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Highlights

- A generalized Freund bivariate model for two-component load-sharing systems is posed.
- New bivariate distributions can be generated by combining baseline distributions.
- Maximum likelihood estimation of this model is implemented by a genetic algorithm.
- A procedure to generate synthetic two-dimensional data from this model is described.
- The proposed model is applied to three real engineering data sets evidencing the load-sharing effect.

A generalized Freund bivariate model for a two-component load sharing system

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Abstract

Motivated by reliability systems whose components fail one by one and share a common load, this work provides a generalized Freund bivariate class of distributions for modeling the two component lifetimes of a parallel redundant system. When a component fails, such load-sharing systems can be repaired meanwhile the surviving one endures the total load, modifying its two-dimensional lifetime model, which is of interest in maintenance and stress-strength reliability modeling. The proposed model is based on the overload of the surviving component after the first failure, causing both the proportional failure rate parameters and the baseline distribution of the component to change. A genetic algorithm is employed to find the maximum likelihood estimation, and a simulation study illustrates its implementation and efficiency. Applications in three real engineering data sets are carried out, revealing the usefulness of the proposed class for modeling the load-sharing effect.

Keywords: load-sharing system, genetic algorithm, lifetime distribution, proportional failure rate, reliability

1. Introduction

In reliability engineering, the stochastic behavior of the lifetimes of load-sharing parallel systems has been widely studied, since the contribution of Daniels [9] in the textile industry. In the real world, electrical generator systems, CPUs in a multiprocessor computer system, cables in a suspension bridge, valves or pumps in a hydraulic system are all load-sharing structures [3, 16, 18]. The components of these dynamic reliability systems fail one by one, and the lifetimes of the surviving ones are influenced by the redistribution of the shared

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load (stress, weight, electrical power, traffic, ...). Constructing reliability models which incorporate stochastic dependence among the components is crucial in reliability engineering [5, 29], inasmuch few options are available for modeling dependent systems [16], such as load-sharing configurations. In particular, the system reliability is enhanced using redundancy by connecting several components in parallel [3, 18, 29, 34, 36], and whose dependence may be also modeled by incorporating frailty [2]. Although different approaches and extensions have been considered, Wang et al. [33] point out that most of them assume memoryless load-sharing models and/or all the components follow the same lifetime distribution [20, 37]. Thus, it is essential to look for distributions capable of handling the reliability and dependence of load-sharing systems effectively.

Physically motivated by the reliability of two-component parallel redundant systems, Freund [12] introduced an absolutely continuous bivariate extension (FBE) of the exponential model, which does not satisfy the strong version of bivariate memoryless property given by (2.5) of Marshall and Olkin [22], although it does the relaxed version (2.8) in [22]. In such systems, the exponential residual lifetime of one component depends on the working status of the other one [28], and the first failure induces a higher failure rate of the overloaded component. Considered one of the first prototype distributions for load-sharing structures [32], the system under the FBE model is repaired whenever a component fails, which is of interest in maintenance and stress-strength reliability modeling [6, 7].

A number of papers have been addressed to extend the FBE model in different ways. Some of them are based on functional representations of this distribution to overcome restrictions as the constant failure rates, e.g., by replacing exponential lifetimes with Weibull [21, 28, 30, 35]. Recently, Asha et al. [1] have provided an extension, named the extended Freund's bivariate (EFB) distribution, where the lifetime distributions of the components have proportional failure rate (PFR) with a common baseline distribution. In all these extensions, after the first failure, the total load is imposed on the surviving component which leads the PFR parameter to change, whereas the baseline distribution stays the same.

The main goal of this work is to propose a more flexible generalized Freund bivariate (GFB) model for two-component load-sharing systems. Its construction is based on how the lifetime distributions change after the first failure, given that the occurrence of this event affects the PFR parameter of the surviving component and its baseline distribution. The remaining component has to endure extra load or stress which shortens its residual lifetime. A priori, the lifetimes of both components have the same baseline distribution, but the failure of one of them causes changes on the performance of the surviving component, and consequently, on its underlying model.

From a practical viewpoint of modeling, choosing appropriate initial values always poses a challenge in the estimating procedure. If these values are far from the true parameters, the algorithm may take a large number of steps to converge or even may result in divergence. In order to avoid this issue, a genetic algorithm (GA) [8, 17] is applied to find the maximum likelihood estimations (MLE) of the unknown parameters of this GFB model, encouraging the usefulness of this meta-heuristic search technique in modeling of the system reliability when there are no explicit expressions for the MLEs. The GA proposed to approximate the MLEs (GA-MLE) is an appropriate parameter estimation method for reliability of load-sharing systems, without restrictions on the underlying distributions. In addition, the GA-MLE avoids the weakness of other estimation algorithms pointed out by Park [25].

The rest of the paper is organized as follows. In Section 2, a class of generalized Freund bivariate distribution is derived for modeling the reliability of a load-sharing system of two components. Some distributional properties are also supplied for this lifetime model applicable for the stochastically dynamic load. Section 3 is devoted to find the maximum likelihood estimates of the unknown parameters via a GA. The estimation procedure is described for its implementation by combining the GA and the MLE criterion. In Section 4, a simulation study analyzes the performance of the GA-MLE procedure for the GFB model. Further, the usefulness of the GFB class for modeling the load-sharing effect is illustrated with three real engineering data. Finally, the conclusions are given in Section 5.

2. The generalized Freund bivariate model

This section is devoted to the construction of a generalized class of absolutely continuous bivariate distributions based on the physical interpretation of the FBE model [12]. Furthermore, the joint reliability function and the marginal probability density functions (pdf) are also provided.

2.1. The model construction

Here, a more flexible option to model the stochastic behavior of the lifetimes T_1 and T_2 of two components C_1 and C_2 in a load-sharing parallel system is provided, referred to as the GFB family. The basic idea behind this model is to assume that both components are simultaneously working, their corresponding lifetimes are independent and not necessarily identically distributed with reliability functions from the same PFR class with a common baseline reliability R_B and parameters θ_B and θ_i, i.e., $R_i(t) = P(T_i > t)$ is given by $R_i(t) = (R_B(t, \theta_B))^{\theta_i}$, shortly denoted by $T_i \sim PFR(R_B(\cdot, \theta_B), \theta_i)$, $i = 1, 2$. When one of the components fails, the surviving one is overloaded and the structural dependence can cause the change of its baseline reliability and its PFR parameter. Figure 1 shows the impact of load changing on lifetimes of a two-component parallel redundant system. Without loss of generality, the baseline parameter θ_B is omitted to denote $R_B(\cdot, \theta_B)$ by R_B.

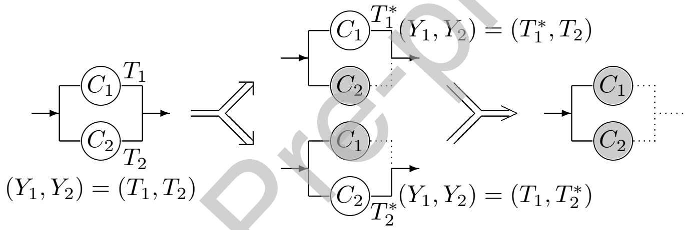


Figure 1: Schematic of failures on a two-component parallel redundant system.

- If C_1 fails before C_2, i.e., $T_1 < T_2$, the lifetime of C_2 changes from T_2 to $T_2^* \sim PFR(R_B^*, \theta_2^*)$. The system fails as soon as C_2 fails, observing (T_1, T_2^*).
- If C_2 fails first, i.e., $T_2 < T_1$, then C_1 changes its lifetime from T_1 to $T_1^* \sim PFR(R_B^*, \theta_1^*)$. The system fails as soon as C_1 fails, observing (T_1^*, T_2).

From now on, the two-dimensional random variable (Y_1, Y_2) denotes the lifetimes of the two components, assuming that $(Y_1, Y_2) = (T_1^*, T_2)$ if $Y_1 > Y_2$, and $(Y_1, Y_2) = (T_1, T_2^*)$ if $Y_1 < Y_2$.

Therefore, let T_1 and T_2 be random variables independently distributed belonging to the PFR class with baseline reliability R_B and parameters θ_1 and θ_2, respectively. Moreover, let T_1^* and T_2^* be random variables belonging to the PFR class with baseline reliability R_B^* and parameters θ_1^* and θ_2^*, respectively. The joint pdf of (Y_1, Y_2) can be expressed as:

$$f(y_1, y_2) = \begin{cases} \theta_1^* \theta_2 f_B^*(y_1) f_B(y_2) \left(R_B^*(y_1)\right)^{\theta_1^*-1} \left(R_B(y_2)\right)^{\theta_1+\theta_2-1} \left(R_B^*(y_2)\right)^{-\theta_1^*}, & \text{if } y_1 > y_2 > 0 \\ \theta_1 \theta_2^* f_B(y_1) f_B^*(y_2) \left(R_B(y_1)\right)^{\theta_1+\theta_2-1} \left(R_B^*(y_2)\right)^{\theta_2^*-1} \left(R_B^*(y_1)\right)^{-\theta_2^*}, & \text{if } y_2 > y_1 > 0 \end{cases} \tag{2.1}$$

being f_B (f_B^*) the pdf associated with R_B (R_B^*). It is denoted by $(Y_1, Y_2) \sim GFB(R_B, R_B^*, \overrightarrow{\theta} = (\theta_1, \theta_2, \theta_1^*, \theta_2^*), \theta_B, \theta_B^*)$, with PFR parameters $\theta_1, \theta_2, \theta_1^*, \theta_2^* > 0$, and baseline parameters θ_B and θ_B^*, respectively. The detailed derivation of (2.1) is provided in Appendix A.1.

Accordingly, when it is analyzed the two-component load-sharing parallel system reliability with initial independent lifetimes while both components are working, the GFB class may well help in modeling the stochastic dependence between Y_1 and Y_2, introduced by the automatic transfer of load to the surviving component after the first failure, allowing its underlying lifetime to change due to the effect of the overload on its residual lifetime.

Note that the bivariate distributions supplied by [1, 12, 21, 30] can be directly derived from the GFB class as particular load-sharing system models. More details can be found in the Supplementary Material S.1. In addition, other particular load-sharing models of the GFB class are those in which also change the baseline distribution of the surviving component, i.e., when $R_B \not\equiv R_B^*$. In this setting, the GFB family can be used to generate new bivariate lifetime models by combining different baseline reliability functions R_B and R_B^*. Some new specific cases of GFB models can be found in the Supplementary Material S.2, which are also considered in Section 4. For simplicity and without loss of generality, let us consider from now on that the scale parameters of the baseline distributions can be equal to 1, since such GFB models are equivalent by a simple reparametrization of their PFR parameters.

2.2. GFB's reliability functions

Now, the joint reliability function for (Y_1, Y_2) of a load-sharing system under the GFB model is provided by integrating (2.1) over the times y_1 and y_2, which can be written as:

$$R(y_1, y_2) = \begin{cases} (R_B(y_1))^{\theta_1+\theta_2} + \theta_2 \left(R_B^*(y_1)\right)^{\theta_1^*} \displaystyle\int_{y_2}^{y_1} f_B(t) \frac{(R_B(t))^{\theta_1+\theta_2-1}}{(R_B^*(t))^{\theta_1^*}} dt, & \text{if } y_1 > y_2 > 0 \\ (R_B(y_2))^{\theta_1+\theta_2} + \theta_1 \left(R_B^*(y_2)\right)^{\theta_2^*} \displaystyle\int_{y_1}^{y_2} f_B(t) \frac{(R_B(t))^{\theta_1+\theta_2-1}}{(R_B^*(t))^{\theta_2^*}} dt, & \text{if } y_2 > y_1 > 0. \end{cases} \tag{2.2}$$

The detailed derivation of (2.2) is provided in Appendix A.2.

In particular, if $R_B^* \equiv R_B$ then the reliability function of the EFB model [1] can be derived from (2.2) as follows:

$$R(y_1, y_2) = \begin{cases} (R_B(y_1))^{\theta_1+\theta_2} + \frac{\theta_2 (R_B(y_1))^{\theta_1^*} \left((R_B(y_2))^{\theta_1+\theta_2-\theta_1^*} - (R_B(y_1))^{\theta_1+\theta_2-\theta_1^*} \right)}{\theta_1+\theta_2-\theta_1^*}, & \text{if } y_1 > y_2 > 0 \\ (R_B(y_2))^{\theta_1+\theta_2} + \frac{\theta_1 (R_B(y_2))^{\theta_2^*} \left((R_B(y_1))^{\theta_1+\theta_2-\theta_2^*} - (R_B(y_2))^{\theta_1+\theta_2-\theta_2^*} \right)}{\theta_1+\theta_2-\theta_2^*}, & \text{if } y_2 > y_1 > 0 \end{cases}$$

when $\theta_1 + \theta_2 = \theta_i^*$, $i = 1, 2$. Otherwise, it can be also rewritten by taking into account, for $i = 1, 2$, that

$$\int_a^b f_B(t) \left(R_B(t)\right)^{\theta_1+\theta_2-\theta_i^*-1} dt = \log R_B(a) - \log R_B(b), \text{ if } \theta_1 + \theta_2 = \theta_i^*.$$

Some reliability properties for the GFB model, which play a important role in policies for preventive maintenance, can be derived from (2.1) and (2.2). For instance, the time to the earliest failure, $Y_{1:2} = \min(Y_1, Y_2)$, is determined by

$$P(Y_{1:2} > y) = \left(R_B(y)\right)^{\theta_1+\theta_2}, \tag{2.3}$$

and the probability that C_1 is the surviving component can be expressed as

$$P(Y_1 > Y_2) = \frac{\theta_2}{\theta_1 + \theta_2}. \tag{2.4}$$

More details can be found in Appendix A.3.

2.3. GFB's marginal distributions

Note that by letting $y_1 = 0$ ($y_2 = 0$) in (2.2), the marginal reliability function of Y_1 (Y_2) can be easily obtained for the $GFB(R_B, R_B^*, \theta_1, \theta_2, \theta_1^*, \theta_2^*, \theta_B, \theta_B^*)$. Furthermore, by integrating (2.1) with respect to y_2 or y_1, the marginal pdfs of Y_1 and Y_2 is given by:

$$f_1(y_1) = \theta_1^* \theta_2 \left(R_B^*(y_1)\right)^{\theta_1^*-1} f_B^*(y_1) \int_0^{y_1} \frac{\left(R_B(t)\right)^{\theta_1+\theta_2-1} f_B(t)}{\left(R_B^*(t)\right)^{\theta_1^*}} dt + \theta_1 \left(R_B(y_1)\right)^{\theta_1+\theta_2-1} f_B(y_1) \tag{2.5}$$

and

$$f_2(y_2) = \theta_1 \theta_2^* \left(R_B^*(y_2)\right)^{\theta_2^*-1} f_B^*(y_2) \int_0^{y_2} \frac{\left(R_B(t)\right)^{\theta_1+\theta_2-1} f_B(t)}{\left(R_B^*(t)\right)^{\theta_2^*}} dt + \theta_2 \left(R_B(y_2)\right)^{\theta_1+\theta_2-1} f_B(y_2).$$

As a particular case, the marginal pdfs of the EFB model [1] can be obtained from (2.5) when $R_B^* \equiv R_B$:

$$f_1(y_1) = \begin{cases} \frac{f_B(y_1)(R_B(y_1))^{\theta_1^*-1}}{\theta_1+\theta_2-\theta_1^*}\left(\theta_1^*\theta_2 - (R_B(y_1))^{\theta_1+\theta_2+\theta_1^*}(\theta_1+\theta_2)(\theta_1^*-\theta_1)\right), & \text{if } \theta_1+\theta_2 = \theta_1^* \\ f_B(y_1)\,(R_B(y_1))^{\theta_1^*-1}\,(\theta_1 - \theta_1^*\theta_2 \log R_B(y_1)), & \text{if } \theta_1+\theta_2 = \theta_1^*. \end{cases}$$

Example 2.1. *The GFB model with baseline Rayleigh switching to exponential after the first failure, has joint pdf given by*

$$f(y_1, y_2) = \begin{cases} 2\theta_1^*\theta_2 y_2 e^{-(\theta_1+\theta_2)y_2^2} e^{-\theta_1^*(y_1-y_2)}, & \text{if } y_1 > y_2 > 0 \\ 2\theta_1\theta_2^* y_1 e^{-(\theta_1+\theta_2)y_1^2} e^{-\theta_1^*(y_2-y_1)}, & \text{if } y_2 > y_1 > 0 \end{cases}$$

and its marginal pdf can be written as

$$\begin{aligned} f_1(y_1) = {} & 2\theta_1 y_1 e^{-(\theta_1+\theta_2)y_1^2} + \frac{\theta_1^*\theta_2}{\theta_1+\theta_2}\left(e^{-\theta_1^* y_1} - e^{-(\theta_1+\theta_2)y_1^2}\right) + \frac{\theta_1^{*2}\theta_2}{\theta_1+\theta_2}\sqrt{\frac{\pi}{(\theta_1+\theta_2)}}\,e^{-\theta_1^* y_1} \\ & + \frac{\theta_1^{*2}}{4(\theta_1+\theta_2)}\left(\Phi\left(\sqrt{2(\theta_1+\theta_2)}\,y_1 - \frac{\theta_1^*}{\sqrt{2(\theta_1+\theta_2)}}\right) - \Phi\left(\frac{-\theta_1^*}{\sqrt{2(\theta_1+\theta_2)}}\right)\right) \end{aligned}$$

where Φ is the cumulative distribution function of the standard normal model.

3. Maximum likelihood estimation of the GFB model

In many studies related to load-sharing system reliability, baseline parameter values are supposed to be known which is a restrictive assumption in practice, since such parameters are usually unknown and have to be computed by using estimation techniques. In this section, the MLEs are obtained for the unknown parameters of the bivariate load-sharing model proposed by applying a GA in the optimization process.

Without assuming any underlying distributions, the MLEs of the parameters of the $GFB(R_B, R_B^*, \overrightarrow{\theta}, \theta_B, \theta_B^*)$ model are derived under each of the three following conditions on θ_B and θ_B^* parameters of the baseline reliability functions R_B and R_B^*, respectively: (i) both are known, (ii) one of them is unknown, and (iii) both are unknown.

For notation convenience, we suppose n independent and identically distributed (iid) load-sharing parallel systems, each consisting of two components put on test simultaneously. From $(Y_1, Y_2) \quad GFB(R_B, R_B^*, \overrightarrow{\theta}, \theta_B, \theta_B^*)$, their failure times $\{(y_{1i}, y_{2i}), i = 1, ..., n\}$ are recorded. It is denoted as $I_1 = \{i : y_{1i} > y_{2i}\}$, $I_2 = \{i : y_{1i} < y_{2i}\}$, $I = I_1 \quad I_2$, $n_1 = \# I_1$ and $n_2 = \# I_2$, with $n_1 + n_2 = n$ since the probability of ties is zero. From the observed sample, the log-likelihood function in terms of the paremeters $\overrightarrow{\theta} = (\theta_1, \theta_2, \theta_1^*, \theta_2^*)$, θ_B and θ_B^*, $l(\overrightarrow{\theta}, \theta_B, \theta_B^*) = \log L(\overrightarrow{\theta}, \theta_B, \theta_B^*)$, can be written as

$$
\begin{aligned}
l(\overrightarrow{\theta}, \theta_B, \theta_B^*) = {} & n_1 \log \theta_1^* + \theta_1^* \sum_{i \in I_1} \log \frac{R_B^*(y_{1i}, \theta_B^*)}{R_B^*(y_{2i}, \theta_B^*)} + n_2 \log \theta_2^* + \theta_2^* \sum_{i \in I_2} \log \frac{R_B^*(y_{2i}, \theta_B^*)}{R_B^*(y_{1i}, \theta_B^*)} \\
& + n_2 \log \theta_1 + n_1 \log \theta_2 + (\theta_1 + \theta_2) \sum_{i \in I} \log R_B(\min(y_{1i}, y_{2i}), \theta_B) \\
& + \sum_{i \in I} \log \frac{f_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)}{R_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)} + \sum_{i \in I} \log \frac{f_B(\min(y_{1i}, y_{2i}), \theta_B)}{R_B(\min(y_{1i}, y_{2i}), \theta_B)}.
\end{aligned} \tag{3.6}
$$

Note that it is assumed that $n_1 > 0$ and $n_2 > 0$, since the MLE of θ_1^* does not exist if $n_1 = 0$. Analogously, the MLE of the θ_2^* does not exist if $n_2 = 0$.

By differentiating (3.6) with respect to the PFR parameters, the following relationships among all the parameters are obtained:

$$
\begin{aligned}
\theta_1(\theta_B, \theta_B^*) &= \frac{-n_2}{\sum_{i \in I} \log R_B(\min(y_{1i}, y_{2i}), \theta_B)}, \quad \theta_2(\theta_B, \theta_B^*) = \frac{-n_1}{\sum_{i \in I} \log R_B(\min(y_{1i}, y_{2i}), \theta_B)} \\
\theta_1^*(\theta_B, \theta_B^*) &= \frac{n_1}{\sum_{i \in I_1} \log \frac{R_B^*(y_{2i}, \theta_B^*)}{R_B^*(y_{1i}, \theta_B^*)}}, \quad \theta_2^*(\theta_B, \theta_B^*) = \frac{n_2}{\sum_{i \in I_2} \log \frac{R_B^*(y_{1i}, \theta_B^*)}{R_B^*(y_{2i}, \theta_B^*)}}.
\end{aligned} \tag{3.7}
$$

It is noteworthy that the two first equations only depend on the initial PFR parameter θ_B, whereas the two rest ones only depend on the altered baseline parameter θ_B^*.

In general, the fitness function to estimate the parameters based on the MLE criterion is the log-likelihood function given by (3.6). However, there are no closed form solutions for the parameters, and the six-dimensional nonlinear optimization problem might be also cumbersome in practice. In this setting, the four PFR parameters are determined by (3.7) in terms of the baseline parameters, and the profile log-likelihood function is obtained by substituting (3.7) into (3.6), which reduces the dimensionality to the baseline parameters. Hence, the MLEs of the baseline parameters can be obtained by maximizing the profile log-likelihood function, and then the MLEs of the remaining parameters can be calculated thereafter. Thereby, such a profile function can be considered the fitness function of the optimization problem.

Unlike the conventional numerical procedures such as the Newton-type methods, GA is a nonlinear optimization approach that may well be considered an appropriate global stochastic search process in practice [11, 17]. In particular, Li et al. [19] suggest its application especially for direct constrained optimization of the profile log-likelihood. For a detailed overview of GA, see [8, among others], and more recent advances and applications of GA can be found in [15, 17, and the references therein].

In the formulation of the GA estimation of the N unkonwn parameters $\boldsymbol{\theta} = (\theta_1, ..., \theta_N)$ subject to the feasibility into the parametric space $\boldsymbol{\Theta}$, the proposed approach is based on the MLE criterion wherein the objective function $Z(\boldsymbol{\theta})$ in the genetic setup of this nonlinear optimization problem is the profile log-likelihood function $p(\boldsymbol{\theta})$. Thereby, the optimal solution is searched by applying iteratively the three genetic operators of selection, crossover and mutation, over the successive stages: evaluation, selection and reproduction, being the result a N-tuple of the real-valued decision variables $\boldsymbol{\theta}$ $\boldsymbol{\Theta}$.

Although MLEs can be approximated using quasi-Newton methods for solving constrained nonlinear optimization problem, such numerical procedures require proper initial values of $\boldsymbol{\theta}_0$ to converge at the optimal solution. However, GA formulation setup does only need a parametric range of the parameter vector $\boldsymbol{\theta}$ $\boldsymbol{\Theta}$, to generate the initial population of possible solutions. Specifically, let $\boldsymbol{\theta}$ $\boldsymbol{\Theta}$ be the vector of unknown parameters of the GFB model, the pseudo-code of the GA-MLE introduced as a parameter estimation approach for reliability of load-sharing parallel systems is summarized in Figure 2.

Initial step:

$t \leftarrow 0$
Initialize the population P_t of random individuals $\boldsymbol{\theta} = (\theta_1, ..., \theta_N) \in \boldsymbol{\Theta} \subset \mathbb{R}^N$, N number of unknown parameters
Evaluate the fitness function on the population P_t (i.e. compute the profile log-likelihood for $\boldsymbol{\theta} \in P_t$)

Iterative step:

While stop condition is not met (i.e. maximum profile log-likelihood is not achieved) do
    $t \leftarrow t + 1$
    Select individuals from P_{t-1} for P_t
    Crossover individuals and save the siblings in P_t
    Mutate individuals in P_t
    Evaluate the fitness function on the population P_t
End while

Output:

Select the best fitness (i.e. $\boldsymbol{\theta}_t = (\theta_{1t}, ..., \theta_{Nt}) \in P_t$ with maximum profile log-likelihood)

Figure 2: Pseudocode GA-MLE

3.1. Known baseline parameters

In this case, known baseline parameters, θ_B and θ_B^*, are assumed, and therefore the baseline distributions are completely determined. Accordingly, by maximizing the log-likelihood function with respect to the unknown PFR parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_1^*, \theta_2^*)$ the MLEs $(\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$ can be explicitly calculated by

$$\begin{aligned}
\widehat{\theta}_1 = \frac{-n_2}{\sum_{i\in I}\log R_B(\min(y_{1i}, y_{2i}))},\ \widehat{\theta}_2 = \frac{-n_1}{\sum_{i\in I}\log R_B(\min(y_{1i}, y_{2i}))} \\
\widehat{\theta}_1^* = \frac{n_1}{\sum_{i\in I_1}\log\frac{R_B^*(y_{2i})}{R_B^*(y_{1i})}},\ \widehat{\theta}_2^* = \frac{n_2}{\sum_{i\in I_2}\log\frac{R_B^*(y_{1i})}{R_B^*(y_{2i})}}.
\end{aligned} \tag{3.8}$$

3.2. One unknown baseline parameter

To deal with the aforesaid second case, θ_B is supposed to be known whereas θ_B^* unknown. In this case, the two first PFR estimates, $\widehat{\theta}_1$ and $\widehat{\theta}_2$, are fully determined by (3.8). However, θ_1^* and θ_2^* depend on θ_B^*, and thence it should be considered the partial derivative of (3.6) with respect to such a parameter θ_B^*. Anyway, there are no explicit form solutions to θ_1^*, θ_2^* and θ_B^* in general. Therefore, by substituting $\theta_1^*(\theta_B^*)$ and $\theta_2^*(\theta_B^*)$ from (3.7) into (3.6), the resulting profile log-likelihood function for θ_B^* is given by

$$\begin{aligned}
p(\theta_B^*) = n_1\log\theta_1^*(\theta_B^*) + n_2\log\theta_2^*(\theta_B^*) + n_2\log\widehat{\theta}_1 + n_1\log\widehat{\theta}_2 - 2n \\
+ \sum_{i\in I}\log\frac{f_B^*(\max(y_{1i},y_{2i}),\theta_B^*)}{R_B^*(\max(y_{1i},y_{2i}),\theta_B^*)} + \sum_{i\in I}\log\frac{f_B(\min(y_{1i},y_{2i}))}{R_B(\min(y_{1i},y_{2i}))}.
\end{aligned} \tag{3.9}$$

In order to maximize the profile log-likelihood function $p(\theta_B^*)$ with respect to θ_B^*, the estimation of θ_B^*, $\widehat{\theta}_B^*$, can be obtained by applying a GA, and then the MLEs of θ_1^* and θ_2^* are calculated by $\theta_1^*(\widehat{\theta}_B^*)$ and $\theta_2^*(\widehat{\theta}_B^*)$, respectively.

Analogously, when θ_B^* is known and θ_B is unknown, the estimations can be obtained by using the profile log-likelihood function for θ_B which can be derived similarly to (3.9).

3.3. Unknown baseline parameters

Let us finally assume that θ_B and θ_B^* are unknown. In general, there are no closed form expressions to obtain estimates for the baseline parameters. However, the MLEs can be approximated by the profile log-likelihood function for unknown θ_B and θ_B^*, which is derived by substituting (3.7) into (3.6), and it can be expressed as

$$p(\theta_B, \theta_B^*) = n_1 \log \theta_1^*(\theta_B^*) + n_2 \log \theta_2^*(\theta_B^*) + n_2 \log \theta_1(\theta_B) + n_1 \log \theta_2(\theta_B) - 2n$$
$$+ \sum_{i \in I} \log \frac{f_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)}{R_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)} + \sum_{i \in I} \log \frac{f_B(\min(y_{1i}, y_{2i}), \theta_B)}{R_B(\min(y_{1i}, y_{2i}), \theta_B)}. \tag{3.10}$$

Similarly to the above subsection, this profile function can be maximized with respect to both baseline parameters by using GA-MLE in order to obtain the MLEs of the baseline and PFR parameters.

3.4. Statistical tests

The existence of a load-sharing effect in a two-component parallel redundant system may be also analyzed by testing whether the baseline lifetimes are affected in a GFB model. For such a purpose, the deviance test statistic d_n based on the likelihood ratio test can be used, which approximately follows a chi-square distribution with m degree of freedom, where m is the difference in the parameter number between the two GFB models tested [29]. The statistic d_n is also suitable from the GA-MLE solution $\widehat{\boldsymbol{\theta}}$ based on two-stage estimation procedure of the profile log-likelihood, because $\widehat{\boldsymbol{\theta}}$ has asymptotic properties similar to that of the MLE [23].

Therefore, assuming the same distribution family for both baselines, the deviance test statistic d_n could be applied to test whether the baseline distribution is not affected by the first failure by comparing its baseline parameters, i.e., $H_0 : \theta_B = \theta_B^*$ *v.s.* $H_1 : \theta_B = \theta_B^*$, which can be derived from the knowledge of the baseline parameters discussed in the previous subsections. Analogously, the significance of a baseline distribution for the GFB model with respect to other particular one (e.g., exponential versus Weibull) can be also testing by d_n.

Nevertheless, it is noteworthy that the components of the GFB model might differ in their PFR parameters even though they come from the same baseline in the same stage. In that case, it is necessary to consider additional assumptions on the PFR parameters in order to test whether they are identical. For instance, the hypothesis that the components are identically distributed before the first failure, i.e., $H_0 : \theta_1 = \theta_2$ *v.s.* $H_1 : \theta_1 = \theta_2$, can be also analyzed by applying the deviance test statistic. To do that, under the restriction of the null hypothesis, the log-likelihood function given by (3.6) can be rewritten as:

$$
\begin{aligned}
l(\overrightarrow{\overline{\theta}}, \theta_B, \theta_B^*) = {} & n_1 \log \theta_1^* + \theta_1^* \sum_{i \in I_1} \log \frac{R_B^*(y_{1i}, \theta_B^*)}{R_B^*(y_{2i}, \theta_B^*)} + n_2 \log \theta_2^* + \theta_2^* \sum_{i \in I_2} \log \frac{R_B^*(y_{2i}, \theta_B^*)}{R_B^*(y_{1i}, \theta_B^*)} \\
& + n \log \theta_1 + 2\theta_1 \sum_{i \in I} \log R_B(\min(y_{1i}, y_{2i}), \theta_B) \\
& + \sum_{i \in I} \log \frac{f_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)}{R_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)} + \sum_{i \in I} \log \frac{f_B(\min(y_{1i}, y_{2i}), \theta_B)}{R_B(\min(y_{1i}, y_{2i}), \theta_B)}. \qquad (3.11)
\end{aligned}
$$

where $\overrightarrow{\overline{\theta}} = (\theta_1, \theta_1^*, \theta_2^*)$ are the PFR parameters, since $\theta_1 = \theta_2$. By differentiating in (3.11) with respect to $\overrightarrow{\overline{\theta}}$, we have the following relationships among all the parameters:

$$
\begin{aligned}
\theta_1(\theta_B) &= \frac{-n/2}{\sum_{i \in I} \log R_B(\min(y_{1i}, y_{2i}), \theta_B)}, \\
\theta_1^*(\theta_B^*) &= \frac{n_1}{\sum_{i \in I_1} \log \frac{R_B^*(y_{2i}, \theta_B^*)}{R_B^*(y_{1i}, \theta_B^*)}}, \ \theta_2^*(\theta_B^*) = \frac{n_2}{\sum_{i \in I_2} \log \frac{R_B^*(y_{1i}, \theta_B^*)}{R_B^*(y_{2i}, \theta_B^*)}}. \qquad (3.12)
\end{aligned}
$$

From now on, the maximum log-likelihood estimation under the null hypothesis can be approximated through the GA-MLE procedure in the profile log-likelihood function, by substituting (3.12) into (3.11). Thus, it is needed to derive such profile functions according to the knowledge of the baseline parameter in a similar way to the former developments in Subsections 3.1-3.3 without the restriction of the null hypothesis, i.e., θ_B and θ_B^* are known, one of them is unknown, or both are unknown.

For instance, when baseline parameters are unknown, the profile log-likelihood function under the null hypothesis can be written as:

$$
\begin{aligned}
p(\theta_B, \theta_B^*) = {} & n_1 \log n_1 + n_2 \log n_2 + n \log(n/2) - 2n - n \log \left(\sum_{i \in I} \log R_B(\min(y_{1i}, y_{2i}), \theta_B) \right) \\
& - n_1 \log \left(\sum_{i \in I_1} \log \frac{R_B^*(y_{2i}, \theta_B^*)}{R_B^*(y_{1i}, \theta_B^*)} \right) - n_2 \log \left(\sum_{i \in I_2} \log \frac{R_B^*(y_{1i}, \theta_B^*)}{R_B^*(y_{2i}, \theta_B^*)} \right) \\
& + \sum_{i \in I} \log \frac{f_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)}{R_B^*(\max(y_{1i}, y_{2i}), \theta_B^*)} + \sum_{i \in I} \log \frac{f_B(\min(y_{1i}, y_{2i}), \theta_B)}{R_B(\min(y_{1i}, y_{2i}), \theta_B)}. \qquad (3.13)
\end{aligned}
$$

which can be maximized with respect to both baseline parameters by using GA-MLE in order to obtain the MLEs of the baseline and PFR parameters under stated terms. Therefore, the deviance test statistic can be computed by using GA-MLE procedure to maximize (3.10) and (3.13, which will approximately follow a chi-square distribution with 1 degree of freedom under the null hypothesis.

Analogously, the testing hypotheses that the components of the GFB model are identically distributed after the first failure ($H_0 : \theta_1^* = \theta_2^*$) or they have identical distribution before and after the first failure but changing the baseline distribution ($H_0 : (\theta_1 = \theta_2, \theta_1^* = \theta_2^*)$), can be derived similarly to the above case.

4. Simulation and real data analysis

In this section, the practical application of the proposed GFB model has been conducted by using different baseline component distributions. Specifically, fourteen load-sharing system models have been considered by combining exponential, Weibull, LFR and Gamma baseline lifetimes in the GFB class. Such particular GFB models are those whose components have the following baseline lifetimes (R_B, R_B^*): (Exp, Exp) given in (S.1), $(W(\theta_B), W(\theta_B))$ in (S.2), $(LFR(\theta_B), LFR(\theta_B))$ in (S.4), $(Exp, W(\theta_B))$ in (S.5), $(Exp, LFR(\theta_B))$ in (S.6), $(W(\theta_B), Exp)$, $(W(\theta_B), W(\theta_B^*))$ in (S.7), $(W(\theta_B), LFR(\theta_B^*))$, $(LFR(\theta_B), W(\theta_B^*))$, $(LFR(\theta_B), LFR(\theta_B^*))$, $(G(\theta_B), G(\theta_B))$, $(G(\theta_B), W(\theta_B^*))$, $(W(\theta_B), G(\theta_B^*))$, and $(G(\theta_B), G(\theta_B^*))$, where $\theta_B = (\theta_{B_1}, \theta_{B_2})$ and $\theta_B^* = (\theta_{B_1}^*, \theta_{B_2}^*)$ for the Gamma distribution.

Two of them are considered in the simulation study for the performance evaluation of the GA-MLE approach from lifetimes of the randomly generated samples, and the fourteen settings of the GFB model have been fitted to three real data sets of the reliability engineering field, revealing the load-sharing effect.

In the genetic formulation setup, the size of initial population is fixed at 50 individuals after testing different sizes with non-significant improvements in the algorithm performance. For each load-sharing model derived from the GFB class, the parameter initialization for genetic search is done according to their ranges so as to generate the initial population of possible solutions. Specifically, broad search domains are considered for the parameters, intending to give more flexibility to the optimization process which constitutes itself a clear advantage over the other estimation methods, as mentioned before. Regarding the selection operator, the fitness proportional selection with the fitness linear scaling is used to avoid the stalled evolutions and premature convergence which may be caused by assuming that the probability of individual selection for reproduction is proportional to its fitness. The fitness linear scaling prevents the dominance in the initial generation, helps the examination of the whole search space and strengthen the selection pressure to converge at the exact optimum

[14]. For effective genetic operation, blend crossover and uniform random mutation operators are applied to the solution population at each iteration. In particular, crossover and mutation probabilities for the GA-MLE are experimentally set to 0.8 and 0.01, respectively, because these values performed the best than others. The blend crossover operator applied chooses a value for each offspring from two parent solutions using the uniform distribution inside the range and tunable parameter 0.5 according to the recommendations made by [10]. The elitist strategy is considered in which the top 5% individuals survive at each iteration. Furthermore, the termination criterion adopted herein has been 50 generations without improving the best fitness value or the maximum number of 2000 generations is reached.

For both simulated and real data, the GA-MLE is proposed as parameter estimation method to find the MLEs of the unknown parameters of the GFB model following the procedure shown in Figure 2. The MLEs of the PFR parameters are calculated for each GFB model analyzed. The R package 'GA' [27] has been used to apply the GA procedure along with the R package 'maxLik' [13], for estimating all the load-sharing system models.

4.1. Simulation study

In order to analyze the GA-MLE approach performance in the parameter estimations by the previously listed fourteen load-sharing models, we assume two different baseline Weibull distributions to generate 1000 two-dimensional random samples (y_{1i}, y_{2i}), $i = 1, ..., n$, of sizes n=25, 50, 100, 200 and 500, from the GFB model given in (S.7), for the following values of parameters $\vec{\vec{\theta}} = (\theta_1 = 1, \theta_2 = 0.75, \theta_1^* = 0.5, \theta_2^* = 2)$, $\theta_B = 0.5$, and $\theta_B^* = 2$. The simulation procedure from a $GFB(R_B, R_B^*, \vec{\vec{\theta}}, \theta_B, \theta_B^*)$ model is based on a modified version of the algorithm suggested by [1, 2], see Figure 3.

From the simulated samples, we evaluate the standard error (SE) and mean squared error (MSE) of the estimations.

Table 1 summarizes the parameters estimated along with the empirical SE and MSE for the GFB model with baseline lifetimes (R_B, R_B^*) $(W(\theta_B), W(\theta_B^*))$ given by (S.7), and Figure 4 depicts the curves of the empirical SE and MSE of the parameters for the fitted GFB models. Analogously, Appendix A.4 presents the experimental results obtained for baseline lifetimes (R_B, R_B^*) $(W(\theta_B), G(\theta_{B_1}^*, \theta_{B_2}^*))$ given by (S.8).

From Table 1 and Figure 4a, we observe that as the sample size increases, standard errors

Initial step:

Identify the baseline distributions (e.g. $R_B \sim W(\theta_B)$, and $R_B^* \sim W(\theta_B^*)$)
Identify the PFR parameters $(\theta_1, \theta_2, \theta_1^*, \theta_2^*)$
Generate three independent random samples, u_i, v_i and w_i, $i = 1, 2, ..., n$, of size n from the uniform distribution, $\mathcal{U}(0, 1)$

Structure of load-sharing system step:

For each i, u_i defines the first failure:

If $u_i \leq \dfrac{\theta_1}{\theta_1 + \theta_2}$, then $y_{1i} = y_{1:2,i} = R_B^{-1}\left((1 - v_i)^{1/(\theta_1+\theta_2)}\right)$ and $y_{2i} = (R_B^*)^{-1}\left((1 - w_i)^{1/\theta_2^*} R_B^*(y_{1i})\right)$

If $u_i > \frac{\theta_1}{\theta_1+\theta_2}$, then $y_{2i} = y_{1:2,i} = R_B^{-1}\left((1 - w_i)^{1/(\theta_1+\theta_2)}\right)$ and $y_{1i} = (R_B^*)^{-1}\left((1 - v_i)^{1/\theta_1^*} R_B^*(y_{2i})\right)$

End for

Output:

Two-dimensional sample (y_{1i}, y_{2i}), $i = 1, 2, ..., n$, of size n from the $GFB(R_B, R_B^*, \overrightarrow{\theta} = (\theta_1, \theta_2, \theta_1^*, \theta_2^*), \theta_B, \theta_B^*)$

Figure 3: Simulation procedure of two-dimensional samples from the GFB class

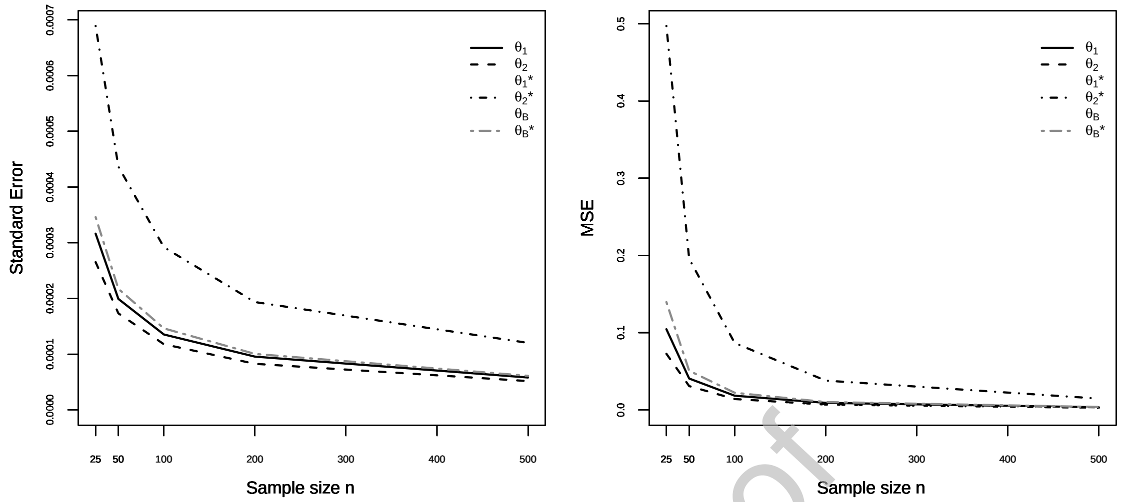
		θ_1	θ_2	θ_1^*	θ_2^*	θ_B	θ_B^*
$n = 25$	Estimates	1.06745848	0.79886184	0.52263955	2.15115350	0.52404280	2.14153692
	St.Error	0.00031634	0.00026525	0.00022217	0.00068904	0.00008638	0.00034599
	MSE	0.10452455	0.07267485	0.04982437	0.49715451	0.00803268	0.13961923
$n = 50$	Estimates	1.03147675	0.77885149	0.51521476	2.07172413	0.51431082	2.05935779
	St.Error	0.00019920	0.00017330	0.00014128	0.00043671	0.00006010	0.00021731
	MSE	0.04063364	0.03083454	0.02017269	0.19566713	0.00381342	0.05070059
$n = 100$	Estimates	1.00890580	0.76434516	0.50927975	2.03899785	0.50539974	2.02895728
	St.Error	0.00013535	0.00011788	0.00010273	0.00029190	0.00003981	0.00014628
	MSE	0.01837988	0.01408698	0.01062950	0.08664137	0.00161203	0.02221417
$n = 200$	Estimates	1.00218613	0.75690113	0.50791328	2.02300836	0.50301511	2.00879176
	St.Error	0.00009565	0.00008273	0.00006900	0.00019353	0.00002774	0.00010046
	MSE	0.00914433	0.00688512	0.00481854	0.03794669	0.00077799	0.01015936
$n = 500$	Estimates	1.00346712	0.75200567	0.50141798	2.01439874	0.50145609	2.00520707
	St.Error	0.00005815	0.00005173	0.00004336	0.00012028	0.00001713	0.00006115
	MSE	0.00339033	0.00267745	0.00188051	0.01465985	0.00029543	0.00376314

Table 1: Estimates, SEs and MSEs of the parameters based on 1000 simulated samples for sizes n =25, 50, 100, 200 and 500, when the true load-sharing system model is GFB with different baseline Weibull distributions, initial $W(0.5)$ switching to $W(2)$ after the first failure, and PFR parameters $\vec{\theta} = (1, 0.75, 0.5, 2)$.

of the parameter estimates decrease, leading to increase the precision in the estimation of the GFB model. Analogous trends are observed for MSE in estimating the baseline parameters and all the PFR parameters (Figure 4b). Such a tendency is an evidence of the appropriateness of the proposed simulation procedure to generate two-dimensional data from a GFB class with known baseline distributions and PFR parameters. The results and plots empirically display as MSE's quickly converge to zero when the sample size increases.

4.2. Real data analysis

The three real data sets used for illustrative purposes are available in the reliability engineering literature. In order to assess the GFB models adequacy to the data, the performance



(a) Plots of trend in standard error.

(b) Plots of trend in mean squared error.

Figure 4: Plots of the parameter estimates $\hat{\theta}$'s for varying the sample size n =25, 50, 100, 200 and 500.

evaluation is reported by using the Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) as appropriate goodness-of-fit criteria for the selection of the fitted models with different number of parameters. Furthermore, we also analyze whether there exists a load-sharing tendency in the data by testing whether the component lifetimes are affected in a system under the GFB model, using the deviance test statistic.

For comparison purposes, it is noteworthy that the results of the FBE model are also displayed in all the tables, as a basic model with underlying exponential lifetimes, and whose MLEs are explicitly determined by (3.8). Moreover, Appendix A.5 presents a summary table of the following real data analysis along with the experimental results by the Broyden-Fletcher-Goldfarb-Shanno algorithm for maximizing the log-likelihood function, disclosing that the GA-MLE procedure performed as well or slightly better than such a quasi-Newton method for solving this constrained nonlinear optimization problem.

4.2.1. Failure times of a two-motor system

The first reliability data set used to illustrate the performance of the GFB family was studied by [31]. Such a set consists of the chronological failure times of both components in 18 parallel redundant systems with two identical motors. As long as both motors are functioning, the total load is equally shared between them, but after the first motor failure the entire load is shifted to the working motor. Based on the underlying LFR distribution, [31] concluded the existence of such a load-sharing effect for this motor data set.

The fourteen load-sharing system models derived from the GFB class are fitted to the motor lifetimes. The estimates of the unknown parameters are reported in Table 2, along with the log-likelihood value, and AIC and BIC criteria only for those GFB models which achieved the best fit in terms of the log-likelihood or AIC values for this two-motor system data set.

	θ_1	θ_2	θ_1^*	θ_2^*	θ_B	θ_B^*	$\log L$	AIC	BIC
E	0.003110420	0.002488336	0.02179837	0.01893939			-211.97	431.94	435.50
WE	0.000000007	0.000000006	0.02179837	0.01893939	3.42512		-199.83	409.65	414.10
2W	0.000000007	0.000000006	0.00000002	0.00000001	3.42512	1.82975	-199.13	410.27	415.61

Table 2: Summary of fitted models for the two-motor systems. E model represents a GFB with identical baseline exponential, WE model denotes a GFB with initial baseline Weibull switching to exponential after the first failure, and 2W model is a GFB with different baseline Weibull distributions.

According to the information criterion values displayed in Table 2, both WE model and 2W model are more appropriate than FBE model for the motor data. In addition, the values of the deviance test statistics are significant for both FBE model *vs.* WE model (d_n = 24.29204, $d.f.$ = 1, p = 8.27806e^{-7}) and FBE model *vs.* 2W model (d_n = 25.67368, $d.f.$ = 2, p = 2.660916e^{-6}), i.e., $\{H_0 : \theta_B = 1$ *vs.* $H_1 : \theta_B = 1\}$ and $\{H_0 : \theta_B = \theta_B^* = 1$ *vs.* $H_1 : \theta_B = \theta_B^*\}$, respectively, which strongly suggest that FBE model is rejected to model the reliability of this two-motor system. However, the value of the deviance test statistics for WE model ($H_0 : \theta_B = \theta_B^*$) *vs.* 2W model ($H_1 : \theta_B = \theta_B^*$) is not significant ($d_n$ = 1.381644, $d.f.$ = 1, p = 0.2398213).

Thereby, we can conclude that the extra load placed on the surviving motor after the first failure affects its residual lifetime since its underlying lifetime itself changes, as it can be seen in Figure 5, supporting the usefulness of the proposed GFB class to model such a load-sharing effect in this two-motor system. From Figure 5, the difference between two reliability curves can be ignored before the first failure, and after that, Motor A (C_1) will be lightly more affected by the overload.

4.2.2. Failure and repair times of a nuclear power plant reactor

The second real data set represents 30 paired observations of failure time and repair time for a nuclear reactor obtained from an operational performance system for nuclear

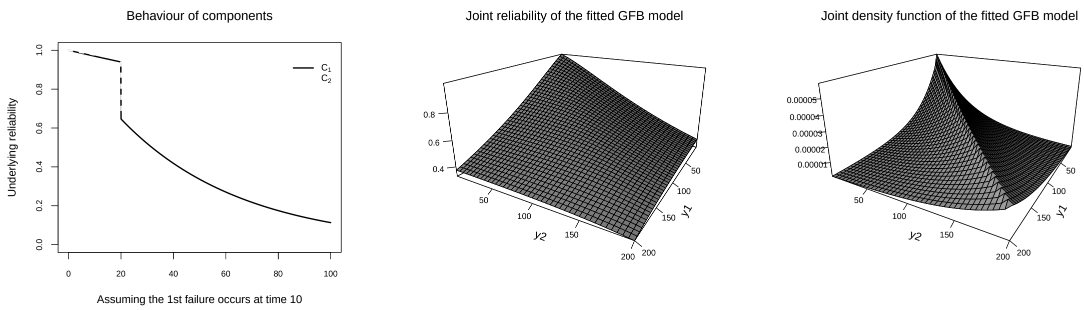


Figure 5: Change in the behaviour of the underlying reliability of the surviving motor, and bivariate reliability and density functions of the GFB model fitted with identical baseline exponential.

power plants [26]. After concluding that these two variables were significantly dependent by Kendall's τ method, they were employed as factors for warranty policy to conduct a warranty cost analysis [26]. Namely, a cost model where such two factors were stochastically correlated by the FBE model was investigated in [26]. From the fourteen GFB models considered, Table 3 displays the estimates of the parameters for those which are better in fitting the data set because they reported the highest log-likelihood or the smallest AIC values. Thus, both WG model and 2G migth be more suitable than FBE model for the nuclear power plant data.

	θ_1	θ_2	θ_1^*	θ_2^*	θ_{B_1}	θ_{B_2}	$\theta_{B_1}^*$	$\theta_{B_2}^*$	$\log L$	AIC	BIC
E	0.1180312	0.236063	0.0755684	0.3429747					-172.60	353.19	358.80
WG	0.1210064	0.242013	0.0215423	0.0976824	0.9827450		0.285233	7.16439	-170.95	355.89	365.70
2G	0.7937402	1.587481	0.5309025	2.3751090	0.9591324	7.16439	0.285224	7.16439	-170.93	357.86	369.07

Table 3: Summary of fitted models for the nuclear power plant data. WG model denotes a GFB with initial baseline Weibull switching to Gamma after the first failure, and 2G model is a GFB with different baseline Gamma distributions.

Nevetheless, the values of the deviance test statistics are significant neither for FBE model *vs.* WG model nor FBE model *vs.* 2G model, i.e., $\{H_0 : \theta_B = \theta_{B_1}^* = \theta_{B_2}^* = 1$ *vs.* H_1 : not all $\theta's$ are $1\}$ and $\{H_0 : \theta_{B_1} = \theta_{B_2} = \theta_{B_1}^* = \theta_{B_2}^* = 1$ *vs.* H_1 : not all $\theta's$ are $1\}$. Concretely, these experimental values achievable from Table 3 are $\{d_n = 3.301513$, $d.f. = 3$, $p = 0.3474322\}$ and $\{d_n = 3.333169$, $d.f. = 4$, $p = 0.5036942\}$, respectively.

Therefore, there are no significant differences with respect to the GFB model with identical baseline exponential, which was studied in [26]. Thus, the GFB distribution is also useful for modeling the stochastic dependence shown between both times, reflecting the overload of the warranty services due to the failure times of the reactor. From Figure 6, we can see that the repair times (C_2) will be strongly affected by the overload due to the failure times of the nuclear reactor.

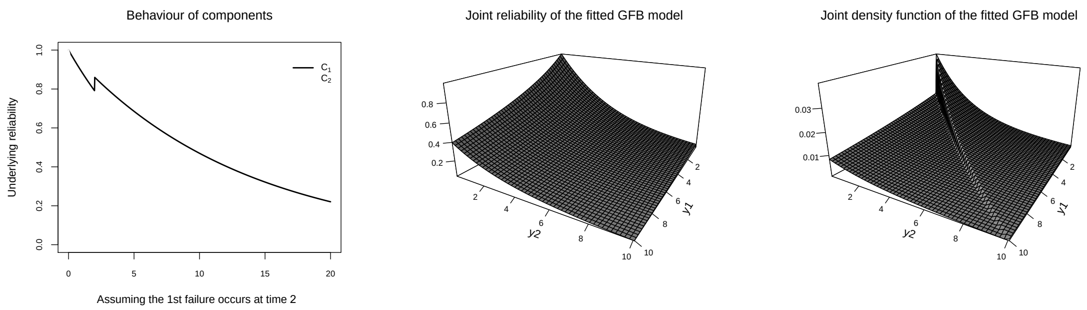


Figure 6: Change in the behaviour of the underlying reliability of the failure and repair times, and bivariate reliability and density functions of the GFB model fitted with identical baseline exponential.

4.2.3. Failure data of caterpillar tractors

Finally, we considered the real data set on paired first failure times of the transmission and the transmission pump on 15 DQG-66A caterpillar tractors [4]. These components might be positively dependent as it was indicated in [4], which also were analyzed by a FBE model by [24]. For that reason, its failure dependent structure could be modeled by the proposed GFB class. Thus, the fourteen particular GFB models are applied to describe the behavior of such paired first failure times. Table 4 presents the parameter estimations for those models achieving the best values of the log-likelihood, AIC and BIC criteria.

	θ_1	θ_2	θ_1^*	θ_2^*	θ_B	θ_B^*	$\log L$	AIC	BIC
E	0.002633311	0.003949967	0.00970481	0.01450151			-182.56	373.12	375.95
W	0.000000479	0.000000719	0.00000049	0.00000078	2.656673		-173.80	357.60	361.14
2W	0.000000241	0.000000362	0.00000023	0.00000037	2.787074	2.3375886	-173.71	359.43	363.68

Table 4: Summary of fitted models for the failure data of caterpillar tractors. W model represents a GFB with identical baseline Weibull distributions.

Thereby, both W and 2W models are more appropriate than the FBE distribution for modeling the failure dependent structure on caterpillar tractors. Indeed, the values of the deviance test statistics are significant both for FBE model *vs.* W model and FBE model *vs.* 2W model, i.e., $\{H_0 : \theta_B = 1 \ vs. \ H_1 : \theta_B = 1\}$ and $\{H_0 : \theta_B = \theta_B^* \ vs. \ H_1 : \theta_B = \theta_B^*\}$. These experimental values calculable from Table 4 are $\{d_n = 17.52357,\ d.f. = 1,\ p = 2.837679e^{-5}\}$ and $\{d_n = 17.6931,\ d.f. = 2,\ p = 1.438772e^{-4}\}$, respectively, which strongly suggest that FBE model is rejected to model the failure time of transmission and transmission pump.

Nevertheless, there is no significant evidence against the GFB model with the same baseline Weibull, i.e., for testing W model ($H_0 : \theta_B = \theta_B^*$) $vs.$ 2W model ($H_1 : \theta_B = \theta_B^*$), being its experimental results $\{d_n = 0.1695299,\ d.f. = 1,\ p = 0.6805299\}$. In Figure 7, it can be seen the change in the behaviour of the surviving component after the first failure. In initial period, the difference between two reliability curves of the transmission and transmission pump can be ignored but after the first failure the transmission pump (C_2) will be more affected by the load-sharing effect than the transmission (C_1) of this caterpillar tractors. Furthermore, it is worthy to notice that both components of this load-sharing system are not equally reliable, since $P(Y_1 > Y_2)$ is approximately 0.6 for the three GFB models of Table 4.

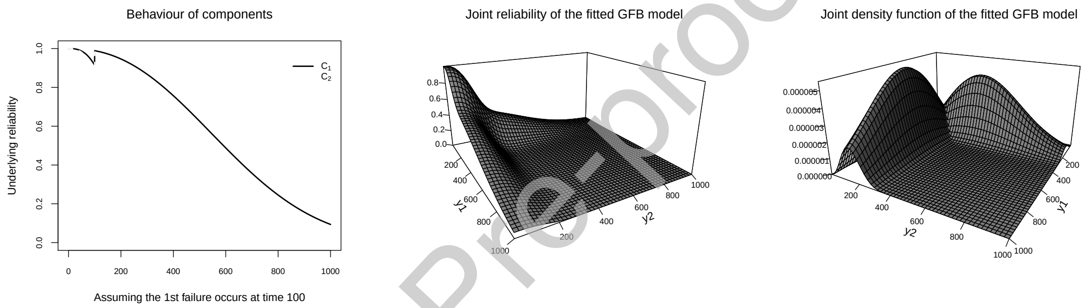


Figure 7: Change in the behaviour of the underlying reliability of the transmission and transmission pump, and bivariate reliability and pdf of the GFB model fitted with two different baseline Weibull distributions.

5. Concluding remarks

In this study, a generalized Freund bivariate (GFB) family has been proposed to model the stochastic dependence caused by the load-sharing effect in a two-component system. While both components are working, their lifetimes are assumed independent and not necessarily identically distributed with reliability functions from the same PFR class. However, upon the first failure, the GFB model permits changes in the PFR parameter and the base lifetime of the surviving component to reflect the undergone alterations in its failure pattern.

Load-sharing system models supplied by [1, 12, 21, 30] may be derived from the GFB model. New bivariate models can be generated from the GFB class by combining different baseline distributions, such as it has been illustrated by four particular examples available in the Supplementarty Material.

The GA-MLE introduced as a parameter estimation approach of the load-sharing system reliability can be applied under any baseline lifetimes of the components, which constitutes a clear edge over other estimation algorithms developed. In particular, the GA-MLE may well be appropriate to approximate the unknown parameters when the MLEs do not have closed form expressions, such as it happens when at least one of the baseline parameters of the GFB model is unknown. In practice, this estimating procedure performed as well or slightly better than a quasi-Newton method for solving this constrained nonlinear optimization problem.

A procedure to generate synthetic two-dimensional data from a GFB model is described, which allows one to simulate samples of a two-component load-sharing parallel system with predetermined baseline distributions and PFR parameters. Simulation outcomes revealed that the performance both of the GFB model and the GA-MLE estimating procedure are quite satisfactory.

Experimental results showed that the proposed GFB model has been well suited to the three real data sets of the reliability engineering field, which support the existence of such a load-sharing effect in such applications. Moreover, the most of comparisons reflected that the extra load on the surviving component affected its underlying lifetime distribution in addition to its PFR parameter, which evidence the usefulness of the GFB family to model two-component load-sharing systems.

However, further study is needed to develop inferential issues of the GFB model for censored data as well as to incorporate the information about covariates shared by two components. Another point of interest would be to extend the GFB model to multi-component load-sharing systems and its application to step-stress models, although its extensibility does not seem easy due to the number of parameters and the complexity of the likelihood function.

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Appendix A. Appendix

A.1. Joint pdf

The joint pdf of the two-dimensional lifetime (Y_1, Y_2) is defined by

$$f(y_1, y_2) = \lim_{(h_1,h_2)\to(0,0)} \frac{P(y_1 \leq Y_1 < y_1 + h_1, y_2 \leq Y_2 < y_2 + h_2)}{h_1 h_2}. \tag{A.1}$$

Thus, when (Y_1, Y_2) follows a GFB model, by using the law of total probability, we have for $y_1 > y_2$ that the numerator of (A.1) can be rewritten as:

$$\begin{aligned} P(y_1 \leq T_1^* < y_1 + h_1, y_2 \leq T_2 < y_2 + h_2) &= P(y_1 \leq T_1^* < y_1 + h_1, y_2 \leq T_{1:2} < y_2 + h_2, T_1 > T_2) \\ &= A \times B \end{aligned} \tag{A.2}$$

where $A = P(y_2 \leq T_{1:2} \leq y_2 + h_2, T_1 > T_2)$, $B = P(y_1 \leq T_1^* \leq y_1 + h_1 \mid y_2 \leq T_{1:2} \leq y_2 + h_2, T_1 > T_2)$ and $T_{1:2} = \min\{T_1, T_2\}$.

Now let us determine each of these probabilities denoted as A and B, respectively. Firstly, the probability A can be expressed as

$$\begin{aligned} A = P(y_2 \leq T_2 \leq y_2 + h_2, T_1 > T_2) &= \int_{t_2=y_2}^{y_2+h_2} \int_{t_1=t_2}^{\infty} f_{T_1}(t_1) f_{T_2}(t_2) dt_1 dt_2 \\ &= \frac{\theta_2}{\theta_1 + \theta_2} \left((R_B(y_2))^{\theta_1+\theta_2} - (R_B(y_2 + h_2))^{\theta_1+\theta_2} \right). \end{aligned} \tag{A.3}$$

Secondly, the probability B can be rewritten as

$$\begin{aligned} B &= P(y_1 \leq T_1^* \leq y_1 + h_1 \mid y_2 \leq T_2 \leq y_2 + h_2, T_1^* > y_2) \\ &= \int_{t_1=y_1}^{y_1+h_1} f_{T_1^* \mid T_1^* > y_2, y_2 \leq T_2 \leq y_2+h_2}(t_1) dt_1 = \frac{(R_B^*(y_1))^{\theta_1^*} - (R_B^*(y_1 + h_1))^{\theta_1^*}}{(R_B^*(y_2))^{\theta_1^*}}. \end{aligned} \tag{A.4}$$

Therefore, by plugging (A.3) and (A.4) into (A.2), we have that

$$f(y_1, y_2) = \lim_{(h_1,h_2)\to(0,0)} \frac{A \times B}{h_1 h_2} = \frac{\theta_2}{(\theta_1 + \theta_2)(R_B^*(y_2))^{\theta_1^*}} \frac{d}{dy_2} \left((R_B(y_2))^{\theta_1+\theta_2} \right) \frac{d}{dy_1} \left((R_B^*(y_1))^{\theta_1^*} \right)$$

and hence, the joint pdf of the GFB model for $y_1 > y_2$ is given by (2.1).

Analogously, the expression of the joint pdf can be obtained for $y_2 > y_1$.

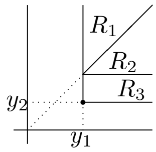
A.2. Joint reliability

The joint reliability function of (Y_1, Y_2), for $y_1 > y_2$, can be expressed as

$$R(y_1, y_2) = P(Y_1 > y_1, Y_2 > y_2) = \int_{R_1 \cup R_2 \cup R_3} f(t_1, t_2) dt_1 dt_2 = A_1 + A_2 + A_3 \tag{A.5}$$

where

$$R_1 = \{(t_1, t_2) : y_1 \le t_1 < \infty, t_1 < t_2 < \infty\}$$
$$R_2 = \{(t_1, t_2) : y_1 \le t_2 < \infty, t_2 < t_1 < \infty\}$$
$$R_3 = \{(t_1, t_2) : y_1 \le t_1 < \infty, y_2 < t_2 < y_1\}$$



and $A_i = P((Y_1, Y_2) \quad R_i)$, $i = 1, 2, 3$. In this way,

$$\begin{aligned} A_i &= \int_{t_i = y_1}^{\infty} \int_{t_j = t_i}^{\infty} \theta_i \theta_j^* f_B(t_i) f_B^*(t_j) \frac{(R_B(t_i))^{\theta_1 + \theta_2 - 1} (R_B^*(t_j))^{\theta_j^* - 1}}{(R_B^*(t_i))^{\theta_j^*}} dt_i dt_j \\ &= \frac{\theta_i}{\theta_1 + \theta_2} (R_B(y_1))^{\theta_1 + \theta_2}, \end{aligned}$$

for $i = j \quad \{1, 2\}$, and

$$\begin{aligned} A_3 &= \int_{t_1 = y_1}^{\infty} \int_{t_2 = y_2}^{y_1} \theta_1^* \theta_2 f_B^*(t_1) f_B(t_2) \frac{(R_B^*(t_1))^{\theta_1^* - 1} (R_B(t_2))^{\theta_1 + \theta_2 - 1}}{(R_B^*(t_2))^{\theta_1^*}} dt_1 dt_2 \\ &= (R_B^*(y_1))^{\theta_1^*} \int_{t_2 = y_2}^{y_1} \theta_2 f_B(t_2) \frac{(R_B(t_2))^{\theta_1 + \theta_2 - 1}}{(R_B^*(t_2))^{\theta_1^*}} dt_2. \end{aligned}$$

By plugging A_1, A_2 and A_3 into (A.5), it is obtained the joint reliability function of the GFB model given by (2.2) for $y_1 > y_2$.

Analogously, the expression of the joint reliability function can be obtained for $y_2 > y_1$.

A.3. First failure probability

Taking into account that $P(Y_{1:2} > y) = \lim_{h \to 0} R(y + h, y)$, and the above probability A_3 converges to 0 as $y_1 = y + h \to y_2 = y$, (2.3) is obtained from (A.5).

Finally, (2.4) is also obvious from the above A_2, since $P(Y_1 > Y_2) = \lim_{y_1 \to 0} P((Y_1, Y_2) \quad R_2)$.

A.4. Simulation results from different baseline distributions

Table A.5 displays the parameter estimates and some empirical measures from the simulation of the GFB model with baseline lifetimes $(R_B, R_B^*) \quad (W(\theta_B), G(\theta_{B_1}^*, \theta_{B_2}^*))$ given by (S.8), and in Figure A.8 it can be seen the empirical curves of the SE and MSE of the parameters, respectively. These results are based on 1000 simulated samples for sizes n =25, 50, 100, 200 and 500, which were generated from a GFB model with baseline Weibull, $R_B \quad W(0.5)$, switching to Gamma distribution, $R_B^* \quad G(2, 1)$, after the first failure, and PFR parameters $\vec{\theta} = (1, 0.75, 0.5, 2)$.

	θ_1	θ_2	θ_1^*	θ_2^*	θ_B	$\theta_{B_1}^*$	$\theta_{B_2}^*$
$n = 25$							
Estimates	1.06067105	0.79381635	0.7892713	1.77941909	0.51487137	3.98125896	4.40551394
St.Error	0.00031505	0.00026458	0.0005457	0.00116679	0.00011070	0.00329859	0.00458082
MSE	0.10284085	0.07185368	0.3811723	1.40868232	0.01246248	14.79522856	32.56047263
$n = 50$							
Estimates	1.02574866	0.77437093	0.68565022	1.5633098	0.50598615	3.45563581	4.05985245
St.Error	0.00020104	0.00017314	0.00043763	0.0009161	0.00008588	0.00291258	0.00440812
MSE	0.04103900	0.03054037	0.22579442	1.0290949	0.00740414	10.59352188	28.77480447
$n = 100$							
Estimates	1.00606320	0.76200902	0.62685574	1.4372523	0.49846533	3.07422560	3.28520440
St.Error	0.00013735	0.00011776	0.00035722	0.0007218	0.00005503	0.00239965	0.00399073
MSE	0.01888287	0.01399660	0.14357158	0.8371583	0.00302743	6.90650442	21.13214863
$n = 200$							
Estimates	0.99965538	0.75506522	0.54334429	1.27233871	0.49759339	2.50198580	2.82324852
St.Error	0.00009510	0.00008303	0.00023016	0.00048824	0.00003876	0.00153309	0.00346754
MSE	0.00903587	0.00691299	0.05480141	0.76762803	0.00150689	2.59999418	15.33606588
$n = 500$							
Estimates	1.00202960	0.75093341	0.49339840	1.17587629	0.49778216	2.1753039	1.95674867
St.Error	0.00005846	0.00005194	0.00012094	0.00026047	0.00002490	0.0007514	0.00240532
MSE	0.00341788	0.00269542	0.01465653	0.74695571	0.00062409	0.5947682	6.69513598

Table A.5: Estimates, SEs and MSEs of the parameters based on 1000 simulated samples for sizes n =25, 50, 100, 200 and 500, according to a load-sharing system from a GFB model with baseline Weibull switching to Gamma distribution after the first failure, and PFR parameters $\vec{\theta} = (1, 0.75, 0.5, 2)$.

A.5. Additional experimental results

The three reliability data analysis presented in Section 4.2 are gathered in Table A.6, along with the experimental results obtained by the Broyden-Fletcher-Goldfarb-Shanno algorithm for maximizing the log-likelihood function. It can be seen that the GA-MLE procedure performed as well or slightly better than such a quasi-Newton method for solving this constrained nonlinear optimization problem.

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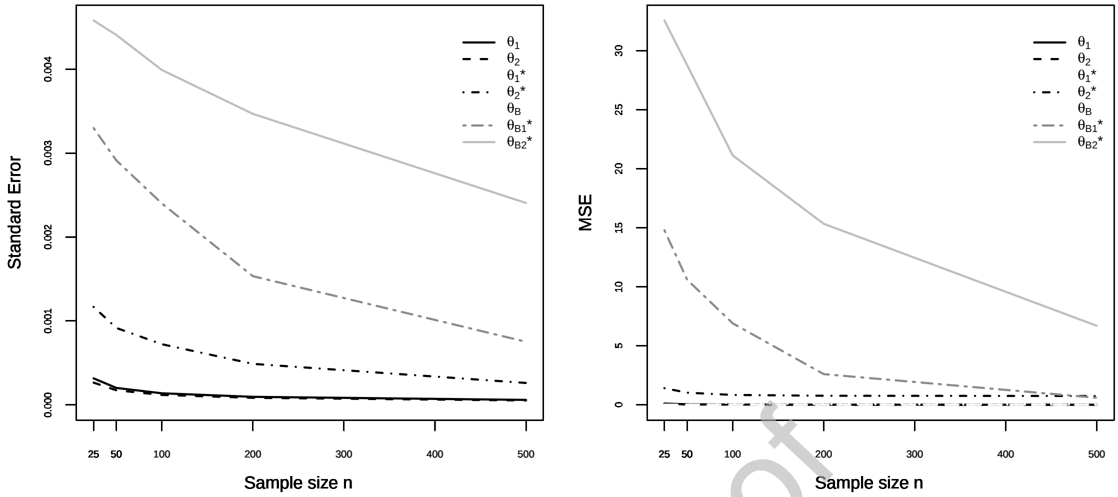

(a) Plots of trend in standard error. (b) Plots of trend in mean squared error.

Figure A.8: Plots of the parameter estimates $\hat{\theta}$'s for varying the sample size n =25, 50, 100, 200 and 500.

	θ_1	θ_2	θ_1^*	θ_2^*	θ_{B_1}	θ_{B_2}	$\theta_{B_1}^*$	$\theta_{B_2}^*$	$\log L$	AIC	BIC
Motors A & B											
E											
MLE	0.003110994	0.002488436	0.02179863	0.01894344					-211.97	431.94	435.50
WE											
BFGS	0.000001604	0.000001258	0.02182493	0.01885467	2.428203				-201.14	412.28	416.73
GA-MLE	0.000000007	0.000000006	0.02179837	0.01893939	3.42512				-199.83	409.65	414.10
2W											
BFGS	0.000001135	0.000000939	0.02001849	0.01638703	2.490958		1.010757		-200.98	413.96	419.29
GA-MLE	0.000000007	0.000000006	0.00000002	0.00000001	3.42512		1.82975		-199.13	410.27	415.61
Nuclear reactor											
E											
MLE	0.1180323	0.236063	0.0755688	0.3429770					-172.59	353.19	358.79
WG											
BFGS	0.1209730	0.241942	0.1700659	0.5940733	0.9829146		0.001249	2.84839	-170.88	355.77	365.58
GA-MLE	0.1210064	0.242013	0.0215423	0.0976824	0.9827450		0.285233	7.16439	-170.95	355.89	365.70
2G											
BFGS	1.2877660	2.575524	0.1705124	0.5951608	0.9660629	11.63064	0.001231	2.85734	-170.87	357.75	368.96
GA-MLE	0.7937402	1.587481	0.5309025	2.3751090	0.9591324	7.16439	0.285224	7.16439	-170.93	357.86	369.07
Caterpillar tractors											
E											
MLE	0.002633354	0.003950023	0.00970491	0.01450175					-182.56	373.12	375.95
W											
BFGS	0.000011906	0.000017846	0.00001796	0.00002743	2.043840				-174.67	359.34	362.88
GA-MLE	0.000000479	0.000000719	0.00000049	0.00000078	2.656673				-173.80	357.60	361.14
2W											
BFGS	0.000000197	0.000002899	0.00009325	0.00014309	2.391356		1.7619374		-174.17	360.35	364.59
GA-MLE	0.000000241	0.000000362	0.00000023	0.00000037	2.787074		2.3375886		-173.71	359.43	363.68

Table A.6: Summary of fitted GFB models for the three real data. MLE corresponds to estimations determined by (3.8) for the standard Freund bivariate model. GA-MLE rows are the results summarized in Tables 2, 3 and 4, and BFGS rows correspond to the results obtained by applying the Broyden-Fletcher-Goldfarb-Shanno algorithm for maximizing the log-likelihood function.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.