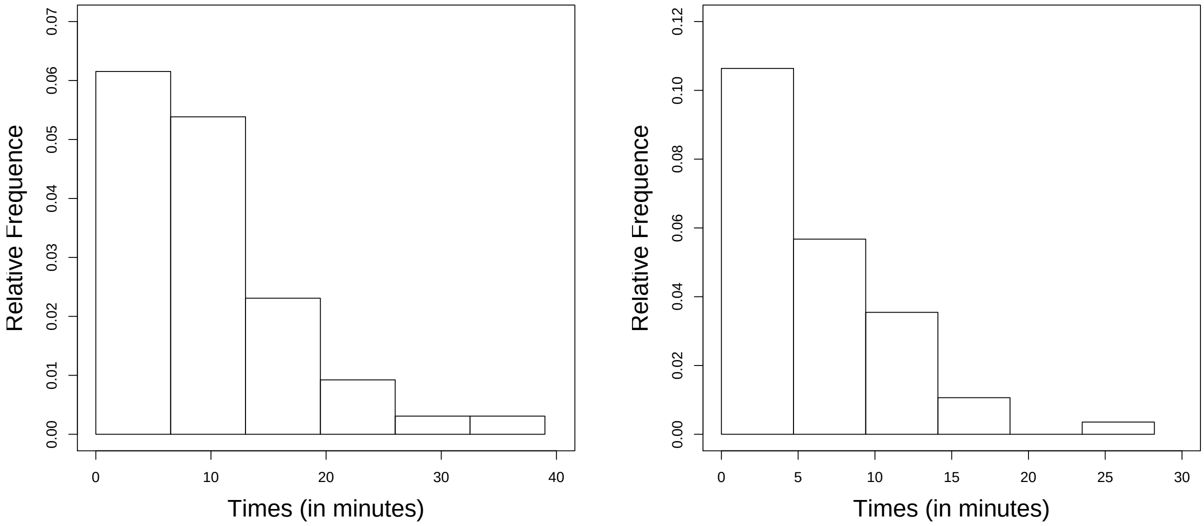
of the distribution for a root of the equation

$$F(x;\hat{\theta}) - \frac{i}{n+1} = 0, \qquad i = 1, 2, \ldots, n.$$

We have used the `uniroot` function in the `R` package to locate the root of the last equation on the interval (0, 1000).

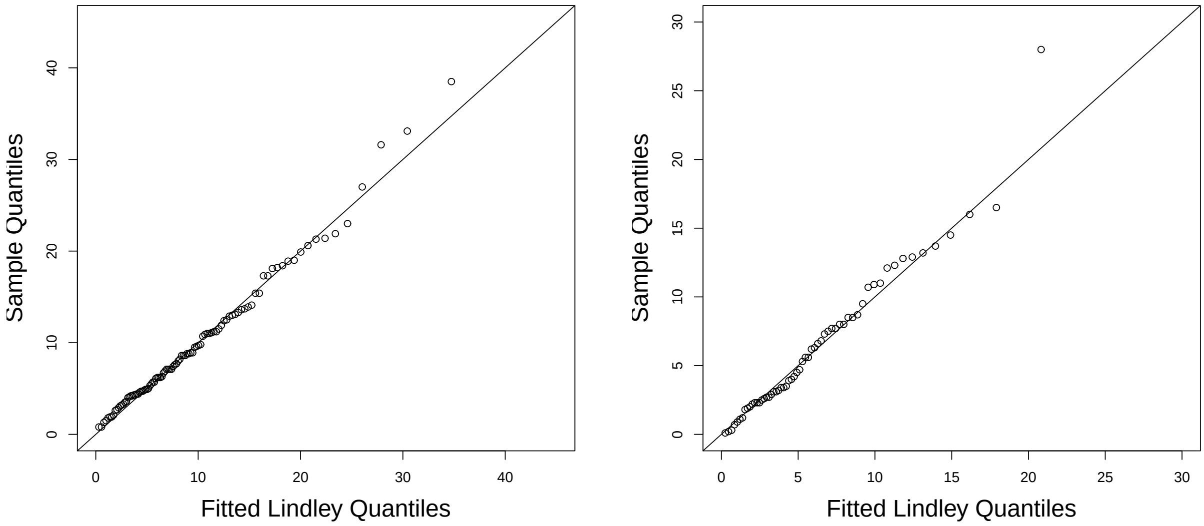
For the given data sets, we have the MLEs $\hat{\theta}_1 = 0.187$ and $\hat{\theta}_2 = 0.280$. Figure 2 shows the $Q-Q$ plots for each of the given data sets.

Based on these plots, we conclude that the Lindley distributions provide good fit for the given data sets. This conclusion is also supported by the Kolmogorov-Smirnov tests as given in Table 13.



(a) (b)

Figure 1: Relative histogram of data set from (a) Bank A and (b) Bank B.



(a) (b)

Figure 2: Q-Q plot of the fitted Lindley distribution for (a) Bank A and (b) Bank B.

Table 13: Kolmogorov-Smirnov distances and the associated $p-$values.

Data set	K-S statistic	p-value
Bank A	0.068	0.750
Bank B	0.080	0.840

Now we compute the UMVUE, MLE and Bayes estimates of R. Since we do not have any prior information, we have used non-informative prior only. For the given data, $\widehat{R}_{UMVUE} = 0.647$, $\widehat{R}_{MLE} = 0.645$, Bayes-1 estimate of R is 0.631 under the squared error loss function (0.637 under the absolute error loss function), and Bayes-2 estimate of R is 0.629 under the squared error loss function (0.633 under the absolute error loss function).

Different 95% confidence intervals and the credible interval of R based on logit transformation are provided in Table 14. It is quite interesting to see that all the intervals are virtually indistinguishable. To explore an explanation for this observation, we provide the histogram of the bootstrap replicates for the logit of R based on B=5000 in Figure 3. Clearly, the histogram resembles that of a normal distribution with approximate mean 0.6031 and standard deviation 0.1752. A formally K-S test yields distance statistic 0.009 and p-value = 0.854, confirming the normality of the bootstrap replicates for the logit of R. This can be the reason that the confidence and credible intervals are almost indistinguishable for the given data.

It is clear that the MLE and Bayes estimator with respect to non-informative prior behave quite similarly, although the length of the credible interval is slightly shorter than the corresponding confidence intervals obtained by different methods.

Table 14: Different 95% confidence intervals and the credible interval R.

	Parametric			Bayes
MLE	p-boot	t-boot	BC_a-boot	Credible Interval
(0.565, 0.720)	(0.567, 0.721)	(0.567, 0.718)	(0.562, 0.718)	Bayes-1: (0.573, 0.706) Bayes-2: (0.569, 0.704)

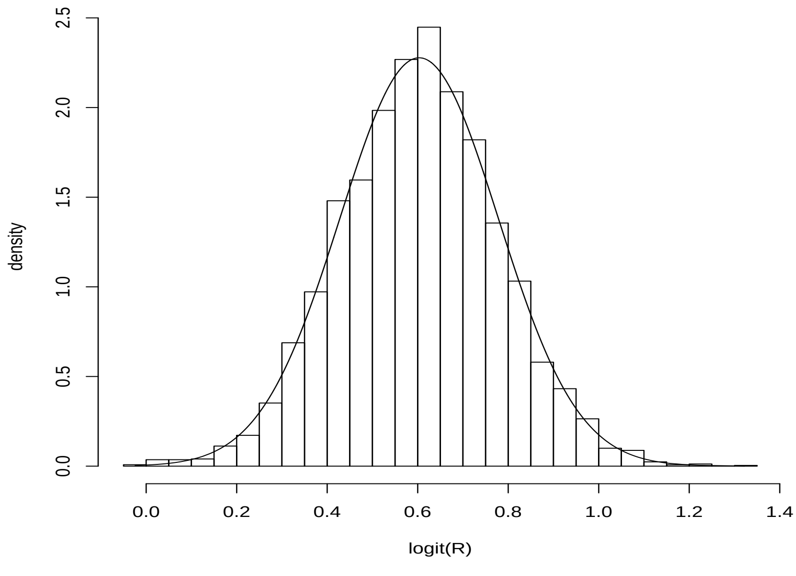


Figure 3: Histogram of bootstrap replicates for the logit of R with normal curve imposed.

7. Conclusions

In this paper we have studied several point and interval estimation procedures of the stress-strength parameter of the Lindley distribution. We have obtained the UMVUE of the stress-strength parameter, however the exact or asymptotic distribution of it is very difficult to obtain. We have derived the MLE of R and its asymptotic distribution. Also, different parametric bootstrap confidence intervals are proposed and it is observed that the p-boot estimate works the best even for small sample sizes.

We have computed the Bayes estimate of R based on the independent gamma priors and using squared and absolute error loss functions. Since the Bayes estimate cannot be obtained in explicit form, we have used the MCMC technique to compute the Bayes estimate and also the associated credible interval. Simulation results suggest that the performance of the Bayes estimator based on the non-informative prior works very well and it can be used for all practical purposes.

Acknowledgements

The authors would like to thank the Associate editor and the reviewer for their helpful comments which improved an earlier draft of the manuscript. The first author would like to thank Kuwait Foundation for the Advancement of Sciences (KFAS) for financial support. Part of the work of the third author has been supported by a grant from the Department of Science and Technology, Government of India.

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